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RESEARCH

Exact Solutions and Conservation Laws of A (2+1)-dimensional Combined Potential Kadomtsev-Petviashvili-B-type Kadomtsev-Petviashvili Equation

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Abstract
This article investigates a sixth order integrable nonlinear partial differential equation model that fulfills the Hirota N-soliton. Space and time-dependent shift, rotation and space-dependent shift, time and space translations, and time and space dilations Lie point symmetries are presented methodically. Under a specific point symmetries, the Lie point symmetries lead to group invariant solutions. The significance of conservation laws of the underlying equation are shown. The results are quite accurate in recreating complex waves and the dynamics of their interactions.

Keywords An integrable nonlinear partial differential equation model of the sixth-order · Lie symmetry analysis · Conservation laws

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1 Introduction

The dynamics of solitary waves are governed by the well-known Korteweg-de Vries equation

$$\Theta_t + 6\Theta\Theta_x + \Theta_{xxx} = 0. \tag{1.1}$$

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Long wavelength, low amplitude shallow water waves are what gave rise to its creation. Due to its unlimited number of conservation laws, multiple-soliton solutions, bi-Hamiltonian structures, Lax pair, and other physical features, it is significant from the perspective of integrable systems. The Korteweg-de Vries equation (1.1) in two dimensions is known as the Kadomtsev-Petviashvili equation

$$\left(\Theta_t + 6\Theta\Theta_x + \Theta_{xxx}\right)_x + \Theta_{yy} = 0. \tag{1.2}$$

It (1.2) serves as a model for shallow long waves with dispersion in the x and mild y directions and is totally integrable and produces multiple-soliton solutions.

An integrable nonlinear partial differential equation model of the sixth-order known as a combined potential Kadomtsev-Petviashvili-B-type Kadomtsev-Petviashvili equation in two dimensions

$$\begin{aligned}&\varepsilon_1\left(15\Phi_x^3 + 15\Phi_x\Phi_{xxx} + \Phi_{xxxxx}\right)_x + \varepsilon_2\left(6\Phi_x\Phi_{xx} + \Phi_{xxxx}\right) + \varepsilon_3\left(\Phi_{xxxy} + 3\left(\Phi_x\Phi_y\right)_x\right)\\&+ \varepsilon_4\Phi_{xx} + \varepsilon_5\Phi_{xt} + \varepsilon_6\Phi_{yy} = 0\end{aligned} \tag{1.3}$$

was established in [1] and it was shown that (1.3) satisfies the Hirota N-soliton condition which implied that (1.3) possesses an N-soliton solution. Selecting certain values of the parametes in (1.3) leads to a series of equations of interest as follows. Setting $\varepsilon_1 = \varepsilon_5 = 1, \varepsilon_3 = \varepsilon_6 = -5, \varepsilon_2 = \varepsilon_4 = 0$, with the aid of $\Theta = \Phi_x$, (1.3) is reduced to the $(2+1)$-dimensional with B-type Kadomtsev-Petviashvili equation

$$\Theta_t + \Theta_{xxxxx} - 5(\Theta_{xxy} + \partial_x^{-1}\Theta_{yy}) + 15(\Theta_x\Theta_{xx} + \Theta\Theta_{xxx} - \Theta\Theta_y - \Theta_x\partial_x^{-1}\Theta_y) + 45\Theta^2\Theta_x = 0 \tag{1.4}$$

which depicts weakly dispersive waves propagating in quasi-medium and fluid mechanics, or the electrostatic wave potential in plasmas. The dependent variable Φ denotes the wave amplitude function, x, y space coordinates, t denotes the time coordinate and ∂_x^{-1} denotes the integral with respect to x. By setting $\varepsilon_5 = 36, \varepsilon_2 = \varepsilon_4 = 0$ courtesy of $\Theta = \Phi_x$, Eq. (1) is reduced to a generalized $(2+1)$-dimensional Caudrey-Dodd-Gibbon- Kotera-Sawada equation

$$\varepsilon_1(15\Theta^3 + 15\Theta\Theta_{xx} + \Theta_{xxxx})_x + \varepsilon_3(\Theta_{xxy} + 3(\Theta\partial_x^{-1}\Theta_y)_x) + 36\Theta_t + \varepsilon_6\partial_x^{-1}\Theta_{yy} = 0 \tag{1.5}$$

that models a series of nonlinear dispersion physical phenomena. Setting $\varepsilon_1 = \varepsilon_5 = 1, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6 = 0$ through the transformation $\Theta = \Phi_x$, (1.3) degenerates to a fifth-order Sawada-Kotera equation

$$\Theta_t + \Theta_{xxxxx} + 15(\Theta\Theta_{xxx} + \Theta_x\Theta_{xx}) + 45\Theta^2\Theta_x = 0 \tag{1.6}$$

that accounts for the long waves in the shallow water under the gravity and in a one-dimensional nonlinear lattice. Finally in the case where $\varepsilon_2 = \varepsilon_5 = 1, \varepsilon_6 = 1, \varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 0$ and $\varepsilon_2 = \varepsilon_5 = 1, \varepsilon_1 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6 = 0$ through potential $\Theta = \Phi_x$ (1.3) disintegrates into (1.2) and (1.1) respectively.

Differential equations [2–6] such as (1.3) are used to explain a wide range of physical processes and as such plethora of methods have been devised to extract closed-form solutions [7–11]. In [7] soliton solutions were examined, and the Hirota N-soliton condition for the B-type Kadomtsev-Petviashvili equation within the Hirota bilinear formulation was established. A weight number was employed in an algorithm to assess the Hirota condition while converting the Hirota function in N wave vectors to a homogeneous polynomial, and soliton solutions were presented under generic dispersion relations. The (2+1)-dimensional Burgers equation served as the foundation for the introduction of a generalized Burgers equation with variable coefficients [8] and the authors found lump solutions to the generalized Burgers equation with variable coefficients by combining the test function method with the bilinear form. In [9] a proposed (2+1)-dimensional nonlinear model and localized wave interaction solutions, including lump-kink and lump-soliton types were examined. The authors in [10] examined the generalized Kadomtsev-Petviashvili equation in (3+1) dimensions using the Hirota bilinear approach and symbolic computation. The bilinear Bäcklund transformation was built using its bilinear form and the determinant's properties were used to derive the Pfaffian, Wronskian, and Grammian form solutions. A generalized (3+1)-dimensional Kadomtsev-Petviashvili type problem was developed [11] based on the prime number $p = 3$, and certain accurate solutions were achieved by redefining the bilinear operators using specific prime numbers.

