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Group-Covariant Stochastic Products and Phase-Space Convolution Algebras

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Abstract

A quantum stochastic product is defined as a binary operation on the convex set of quantum states that preserves the convex structure. We discuss a class of group-covariant, associative stochastic products, the twirled products, having remarkable connections with quantum measurement theory and with the theory of open quantum systems. By extending this binary operation from the density operators to the full Banach space of trace class operators, one obtains a Banach algebra. In the case where the covariance group is the group of phase-space translations, one has a quantum convolution algebra. The expression of the quantum convolution in terms of Wigner distributions and of the associated characteristic functions is analyzed.

Keywords Quantum state · Quantum stochastic product · Quantum measurement · Operator algebra · Convolution algebra · Square integrable representation

1 Introduction

Operator algebras are ubiquitous in physical theories involving a Hilbert space structure; especially — just to mention the most relevant examples — in quantum mechanics, quantum information science, quantum field theory, quantum statistical mechanics and noncommutative geometry [1–8]. Within the operator algebra framework, the physical *states* of a quantum system can be introduced as, suitably normalized, *positive functionals* on the C*-algebra of all bounded *observables* [1, 4, 5, 8]. In the most elementary case (say, in 'ordinary quantum mechanics'), this abstract algebra is isometrically $*$-isomorphic to — and, thus, can be identified with — the Banach space $\mathcal{B}(\mathcal{H})$ of all bounded operators on a separable complex Hilbert space $\mathcal{H}$, endowed with the usual composition of operators $(A, B) \mapsto AB \equiv A \circ B$, the algebra product, and with the adjoining operation $A \mapsto A^*$, i.e., the algebra involution.

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In several applications however — e.g., in the context of quantum information theory, quantum control and quantum measurement theory [6, 7] — one actually restricts to a distinguished class of states, the so-called σ*-additive* states [1, 8] (which is analogous to restricting, in classical statistical mechanics, to σ-additive probability measures). These states can be realized as normalized, positive *trace class* operators; namely, the so-called *density operators* (or density states), that form a *convex subset* $\mathcal{D}(\mathcal{H})$ of the complex Banach space $\mathcal{T}(\mathcal{H})$ of trace class operators on $\mathcal{H}$.

The *selfadjoint component* $\mathcal{B}(\mathcal{H})_{\mathbb{R}}$ of the C*-algebra $\mathcal{B}(\mathcal{H})$, that coincides with the set of all *true* bounded observables, in its own respect, can be endowed with a *two-fold* algebraic structure. In fact, the *real* Banach space $\mathcal{B}(\mathcal{H})_{\mathbb{R}}$ can be endowed with the pair of binary operations

$$A \bullet B := \frac{1}{2}(AB + BA) \quad \text{and} \quad A \diamond B := \frac{1}{2\mathrm{i}}(AB - BA), \tag{1}$$

the so-called *Jordan* (symmetric) and *Lie* (skew-symmetric) products. The triple

$$(\mathcal{B}(\mathcal{H})_{\mathbb{R}}, (\cdot) \bullet (\cdot), (\cdot) \diamond (\cdot)) \tag{2}$$

is a *Jordan-Lie Banach algebra*, where the two products — that determine a Jordan and a Lie algebra structure, respectively — are mutually related by the Leibniz rule and by the associator identity; conversely, via complexification, every Jordan-Lie Banach algebra can be promoted to a C*-algebra [1, 5].

We stress that, on the one hand, *states* do not directly fit in the Jordan-Lie Banach algebra structure (2). In fact, for every pair of density operators $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, the Jordan product $\rho \bullet \sigma$ is a selfadjoint trace class operator, *but* it is a density operator iff $\rho = \sigma \equiv P$, where P is a *pure* state [9], i.e., a rank-one orthogonal projection; the Lie product $\rho \diamond \sigma$, moreover, is a selfadjoint trace class operator too, *but* it simply cannot be a density operator because $\mathrm{tr}(\rho \diamond \sigma) = 0$. On the other hand, one may ask [9–12] whether it is possible to endow the Banach space $\mathcal{T}(\mathcal{H})$ with some binary operation

$$(\cdot) \boxdot (\cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}) \tag{3}$$

satisfying suitable assumptions, that should be consistent with the notion of physical state. Therefore, as a starting point, we suppose that

(A1) The pair $(\mathcal{T}(\mathcal{H}), (\cdot) \boxdot (\cdot))$ defines an *algebra* — i.e., the product (3) is bilinear — and, in addition, the algebra product is *state-preserving*. Namely, we suppose that *the product of two states* $\rho \boxdot \sigma$ *is a state too*.

If we wish to obtain a — both mathematically and physically — interesting structure, we should further require that

(A2) This algebra $(\mathcal{T}(\mathcal{H}), (\cdot) \boxdot (\cdot))$ is *associative*.
(A3) The algebra product $(\cdot) \boxdot (\cdot)$ is *continuous* w.r.t. some topology suitably consistent with physics (quantum mechanics).

Owing to the parallelism between classical and quantum physics [5], if such an algebraic structure does exist in the quantum setting, one should expect that an analogous structure may exist in the classical setting too. Indeed, let us consider the *convolution* $\mu \circledast \nu$ of a pair μ, ν of *complex Radon measures*, defined on a *locally compact topological group* G. It is well known that all measures of this kind form a Banach space $\mathcal{M}(G)$, which endowed with the convolution product, becomes a *Banach algebra* (containing the smaller group algebra $\mathrm{L}^1(G)$) [13]. If μ, ν are *probability measures* on G — here regarded as classical states — then $\mu \circledast \nu$ is a probability measure (i.e., a classical state) too. From the physical point of view, the most relevant case is that of the vector group $G = \mathbb{R}^n \times \mathbb{R}^n$ — regarded as the group of translations on phase space — and the associated *phase-space convolution algebra* $\mathcal{M}(\mathbb{R}^n \times \mathbb{R}^n)$.

The observation that convolution is a group-theoretical operation and the central role played in quantum mechanics by the *symmetry transformations* — see [14, 15], and references therein — lead us to consider a group-theoretical framework in the quantum setting, as well. Therefore, we will further assume that

(A4) One can define a whole class of associative, state-preserving bilinear products on $\mathcal{T}(\mathcal{H})$, and a generic product $(\cdot) \boxdot (\cdot)$ of this class is — except for special or trivial cases — a *genuinely binary* operation (the map $(\cdot) \boxdot (\cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ should depend, in general, on *both* its arguments).

(A5) The construction of each product involves a *symmetry group* G and possesses some *covariance* property w.r.t. a *symmetry action* of G.

(A6) By a natural analogy with the 'classical' convolution algebra $(\mathcal{M}(G), (\cdot) \circledast (\cdot))$, in the case where the symmetry group G is *abelian*, the algebra $(\mathcal{T}(\mathcal{H}), (\cdot) \boxdot (\cdot))$ is commutative.

The assumption (A6) seems to be quite reasonable even if it refers to a quantum setting: Non-commutativity is a natural feature for an algebra of quantum *observables*, whereas one may well expect an algebra involving *states* to behave differently.

