International Journal of Theoretical Physics (2023) 62: (2023) 62:248
https://doi.org/10.1007/s10773-023-05503-9

RESEARCH

Mass-Charge Metric in Curved Spacetime

Espen Gaarder Haug[1] · Gianfranco Spavieri[2]

Received: 3 August 2023 / Accepted: 6 November 2023 / Published online: 21 November 2023

Abstract
In the context of gravitational objects with spherical symmetry, we derive a solution to Einstein's field equations using two methods leading to the same result. The first is based on a stress-energy tensor that takes into account both the electric field energy of the charge and the gravitational field energy of the mass. The second is based on the mass-energy equivalence and has more general validity. We show that the metric falls within the Weyl class of metrics, representing a static and axisymmetric solution to Einstein's field equations. The metric, which has a form similar to that of Reisser-Nordström, is used for predictions in strong fields and possibly shows better agreement with observation in high z quasars.

Keywords Mass-charge · General relativity · Strong gravitational fields · Black holes · Reisser-Nordström metric · Weyl class metrics

1 Introduction

Analogies between Newton's law of gravitation and Coulomb's law of electrostatics have been pointed out and discussed by many authors. Already in 1893 Oliver Heaviside [1] considered an approach to gravitation based on an analogy with electromagnetism. We are not interested in extending such an analogy because we discuss our metric in the context of General Relativity. However, there is an aspect of the mentioned analogy that we believe important to highlight and that refers to the energy of the fields with spherical symmetry produced by a localized source, mass, electric charge, or of other physical nature.

In the Schwarzschild metric, the constant in the solution to Einstein's field equations is determined in the weak approximation using Newton's law. In Newton's theory the energy of gravitation fields produced by the mass M is unrelated to the energy of the fields produced by the source M, which is the same M appearing in Newton's law. Similarly, in Coulomb's law the energy of the electric fields is unrelated to the charge Q representing the source of the fields.

✉ Espen Gaarder Haug
espenhaug@mac.com

Gianfranco Spavieri
gspavieri@gmail.com

1 Norwegian University of Life Sciences, Christian Magnus Falsensvei 18, 1430 Ås, Norway

2 Centro de Física Fundamental, Universidad de Los Andes, La Hechicera, 5101 Mérida, Venezuela

In General Relativity spacetime curvature arises as an effect due to gravitational mass. Our claim is that, because of the equivalence mass-energy, the energy produced by fields generated by sources of different nature, is expected to contribute to the effect of spacetime bending regardless of the different origins of the fields.

Nevertheless, in the case of a mass M possessing an electric charge Q, in the corresponding Reissner-Nordström metric, the charge Q is present in the term $Q^2 r^{-2}$, different from the r^{-1} dependence of the Schwarzschild term with the mass M given by Mr^{-1}. Our aim is to show, using two different approaches leading to the same result, that a metric containing the additional term $M^2 r^{-2}$ is a solution to Einstein's field equations. For the first approach, discussed in Sections 2 and 3, we introduce the appropriate stress-energy tensor related to the field energy of the sources. For the second approach of Section 4 we exploit directly the mass-energy equivalence and obtain the result indicating that the terms $Q^2 r^{-2}$ and $M^2 r^{-2}$ represent in general the contribution to the Schwarzschild term Mr^{-1} due to the energy of the fields whose source is Q, M. Similarly, for any other source P of different nature but with the same radial dependence, the contributing term is $P^2 r^{-2}$, as for example in the case of the magnetic monopole. In Section 5 we show that our new metric falls within the Weyl class of metrics, which are a class of static and axisymmetric solutions to Einstein's field equations. In Sections 6 and 7 we shortly discuss possible physical implications of our metric.

2 Newton's and Coulomb's Laws and the Stress-Energy Tensor

In pre-relativistic classical mechanics and Maxwell's electromagnetism we have the relations,

$$\frac{F_{New}}{m} = G\frac{M}{r^2} \qquad \frac{F_{Coul}}{q} = k\frac{Q}{r^2}, \tag{1}$$

where the first equation represents Newton's law of gravity with G the constant of gravity, m and M the interacting masses and F_{New} the interacting force, while the second equation represents Coulomb's law with $k = (4\pi\varepsilon_0)^{-1}$ the corresponding constant, q and Q the interacting electric charges and F_{Coul} the interacting force.