A class of nonlinear partial differential equations describing physical systems gives rise to soliton solutions. Solitons are solitary waves having the ability to scatter elastically and they maintain their structures and speed even when they collide. Finding solutions to these nonlinear partial differential equations is thus inevitable. However, finding closed-form solutions is a highly challenging endeavour, and closed-form solutions solutions are only possible in a small number of situations. Several techniques for achieving closed-form solutions have been put forward in recent years. The Lie symmetry analysis method [12], the tanh method [12, 13], the Hirota bilinear method [14], the Darboux transformation method [15], and the inverse Hirota's bilinear approach [16, 17] are a few of the primary techniques used to do the integration of nonlinear partial differential equations. In addition to soliton solution extraction, conservation laws are crucial. The procedure for solving nonlinear partial differential equations involves conservation laws in a significant way. The initial step in solving a problem is often to discover the conservation laws of a system of nonlinear partial differential equations. A system of nonlinear partial differential equation's conservation laws is a powerful indicator of the system's integrability.

The Lie symmetry approach, commonly known as the Lie group method, is one of the most effective techniques for finding solutions to nonlinear partial differential equations among the techniques discussed above. It is based on research into the invariance of point transformations on the one-parameter Lie group. Lie symmetry techniques are heavily algorithmic and were first created by Sophus Lie in the second half of the 19th century. These approaches provide explicit solutions for differential equations, particularly nonlinear differential equations, by methodically combining and extending well-known ad hoc techniques. The quantity of academic articles, books, and new symbolic software dedicated to symmetry approaches for differential equations shows that there have been significant breakthroughs in recent years.

This paper examines (1.3), where $\Phi = \Phi(x, y, t)$ is a real function of x, y and t and ε_i $i = 1 \cdots 6$ are real parameters. The Lie point symmetries of (1.3) are computed and it gives rise to group invariant solutions and symmetry reductions under a certain point symmetries and conserved vectors are illustrated courtesy of the multiplier method.

The outline of the paper is as follows. In Section 2, we obtain Lie point symmetries and the commutator table of the Lie algebra of (1.3). Section 3 illustrates symmetry reductions and associated group invariant solutions. Then in Section 4 we construct conservation laws for (1.3) using the multiplier method and discuss the significance of the computed conservation laws. Finally, in Section 5 concluding remarks are presented.

2 Lie Point Symmetries of (1.3)

A differential equation's Lie point symmetry [18] is an invertible transformation of the dependent and independent variables that does not modify the equation itself. A differential equation's symmetries must be determined, which is a difficult task. However, Sophus Lie (1842-1999), a Norwegian mathematician, discovered that if one limits oneself to symmetries that form a group (continuous one-parameter group of transformations) and depend continuously on a small parameter, one can linearize the symmetry conditions and come up with an algorithm for calculating continuous symmetries.

The vector field

$$\Delta = \Xi^1(t,x,y,\Phi)\frac{\partial}{\partial t} + \Xi^2(t,x,y,\Phi)\frac{\partial}{\partial x} + \Xi^3(t,x,y,\Phi)\frac{\partial}{\partial y} + \Psi(t,x,y,\Phi)\frac{\partial}{\partial \Phi} \tag{2.1}$$

is a Lie point symmetry of (1.3) if

$$\begin{aligned}\Delta^{[6]}\Big\{ & \varepsilon_1\left(15\Phi_x^3 + 15\Phi_x\Phi_{xxx} + \Phi_{xxxxx}\right)_x + \varepsilon_2\left(6\Phi_x\Phi_{xx} + \Phi_{xxxx}\right) + \varepsilon_3\Big(\Phi_{xxxy} + 3\left(\Phi_x\Phi_y\right)_x\Big) \\ & + \varepsilon_4\Phi_{xx} + \varepsilon_5\Phi_{xt} + \varepsilon_6\Phi_{yy}\Big\}|_{(1.3)} = 0,\end{aligned} \tag{2.2}$$

where $\Delta^{[6]}$ is the sixth prolongation of (2.1).

$$\Xi^2_\Phi = 0, \tag{2.3}$$

$$\Xi^3_\Phi = 0, \tag{2.4}$$

$$\Xi^3_x = 0, \tag{2.5}$$

$$\Xi^1_\Phi = 0, \tag{2.6}$$

$$\Xi^1_x = 0, \tag{2.7}$$

$$\Xi^1_y = 0, \tag{2.8}$$

$$\Xi^2_{xx} = 0, \tag{2.9}$$

$$\Psi_{\Phi\Phi} = 0, \tag{2.10}$$

$$\Psi_{x\Phi} = 0, \tag{2.11}$$

$$3\,\Xi^2_x - \Xi^3_y = 0, \tag{2.12}$$

$$3\,\Xi^2_x - \Xi^3_y = 0, \tag{2.13}$$

$$\Psi_\Phi + \Xi^2_x = 0, \tag{2.14}$$

$$\Psi_\Phi + \Xi^2_x = 0, \tag{2.15}$$

$$-\,\Xi^1_t + 5\,\Xi_{x;x} = 0, \tag{2.16}$$

$$\Psi_\Phi + \Xi^2_x = 0, \tag{2.17}$$

$$2\,\Xi^2_x\varepsilon_2 + 15\,\Psi_x\varepsilon_1 - \varepsilon_3\Xi^2_y = 0, \tag{2.18}$$

$$15\,\psi_{\Phi xx}\varepsilon_1 + \varepsilon_3\Psi_{y\Phi} - 3\,\varepsilon_3\Xi^2_{xy} = 0, \tag{2.19}$$

$$3\,\Psi_{xx}\varepsilon_3 - \varepsilon_6\Xi^3_{yy} + 2\,\varepsilon_6\Psi_{y\Phi} = 0, \tag{2.20}$$