We conclude with the following further ansatz:

(A7) In the remarkable case where the symmetry group is the phase-space translation group — i.e., for $G = \mathbb{R}^n \times \mathbb{R}^n$ — one should get a *quantum convolution algebra* that may be regarded, in some suitable sense, as a quantum counterpart of the classical phase-space convolution algebra.

A binary operation on the trace class $\mathcal{T}(\mathcal{H})$ satisfying our first assumption (A1) is called a *stochastic product*, and the Banach space $\mathcal{T}(\mathcal{H})$, endowed with a stochastic product verifying (A2) too, is called a *stochastic algebra* [10]. For such an algebra, condition (A3) holds automatically w.r.t. the norm topology on $\mathcal{T}(\mathcal{H})$ (in fact, (A3) turns out to be a consequence of (A1) alone). Moreover, one can introduce, by means of a group-theoretical construction, a class of associative stochastic products — the so-called *twirled products* [9–12] — that satisfy assumptions (A4)–(A7), as well.

The paper is organized as follows. In Section 2, we introduce the notion of stochastic product of quantum states. Next, in Section 3, we provide an explicit group-theoretical construction of the twirled products. These products induce a class of Banach algebras, the *twirled stochastic algebras* — see Section 4 — and admit a nice physical interpretation (Section 5). In Section 6, we consider two remarkable examples: the compact groups and the group of phase-space translations. Finally, in Section 7, a few conclusions are drawn, together with a glance at some future prospects.

2 The Notion of Stochastic Product of Quantum States

A map $\mathfrak{S} \colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ which is *convex-linear* — namely, that preserves the natural convex structure of the space of density operators $\mathcal{D}(\mathcal{H})$ — is called a *quantum stochastic map*. It can be shown that such a map admits a unique (trace-preserving, positive) *linear extension* $\mathfrak{S}_{\text{ext}} \colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ [10], a *linear stochastic map* on the trace class $\mathcal{T}(\mathcal{H})$.

Analogously, a *quantum stochastic product* is defined as as a binary operation on $\mathcal{D}(\mathcal{H})$,

$$(\cdot) \odot (\cdot) \colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}), \tag{4}$$

that is *convex-linear w.r.t. both its arguments*; namely, for all $\rho, \sigma, \tau, \upsilon \in \mathcal{D}(\mathcal{H})$ and all $\alpha, \epsilon \in [0, 1]$, it is assumed that the following relation is satisfied:

$$\begin{aligned}(\alpha\rho + (1-\alpha)\sigma) \odot (\epsilon\tau + (1-\epsilon)\upsilon) = \alpha\epsilon\rho \odot \tau + \alpha(1-\epsilon)\rho \odot \upsilon \\ + (1-\alpha)\epsilon\sigma \odot \tau + (1-\alpha)(1-\epsilon)\sigma \odot \upsilon.\end{aligned} \tag{5}$$

We will also use the following notion: A binary operation $(\cdot) \boxdot (\cdot)$ on the trace class $\mathcal{T}(\mathcal{H})$ is called *state-preserving* if it is such that $\mathcal{D}(\mathcal{H}) \boxdot \mathcal{D}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})$.

Every quantum stochastic product is continuous w.r.t. the natural topology on $\mathcal{D}(\mathcal{H})$ — see Remark 1 below — and admits a (state-preserving) bilinear extension. In fact, the following result holds [10]:

Proposition 1 *Every quantum stochastic product on $\mathcal{D}(\mathcal{H})$ is continuous w.r.t. the norm topology inherited from $\mathcal{T}(\mathcal{H})$; namely, denoting by $\|\cdot\|_1 \equiv \|\cdot\|_{\mathrm{tr}}$ the trace norm on $\mathcal{T}(\mathcal{H})$, w.r.t. the topology on $\mathcal{D}(\mathcal{H})$ and on $\mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H})$ induced, respectively, by the metrics*

$$\mathsf{d}_1(\rho, \sigma) := \|\rho - \sigma\|_1 \quad \text{and} \quad \mathsf{d}_{1,1}((\rho, \tau), (\sigma, \upsilon)) := \max\{\|\rho - \sigma\|_1, \|\tau - \upsilon\|_1\}. \tag{6}$$

For every quantum stochastic product $(\cdot) \odot (\cdot) \colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$, there is a unique bilinear stochastic map (or stochastic product) $(\cdot) \boxdot (\cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ — i.e., a unique state-preserving bilinear map on $\mathcal{T}(\mathcal{H})$ — such that $\rho \odot \sigma = \rho \boxdot \sigma$, for all $\rho, \sigma \in \mathcal{D}(\mathcal{H})$.

Remark 1 It can be shown that the weak and the strong topologies on $\mathcal{D}(\mathcal{H})$ (inherited from $\mathcal{B}(\mathcal{H})$), as well as the topologies induced on $\mathcal{D}(\mathcal{H})$ by the metrics associated with Schatten p-norms $\|\cdot\|_{\mathsf{p}}$, $1 \le \mathsf{p} \le \infty$, all coincide [10]. This unique topology on $\mathcal{D}(\mathcal{H})$ is called the *standard topology*. Thus, the topology induced on $\mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H})$ by the metric $\mathsf{d}_{1,1}$ is precisely the product topology associated with the standard topology on $\mathcal{D}(\mathcal{H})$.

Definition 1 The space $\mathcal{T}(\mathcal{H})$, endowed with a map $(\cdot) \boxdot (\cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ that is bilinear, state-preserving and *associative*, is called a *stochastic algebra*.

It is worth observing that, by Proposition 1, every quantum stochastic product $(\cdot) \odot (\cdot)$ on $\mathcal{D}(\mathcal{H})$ can be regarded as the restriction of a uniquely determined bilinear stochastic map $(\cdot) \boxdot (\cdot)$ on $\mathcal{T}(\mathcal{H})$, which is *associative* iff the product $(\cdot) \odot (\cdot)$ is associative. As a consequence, a stochastic algebra can also be defined as a Banach space of trace class operators $\mathcal{T}(\mathcal{H})$, together with an *associative* quantum stochastic product $(\cdot) \odot (\cdot) \colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$.

