International Journal of Theoretical Physics (2023) 62:98
https://doi.org/10.1007/s10773-023-05355-3

RESEARCH

Generalized $\mathbb{XOR}$ Operation and the Categorical Equivalence of the Abbott Algebras and Quantum Logics

Dominika Burešová[1]

Received: 24 December 2022 / Accepted: 4 April 2023 / Published online: 4 May 2023

Abstract

Considering the inference rules in generalized logics, J.C. Abbott arrives to the notion of orthoimplication algebra (see Abbott (1970) and Abbott (Stud. Logica. 2:173–177, XXXV)). We show that when one enriches the Abbott orthoimplication algebra with a falsity symbol and a natural $\mathbb{XOR}$-type operation, one obtains an orthomodular difference lattice as an enriched quantum logic (see Matoušek (Algebra Univers. 60:185–215, 2009)). Moreover, we find that these two structures endowed with the natural morphisms are categorically equivalent. We also show how one can introduce the notion of a state in the Abbott $\mathbb{XOR}$ algebras strenghtening thus the relevance of these algebras to quantum theories.

Keywords Boolean algebra · Abbott orthoimplication algebra · Orthomodular lattice · Symmetric difference · Categorical equivalence

1 Introduction and Basic Notions

Recently there has been an effort to soundly introduce and study the notion of a symmetric difference in orthomodular lattices and posets (see [3, 4, 8, 9, 12], etc.). In the relation to quantum axiomatics, the idea has been to enrich the "quantum logics" with a kind of a $\mathbb{XOR}$ operation. There are several non-Boolean orthomodular lattices that allow for this operation and become thus "nearly Boolean".

In this note, we introduce a $\mathbb{XOR}$ operation by extending the language of the Abbott implication algebras. The technical side overlaps to a certain extent with a synthesis of [1, 2], the presence of the falsity symbol 0 and the $\mathbb{XOR}$ operation makes a nead for a modified formulation in places. Also, we provide a direct proof of the journey "from orthomodular lattices to Abbott algebras" making thus the envisaged equivalence more insightful. At that, as a by-product, this introduces an "Abbott operation" in the class of orthomodular lattices and may allow for another algebraic investigation (see also [6]).

Thought we consider the algebras endowed with a $\mathbb{XOR}$ operation, it may be noted that we also provide another proof of the equivalence of the Abbott algebras with the orthomodular

✉ Dominika Burešová
buresdo2@fel.cvut.cz

1 Czech Technical University in Prague, Department of Cybernetics, Faculty of Electrical Engineering, Prague, Czech Republic

lattices (forgetting the operation $\mathbb{XOR}$). As commented at the end of this paper, this allows us to translate all algebraic and state space features of the orthomodular lattices into the Abbott algebras.

For a last introductory remark, recall that the Abbott algebras are originally mostly studied without the falsity nullary element. It should be noted that we could also introduce "a partial $\mathbb{XOR}$ operation" into the algebras. But a potential interpretation of such a notion does not seem naturally possible in the quantum logic theory.

Let us take up the subject proper. Our basic definition reads as follows.

Definition 1.1 Let $(A,\ 0,\ \cdot,\ \Delta)$ be an algebra with a nullary operation 0 and two binary operations $\cdot$ and Δ. Let the operations fulfill the following requirements (we omit the symbol $\cdot$ writing simply ab instead of $a \cdot b$; let $a, b, c \in A$).

1. $(ab)a = a$,
2. $(ab)b = (ba)a$,
3. $a\big((ba)c\big) = ac$,
4. $0a = bb$,
5. $(a\Delta b)\Delta c = a\Delta(b\Delta c)$,
6. $a\Delta bb = a0$,
7. $bb\Delta a = a0$,
8. $(a\Delta b)\big((ab)b\big) = aa$.

Then $(A,\ 0,\ \cdot,\ \Delta)$ is said to be **the Abbott $\mathbb{XOR}$ algebra**.

2 Results

Before we pass to our results, let us recall a few properties of the operation $\cdot$ of the algebras studied.