If a point charge q is brought from $\mathbf{x} = \infty$ to point $\mathbf{x}$ in a region where there is an electric scalar potential $\Phi(\mathbf{x})$, the work done on the charge (representing its potential energy) is $W = q\Phi(\mathbf{x})$. For a system composed of point charges of total charge Q localized on the mass M, the electric potential energy and corresponding energy density are,

$$W_Q = \frac{1}{8\pi}\int |\nabla\Phi|^2\, d^3x = \frac{1}{8\pi}\int |\mathbf{E}|^2\, d^3x \qquad w_Q = \frac{1}{8\pi}|\mathbf{E}|^2\,, \tag{2}$$

where for spherical symmetry and at the distance r from the localized mass M, the electric field is $E_r = Q/r^2$. Considering that analogous results are obtained for the gravitational potential energy, the total energy density of the system $M + Q$ can be conveniently expressed as,

$$w_{Q_M} = \frac{1}{8\pi}\left|\mathbf{E}_{Q_M}\right|^2 = \frac{1}{8\pi}\frac{Q_M^2}{r^4} = \frac{1}{8\pi}G\frac{M^2}{r^4} + \frac{1}{8\pi}k\frac{Q^2}{r^4}, \tag{3}$$

where

$$Q_M^2 = GM^2 + kQ^2 \qquad Q_M = \sqrt{GM^2 + kQ^2}. \tag{4}$$

The contribution to the total energy densities in (3) that can never disappear is the gravitational, because present even for a perfectly neutral body. Since electromagnetic is much stronger than gravitational interaction, the term with GM^2 in (3) and (4) is generally neglected in most physical situations where electric fields are present. However, in the context of black holes, astrophysical objects with a large mass M are generally almost neutral and in proximity of the mass M where r is small the energy density due to M may exceed the one due to Q.

3 The Stress-Energy Tensor $T^{\mu\nu}$

With the total energy density given by (3) we can form a stress-energy tensor $T^{\mu\nu}$ that takes into account the field energy density due to M and Q. Stress-energy tensors that take into account different forms of energy are common in physics in several contexts outside or within general relativity. As an example, we may consider the tensor $T^{\mu\nu} = \theta^{\mu\nu} + S^{\mu\nu} + \delta_0 U^\mu U^\nu$, used in the literature in the context of electrodynamics of charged bodies with internal stresses (see [2, 3, 5]). This tensor is complemented by the continuity equation, $\partial_\mu T^{\mu\nu} = 0$ while the term $\theta^{\mu\nu}$ represents the usual electromagnetic tensor, $S^{\mu\nu}$ the stress tensor, and δ_0 the proper density of the proper mass.

In our case, we do not consider the stresses inside the mass M as we are interested in the solution of Einstein's field equations outside M. For our purpose we can simply exploit the fact that, for spherical symmetry, the form of the gravitational and electric fields is the same, as shown in (3). Hence, we can formally adopt the stress-energy tensor $T^{\mu\nu} = \theta^{\mu\nu}(Q_M)$ where Q_M is given by (4). Thus, our approach reflects the one by Reisser-Nordström [6, 7] and we use here the formalism adopted by Zee [8] using geometric units $c = G = k = 1$. However, we express our results using also the SI units to show explicitly c, G, k and make evident the contribution due to the field energy.

The formal properties of $T_{\mu\nu}$ and the related field tensor $F_{\mu\gamma}$ are unchanged, being,

$$T_{\mu\nu} = \frac{1}{\mu_0}(F_{\mu\gamma} F_{\nu\alpha} g^{\alpha\gamma} - \frac{1}{4} g_{\mu\gamma} F_{\gamma\delta} F^{\gamma\delta}).$$

The corresponding Einstein's [9] field equations are,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + 8\pi T^{\mu\nu} = 0.$$

In the context of black holes and for spherically symmetric spacetime the metric has the form,

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \tag{5}$$

In the solution of the Reisser-Nordström form, the relevant component of $T^{\mu\nu}$ is $T_{22} = (1/2\mu_0) r^2 F_{01} F^{01}$. As we have only a radial component of the electric field, we may write: $E_r = F_{01} = F_{10} = Q_M / r^2$. Then, $T_{22} = (1/2\mu_0) r^2 (Q_M^2 / r^4)$ and we find that $A(t,r)$ and $B(t,r)$ obey the equations ([8]),

$$A(r,t) = f(t) + \frac{C(t)}{r} + \frac{Q_M^2}{r^2}$$
$$AB = f(t).$$

So far, the solution for (5) is expressed in terms of mere algebraic quantities and does not represent physical reality. In order to be used to reflect physical reality and provide a falsifiable theory that can be tested experimentally, we need to relate the algebraic quantities to observable physical quantities.