Now, let $\mathsf{BL}(\mathcal{H})$ be the complex vector space of all bounded bilinear maps on $\mathcal{T}(\mathcal{H})$. $\mathsf{BL}(\mathcal{H})$ becomes a Banach space when endowed with the norm $\|\cdot\|_{(1)}$ defined as follows. For every $\beta(\cdot, \cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ in $\mathsf{BL}(\mathcal{H})$, we put

$$\|\beta(\cdot, \cdot)\|_{(1)} := \sup\{\|\beta(A, B)\|_1 \colon \|A\|_1, \|B\|_1 \le 1\}. \tag{7}$$

Denoting by $\mathcal{T}(\mathcal{H})_{\mathbb{R}} \subset \mathcal{T}(\mathcal{H})$ the *real* Banach space of all *selfadjoint* trace class operators on $\mathcal{H}$, one can prove the following result [10]:

Proposition 2 *Every bilinear stochastic map $(\cdot) \boxdot (\cdot) \colon \mathcal{T}(\mathcal{H}) \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is bounded and its norm is such that $\|(\cdot) \boxdot (\cdot)\|_{(1)} \le 2$, whereas, for its restriction $(\cdot) \boxminus (\cdot)$ to a bilinear map on the real Banach space $\mathcal{T}(\mathcal{H})_{\mathbb{R}}$, we have that $\|(\cdot) \boxminus (\cdot)\|_{(1)} = 1$. It follows that, whenever a stochastic product $(\cdot) \boxdot (\cdot)$ on $\mathcal{T}(\mathcal{H})$ is associative, the pair $(\mathcal{T}(\mathcal{H})_{\mathbb{R}}, (\cdot) \boxminus (\cdot))$ is a real Banach algebra, since, for all $A, B \in \mathcal{T}(\mathcal{H})_{\mathbb{R}}$, $\|A \boxminus B\|_1 \le \|A\|_1 \|B\|_1$.*

Remark 2 With regard to the previous proposition, note that the restriction of a stochastic product $(\cdot) \boxdot (\cdot)$ on $\mathcal{T}(\mathcal{H})$ to a bilinear map on $\mathcal{T}(\mathcal{H})_{\mathbb{R}}$ is well defined, because the fact that the map $(\cdot) \boxdot (\cdot)$ is bilinear and state-preserving implies that it is also *adjoint-preserving*; hence, $\mathcal{T}(\mathcal{H})_{\mathbb{R}} \boxdot \mathcal{T}(\mathcal{H})_{\mathbb{R}} \subset \mathcal{T}(\mathcal{H})_{\mathbb{R}}$. Also note that the inequality $\|(\cdot) \boxdot (\cdot)\|_{(1)} \leq 2$ may *not* be saturated. E.g., the algebra $(\mathcal{T}(\mathcal{H}), (\cdot) \boxdot (\cdot))$ may well be a Banach algebra too; see the final claim of Theorem 1 below.

3 Explicit Construction of Stochastic Products: The Twirled Products

We will now show that a suitable group-theoretical construction leads us to an interesting class of quantum stochastic products, the so-called *twirled products* [10]. A fundamental mathematical tool for this construction are the so-called *square integrable* representations.

3.1 The Main Tool: Square Integrable Representations

Let G be a locally compact, second countable Hausdorff topological group; in short, a *l.c.s.c. group*. Denoting by $\mathcal{U}(\mathcal{H})$ the unitary group of a separable complex Hilbert space $\mathcal{H}$, let the map $U: G \to \mathcal{U}(\mathcal{H})$ be an *irreducible projective representation* of G. The fact that the representation U is supposed, in general, to be *projective* entails that

$$U(gh) = \gamma(g,h)U(g)U(h), \tag{8}$$

where $\gamma: G \times G \to \mathbb{T}$ is a Borel function, the *multiplier* associated with U [16]. In particular, we say that U is *unitary* if $\gamma \equiv 1$.

We will assume that the scalar product $\langle\cdot,\cdot\rangle$ in $\mathcal{H}$ is *conjugate-linear* in its *first* argument, and we fix a normalization μ_G of the left *Haar measure* [13, 16] on G. Then, for every pair of vectors $\psi, \phi \in \mathcal{H}$, we can define the *coefficient*

$$\kappa_{\psi\phi}: G \ni g \mapsto \langle U(g)\,\psi, \phi\rangle \in \mathbb{C}, \tag{9}$$

which is a bounded Borel function, and we consider, in particular, the distinguished set of coefficient functions

$$\mathcal{A}(U) := \left\{\psi \in \mathcal{H}\colon \exists\phi \in \mathcal{H},\ \text{with } \phi \neq 0, s.t.\ \kappa_{\psi\phi} \in \mathrm{L}^2(G, \mu_G; \mathbb{C})\right\}. \tag{10}$$

The set $\mathcal{A}(U)$ — consisting of all *admissible vectors* for the representation U — is a linear subspace of $\mathcal{H}$, that is either trivial or dense in $\mathcal{H}$.

Definition 2 The projective representation U is called *square integrable* if $\mathcal{A}(U) \neq \{0\}$; equivalently, if $\mathcal{A}(U)$ is a *dense* linear subspace of $\mathcal{H}$.

For further information on the theory of square integrable — unitary or, more generally, projective — representations, the associated harmonic analysis, the numerous applications and related topics, see [9, 17–25], and the bibliography therein.

Here, for the sake of simplicity, we will briefly outline a few facts, focusing, in particular, on the case where the l.c.s.c. group G is *unimodular* [13]. Namely, we will suppose that the Haar measure μ_G is both left and right invariant. In this case, if $U: G \to \mathcal{U}(\mathcal{H})$ is a square integrable projective representation, then it can be shown that *all* vectors in $\mathcal{H}$ are admissible; i.e., $\mathcal{A}(U) = \mathrm{L}^2(G, \mu_G; \mathbb{C})$. Moreover, *all* coefficient functions — i.e., all functions of the form (9) — are square integrable w.r.t. the Haar measure μ_G and satisfy the *orthogonality relations*

$$\int_G \mathrm{d}\mu_G(g)\, \overline{\kappa_{\phi\eta}(g)}\kappa_{\psi\chi}(g) = \mathsf{c}_U \langle \eta, \chi\rangle\langle\psi, \phi\rangle, \quad \forall \eta, \chi, \psi, \phi \in \mathcal{H}, \tag{11}$$

where c_U is a *strictly positive constant* depending on the representation U and on the given normalization of μ_G (but *not* on the choice of $\eta, \chi, \psi, \phi \in \mathcal{H}$).

Remark 3 The orthogonality relations in the *non-unimodular case* — consider, e.g., the square integrable unitary representations of the semidirect product $\mathbb{R} \rtimes \mathbb{R}_*^+$ (or, also, the group $\mathbb{R} \rtimes \mathbb{R}_*$), i.e., of the one-dimensional *affine group*, that are fundamental in *wavelet analysis* (see [18, 20, 21, 24, 25], and references therein) — are somewhat more complicated. Precisely, the orthogonality relations of a square integrable representation $U\colon G \to \mathcal{U}(\mathcal{H})$ involve, in general, a *positive selfadjoint operator* D_U on $\mathcal{H}$ (which is sometimes called the *Duflo-Moore operator* [9, 20, 22–25]), whose domain coincides with the dense linear subspace $\mathcal{A}(U)$ of $\mathcal{H}$, and that is bounded iff G is unimodular. In the latter case, D_U is simply a positive multiple of the identity, i.e.,

$$D_U = \mathsf{d}_U \mathrm{Id}, \quad \text{where } \mathsf{d}_U \equiv \mathsf{c}_U^{1/2}, \tag{12}$$

whence we recover the simplified form (11) of orthogonality relations.

3.2 Remarkable Cases

In Section 6, explicit examples of quantum stochastic products will involve two remarkable types of square integrable representations of (in both cases, unimodular) l.c.s.c. groups:

(T1) We will consider, at first, the case of a *compact* group G. As is well known [13], such a group is always unimodular, and all its irreducible *unitary* representations are finite-dimensional. They are square integrable, as well, since in the compact case the Haar measure μ_G is *finite*. In this case, moreover, relations (11) are nothing but the classical *Schur orthogonality relations* and, according to the *Peter-Weyl theorem*, if μ_G is normalized as a probability measure ($\mu_G(G) = 1$), then $\mathsf{c}_U = \dim(\mathcal{H})^{-1}$ [13].