Proposition 2.1 *Let $(A,\ 0,\ \cdot,\ \Delta)$ be the Abbott $\mathbb{XOR}$ algebra. Then the following statements hold true $(a, b, c \in A)$:*

(i) $aa = 1$,
(ii) $1a = a$,
(iii) $a1 = 1$,
(iv) $ab = ba \implies a = b$,
(v) $a(ba) = 1$,
(vi) $ab = 1 \implies a(bc) = ac$,
(vii) $ab = 1 \implies (ba)(ac) = 1$.

Proof The proof of the *Proposition 1.2* can be obtained as an interplay of the results of [1] and [2]. Since the calculus in the Abbott algebras is rather non-standard and since we want to preserve the self-containedness, let us very briefly recall the proofs. One uses the adequate axioms of *Definition 1.1*. *Ad (i):* $a(ab) = \big((ab)a\big)(ab) = ab$, so $\big((ab)a\big)a = \big(a(ab)\big)(ab) = (ab)(ab)$ and therefore $(aa) = (ab)(ab) = \big((ab)b\big)\big((ab)b\big) = \big((ba)a\big)\big((ba)a\big) = (ba)(ba) = bb$. *Ad (ii):* $1a = (aa)a = a$. *Ad (iii):* $a1 = a(aa) = aa = 1$. *Ad (iv):* $ab = ba \implies a = (ab)a = (ba)a = (ab)b = b$. *Ad (v):* $a(ba) = a\big((ba)(ba)\big) = a1 = 1$. *Ad (vi):* $ab = 1 \implies a(bc) = a\big((1b)c\big) = a\Big(\big((ab)b\big)c\Big) = a\Big(\big((ba)a\big)c\Big) = ac$. *Ad (vii):* $ab = 1 \implies (bc)(ac) = bc\big(a(bc)\big) = 1$. □

Let us now introduce another algebraic structure (see [5]-the structure of orthomodular lattices alias quantum logics). As known, the orthomodular lattices found its application in quantum theories. We enrich them with another operation, Δ.

Definition 2.2 Let us consider a 7-tuple $(D, 0, 1, \wedge, \vee, ^{\perp}, \Delta)$ where $(D, 0, 1, \wedge, \vee, ^{\perp})$ is an orthomodular lattice, and the binary operation Δ has the following properties ($a, b \in D$):

1. the operation Δ is associative,
2. $1\Delta a = a^{\perp}, a\Delta 1 = a^{\perp}$,
3. $a\Delta b \leq a \vee b$.

Then $(D, 0, 1, \wedge, \vee, ^{\perp})$ is said to be **the orthomodular difference lattice** (see [9]).

Theorem 2.3 *Let $\mathcal{A}$ be the category of Abbott $\mathbb{XOR}$ algebras with the corresponding (universal algebra) morphisms, and let $\mathcal{D}$ be the category of orthomodular difference lattices with the corresponding (universal algebra) morphisms. Then the categories $\mathcal{A}$ and $\mathcal{D}$ are equivalent.*

Proof Let $A \in \mathcal{A}$ and let us see how we can view A as an object of $\mathcal{D}$. Let us first endow A with a partial ordering. Let us introduce the partial ordering in A by requiring $a \leq b$ if $ab = 1$. Then $0 \leq a \leq 1$ for all $a \in A$ because $0a = 1$ and $a1 = 1$. Let us show that $\leq$ is a partial ordering with a least (resp. greatest) elements 0 (resp. 1). Indeed, $a \leq a$ since $aa = 1$, and if $a \leq b$ and $b \leq a$, then $ab = 1 = ba$ and therefore $a = b$. Further, if $a \leq b$ and $b \leq c$ then $ab = 1$ and $bc = 1$. It follows from *Proposition 1.2 (vii)* that $(ba)(ac) = 1$. Then $bc \leq ac$ but $bc = 1$ and therefore $1 \leq ac$. So $ac = 1$ and therefore $a \leq c$.