Showing explicitly c, G, k, the quantities f and C can be determined by taking the limit to the Schwarzschild metric $Q_M \to 0$ and,

$$g_{00} = 1 - \frac{2GM}{c^2 r} \quad f(t) = 1 \quad C(t) = -\frac{2GM}{c^2} \equiv |r_s|\,.$$

Finally, by means of (4) we find the *horizon function* $\Delta = A(r)$,

$$\Delta = A(r) = 1 - \frac{2GM}{c^2 r} + \frac{G^2 M^2}{c^4 r^2} + \frac{GkQ^2}{c^4 r^2} \tag{6}$$

and the resulting metric is,

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2. \tag{7}$$

4 Deriving the Same Metric by Exploiting the Mass-Energy Equivalence Relation for the Fields Energy

Integrating the energy density given by (3) starting from the observer position r up to $r = \infty$ we find,

$$E_{Q_M} = E_M + E_Q = 4\pi \int_r^{r=\infty} (\frac{1}{8\pi}\frac{Q_M^2}{r'^4}) r'^2 dr = \frac{GM^2}{2r} + \frac{kQ^2}{2r} = (\delta M_M)c^2 + (\delta M_Q)c^2, \tag{8}$$

which is the energy stored in the gravitational fields of the sources M and Q outside r. Then, as a consequence of mass-energy equivalence, the stored field energy in the space from r to $r = \infty$ must be equivalent to the mass $(\delta M_M(r)) + (\delta M_Q(r))$. Hence, the field energy from r to $r = \infty$ contributes to spacetime bending, modifying the original spacetime curvature corresponding to the Schwarzschild mass M in the absence of δM_M and δM_Q.

With the contributions due to $\delta M_M + \delta M_Q$ the modified Schwarzschild horizon function can be expressed as,

$$\begin{aligned}\Delta &= 1 - \frac{2GM}{c^2 r} + \frac{G^2M^2}{c^4 r^2} + \frac{GkQ^2}{c^4 r^2} = 1 - \frac{2GM}{c^2 r} + \frac{2G[(\delta M_M) + (\delta M_Q)]}{c^2 r} \\ &= 1 - \frac{2GM}{c^2 r}\left(1 - \frac{(\delta M_M) + (\delta M_Q)}{M}\right).\end{aligned} \tag{9}$$

We can see from (9) that, if the term with (δM_Q) corresponding to the Reisser-Nordström metric is a solution to Einstein's field equations, any other term with (δM_M) with the same r dependence is a solution also.

If in (9) we neglect the contribution (δM_M) due to the gravitational field, the contribution (δM_Q) due to the electromagnetic field appears as a perturbation of the mass M. Similarly, the gravitational field contribution (δM_M) can be considered as a perturbation of the Newtonian mass M, which is derived for the Schwarzschild metric by means of the weak approximation for $r \Rightarrow \infty$ in flat spacetime limit.

Then, for a neutral mass with $Q = 0$,

$$\Delta = 1 - \frac{2GM}{c^2 r} + \frac{G^2M^2}{c^4 r^2} = 1 - \frac{2GM}{c^2 r}(1 - \frac{GM}{2c^2 r})$$

and with our new metric the modified Schwarzschild radius is now $r_{HS} = r_s/2$.

Obviously, r_{HS} is modified if the mass M contains sources of electromagnetic origin or other nature whose field energy contributes to the mass M as indicated in (9). Taking into account results (8) and (9), in geometric units the generalization of the Schwarzschild horizon function for sources of spherically symmetric fields proportional to r^{-2} assumes the form,

$$\begin{aligned}\Delta &= 1 - \frac{2M}{r} + \frac{M^2}{r^2} + \frac{Q^2}{r^2} + \frac{P^2}{r^2}\dots \\ &= 1 - \frac{2M}{r}\left(1 - \frac{(\delta M_M)}{M} - \frac{(\delta M_Q)}{M} - \frac{(\delta M_P)}{M}\dots\right)\end{aligned} \tag{10}$$

where in (10) we have added the contribution due to magnetic monopoles [10, 11] with magnetic charge P. The derivation leading to the generalization (10), obtained exploiting the mass-energy equivalence for point sources of different nature, corroborates the derivation for the term (δM_M) of point 1- based on the analogy gravitation-electromagnetism that makes use of the stress-energy tensor.