(T2) We will also consider the irreducible projective representations of the group $\mathbb{R}^n \times \mathbb{R}^n$ of *phase-space translations* — regarded as a direct product of the subgroups of *position* (with n position degrees of freedom) and *momentum* translations — that are characterized by a *symplectic multiplier* [26] γ_{h}. Explicitly, we have:

$$\gamma_{\mathsf{h}}(q, p; \tilde{q}, \tilde{p}) := \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})/2\mathsf{h}), \quad (q, p), (\tilde{q}, \tilde{p}) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{13}$$

Here, the parameter h ranges over all nonzero real numbers and its modulus can be regarded as Planck's constant $\hbar$; moreover, the product $q \cdot \tilde{p} = q_1\tilde{p}_1 + q_2\tilde{p}_2 + \cdots$ (or $p \cdot \tilde{q}$) should be regarded as a *pairing* between the vector q and the co-vector $\tilde{p}$. Those genuinely projective representations are infinite-dimensional and square integrable, and, for each fixed value of the parameter h, they form precisely a single unitary equivalence class. It is worth observing that, from the physical point of view, one can actually restrict to the positive values of h only. In fact, one can easily check that two representations belonging, respectively, to the unitary equivalence classes associated with h and $-\mathsf{h}$ are *anti*-unitarily — hence, physically — equivalent [25, 26]. It is clear, moreover, that selecting a certain value of $\hbar \equiv \mathsf{h} > 0$ amounts to choosing suitable physical units for Planck's constant (i.e., the physical units of an *action*). An irreducible representation of the group $\mathbb{R}^n \times \mathbb{R}^n$, with multiplier $\gamma_\hbar$, is usually called a ($\hbar$-)*Weyl system* [26].

3.3 The Twirled Products

Let us first summarize the main notations and assumptions for our construction:

- We suppose that G is a *unimodular* l.c.s.c. group, and that G admits square integrable representations.
- We denote by $\mathscr{B}(G)$ the Borel σ-algebra of G and by $\mathscr{P}(G)$ the set of all Borel probability measures on G.
- Next, we select a *square integrable projective representation* $U\colon G \to \mathcal{U}(\mathcal{H})$. The assumption that U is square integrable cannot be dispensed with, because it ensures the validity of Proposition 4 below [10], which is a fundamental step for our construction.
- For the sake of simplicity, it is convenient to suppose henceforth that the Haar measure μ_G is normalized in such a way that $\mathsf{c}_U = 1$; see (11).
- The representation U induces an *isometric* representation $G \ni g \mapsto \mathsf{S}_U(g)$ in the Banach space $\mathcal{T}(\mathcal{H})$, i.e.,
$$\mathsf{S}_U(g)T := U(g)TU(g)^*, \quad T \in \mathcal{T}(\mathcal{H}). \tag{14}$$
The mapping
$$G \times \mathcal{D}(\mathcal{H}) \ni (g, \rho) \mapsto \rho_g \equiv \mathsf{S}_U(g)\rho \in \mathcal{D}(\mathcal{H}) \tag{15}$$
is the standard *symmetry action* [14, 15] of G on the convex set $\mathcal{D}(\mathcal{H})$ of density states. Although U is, in general, *projective* — see (8) — S_U behaves as a group *homomorphism*: $\mathsf{S}_U(gh) = \mathsf{S}_U(g)\,\mathsf{S}_U(h)$.
- We further select a *fiducial density operator* $\upsilon \in \mathcal{D}(\mathcal{H})$ and a *measure* $\varpi \in \mathscr{P}(G)$. Let ϖ^g (ϖ_g) be the left (right) g-translate of ϖ; i.e., for every $g \in G$ and $\mathcal{E} \in \mathscr{B}(G)$,
$$\varpi^g(\mathcal{E}) := \varpi(g^{-1}\mathcal{E}), \quad \varpi_g(\mathcal{E}) := \varpi(\mathcal{E}g). \tag{16}$$
E.g., $\varpi = \delta \in \mathscr{P}(G)$ is the *Dirac measure* at the identity $e \in G$.
- In the following, all integrals of operator-valued (precisely, $\mathcal{T}(\mathcal{H})$-valued) functions on G w.r.t. a probability measure should be regarded as *Bochner integrals*.

With our previous notations and assumptions, we have the following two results [10]:

Proposition 3 *For every probability measure $\mu \in \mathscr{P}(G)$, the linear map*
$$\mu[U]\colon \mathcal{T}(\mathcal{H}) \ni T \mapsto \int_G \mathrm{d}\mu(g)\,(\mathsf{S}_U(g)T) \in \mathcal{T}(\mathcal{H}) \tag{17}$$
is both positive and trace-preserving. Therefore, we can define the quantum stochastic map
$$\mathcal{D}(\mathcal{H}) \ni \rho \mapsto \mu[U]\rho \in \mathcal{D}(\mathcal{H}). \tag{18}$$

Proposition 4 *For every density state $\rho \in \mathcal{D}(\mathcal{H})$, the mapping*
$$\nu_{\rho,\upsilon}\colon \mathscr{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\,\mathrm{tr}\big(\rho(\mathsf{S}_U(g)\upsilon)\big) \in \mathbb{R}^+ \tag{19}$$
belongs to $\mathscr{P}(G)$.

Taking into account the above facts, we can now define a *binary operation* on $\mathcal{D}(\mathcal{H})$, associated with the triple (U, υ, ϖ); i.e., we set

$$\rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma := \big((\nu_{\rho,\upsilon} \circledast \varpi)[U]\big)\sigma = \int_G \mathrm{d}(\nu_{\rho,\upsilon} \circledast \varpi)(g)\ (\mathsf{S}_U(g)\sigma), \quad \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}). \tag{20}$$

Let us analyze this definition. First, we have considered the fact that $\nu_{\rho,\upsilon}$ is a probability measure associated with the density operators ρ and υ (Proposition 4). Then, we have formed the convolution $\mu \equiv \nu_{\rho,\upsilon} \circledast \varpi \in \mathscr{P}(G)$ of $\nu_{\rho,\upsilon}$ with the previously selected probability measure ϖ. Eventually, we have applied the stochastic map $(\nu_{\rho,\upsilon} \circledast \varpi)[U]$ — constructed according to Proposition 3 — to the density state σ.

We can put this product in a more explicit form:

$$\rho \overset{\upsilon}{\underset{\varpi}{\odot}} \sigma = \int_G \mathrm{d}\mu_G(g) \int_G \mathrm{d}\varpi(h)\ \mathrm{tr}\big(\rho(\mathsf{S}_U(g)\upsilon)\big)(\mathsf{S}_U(gh)\sigma). \tag{21}$$

Definition 3 We call the binary operation on $\mathcal{D}(\mathcal{H})$ defined by (20) the *twirled product* generated by the triple (U, υ, ϖ), where U is called the *inducing representation* of the twirled product, the states ρ, υ and σ are called the *input*, the *probe* and the *whirligig*, respectively, and $\varpi \in \mathscr{P}(G)$ is called the *smearing measure*.