$$3\,\Psi_x\varepsilon_3 - 2\,\varepsilon_6\Xi^2_y - \varepsilon_5\Xi^3_t = 0, \tag{2.21}$$

$$-\,\Xi^3_y + 4\,\Xi^2_x + \Psi_\Phi = 0, \tag{2.22}$$

$$-\,\Xi^3_y + 4\,\Xi^2_x + \Psi_\Phi = 0, \tag{2.23}$$

$$\varepsilon_3\Psi_{y\Phi} + 15\,\Psi_{xx}\varepsilon_1 - \varepsilon_3\Xi^2_{xy} = 0, \tag{2.24}$$

$$\Psi_\Phi\varepsilon_2 + 15\,\Psi_x\varepsilon_1 - \varepsilon_3\Xi^2_y + 3\,\Xi^2_x\varepsilon_2 = 0, \tag{2.25}$$

$$4\,\Xi^2_x\varepsilon_4 + 15\,\Psi_{xxx}\varepsilon_1 + 6\,\Psi_x\varepsilon_2 - \varepsilon_5\Xi^2_t + 3\,\varepsilon_3\Psi_y = 0, \tag{2.26}$$

$$\Psi_{xxxx}\varepsilon_2 + \Psi_{xxxxxx}\varepsilon_1 + \varepsilon_3\Psi_{xxxy} + \varepsilon_6\psi_{\Phi;yy} + \Psi_{xx}\varepsilon_4 + \varepsilon_5\Psi_{tx} = 0, \tag{2.27}$$

$$6\,\Psi_{xx}\varepsilon_2 + 15\,\Psi_{xxxx}\varepsilon_1 - \varepsilon_6\Xi^2_{yy} + \varepsilon_5\Psi_{t\Phi} + 3\,\varepsilon_3\Psi_{xy} - \varepsilon_5\Xi^2_{tx} = 0. \tag{2.28}$$

It can clearly be seen that from (2.3-2.28) consists of 26 linear partial differential equations and the four unknown infinitesimal functions are $\Xi^1(t,x,y,\Phi), \Xi^2(t,x,y,\Phi), \Xi^3(t,x,y,\Phi), \Psi(t,x,y,\Phi)$. This implies that the above system of linear partial differential equations is over-determined. The integration of the above over-determined system of linear partial differential equation leads to the following general solution:

$$\Xi^1(t,x,y,\Phi) = R_1 - 75R_6\varepsilon_3\varepsilon_1\varepsilon_5 t, \tag{2.29}$$

$$\Xi^2(t,x,y,\Phi) = 3R_4\varepsilon_3 g(t) - 15R_5\varepsilon_1\varepsilon_5 y - 15R_6\varepsilon_3\varepsilon_1\varepsilon_5 x, \tag{2.30}$$

$$\Xi^3(t,x,y,\Phi) = R_2 + R_5\left(30\varepsilon_6\varepsilon_1 t - 3{\varepsilon_3}^2 t\right) + R_6\left(6{\varepsilon_3}^2\varepsilon_2 t - 45\varepsilon_3\varepsilon_1\varepsilon_5 y\right), \tag{2.31}$$

$$\begin{aligned}\Psi(t,x,y,\Phi) &= R_3 f(t) + R_4\varepsilon_5\dot{g}(t)y + R_5\left(2\varepsilon_5\varepsilon_2 y - \varepsilon_5\varepsilon_3 x\right)\\ &+ R_6\left(15\varepsilon_5\varepsilon_3\varepsilon_1\Phi + 2\varepsilon_5\varepsilon_2\varepsilon_3 x + 20\varepsilon_5\varepsilon_4\varepsilon_1 y - 4\varepsilon_5{\varepsilon_2}^2 y\right),\end{aligned} \tag{2.32}$$

where $R_i, i = \cdots 6$ are arbitrary constants of integration.

$$\begin{aligned}
&\Delta_1 = \frac{\partial}{\partial t}, \quad \text{time translation}\\
&\Delta_2 = \frac{\partial}{\partial y}, \quad \text{space translation}\\
&\Delta_3 = f(t)\frac{\partial}{\partial \Phi}, \quad \text{space \&time} - \text{dependentshift}\\
&\Delta_4 = \varepsilon_5\dot{g}(t)y\frac{\partial}{\partial \Phi} + 3\varepsilon_3 g(t)\frac{\partial}{\partial x}, \quad \text{time-dependent shift\&space}\\
&\Delta_5 = \left(2\varepsilon_5\varepsilon_2 y - \varepsilon_5\varepsilon_3 x\right)\frac{\partial}{\partial \Phi} - 15\varepsilon_1\varepsilon_5 y\frac{\partial}{\partial x} + \left(30\varepsilon_6\varepsilon_1 t - 3{\varepsilon_3}^2 t\right)\frac{\partial}{\partial y}, \quad \text{rotatation \&space} - \text{dependentshift}\\
&\Delta_6 = \left(15\varepsilon_5\varepsilon_3\varepsilon_1\Phi + 2\varepsilon_5\varepsilon_2\varepsilon_3 x + 20\varepsilon_5\varepsilon_4\varepsilon_1 y - 4\varepsilon_5{\varepsilon_2}^2 y\right)\frac{\partial}{\partial \Phi} - 75\varepsilon_3\varepsilon_1\varepsilon_5 t\frac{\partial}{\partial t}\\
&- 15\varepsilon_3\varepsilon_1\varepsilon_5 x\frac{\partial}{\partial x} + \left(6{\varepsilon_3}^2\varepsilon_2 t - 45\varepsilon_3\varepsilon_1\varepsilon_5 y\right)\frac{\partial}{\partial y}, \quad \text{dilation \&space\&time} - \text{dependentshift}
\end{aligned}$$

The commutator $[\Delta_i, \Delta_j]$ is given by $[\Delta_i, \Delta_j] = \Delta_i\Delta_j - \Delta_j\Delta_i$.