Our metric indicates that, on account of the mass-energy equivalence, the energy of gravitational fields plays the same role as that of electromagnetic fields. Therefore, the additional mass term in our metric has to be considered in a wider context than that of simple analogy between Newton's gravity law and Coulomb's law. In fact, we are going beyond Newton's gravity theory and use here General Relativity Theory, regardless of a possible analogy between features of gravitation and those of electromagnetism.

In conclusion, the added mass term in the exact solution to Einstein's field equation given by our metric (10) is not surprising because it is based essentially on the equivalence between mass and field energy, regardless of whether the field source is the charge Q or the mass M. Hence, our metric could potentially provide new insights in this area and, as shown below, in the interpretation of observational data in the context of strong fields.

5 The New HS Metric Inside the Weyl's Class of Metrics

We show now that our new metric does fall within the Weyl [12] class of metrics, which is a class of static and axisymmetric solutions to Einstein's field equations. The well known Schwarzschild, Reissner-Nordström, Kerr [13], and Kerr-Newman [14, 15] metrics are all within the Weyl class of solutions to Einstein's field equations, which has the generic form given by,

$$ds^2 = -e^{2\psi(\rho,z)}dt^2 + e^{2\gamma(\rho,z)-2\psi(\rho,z)}(d\rho^2 + dz^2) + e^{-2\psi(\rho,z)}\rho^2 d\phi^2 \tag{11}$$

where $\psi(\rho, z)$ and $\gamma(\rho, z)$ are the two metric functions.

Here, we are following Gautreau, Hoffman and Armenti [16] that do a similar derivation for the Reissner-Nordström metric (see also Stephani et al. [17]). The Weyl potentials are given by (in geometric units),

$$\psi = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{(L + M)^2}, \quad \gamma = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{l_+ l_-}. \tag{12}$$

With the units adopted, the M term in (12) seems to have the same formal weight as the Q term. However, they do not, if the units are shown explicitly. In fact, we can see from the result below that the resulting r dependence of M (r^{-1}) is different from that of Q (r^{-2}). Moreover, the fact that Reissner-Nordström, Weyl, and other physicists use mass and charge in their equations, does not imply mixing gravitation with electromagnetism. We clearly show in Section 4 that the term containing Q in the Reissner-Nordström metric represents the mass-energy equivalent of the energy stored in the electromagnetic fields produced by the source Q, as indicated by expression (8) (energy equivalent), which is the source of spacetime bending.

On account of relation (8), the expression of the component T^{22} of the energy-stress tensor of Section 2, and the position $Q_M^2 = GM^2 + kQ^2$ of (4), we set the Weyl potentials generating the new mass-charge metric as given by,

$$\psi_{HS} = \frac{1}{2}\ln\frac{L^2-(M^2-Q_M^2)}{(L+M)^2}, \quad \gamma_{HS} = \frac{1}{2}\ln\frac{L^2-(M^2-Q_M^2)}{l_+l_-}. \tag{13}$$

Obviously, the potentials given in (13) lead to an exact solution to Einstein's field equations if the potentials given in (12) does. For convenience of the reader, we indicate schematically below the procedure leading to the solution.

Let,

$$L = \frac{1}{2}(l_+l_-)$$

and

$$l_+ = \sqrt{\rho^2 + \left(z + \sqrt{M^2-Q_M^2}\right)^2}, \quad l_- = \sqrt{\rho^2 + \left(z - \sqrt{M^2-Q_M^2}\right)^2}$$

where ψ_{HS} and γ_{HS} inserted in the Weyl's metric become

$$ds^2 = -\frac{L^2-(M^2-Q_M^2)}{(L+M)^2}dt^2 + \frac{(L+M)^2}{l_+l_-}(d\rho^2+dz^2) + \frac{(L+M)^2}{L^2-(M^2-Q_M^2)}\rho^2 d\phi^2. \tag{14}$$