E.g., with $\varpi = \delta$ in (21), we get

$$\rho \overset{\upsilon}{\odot} \sigma \equiv \rho \overset{\upsilon}{\underset{\delta}{\odot}} \sigma = \int_G \mathrm{d}\mu_G(g)\ \mathrm{tr}\big(\rho(\mathsf{S}_U(g)\upsilon)\big)(\mathsf{S}_U(g)\sigma), \tag{22}$$

namely, the *un-smeared* twirled product generated by the pair (U, υ) (see Section 5).

4 Banach Algebra Structure, Covariance, Invariance and Equivariance

By the following results (see [10] for their proof), the twirled product turns out to satisfy our initial assumptions (A1)–(A6); see Section 1. The further assumption (A7), concerning the case where $G = \mathbb{R}^n \times \mathbb{R}^n$, will be discussed in Section 6.

Theorem 1 *The twirled product*

$$(\cdot) \overset{\upsilon}{\underset{\varpi}{\odot}} (\cdot) \colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}), \tag{23}$$

generated by the triple (U, υ, ϖ) — for any square integrable projective representation $U\colon G \to \mathcal{U}(\mathcal{H})$, any probe $\upsilon \in \mathcal{D}(\mathcal{H})$ and any smearing measure $\varpi \in \mathscr{P}(G)$ — is an associative quantum stochastic product. Extending this product to a state-preserving bilinear map on the space $\mathcal{T}(\mathcal{H})$ of trace class operators, one obtains a Banach algebra; i.e., a stochastic Banach algebra. In particular, in the case where the l.c.s.c. group G is abelian, this algebra is commutative.

We will now argue that every twirled product enjoys a natural property of *covariance* w.r.t. the action of the relevant group G on the input state of the product.

Besides, two further properties regarding, instead, *families* of twirled products — i.e., *invariance* and *equivariance* — are also satisfied. To define these properties, we consider a G-space [16] X endowed with a (left) group action

$$(\cdot)[\cdot]\colon G \times X \ni (g, x) \mapsto g[x] \in X. \tag{24}$$

Definition 4 Let the points of X label a family of quantum stochastic products, namely,

$$\left\{(\cdot) \overset{x}{\odot} (\cdot) \colon \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})\right\}_{x \in X}. \tag{25}$$

This family is called *invariant* w.r.t. the action $(\cdot)[\cdot] \colon G \times X \to X$ if

$$\rho \overset{x}{\odot} \sigma = \rho \overset{g[x]}{\odot} \sigma, \quad \forall g \in G,\ \forall x \in X,\ \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}). \tag{26}$$

Moreover, we say that the family of products (25) is *right inner equivariant* w.r.t. the pair $((\cdot)[\cdot] \colon G \times X \to X, U)$, where $U \colon G \to \mathcal{U}(\mathcal{H})$ is a projective representation, if

$$\rho \overset{x}{\odot} (\mathsf{S}_U(g^{-1})\sigma) = \rho \overset{g[x]}{\odot} \sigma, \quad \forall g \in G,\ \forall x \in X,\ \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}). \tag{27}$$

Theorem 2 *The twirled product generated by the triple (U, υ, ϖ) is left-covariant w.r.t. the representation U, namely, it satisfies the relation*

$$\rho_g \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \left(\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma\right)_g, \quad \forall g \in G,\ \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}), \tag{28}$$

where we have set $\rho_g \equiv \mathsf{S}_U(g)\rho$.

Moreover, the family of twirled products

$$\left\{(\cdot) \underset{\varpi}{\overset{\upsilon}{\odot}} (\cdot) \colon \upsilon \in \mathcal{D}(\mathcal{H}),\ \varpi \in \mathscr{P}(G)\right\} \tag{29}$$

is invariant w.r.t. the group action $(\cdot)[\cdot] \colon G \times (\mathcal{D}(\mathcal{H}) \times \mathscr{P}(G)) \to \mathcal{D}(\mathcal{H}) \times \mathscr{P}(G)$, where

$$g[(\upsilon, \varpi)] := (\upsilon_g \equiv \mathsf{S}_U(g)\upsilon, \varpi^g); \tag{30}$$

namely, we have:

$$\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \rho \underset{\varpi^g}{\overset{\upsilon_g}{\odot}} \sigma, \quad \forall g \in G,\ \forall \rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H}),\ \forall \varpi \in \mathscr{P}(G). \tag{31}$$

Finally, the family of twirled products (29) is right inner equivariant w.r.t. the pair $((\cdot)[\cdot], U)$, where this time the group action $(\cdot)[\cdot] \colon G \times (\mathcal{D}(\mathcal{H}) \times \mathscr{P}(G)) \to \mathcal{D}(\mathcal{H}) \times \mathscr{P}(G)$ is of the form

$$g[(\upsilon, \varpi)] := (\upsilon, \varpi_g); \tag{32}$$

namely, we have:

$$\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma_{g^{-1}} = \rho \underset{\varpi_g}{\overset{\upsilon}{\odot}} \sigma, \quad \forall g \in G,\ \forall \rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H}),\ \forall \varpi \in \mathscr{P}(G). \tag{33}$$

5 Physical Contents of the Construction

Let us now briefly comment about the physical meaning of some mathematical objects used in the construction of twirled products:

1. The linear map $\mu[U] \colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ defined by (17) in Proposition 3 — which is often called *twirling (super-)operator* — plays a remarkable role in the theory of open quantum systems and quantum decoherence [27–32].

2. Taking into account that quantum measurements can be described in terms of positive operator-valued measures (POVMs, or quantum observables) and of quantum instruments [7], one can easily check that, for every probe $\upsilon \in \mathcal{D}(\mathcal{H})$, the mapping

$$\mathscr{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\ (\mathsf{S}_U(g)\upsilon) =: \mathsf{E}_\upsilon(\mathcal{E}), \tag{34}$$

where U is square integrable, is a *group-covariant quantum observable* — see [10, 33], and references therein — namely, a POVM which is covariant w.r.t. the projective representation U:

$$\mathsf{E}_\upsilon(g\mathcal{E}) = \mathsf{S}_U(g)\mathsf{E}_\upsilon(\mathcal{E}), \quad \forall g \in G,\ \forall \mathcal{E} \in \mathscr{B}(G). \tag{35}$$

3. By the preceding point, denoting by $\mathrm{L}^1(G)^+_{\mathrm{n}}$ the convex set of all normalized positive elements of $\mathrm{L}^1(G)$ (regarded as probability densities w.r.t. the Haar measure), the probability density

$$\mathsf{p}_{\rho,\upsilon} = \mathrm{tr}\big(\rho(\mathsf{S}_U(\cdot)\upsilon)\big) \in \mathrm{L}^1(G)^+_{\mathrm{n}}, \tag{36}$$

that is involved in the construction of the un-smeared twirled product (20), is nothing but the *probability distribution* on G of the quantum observable E_υ w.r.t. the state ρ. Namely, the Borel function $\mathsf{p}_{\rho,\upsilon}$ is the Radon-Nikodym derivative of the probability measure $\nu_{\rho,\upsilon}\colon \mathscr{B}(G) \ni \mathcal{E} \mapsto \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\ \mathrm{tr}\big(\rho(\mathsf{S}_U(g)\upsilon)\big) \in \mathbb{R}^+$ — see Proposition 4 — w.r.t. the Haar measure μ_G.