The commutator table of the Lie point symmetries of (1.3) is given in Table 1 and the associated relations are given as follows:

Table 1 Commutator table of the Lie algebra of system (1.1)

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
Δ_1	0	0	$\mathbf{F}_1^\infty$	$\mathbf{F}_2^\infty$	$\alpha\Delta_2$	$\mathbf{F}_3$
Δ_2	0	0	0	$\mathbf{F}_4^\infty$	$\mathbf{F}_5$	$\mathbf{F}_6$
Δ_3	$-\mathbf{F}_1^\infty$	0	0	0	0	$\mathbf{F}_7^\infty$
Δ_4	$-\mathbf{F}_2^\infty$	$-\mathbf{F}_4^\infty$	0	0	$\mathbf{F}_8^\infty$	$\mathbf{F}_9^\infty$
Δ_5	$-\alpha\Delta_2$	$-\mathbf{F}_5$	0	$-\mathbf{F}_8^\infty$	0	$\mathbf{F}_{10}$
Δ_6	$-\mathbf{F}_3$	$-\mathbf{F}_6$	$-\mathbf{F}_7^\infty$	$-\mathbf{F}_9^\infty$	$-\mathbf{F}_{10}$	0

$$
\begin{aligned}
&\alpha = 30\varepsilon_1\varepsilon_6 - 3\varepsilon_3^2,\\
&\mathbf{F}_\mathbf{1}^\infty = \dot{f}(t)\frac{\partial}{\partial \Phi},\\
&\mathbf{F}_\mathbf{2}^\infty = \varepsilon_5\ddot{g}(t)y\frac{\partial}{\partial \Phi} + 3\varepsilon_3\dot{g}(t)\frac{\partial}{\partial x},\\
&\mathbf{F_3} = -75\varepsilon_3\varepsilon_1\varepsilon_5\frac{\partial}{\partial t} + 6{\varepsilon_3}^2\varepsilon_2\frac{\partial}{\partial y},\\
&\mathbf{F}_\mathbf{4}^\infty = \varepsilon_5\dot{g}(t)\frac{\partial}{\partial \Phi},\\
&\mathbf{F_5} = 2\varepsilon_5\varepsilon_2\frac{\partial}{\partial \Phi} - 15\varepsilon_1\varepsilon_5\frac{\partial}{\partial x},\\
&\mathbf{F_6} = -45\varepsilon_1\varepsilon_3\varepsilon_5\frac{\partial}{\partial y} - \left(4\varepsilon_2^2\varepsilon_5 - 20\varepsilon_1\varepsilon_4\varepsilon_5\right)\frac{\partial}{\partial \Phi},\\
&\mathbf{F}_\mathbf{7}^\infty = \left(15\varepsilon_1\varepsilon_3\varepsilon_5 f(t) + 75\varepsilon_1\varepsilon_3\varepsilon_5\ddot{f}(t)\right)\frac{\partial}{\partial \Phi},\\
&\mathbf{F}_\mathbf{8}^\infty = \left(-30\varepsilon_1\varepsilon_5\varepsilon_6 t\dot{g}(t) + 3\varepsilon_3^2\varepsilon_5 t\dot{g}(t) - 3\varepsilon_3^2\varepsilon_5 g(t)\right)\frac{\partial}{\partial \Phi},\\
&\mathbf{F}_\mathbf{9}^\infty = \left(225\varepsilon_1\varepsilon_3^2\varepsilon_5 t\dot{g}(t) - 45\varepsilon_1\varepsilon_3^2\varepsilon_5 g(t)\right)\frac{\partial}{\partial x} + \left(75\varepsilon_1\varepsilon_3\varepsilon_5^2 ty\ddot{g}(t) + 60\varepsilon_1\varepsilon_3\varepsilon_5^2 y\dot{g}(t) - 6\varepsilon_2\varepsilon_3^2\varepsilon_5 t\dot{g}(t) + 6\varepsilon_2\varepsilon_3^2\varepsilon_5 g(t)\right)\frac{\partial}{\partial \Phi},\\
&\mathbf{F_{10}} = \left(90\varepsilon_1\varepsilon_2\varepsilon_3^2\varepsilon_5 t - 450\varepsilon_1^2\varepsilon_3\varepsilon_5^2 y\right)\frac{\partial}{\partial x} + \left(900\varepsilon_1^2\varepsilon_3\varepsilon_5\varepsilon_6 t - 90\varepsilon_1\varepsilon_3^3\varepsilon_5 t\right)\frac{\partial}{\partial y}\\
&\quad + \left(600\varepsilon_1^2\varepsilon_4\varepsilon_5\varepsilon_6 t - 120\varepsilon_1\varepsilon_2^2\varepsilon_5\varepsilon_6 t + 90\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_5^2 y - 60\varepsilon_1\varepsilon_3^2\varepsilon_4\varepsilon_5 t - 30\varepsilon_1\varepsilon_3^2\varepsilon_5^2 x\right)\frac{\partial}{\partial \Phi}.
\end{aligned}
$$

3 Symmetry Reductions and Exact Solutions

In this segment, we illustrate symmetry reductions and closed form solutions. One must solve the corresponding Lagrange equations

$$
\frac{dt}{\Xi^1(t,x,y,\Phi)} = \frac{dx}{\Xi^2(t,x,y,\Phi)} = \frac{dy}{\Xi^3(t,x,y,\Phi)} = \frac{d\Phi}{\Psi(t,x,y,\Phi)} \tag{3.1}
$$

to achieve symmetry reductions and exact solutions. We consider the following cases as illustrated below.