Next, we apply the following transformation (similar to the one used when working with the Reissner-Nordström metric):

$$L + M = r, \qquad z = (r-M)\cos\theta \tag{15}$$

with,

$$l_+ - l_- = 2\sqrt{M^2-Q_M^2}\cos\theta \tag{16}$$

and,

$$\rho = \sqrt{r^2-2Mr+Q_M^2}\sin\theta, \quad l_+l_= (r-M)^2-(M^2-Q_M^2)\cos^2\theta. \tag{17}$$

Then, with $Q_M^2 = M^2+Q^2+P^2$ as in (10), we obtain, as expected, the new mass-charge metric,

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}+\frac{Q_M^2}{r^2}\right)c^2dt^2 + \left(1-\frac{2M}{r}+\frac{Q_M^2}{r^2}\right)^{-1}dr^2 + r^2\Omega^2 \\ &= -\left(1-\frac{2M}{r}+\frac{M^2}{r^2}+\frac{Q^2}{r^2}+\frac{P^2}{r^2}\right)c^2dt^2 \\ &\quad + \left(1-\frac{2M}{r}+\frac{M^2}{r^2}+\frac{Q^2}{r^2}+\frac{P^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \end{aligned} \tag{18}$$

where $d\Omega^2 = (d\theta^2 + \sin^2\theta d\varphi^2)$.

With $Q_M^2 = Q^2$, expression (18) represents the Weyl solution for the nonextremal Reissner-Nordström metric ($M > |Q|$). Lastly, with $Q_M^2 = M^2 + Q^2 + P^2$ our metric can be considered to fall within the Weyl class of metrics.

6 Horizons of the Metric

The metric gives the following horizons (geometric units) :

$$r_{\pm} = M \pm \sqrt{M^2 - Q_M^2} = M \pm \sqrt{M^2 - M^2 - Q^2 - P^2} = M \pm \sqrt{-Q^2 - P^2}. \quad (19)$$

In the case of a neutral charge ($Q = 0$) and also no magnetic charge ($P = 0$), the metric only yields one horizon, which is equal to half the Schwarzschild radius, i.e., $r = M$, instead of $r = r_s = 2M$. For comparison, the horizons in the Reissner-Nordström metric are given by $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, as shown in [18–20] for example. The well-known extremal solution of the Reissner-Nordström metric (see [8]) occurs when $Q = M$. In this case, it predicts the same outcome as our metric when the charge is neutral in our metric.

7 Predictions of the Metric in Strong Fields

The predictions of a metric, exact solution to Einstein's field equations, need to be verified experimentally. The predictions of the Reissner-Nordström metric cannot be easily verified because in general massive astrophysical objects are neutral. However, the HS metric (10) predicts that the event horizon for black holes without charge is only half of what is predicted by the Schwarzschild metric. This would result in a stronger predicted gravitational redshift for light emitted just outside the horizon (from the accretion disk). The accretion disk will be able to approach closer to the center of the black hole compared to the Schwarzschild metric. This could potentially help us better understand how accretion disks, in some cases, are assumed to extend below the limit given in analysis based on the Schwarzschild metric, where the extremal Kerr and Kerr-Newman solutions, so far, have been one of the few alternatives where this is physically possible (see [21]). Additionally, our new mass-charge metric predicts a black hole horizon that is half of the Schwarzschild radius. Our new metric could potentially, in addition, help explain the absence of observed velocity time dilation in high-z quasars (see [22, 23]), as more of the redshift could be simply due to gravitational redshift, which is possible if the black hole with the same mass as predicted from observations is inside a smaller volume, and thereby the photons can be more gravitationally redshifted than before.

Although a detailed study would naturally be necessary before drawing any conclusions concerning if this would help explaining this observed phenomena, it is not unlikely that the modifications to the Schwarzschild metric introduced by the HS metric could be observed for physical effects originating near the black body singularity where the term with (δM_M) becomes relevant.

Conclusion

We have derived a new exact solution to Einstein's field equations, where the stress-energy tensor takes into account both the electric field energy of the charge and the gravitational field energy of the mass. We have shown that the energy fields generated by sources of different natures, all contribute to the effect of spacetime bending. This has multiple implications; for instance, the event horizon is predicted to be half of that predicted by the Schwarzschild metric for a non-charged black hole. The new term could potentially have implications for both predicted red-shifts from light emitted from close to the horizon as well as for accretion disk theory, as the accretion disk now could be closer to the black hole. Ideally, one should find testable hypotheses that can determine if our new metric is a more accurate model for certain phenomena related to black holes than other well-known metrics. This is left up to future research to find out.