Let us see that A is a lattice with respect to $\leq$. We claim that $a \vee b = (ab)b$. To see that, we have $a\big((ab)b\big) = a\big((ba)a\big) = 1$ and therefore $a \leq (ab)b$ which means that $a \leq a \vee b$. Analogously, $b \leq a \vee b$. Moreover, if $a \leq c$ and $b \leq c$ then $ac = 1$ and $bc = 1$. Considering $ac = 1$ (and correcting [1]), we infer that $(cb)(ab) = 1$ (Proposition 1.2, (vii)). This implies that $\big((ab)b\big)\big((cb)b\big) = 1$ and therefore $(ab)b \leq (cb)b$. So $a \vee b = (ab)b \leq (cb)b = (bc)c = 1c = c$ and hence $a \vee b \leq c$. This shows that A is a lattice.

With the intention to restructure A to make it an orthocomplemented lattice, let us set $a^{\perp} = (a0)$. We are to verify that $(a^{\perp})^{\perp} = a, a \leq b \implies b \leq a$ and that both equalities $a \vee a^{\perp} = 1, a \wedge a^{\perp} = 0$ are valid. Obviously, $(a^{\perp})^{\perp} = (a0)0 = a \vee 0 = a$. Further, if $a \leq b$ then $b^{\perp} = (b0) \leq (a0) = a^{\perp}$. Let us also see that $a \vee a^{\perp} = 1$ and $a \wedge a^{\perp} = 0$. We have $a \vee a^{\perp} = a^{\perp} \vee a = \big(a0)a\big)a = aa = 1$. As regards the condition on the infimum, one uses the de Morgan law to obtain $a \wedge a^{\perp} = a^{\perp} \wedge a = (a \vee a^{\perp})^{\perp} = (a \vee a^{\perp})0 = \Big(\big(a(a0)\big)(a0)\Big)0 = (a \vee a^{\perp})0 = (10) = 0$.

It remains to verify the orthomodular law. Suppose that $a \leq b$. So we have $(b0) \leq (a0)$ and we see (*by Definition 1.1, 3.*) that $b = b \vee 0 = \big((b0)0\big) = (b0)\big((a0)0\big) = (b0)(a \vee 0) = (b0)a$. Since, $a(ba) = 1$ by *Proposition 2.1 (v)*, then $a \leq (ba)$ and therefore we have $(ba) = \big((ba)0\big)a$. In order to verify the orthomodular law, we are to prove that $b = a \vee (a^{\perp} \wedge b)$. Let us consider the right-hand side of this equality. We obtain $a \vee (a^{\perp} \wedge b) = a \vee \big((a0) \wedge b\big) = a \vee \Big(\big(a \vee (b0)\big)0\Big) = a \vee \Big(\big((b0) \vee a\big)0\Big) = \Big(a \vee \big((b0)a\big)a\Big)0 = a \vee \big((ba)0\big) = \Big(\big((ba)0\big)a\Big)a = (ba)a = b \vee a = b$.

Finally, let us check the conditions of the operation Δ. The operation Δ is associative by definition. Further, $a\Delta(bb) = a\Delta 1 = (a0) = x^{\perp}$ and $(bb)\Delta a = (1\Delta a) = (a0) = a^{\perp}$. To end up the verification, we use $(ab)b = a \vee b$ and we obtain $aa = 1 = (a\Delta b)\big((ab)b\big) = (a\Delta b) \vee (a \vee b)$. Therefore we infer that $a\Delta b \leq a \vee b$.