3.1 Case (I).

The group-invariant solution

$$
\begin{aligned}
u = &-\frac{1}{225{\varepsilon_1}^2\varepsilon_5\varepsilon_3\sqrt[5]{t}}\Big\{5\,\varepsilon_2\varepsilon_3 t^{6/5}\varepsilon_4\varepsilon_1 - {\varepsilon_2}^3\varepsilon_3 t^{6/5} + 15\,\varepsilon_2 x\sqrt[5]{t}\varepsilon_1\varepsilon_5\varepsilon_3 + 75\,\sqrt[5]{t}y\varepsilon_5{\varepsilon_1}^2\varepsilon_4\\
&- 15\,\sqrt[5]{t}y\varepsilon_5\varepsilon_1{\varepsilon_2}^2 - 225\,P(\chi,\varphi){\varepsilon_1}^2\varepsilon_5\varepsilon_3\Big\}
\end{aligned} \tag{3.2}
$$

is generated by the point symmetry Δ_6, where $\chi = \dfrac{x}{\sqrt[5]{t}}, \varphi = \dfrac{5\,y\varepsilon_5\varepsilon_1 + t\varepsilon_3\varepsilon_2}{5t^{3/5}\varepsilon_1\varepsilon_5}$ are invariants of the symmetry Δ_6 and $P(\chi,\varphi)$ satisfies the sixth-order nonlinear partial differential equation

$$\begin{aligned}&25P_{\chi\chi\chi\chi\chi\chi}{\varepsilon_1}^2+1125P_{\chi\chi}{P_\chi}^2{\varepsilon_1}^2+375P_{\chi\chi}P_{\chi\chi\chi}{\varepsilon_1}^2+375P_{\chi\chi\chi\chi}P_\chi{\varepsilon_1}^2\\&+75P_{\chi\chi}P_\varphi\varepsilon_3\varepsilon_1+25\varepsilon_3P_{\chi\chi\chi\varphi}\varepsilon_1+25\varepsilon_6P_{\varphi\varphi}\varepsilon_1+75P_{\chi\varphi}P_\chi\varepsilon_3\varepsilon_1-5\varepsilon_5P_{\chi\chi}\chi\varepsilon_1\\&-10\varepsilon_5P_\chi\varepsilon_1-15P_{\chi\varphi}\varphi\varepsilon_1\varepsilon_5=0.\end{aligned} \quad (3.3)$$

Using the point symmetries of (3.3) we now further reduce the nonlinear partial differential equation to an ordinary differential equation. It should be noted that symmetries of (3.3) are

$$\Upsilon_1=\frac{\partial}{\partial P}, \quad (3.4)$$

$$\Upsilon_2=\varepsilon_5\varphi\frac{\partial}{\partial P}+15\varepsilon_3\frac{\partial}{\partial\chi}. \quad (3.5)$$

The symmetry $\Upsilon_1+\Upsilon_2$ gives rise to the group-invariant solution

$$P(\chi,\varphi)=\frac{\chi+\chi\,\varepsilon_5\varphi+15\,Q(\varphi)\varepsilon_3}{15\varepsilon_3}, \quad (3.6)$$

where $\varphi=\vartheta$,is an invariant of the symmetry $\Upsilon_1+\Upsilon_2$ and $Q(\varphi)$ satisfies the second-order linear ordinary differential equation

$$75\varepsilon_3\varepsilon_6Q_{\vartheta\vartheta}-4{\varepsilon_5}^2\vartheta-\varepsilon_5=0. \quad (3.7)$$

whose solution is

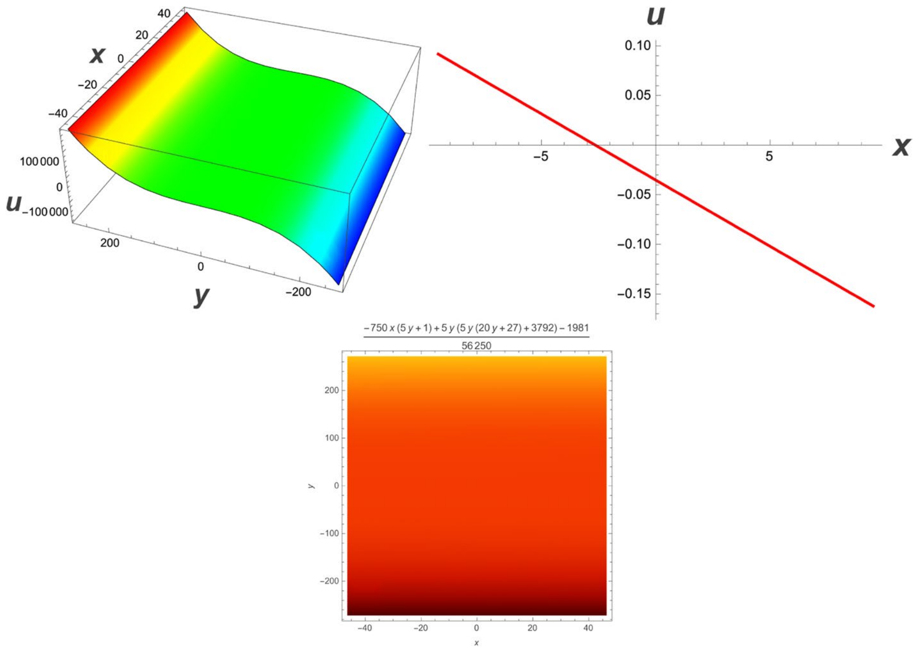

Fig. 1 Graphical simulation of (3.2)

$$Q(\vartheta)=\frac{3\,\varepsilon_5\vartheta^2+4\,{\varepsilon_5}^2\vartheta^3+450\,C_1\,\vartheta\,\varepsilon_6\varepsilon_3+450\,C_2\,\varepsilon_6\varepsilon_3}{450\varepsilon_6\varepsilon_3} \tag{3.8}$$

Finally (3.8) therefore completes group invariant solution (3.2) and the graphical simulation of (3.2) is given in Fig. 1.

3.2 Case (II).

The infinitesimal generator Δ_5 leads to

$$u=\frac{15\big(3\big(10\varepsilon_1^2\varepsilon_3\varepsilon_6-\varepsilon_1\varepsilon_3^3\big)t^4-4y\varepsilon_1\varepsilon_3t^2\big)\varepsilon_5^2}{8}+\big(2\varepsilon_2\big(-10\varepsilon_1\varepsilon_6+\varepsilon_3^2\big)t^3+\big(-x\varepsilon_3+2y\varepsilon_2\big)t\big)\varepsilon_5+P(\chi,\varphi) \tag{3.9}$$

with invariants $\chi=-15t^2\varepsilon_1\varepsilon_6+\frac{3}{2}t^2\varepsilon_3^2+y$, $\varphi=-150\varepsilon_1^2\varepsilon_5t^3\varepsilon_6+15t\varepsilon_5\big(t^2\varepsilon_3^2+y\big)\varepsilon_1+x$ and $P(\chi,\varphi)$ satisfies nonlinear partial differential equation