Acknowledgements We would like to thank an anonymous referee for useful comments on this paper.

Author Contributions Both authors contributed to the research writing and review of the manuscript.

Funding Open access funding provided by Norwegian University of Life Sciences.

Availability of data and materials The study contains no new data.

Declarations

Competing interests The authors have no competing interests to report.

References

1. Heaviside, O.: Gravitational and electromagnetic analogy. Electrician **1**, 455 (1893)
2. Møller, C.: The Theory of Relativity. Clarendon Press, Oxford (1972)
3. Spavieri, G.: A non-standard expression for the force and torque on a magnetic dipole. Nuovo Cim. B **109**, 45 (1994). https://doi.org/10.1007/BF02723729
4. Aharonov, Y., Pearle, P., Vaidman, L.: Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag". Phys. Rev. A **37**, 4052 (1988). https://doi.org/10.1103/PhysRevA.37.4052
5. Spavieri, G., Gillies, G.T., Rodriguez, M., Perez, M.: Effective interaction force between an electric charge and a magnetic dipole and locality (or nonlocality) in quantum effects of the Aharonov-Bohm type. Chin. Phys. Lett. **38**, 034101 (2021). https://doi.org/10.1088/0256-307X/38/3/034101
6. Reissner, H.: Über die eigengravitation des elektrischen feldes nach der Einsteinschen theorie. Ann. Phys. **355**, 106 (1916). https://doi.org/10.1002/andp.19163550905
7. Nordström, G.: On the energy of the gravitation field in Einstein's theory. K. Ned. Akad. Wet., Proc. **20**, 1238 (1918)
8. Zee, A.: Einstein gravity in a nutshell. Princeton University Press, Princeton, NJ (2013)
9. Einstein, A.: Näherungsweise integration der feldgleichungen der gravitation. In: Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin (1916)
10. Lee, K., Nair, V.P., Weinberg, E.J.: Black holes in magnetic monopoles. Phys. Rev. D **45**, 2751 (1991). https://doi.org/10.1103/PhysRevD.45.2751
11. Mammadov, G.: Reissner-Nordström metric. Res. Gate (2009). https://www.researchgate.net/publication/336304500_Reissner-Nordstrom_metric
12. Weyl, H.: Zur gravitationstheorie. Ann. Phys. **54**, 117 (1917). https://doi.org/10.1002/andp.19173591804
13. Kerr, R.P.: Gravitational field of a spinning mass as an example of algebraically special metrics. Phys. Rev. Lett. **11**, 237 (1963). https://doi.org/10.1103/PhysRevLett.11.237
14. Newman, E.T., Janis, A.I.: Note on the Kerr spinning-particle metric. J. Math. Phys. **6**, 915 (1965). https://doi.org/10.1063/1.1704350
15. Newman, E., Couch, E., Chinnapared, K., Exton, A., Prakash, A., Torrence, R.: Metric of a rotating, charged mass. J. Math. Phys. **6**, 918 (1965). https://doi.org/10.1063/1.1704351

16. Gautreau, R., Hoffman, R.B., Armenti, A.: Static multiparticle systems in general relativity. lI Nuovo Cim. B **7**, 71 (1972). https://doi.org/10.1007/BF02827038
17. Stephani, H., Kramer, D., Maccallum, M., Herlt, E.: Exact Solutions of Einstein's Field Equations, 2nd edn. Cambridge Monographs of Mathematical Physics, Cambridge (2009)
18. Hobson, M.P., Efstathiou, G., Lasenby, A.N.: General Relativity. An Introduction for Physicists. Cambridge University Press, Cambridge (2014)
19. Bambi, C.: General Relativity. Springer, Berlin (2018)
20. Aretakis, S.: Dynamics of Extremal Black Holes. Springer Verlag, Berlin (2018)
21. Bromley, B.C., Chen, K., Miller, W.A.: Line emission from an accretion disk around a rotating black hole: Toward a measurement of frame dragging. Astrophys. J. **457**, 1037 (1997). https://doi.org/10.1086/303505
22. Hawkins, M.R.S.: Time dilation and quasar variability. Astrophys. J. **553**, 97–100 (2001). https://doi.org/10.1086/320683
23. Hawkins, M.R.S.: On time dilation in quasar light curves. Mon. Not. R. Astron. Soc. **405**, 1940–1946 (2010). https://doi.org/10.1111/j.1365-2966.2010.16581.x