4. For every Borel set $\mathcal{E} \in \mathscr{B}(G)$, the map $\mathscr{I}^{\upsilon,\sigma}_{\mathcal{E}}\colon \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ defined by

$$\mathscr{I}^{\upsilon,\sigma}_{\mathcal{E}} T := \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\ \mathrm{tr}\big(T(\mathsf{S}_U(g)\upsilon)\big)(\mathsf{S}_U(g)\sigma), \quad T \in \mathcal{T}(\mathcal{H}) \tag{37}$$

— where U is square integrable — is a *quantum operation* [7]; in particular, for $\mathcal{E} = G$, a trace-preserving, positive linear map (i.e., a linear stochastic map, or quantum channel). The associated mapping

$$\mathscr{I}^{\upsilon,\sigma}_{(\cdot)}\colon \mathscr{B}(G) \ni \mathcal{E} \mapsto \mathscr{I}^{\upsilon,\sigma}_{\mathcal{E}} \tag{38}$$

is a quantum instrument; more precisely, a *U-covariant quantum instrument* based on G [10, 34], i.e.,

$$\mathscr{I}^{\upsilon,\sigma}_{g\mathcal{E}}(\mathsf{S}_U(g)T) = \mathsf{S}_U(g)\big(\mathscr{I}^{\upsilon,\sigma}_{\mathcal{E}} T\big), \quad \forall g \in G,\ \forall \mathcal{E} \in \mathscr{B}(G),\ \forall T \in \mathcal{T}(\mathcal{H}). \tag{39}$$

5. A connection between the quantum instrument $\mathscr{I}^{\upsilon,\sigma}_{(\cdot)}$ and the covariant POVM (34) is provided by the following relation:

$$\mathrm{tr}(\rho\mathsf{E}_\upsilon(\mathcal{E})) = \mathrm{tr}\big(\mathscr{I}^{\upsilon,\sigma}_{\mathcal{E}}\rho\big), \quad \forall \rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H}),\ \forall \mathcal{E} \in \mathscr{B}(G). \tag{40}$$

Namely, for every whirligig σ, the quantum instrument $\mathscr{I}^{\upsilon,\sigma}_{(\cdot)}$ is *compatible* [7] with the POVM E_υ associated with the same probe υ. From the physical point of view, this observation entails that the instrument and the POVM describe the same measurement outcome probabilities. More precisely, every E_υ-compatible quantum instrument $\mathscr{I}^{\upsilon,\sigma}_{(\cdot)}$, with $\sigma \in \mathcal{D}(\mathcal{H})$, describes a certain way of measuring the observable E_υ, which produces a certain type of state transformation depending on the whirligig σ.

6. Clearly, the un-smeared twirled product generated by the pair (U, υ) can be recovered from the quantum operation (37) simply putting $\mathcal{E} = G$; i.e., it can be expressed in terms of the quantum channel $\mathscr{I}^{\upsilon,\sigma}_{G}$:

$$\rho \overset{\upsilon}{\odot} \sigma = \mathscr{I}^{\upsilon,\sigma}_{G}\rho. \tag{41}$$

7. Interestingly, the *associativity* of the (un-smeared) twirled product entails the following relation involving the composition of the pair of quantum channels $\mathscr{I}_G^{\upsilon,\rho}$ and $\mathscr{I}_G^{\upsilon,\sigma}$:

$$\mathscr{I}_G^{\upsilon,\sigma} \circ \mathscr{I}_G^{\upsilon,\rho} = \mathscr{I}_G^{\upsilon,\tau}, \quad \text{where } \tau = \rho \overset{\upsilon}{\odot} \sigma. \tag{42}$$

8. In the case where $\varpi \neq \delta$, the quantum observable E_υ defined by (34) is replaced with the *smeared observable* $\mathsf{E}_{\upsilon|\varpi} = \mathsf{E}_\upsilon \circledast \varpi$; i.e., with the convolution of the covariant POVM E_υ with the smearing measure ϖ:

$$\mathsf{E}_\upsilon \circledast \varpi(\mathcal{E}) := \int_G \mathrm{d}\varpi(h)\, \mathsf{E}_\upsilon(\mathcal{E}h) = \int_G \mathrm{d}\varpi(h) \int_{\mathcal{E}} \mathrm{d}\mu_G(g)\, (\mathsf{S}_U(gh^{-1})\upsilon). \tag{43}$$

Accordingly, the probability density $\mathsf{p}_{\rho,\upsilon}$ is replaced with the *smeared density*

$$\mathsf{p}_{\rho,\upsilon|\varpi} = \mathsf{p}_{\rho,\upsilon} \circledast \varpi = \int_G \mathrm{d}\varpi(h)\, \mathsf{p}_{\rho,\upsilon}((\cdot)h^{-1}). \tag{44}$$

9. A similar smearing occurs, in the case where $\varpi \neq \delta$, for the quantum instrument $\mathscr{I}_{(\cdot)}^{\upsilon,\sigma} : \mathscr{B}(G) \ni \mathcal{E} \mapsto \mathscr{I}_{\mathcal{E}}^{\upsilon,\sigma}$. Explicitly, the un-smeared instrument $\mathscr{I}_{(\cdot)}^{\upsilon,\sigma}$ is replaced with the following:

$$\mathscr{I}_{(\cdot)}^{\upsilon|\varpi,\sigma} = \mathscr{I}_{(\cdot)}^{\upsilon,\sigma} \circledast \varpi = \int_G \mathrm{d}\varpi(h)\, \mathscr{I}_{\mathcal{E}h^{-1}}^{\upsilon,\sigma}. \tag{45}$$

6 Two Remarkable Examples

We will now focus on the explicit examples (T1) and (T2) considered in Section 3.2.