$$\begin{aligned}&2P_{\varphi\varphi\varphi\varphi\varphi\varphi}\varepsilon_1+2\big(15P_\varphi\varepsilon_1+\varepsilon_2\big)P_{\varphi\varphi\varphi\varphi}+2\varepsilon_3P_{\chi\varphi\varphi\varphi}+30P_{\varphi\varphi}P_{\varphi\varphi\varphi}\varepsilon_1\\&+2\Big(15\chi\varepsilon_1\varepsilon_5^2+45P_\varphi^2\varepsilon_1+3P_\chi\varepsilon_3+6P_\varphi\varepsilon_2+\varepsilon_4\Big)P_{\varphi\varphi}+6\varepsilon_3P_\varphi P_{\chi\varphi}\\&-2\varepsilon_5^2\varepsilon_3+2\varepsilon_6P_{\chi\chi}=0.\end{aligned} \tag{3.10}$$

Employing the point symmetries of (3.10) one can further reduce the nonlinear partial differential equation to an nonlinear ordinary differential equation and symmetries of (3.10) are

$$\Upsilon_1=\frac{\partial}{\partial P}, \tag{3.11}$$

$$\Upsilon_2=\frac{\partial}{\partial\varphi}, \tag{3.12}$$

$$\Upsilon_3=5\chi\frac{\partial}{\partial P}\varepsilon_5^2\varepsilon_1-\varepsilon_3\frac{\partial}{\partial\chi}\quad . \tag{3.13}$$

The symmetry $\Upsilon_1+\Upsilon_2+\Upsilon_3$ leads to

$$P(\chi,\varphi)=\frac{-2\chi-5\varepsilon_5^2\chi^2\varepsilon_1+2Q(\vartheta)\varepsilon_3}{2\varepsilon_3} \tag{3.14}$$

with invariants $\vartheta=\dfrac{\varepsilon_3\varphi+\chi}{\varepsilon_3}$ and $Q(\vartheta)$ satisfies the nonlinear ordinary differential equation

$$\begin{aligned}&45Q_{\vartheta\vartheta}Q_\vartheta^2\varepsilon_1\varepsilon_3^2+6\varepsilon_2Q_{\vartheta\vartheta}Q_\vartheta\varepsilon_3^2+15Q_{\vartheta\vartheta\vartheta\vartheta}Q_\vartheta\varepsilon_1\varepsilon_3^2+15Q_{\vartheta\vartheta}Q_{\vartheta\vartheta\vartheta}\varepsilon_1\varepsilon_3^2-5\varepsilon_6\varepsilon_5^2\varepsilon_1\varepsilon_3\\&-\varepsilon_5^2\varepsilon_3^3+6Q_\vartheta Q_{\vartheta\vartheta}\varepsilon_3^2+\varepsilon_4Q_{\vartheta\vartheta}\varepsilon_3^2+\varepsilon_2Q_{\vartheta\vartheta\vartheta\vartheta}\varepsilon_3^2+Q_{\vartheta\vartheta\vartheta\vartheta\vartheta\vartheta}\varepsilon_1\varepsilon_3^2-3Q_{\vartheta\vartheta}\varepsilon_3^2\\&+Q_{\vartheta\vartheta\vartheta\vartheta}\varepsilon_3^2+\varepsilon_6Q_{\vartheta\vartheta}=0.\end{aligned} \tag{3.15}$$

Finally any solution $Q(\vartheta)$ of (3.15) completes group invariant solution (3.9).

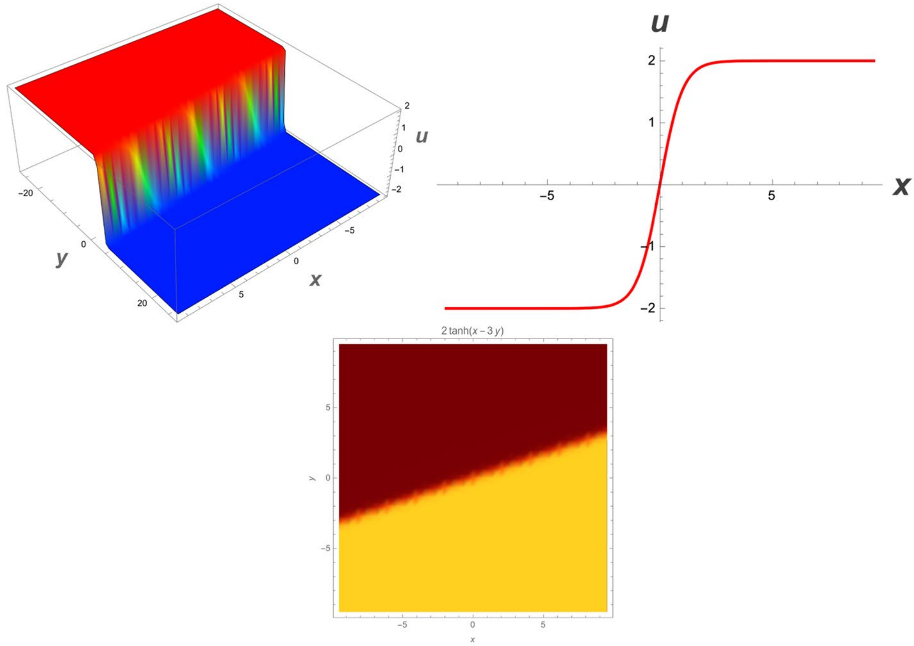

Fig. 2 Evolution of travelling wave solution of (3.16)

3.3 Case (III).

The infinitesimal generator $\Delta_1 + \Delta_2$ leads to the group-invariant solution

$$u = P(\chi, \varphi) \tag{3.16}$$

with invariants $\chi = x, \quad \varphi = -t + y$ and $P(\chi, \varphi)$ satisfies the nonlinear partial differential equation

$$\begin{aligned} &45P_\chi^2 P_{\chi\chi}\varepsilon_1 + 6\varepsilon_2 P_\chi P_{\chi\chi} + 3\varepsilon_3 P_\chi P_{\chi\varphi} + 15P_\chi P_{\chi\chi\chi\chi}\varepsilon_1 + 3\varepsilon_3 P_{\chi\chi} P_\varphi + 15P_{\chi\chi} P_{\chi\chi\chi}\varepsilon_1 \\ &+ \varepsilon_4 P_{\chi\chi} - \varepsilon_5 P_{\chi\varphi} + \varepsilon_6 P_{\varphi\varphi} + \varepsilon_2 P_{\chi\chi\chi\chi} + \varepsilon_3 P_{\chi\chi\chi\varphi} + P_{\chi\chi\chi\chi\chi\chi}\varepsilon_1 = 0. \end{aligned} \tag{3.17}$$