In the considerations above, we have defined an assignement $F : \mathcal{A} \to \mathcal{D}$ as a potential functor on the objects of $\mathcal{A}$ (the assignement F preserves the underlying set). Let us see that this assignement is functorial. Suppose that $f : A \to B$ is a morphism in $\mathcal{A}$. So $f(ab) = f(a)f(b)$, $f(0) = 0$ and $f(a\Delta b) = f(a)\Delta f(b)$. We have to check that f is a morphism in $\mathcal{D}$. For that, suppose that $a \vee b = c$ in $\mathcal{A}$. So it means that $c = (ab)b$. Thus $f(c) = \big(f(a)f(b)\big)f(b)$. This implies that $f(c) = f(a) \vee f(b)$. Further, we have to check that $f(a^\perp) = f(a)^\perp$. But $a^\perp = (a0)$ and therefore $f(a^\perp) = f(a0) = f(a)f(0) = f(a)0$, and hence $f(a)^\perp = f(a^\perp)$. Thus we have checked that F is indeed a functor from $\mathcal{A}$ to $\mathcal{D}$.

We shall now construct a functor, G, $G : \mathcal{D} \to \mathcal{A}$. Let us take $D \in \mathcal{D}$. Then $G(D)$ remains with the same underlying set. We define the object $G(D)$ as follows: If $a \in G(D)$ and $b \in G(D)$, then $ab = (a \vee b)^\perp \vee b$, and $a\Delta b$ is copied from D. Let us check that $G(D)$ sends a morphism of $\mathcal{D}$ into a morphism of $\mathcal{A}$. We first have to check that the axioms of $G(D)$ make it an Abbott $\mathbb{XOR}$ algebra.

1. $(ab)a = a$; we have $\Big(\big((a \vee b)^\perp \vee b\big) \vee a\Big)^\perp \vee a = \big((a \vee b) \wedge b^\perp \wedge a^\perp\big) \vee a = \big((a \vee b) \wedge (a \vee b)^\perp\big) \vee a = 0 \vee a = a$.
2. $(ab)b = (ba)a$; we have $(ab)b = \big((a\vee b)^\perp \vee b\big)^\perp \vee b = \big((a\vee b)\wedge b^\perp\big)\vee b$. Since the triple $b, b^\perp, a \vee b$ is compatible in D, we can use distributivity (see e.g. [5]). Hence, the latter formula gives us $(a\vee b\vee b)\wedge(b^\perp\vee b) = a\vee b$. Analogously, $\big((b\vee a)^\perp\vee a\big)^\perp\vee a = b\vee a$ and so the equality is valid.
3. $a\big((ba)c\big) = ac$. Prior to verifying this condition, let us make a preliminary observation. Consider the elements x and $y^\perp \wedge x$. Then $x \geq y^\perp \wedge x$. So the orthomodular law gives us $x = (y^\perp \wedge x) \vee \big((y^\perp \wedge x)^\perp \wedge x\big) = (y^\perp \wedge x) \vee \big((y \vee x^\perp) \wedge x\big)$.

   Let us verify the axiom proper. We have $a\big((ba)c\big) = a\Big(\big((b \vee a)^\perp \vee a\big)c\Big) = a\Big(\big((b^\perp \wedge a^\perp) \vee a\big)c\Big)$. For the sake of transparency, let us write $y = b^\perp \wedge a^\perp$.

   Hence we have $a\big((ba)c\big) = a\big((y \vee a)c\big) = a\Big(\big((y \vee a) \vee c\big)^\perp \vee c\Big) = \Bigg(a\vee\Big(\big((y \vee a)^\perp\big) \wedge c^\perp\Big)\vee c\Bigg)^\perp \vee\Bigg(\Big(\big((y\vee a)^\perp\big)\wedge c^\perp\Big)\vee c\Bigg) = \Bigg(a^\perp\wedge\Big(\big((y\vee a)^\perp\wedge c^\perp\big)\vee c\Big)^\perp\Bigg)\vee\Big(\big((y\vee a)^\perp\wedge c^\perp\big)\vee c\Big) = \Bigg(a^\perp\wedge\Bigg(\Big(\big((y\vee a)^\perp\wedge c^\perp\big)\vee c^\perp\Big)^\perp\wedge c^\perp\Bigg)\Bigg)\vee\Big(\big((y\vee a)^\perp\wedge c^\perp\big)\vee c\Big) = \Big((a^\perp\wedge c^\perp)\wedge \big((y\vee a)^\perp\wedge c^\perp\big)^\perp\Big)\vee\big((y^\perp\wedge a^\perp\wedge c^\perp)\vee c\big) = \Big((a^\perp\wedge c^\perp)\wedge(y\vee a\vee c)\vee(y^\perp\wedge a^\perp\wedge c^\perp)\Big)\vee c$.
   So we have