6.1 The Compact Case: Stochastic Products in Any Finite Dimension

Suppose that $U : G \to \mathcal{U}(\mathcal{H})$ is an *irreducible unitary representation* of a *compact* group. Then, in relation (8), $\gamma \equiv 1$ — i.e., the multiplier is trivial — and $\dim(\mathcal{H}) = \mathsf{N} < \infty$. As previously observed, the representation U is square integrable, and choosing the normalization of the Haar measure so that $\mu_G(G) = \mathsf{N}$, by the Peter-Weyl theorem we have that $\mathsf{c}_U = 1$ in the Schur orthogonality relations (11). Therefore, the twirled product generated by the triple (U, υ, ϖ), for some $\upsilon \in \mathcal{D}(\mathcal{H})$ and $\varpi \in \mathscr{P}(G)$, is precisely of the form (21). If we choose, in particular, the input, the probe or the whirligig of the twirled product to be the *maximally mixed state* [32] — namely, the unit-trace multiple of the identity $\Omega := \mathsf{N}^{-1}\mathrm{Id} \in \mathcal{D}(\mathcal{H})$ — we obtain the following interesting relations [10]:

$$\Omega \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \Omega, \quad \rho \underset{\varpi}{\overset{\Omega}{\odot}} \sigma = \Omega, \quad \rho \underset{\varpi}{\overset{\upsilon}{\odot}} \Omega = \Omega, \quad \forall \rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H}),\ \forall \varpi \in \mathscr{P}(G). \tag{46}$$

Note that, according to the second of these relations, by choosing Ω as a *probe* we *trivialize* the stochastic product; i.e., the associated twirled product does not depend on its arguments. Similarly, by choosing the invariant measure $\varpi \equiv \nu_G = \mathsf{N}^{-1}\mu_G \in \mathscr{P}(G)$ as a *smearing measure*, one obtains a trivial stochastic product too [10]:

$$\rho \underset{\nu_G}{\overset{\upsilon}{\odot}} \sigma = \Omega, \quad \forall \rho, \upsilon, \sigma \in \mathcal{D}(\mathcal{H}). \tag{47}$$

We stress that twirled products exist for every *finite* Hilbert space dimension. In fact, using, e.g., the irreducible unitary representations of the group $\mathrm{SU}(2)$, one can construct nontrivial products for $\dim(\mathcal{H}) \geq 2$.

6.2 Phase-Space Translations: The Quantum Convolution

As anticipated, the twirled product satisfies our final ansatz (A7) in Section 1, as well. To illustrate this claim, let G be the *group of translations on phase space* with, say, n position degrees of freedom (i.e., $G = \mathbb{R}^n \times \mathbb{R}^n$). In this case, the relevant Hilbert space $\mathcal{H}$ will be identified with $\mathrm{L}^2(\mathbb{R}^n)$, and — putting $\hbar \equiv \mathsf{h} = 1$ — the (genuinely projective) representation U is the Weyl system [24–26, 35]:

$$\big(U(q,p)f\big)(\tilde{q}) := \mathrm{e}^{-\mathrm{i}q\cdot p/2}\mathrm{e}^{\mathrm{i}p\cdot\tilde{q}} f(\tilde{q}-q), \quad (q,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{48}$$

Otherwise stated, in terms of the standard (vector) position and momentum operators $\hat{q}$ and $\hat{p}$ in $\mathrm{L}^2(\mathbb{R}^n)$, we have:

$$U(q,p) = \mathrm{e}^{-\mathrm{i}q\cdot p/2}\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}}. \tag{49}$$

This irreducible representation is characterized by the multiplier $\gamma\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}$ — with $\gamma(q,p;\tilde{q},\tilde{p}) = \exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})/2)$ — and is square integrable. Moreover, if we set $\mathrm{L}^2(G) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\,\mathrm{d}^n p; \mathbb{C})$, we have that $\mathsf{c}_U = 1$. Therefore, in this case we get the following expression for the twirled product:

$$\begin{aligned}\tau = \rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \int \frac{\mathrm{d}^n q\,\mathrm{d}^n p}{(2\pi)^n}\; \mathrm{tr}\big(\rho(\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}}\upsilon\,\mathrm{e}^{\mathrm{i}q\cdot\hat{p}}\mathrm{e}^{-\mathrm{i}p\cdot\hat{q}})\big) \\ \times \int \mathrm{d}\varpi(\tilde{q},\tilde{p})\,\big(\mathrm{e}^{\mathrm{i}(p+\tilde{p})\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}(q+\tilde{q})\cdot\hat{p}}\sigma\,\mathrm{e}^{\mathrm{i}(q+\tilde{q})\cdot\hat{p}}\mathrm{e}^{-\mathrm{i}(p+\tilde{p})\cdot\hat{q}}\big).\end{aligned} \tag{50}$$

This product is called the *phase-space quantum stochastic product* [10]. By the last claim of Theorem 1, it is *commutative*, because it arises from an *abelian* group. E.g., by choosing the measure $\varpi \equiv \delta$ (the Dirac measure at the origin), we get the *un-smeared* phase-space quantum stochastic product, or *quantum convolution* [10]:

$$\begin{aligned}\tau &= \rho \overset{\upsilon}{\odot} \sigma \\ &= \int \frac{\mathrm{d}^n q\,\mathrm{d}^n p}{(2\pi)^n}\; \mathrm{tr}\big(\rho(\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}}\upsilon\,\mathrm{e}^{\mathrm{i}q\cdot\hat{p}}\mathrm{e}^{-\mathrm{i}p\cdot\hat{q}})\big)\big(\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}}\sigma\,\mathrm{e}^{\mathrm{i}q\cdot\hat{p}}\mathrm{e}^{-\mathrm{i}p\cdot\hat{q}}\big).\end{aligned} \tag{51}$$

Endowed with this product, the trace class $\mathcal{T}\big(\mathrm{L}^2(\mathbb{R}^n)\big)$ becomes a Banach algebra, the so-called *quantum phase-space convolution algebra* associated with the probe υ.

6.3 The Quantum Convolution in Terms of Wigner Distributions

The reader will have noticed that both the commutativity of the product (51), as well as the adopted term of 'quantum convolution', do not emerge clearly from its expression. To properly clarify this point, we can then suitably re-elaborate this expression in terms of the *Wigner distributions* [9, 18, 24, 25, 35–38] $\mathscr{W}_\rho, \mathscr{W}_\upsilon, \mathscr{W}_\sigma, \mathscr{W}_\tau$ associated, respectively, with the states ρ (the input), υ (the probe), σ (the whirligig) and τ (the output).

Setting $\widehat{\mathscr{W}_\upsilon}(x) := \mathscr{W}_\upsilon(-x)$, $x \equiv (q,p) \in \mathbb{R}^{2n}$, one can derive the following expression:

$$\mathscr{W}_\tau(z) = \int \mathrm{d}^{2n}x\,\left(\int \mathrm{d}^{2n}y\;\mathscr{W}_\rho(y)\,\widehat{\mathscr{W}_\upsilon}(x-y)\right)\mathscr{W}_\sigma(z-x), \quad x,y,z \in \mathbb{R}^{2n}. \tag{52}$$

It is now clear that

- On the r.h.s. of (52), we can note a *double* convolution of Wigner 'quasi-probability' distributions.
- The function $\mathscr{W}_\upsilon$ — the Wigner distribution of the probe state υ — here plays a peculiar role that has no natural parallel in the classical case.
- Moreover, the function $(q,p)\mapsto \int \mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p}\;\mathscr{W}_\rho(\tilde{q},\tilde{p})\,\widehat{\mathscr{W}_\upsilon}(q-\tilde{q},p-\tilde{p})$ is a genuine *probability distribution* w.r.t. the Lebesgue measure on $\mathbb{R}^n\times\mathbb{R}^n$ (whereas a Wigner distribution, in general, is not).
- If we put, say, $n=1$ and we take, as a probe υ, the (pure) Gaussian state $|\psi\rangle\langle\psi|$, with $\psi(q)=(2\pi)^{-1/4}\mathrm{e}^{-q^2/4}$ — recall that the associated Wigner distribution is of the form $\mathscr{W}_\upsilon(q,p)\equiv\mathscr{W}_\psi(q,p)=\pi^{-1}\,\mathrm{e}^{-(q^2+p^2)}$ — we obtain the following probability distribution on phase space:

$$\Big((q,p)\mapsto \mathscr{Q}_\rho(q,p):=\frac{1}{\pi}\int \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p}\;\mathscr{W}_\rho(\tilde{q},\tilde{p})\,\mathrm{e}^{-(q-\tilde{q})^2-(p-\tilde{p})^2}\Big)\in \mathrm{L}^1(\mathbb{R}\times\mathbb{R})^+_{\mathrm{n}}. \quad (53)$$