A special solution to (3.17) is the following travelling wave solution

$$P(\chi, \varphi) = 2C_2 \tanh\Bigg\{ C_2\chi + C_1 + \frac{\left(-4\varepsilon_3 {C_2}^2 + \varepsilon_5 + \sqrt{-64\varepsilon_6 {C_2}^4\varepsilon_1 + 16\varepsilon_3^2 {C_2}^4 - 16{C_2}^2\varepsilon_6\varepsilon_2 - 8\varepsilon_3 {C_2}^2\varepsilon_5 - 4\varepsilon_4\varepsilon_6 + \varepsilon_5^2}\right) C_2\varphi}{2\varepsilon_6} \Bigg\} + C_4.$$

The evolution of the travelling wave solution (3.16) is illustrated in Fig. 2.

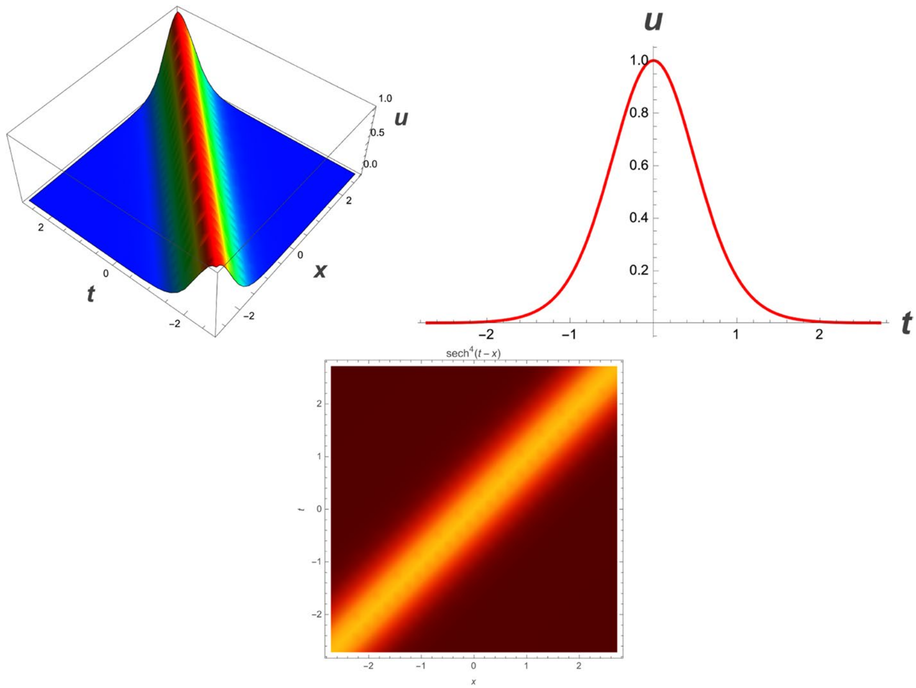

Fig. 3 Profile of (3.18)

3.4 Case (IV).

The point symmetry $\Delta_3 + \Delta_4$ results in the group invariant solution

$$u = \frac{xf(t) + xy\varepsilon_5\dot{g}(t) + 3P(\chi, \varphi)g(t)\varepsilon_3}{3g(t)\varepsilon_3}, \tag{3.18}$$

with invariants $\chi = t,\ \ \varphi = y$ and $P(\chi, \varphi)$ the following linear partial differential equation

$$3gP_{\varphi\varphi}\varepsilon_3\varepsilon_6 + \varphi\varepsilon_5^2 g_{\chi\chi} + \varepsilon_5 f_\chi = 0. \tag{3.19}$$

The integration (3.19) leads to

$$P(\chi, \varphi) = -\frac{\varepsilon_5\left(\frac{1}{2}f_\chi\varphi^2 + \frac{1}{6}\varphi^3\varepsilon_5 g_{\chi\chi}\right)}{3\varepsilon_6\varepsilon_3 g} + F(\chi)\varphi + G(\chi) \tag{3.20}$$

and finally (3.20) completes the group invariant solution (3.18) and a graphical simulation of is given in Fig. 3.

The limiting behavior of problems that are far from their beginning or boundary conditions is captured by group invariant solutions in numerous applications.

4 Conservation Laws of (1.3)

Conservation laws are an important concept in the study of partial differential equations (PDEs). They describe the mathematical relationships between the properties of physical systems and how they change over time. The basic idea behind conservation laws is that the total amount of a particular property, such as mass, energy, or momentum, remains constant within a closed system. In PDEs, conservation laws are represented as equations that describe how certain physical quantities change in response to changes in other quantities. For example, the conservation of mass equation for a fluid is described by the continuity equation, which states that the rate of change of the fluid's density is proportional to the rate of change of its volume. Similarly, the conservation of energy equation for a thermodynamic system is described by the first law of thermodynamics, which states that the change in the internal energy of a system is equal to the heat added to the system minus the work done on the system. Another important concept in conservation laws is the concept of the flux of a physical property. The flux is a measure of the flow of a property from one region of a system to another. For example, in fluid dynamics, the flux of mass is represented by the velocity of the fluid, while the flux of energy is represented by the heat transfer rate. There are several methods used to derive conservation laws from PDEs. One common method is the use of symmetry considerations, where the conservation law is derived from the symmetries of the physical system. Another method is the use of Noether's theorem, which states that the conservation laws can be derived from the invariances of the system under certain transformations.

In summary, conservation laws play a crucial role in the study of partial differential equations. They provide mathematical descriptions of the relationships between physical quantities and how they change over time, and they are derived from the symmetries and invariances of physical systems. Understanding conservation laws is essential for a wide range of scientific and engineering applications, including fluid dynamics, thermodynamics, and solid mechanics, among others.