   $$a\big((ba)c\big) = \Bigg((a^\perp \wedge c^\perp) \wedge \Big(\big((b^\perp \wedge a^\perp) \vee a\big) \vee c)\Big)\Bigg) \vee \Big(\big((b^\perp \wedge a^\perp)^\perp \wedge a^\perp\big) \wedge c^\perp\Big) \vee c$$

   Let us set $u = (a^\perp \wedge c^\perp) \wedge \big((b^\perp \wedge a^\perp) \vee a \vee c)\big) \vee \big((b^\perp \wedge a^\perp)^\perp \wedge a^\perp \wedge c^\perp\big)$.
   Writing $x = a^\perp \wedge c^\perp$ and $y = b^\perp \vee a^\perp$, let us use the orthomodular law formula derived at the beginning of this proof. We obtain $a^\perp \wedge c^\perp = x = (y^\perp \vee x) \vee \big((y^\perp \wedge x)^\perp \wedge x\big) = u$.
   As a result, we have $a\big((ba)c\big) = (a^\perp \wedge c^\perp) \vee c = ac$, which we wanted to prove.
4. $0a = bb$; we have $(0a) = (0 \vee a)^\perp \vee a = (1 \wedge a^\perp) \vee a = a^\perp \vee a = 1 = bb$.
5. $(a\Delta b)\Delta c = a\Delta(b\Delta c)$, the operation Δ is associative in A as well as in the corresponding orthomodular lattice.
6. $a\Delta bb = a0$; we have $(a0) = (a \vee 0)^\perp \vee 0 = a^\perp = a\Delta 1 = a\Delta bb$.
7. $bb\Delta a = a0$; we have $(a0) = (a \vee 0)^\perp \vee 0 = a^\perp = 1\Delta a = bb\Delta a$.

8. $(a\Delta b)\big((ab)b\big) = (ab)b$; we have $(ab)b = a \vee b$ and therefore $(a\Delta b) \vee (a \vee b) = a \vee b$. So $a\Delta b \leq a \vee b$.

□

Making use of the above equivalence of $\mathcal{A}$ and $\mathcal{D}$ we can express the notion of compatibility in the Abbott $\mathbb{XOR}$ algebras. Suppose that $a, b \in A$, $A \in \mathcal{A}$. We say that **the elements *a,b* are compatible** in A if they generate a Boolean subalgebra of A. This notion is associated with "commonsurability" in a quantum experiment (see e.g. [5]). It can be captured in the Abbott $\mathbb{XOR}$ algebras as well, though not as economically as one would hope for.

Proposition 2.4 *Let A be an Abbott $\mathbb{XOR}$ algebra and let $a, b \in A$. Then a, b are compatible in A if either of the following two conditions is satisfied:*

1. $a = \Big(\Big(\Big(\big((a0)(b0)\big)(b0)\Big)0\Big)\Big(\Big(\big((a0)b\big)b\Big)0\Big)\Big)\Big(\Big(\big((a0)b\big)b\Big)0\Big)$

2. $a\Delta b = \Big(\Big(\Big(\big((a0)b\big)\Big)0\Big)\Big(\Big(\big(a(b0)\big)(b0)\Big)0\Big)\Big)\Big(\Big(\big(a(b0)\big)(b0)\Big)0\Big)$

A corollary: A is a Boolean algebra if and only if either of the above equalities is valid for any $a, b \in A$.