Let us consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{4.1}$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, $\cdots$, kth-order partial derivatives, that is, $u^\alpha_i = D_i(u^\alpha), u^\alpha_{ij} = D_j D_i(u^\alpha), \dots$ respectively, with the *total derivative operator* with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \dots, \quad i = 1, \dots, n, \tag{4.2}$$

where the summation pact is used whenever suitable.

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \qquad \alpha = 1, \dots, m. \tag{4.3}$$

The n-tuple vector $\Omega = (\Omega^1, \Omega^2, \dots, \Omega^n),\ \Omega^j \in \mathcal{A},\ j = 1, \dots, n$, is a *conserved vector* of (4.1) if Ω^i satisfies

$$D_i\Omega^i|_{(4.1)} = 0. \tag{4.4}$$

The equation (4.4) defines a *local conservation law* of system (4.1).

A multiplier $\lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i\Omega^i \tag{4.5}$$

hold identically. Here we will consider multipliers of the zeroth order,

i.e., $\lambda_\alpha = \lambda(t, x, y, \Phi)$. The right hand side of (4.5) is a divergence expression. The determining equation for the multiplier λ_α is

$$\frac{\delta(\lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{4.6}$$

Once the multipliers are obtained the conserved vectors are calculated via a homotopy formula [?]. A space-time divergence is a local conservation law for equations (1.3)

$$D_t\Omega^t + D_x\Omega^x + D_y\Omega^y = 0$$

that holds for all formal solutions of the equation (1.3) where the conserved density Ω^t and the spatial fluxes Ω^x,Ω^y.

If u and its derivatives tend to zero as x, y approaches infinity, the conserved quantities are obtained by $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Omega^t dxdy$. For (1.3), we obtain a zeroth order multiplier λ, that is given by

$$\lambda(t, x, y, \Phi) = F(t)y + G(t),$$

where F and G are arbirary functions of t. It should be pointed out that first, second and third order multipliers do not exist. Thus, corresponding to the above zeroth order multiplier we have the following conservation laws of (1.3):

$$\Omega_1^t = \frac{1}{2}y\varepsilon_5 F(t)\Phi_x, \tag{4.7}$$

$$\begin{aligned}\Omega_1^x = \frac{1}{4}\Big\{ & -3\varepsilon_3 F(t)\Phi_x\Phi - 3y\varepsilon_3 F(t)\Phi_{xy}\Phi - 2y\varepsilon_5 F'\Phi + 60y\varepsilon_1 F(t){\Phi_x}^3 \\ & + 12y\varepsilon_2 F(t){\Phi_x}^2 + 4y\varepsilon_4 F(t)\Phi_x + 9y\varepsilon_3 F(t)\Phi_x\Phi_y + 60y\varepsilon_1 F(t)\Phi_{xxx}\Phi_x + 3y\varepsilon_3 F(t)\Phi_{xxy} \\ & + 4y\varepsilon_2 F(t)\Phi_{xxx} + 4y\varepsilon_1 F(t)\Phi_{xxxxx} - \varepsilon_3 F(t)\Phi_{xx} + 2y\varepsilon_5 F(t)\Phi_t \Big\},\end{aligned} \tag{4.8}$$

$$\begin{aligned}\Omega_1^y = \frac{1}{4}\Big\{ & 3y\varepsilon_3 F(t)\Phi_{xx}\Phi - 4\varepsilon_6 F(t)\Phi + 3y\varepsilon_3 F(t){\Phi_x}^2 + y\varepsilon_3 F(t)\Phi_{xxx} \\ & + 4y\varepsilon_6 F(t)\Phi_y \Big\};\end{aligned} \tag{4.9}$$

$$\Omega_2^t = \frac{1}{2}\varepsilon_5 G(t)\Phi_x, \tag{4.10}$$

$$\Omega_2^x = \frac{1}{4}\Big\{ -3\varepsilon_3 G(t)\Phi_{xy}\Phi - 2\varepsilon_5 G'\Phi(t,x,y) + 9\varepsilon_3 G(t)\Phi_x\Phi_y + 3\varepsilon_3 G(t)\Phi_{xxy} + 60\varepsilon_1 G(t)\Phi_x{}^3 + 12\varepsilon_2 G(t)\Phi_x{}^2 + 4\varepsilon_4 G(t)\Phi_x + 60\varepsilon_1 G(t)\Phi_{xxx}\Phi_x + 4\varepsilon_2 G(t)\Phi_{xxx} + 4\varepsilon_1 G(t)\Phi_{xxxxx} + 2\varepsilon_5 G(t)\Phi_t \Big\}, \tag{4.11}$$

$$\Omega_2^y = \frac{1}{4}\big(3\varepsilon_3 G(t)\Phi_{xx}\Phi + 3\varepsilon_3 G(t)\Phi_x{}^2 + \varepsilon_3 G(t)\Phi_{xxx} + 4\varepsilon_6 G(t)\Phi_y\big). \tag{4.12}$$

Physical rules including the conservation of energy, mass, and momentum are expressed mathematically as conservation laws. In order to solve and reduce partial differential equations, conservation rules are absolutely essential. The study of the existence, uniqueness, and stability of solutions to nonlinear partial differential equations, as well as the creation of numerical integrators for partial differential equations, have both made extensive use of conservation laws. It should be noted that a limitless number of conservation rules may be obtained since the multiplier contains an arbitrary function.

5 Concluding Remarks

Today's article examined a nonlinear partial differential equation model of sixth order that satisfies the Hirota N-soliton. Time and space translations, time and space dilations, rotation and space-dependent shift, and space-dependent shift Lie point symmetries were established. Certain Lie point symmetries resulted in group invariant solutions and infinitely many conservation laws the underlying equation were computed, along with their importance. The employed Lie symmetry approach is distinct from the conventional integrability methods, which also include Hirota's bilinear method, the traveling wave solution, and the Darboux transformation method, among others. In terms of simulating complicated waves and the dynamics of their interaction the results obtained in work can serve as benchmarks against the numerical simulations.

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