Proof If we rewrite the equality *1.* in the corresponding orthomodular lattice, we obtain $a = (a^\perp \vee b^\perp)^\perp \vee (a^\perp \vee b)^\perp = (a \wedge b) \vee (a \wedge b^\perp)$. Analogously, we can derive that $a\Delta b = (a \wedge b^\perp) \vee (b \wedge a^\perp)$. Either of the above formulas for a and $a\Delta b$ guarantee that a, b are compatible (see e.g. [9]). □

It may be noted that if one is allowed to use 3 variables, A is Boolean if and only if $a(bc) = (ab)(ac)$ (see [1]). A two variable formula for A to be Boolean can also be derived from this 3 variable formula.

Let us illustrate Theorem 2.3 on one example.

Example 2.5 Let $P = \{1, 2, 3, 4\}$ and let us consider the orthomodular difference lattice $(L, 0, 1, \vee, \wedge, ^\perp, \Delta)$, where L is the collection of all subsets of P of an even cardinality, 0 is the empty set, $1 = P$, the operation $^\perp$ is the complementation operation and Δ is the symmetric difference in P. By the previous theorem, this orthomodular lattice could be viewed as an Abbott $\mathbb{XOR}$ algebra when we set $(AB) = (A \vee B)^\perp \vee B$. The Cartesian product of L interpreted in the corresponding categories presents a prominent example in the quantum logic theory. It is a merely matter of taste which algebraic language we adopt to it, the technicalities may seem equally complex.

For a potential application within the quantum logic theory, let us introduce the notion of a state.

Definition 2.6 Let A be an Abbott $\mathbb{XOR}$ algebra. Let $s : A \to [0, 1]$ be a mapping that satisfies the following conditions ($a, b, c \in A$):

1. $s(aa) = 1$,
2. if $a(b0) = bb$, then $s\big((ab)b\big) = s(a) + s(b)$,
3. $s(a\Delta b) \leq s(a) + s(b)$.

Then s is said to be **a state in** A.

Proposition 2.7 *Let $\mathcal{A}$ be equivalent to $\mathcal{D}$ in the sense of Theorem 1.4. If $A \in \mathcal{A}$ and s is a state of A then s can be viewed as an "orthomodular" state of $F(A)$, and vice versa.*

Proof Recall ([4]) that a state on $\mathcal{D}$ is defined as follows ($a, b \in \mathcal{D}$):

1. $s(1) = 1$
2. If $a \leq b^{\perp} \implies s(a \vee b) = s(a) + s(b)$
3. $a(a \Delta b) \leq s(a) + s(b)$.

It is easy to see that the state space of A is isomorphic (via the isomorphism of *Theorem 1.4*) with the state space of $D = F(A)$. ◻

Let us summarize main results of our paper. The category $\mathcal{A}$ of the Abbott $\mathbb{XOR}$ algebras is equivalent to the category $\mathcal{D}$ of the orthomodular difference lattices, and the respective state spaces are isomorphic. So the knowledge we have acquired on $\mathcal{D}$ and on its state space can be translated into the corresponding category $\mathcal{A}$. For instance, since we know the characterization of the set-representable objects of $\mathcal{D}$, and these are precisely those that have an "abundance" of two-valued states (see [9]), we easily derive the set-representability characterization of the Abbott $\mathbb{XOR}$ algebras. In a similar vein, we can find Abbott $\mathbb{XOR}$ algebras without any state or with a precisely one state (see [7] and [12]). Also, we find that the free Abbott $\mathbb{XOR}$ algebra over 2 generators contains precisely 128 elements and the free algebra over 3 generators is infinite (see [10, 11], etc.). A specific line of algebraic nature is the notion of modularity in the Abbott algebras. We intend to consider this notion elsewhere.

Acknowledgements The author acknowledges the support by the Austrian Science Fund (FWF): Project I 4579-N and the Czech Science Foundation (GAČR): Project 20-09869L.
The author would like to express her gratitude to P. Pták and M. Matoušek for several valuable suggestions.

Statements and Declarations

This work was supported by the Austrian Science Fund (FWF): Project I 4579-N and the Czech Science Foundation (GAČR): Project 20-09869L. The author has no relevant financial or non-financial interests to disclose.

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