- Recall that $\mathscr{Q}_\rho$ is the *Husimi-Kano function* of the input state ρ [36]. Therefore, expressed in terms of functions on phase space, the quantum convolution with the *Gaussian probe* $\upsilon\equiv|\psi\rangle\langle\psi|$ is in the form of a convolution on the group of phase-space translations:

$$\mathscr{W}_\tau(q,p)=\int \mathrm{d}\tilde{q}\,\mathrm{d}\tilde{p}\;\mathscr{Q}_\rho(\tilde{q},\tilde{p})\mathscr{W}_\sigma(q-\tilde{q},p-\tilde{p}),\quad \text{with } \tau=\rho\overset{|\psi\rangle\langle\psi|}{\odot}\sigma. \quad (54)$$

6.4 The Crucial Role of the Probe State: From Wigner Distributions to Quantum Characteristic Functions

Having unveiled the nature of the quantum convolution, it is now worth highlighting the essential role played by the the probe υ in this product. To this end, note that the twirled product admits a very simple form once expressed in terms of the *covariant symbols* of the states ρ, υ and σ. Let us clarify this point.

Given a square integrable projective representation $U\colon G\to\mathcal{U}(\mathcal{H})$ — where we still assume that the l.c.s.c. group G is unimodular — and a trace class operator $T\in\mathcal{T}(\mathcal{H})$, the (covariant) symbol $\breve{T}$ of T is a complex function on G defined by (recall formula (12))

$$\breve{T}(g):=\mathrm{d}_U^{-1}\,\mathrm{tr}(U(g)^*T). \quad (55)$$

We can write this function as $\breve{T}=\mathfrak{D}T$, where $\mathfrak{D}\colon\mathcal{S}(\mathcal{H})\to\mathrm{L}^2(G)\equiv\mathrm{L}^2(G,\mu_G;\mathbb{C})$ is an isometry that maps the Hilbert space $\mathcal{S}(\mathcal{H})$ of *Hilbert-Schmidt operators* — endowed with the Hilbert-Schmidt scalar product $\langle\cdot,\cdot\rangle_{\mathrm{HS}}\colon\mathcal{S}(\mathcal{H})\times\mathcal{S}(\mathcal{H})\ni(S,T)\mapsto\mathrm{tr}(S^*T)=:\langle S,T\rangle_{\mathrm{HS}}$ — into $\mathrm{L}^2(G)$. This isometry can be thought of as a *dequantization map*, and the operator T can be reconstructed back from its symbol $\breve{T}$ via the *quantization map* $\mathfrak{Q}=\mathfrak{D}^*\colon\mathrm{L}^2(G)\to\mathcal{S}(\mathcal{H})$ [24].

In the case where G is the vector group $\mathbb{R}^n\times\mathbb{R}^n$ and U is the Weyl system, the map $\mathfrak{D}$ is related to the *Wigner transform* [9, 18, 24, 25, 35]. Given a density state $\rho\in\mathcal{D}(\mathcal{H})$, with $\mathcal{H}=\mathrm{L}^2(\mathbb{R}^n)$, the associated symbol $\breve{\rho}$ is also known as the *quantum characteristic function* of ρ. In fact, $\breve{\rho}$ is the (symplectic) Fourier transform of the Wigner function $\mathscr{W}_\rho$, in analogy with the 'classical' characteristic function of a probability measure on a l.c.s.c. abelian group [9, 25, 37, 38].

Precisely, for every $\varpi \in \mathscr{P}(G)$, $G = \mathbb{R}^n \times \mathbb{R}^n$, one can identify the characteristic function of ϖ — its Fourier-Stieltjes transform $\widehat{\varpi}\colon \widehat{G} \to \mathbb{C}$, where $\widehat{G}$ is the *dual* of G *à la Pontryagin* [13]—with $\breve{\varpi}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, where $\breve{\varpi}(q,p) = \int \mathrm{d}\varpi(\tilde{q},\tilde{p})\,\exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q}))$.

At this point, with $\mathrm{L}^2(G) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q\,\mathrm{d}^n p; \mathbb{C})$, the twirled product generated by the triple (U, υ, ϖ) — in terms of the characteristic function $\breve{\varpi}$ of the smearing measure ϖ and of the covariant symbols $(q,p) \mapsto \breve{\rho}(q,p) := \mathrm{tr}(U(q,p)^*\rho)$, $\breve{\upsilon}$, $\breve{\sigma}$ of the states ρ, υ, σ (input, probe, whirligig) — takes an elementary form, i.e.,

$$\Big(\rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma\Big)^{\smile}(q,p) = \breve{\varpi}(q,p)\breve{\rho}(q,p)\overline{\breve{\upsilon}(q,p)}\breve{\sigma}(q,p) =: \Big(\breve{\rho} \underset{\breve{\varpi}}{\overset{\breve{\upsilon}}{\circledcirc}} \breve{\sigma}\Big)(q,p). \tag{56}$$

The r.h.s. of this expression defines a *weighted pointwise product* on phase space. Clearly, the input and the whirligig can be interchanged, since this product is manifestly commutative. Once again, for $\varpi = \delta$ — equivalently, for $\breve{\varpi} \equiv 1$ — we say that the weighted pointwise product is *un-smeared*.

Remark 4 It is interesting to compare the (say, un-smeared) weighted pointwise product, realizing the phase-space stochastic product in terms of characteristic functions, with the — non-commutative and, in the case of two generic states ρ and σ, un-physical — operator product $\rho\sigma$, that, expressed in terms of the associated characteristic functions, has the form of a *twisted convolution à la Grossmann-Loupias-Stein* [24, 39]:

$$\begin{aligned}\big(\mathfrak{D}(\rho\sigma)\big)(q,p) &= \big(\breve{\rho}\,\widehat{\circledast}\,\breve{\sigma}\big)(q,p)\\ &:= \int \frac{\mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p}}{(2\pi)^n}\; \breve{\rho}(\tilde{q},\tilde{p})\breve{\sigma}(q-\tilde{q}, p-\tilde{p})\exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})/2).\end{aligned} \tag{57}$$

In hindsight, the weighted pointwise product (56) may be regarded as a direct way to define a commutative stochastic product. Indeed, one can show that the standard pointwise product of two quantum characteristic functions is not, in general, a function of the same kind, *but* the pointwise product of a ‘classical’ characteristic function on phase space — namely, the Fourier-Stieltjes transform of a probability measure on the group $\mathbb{R}^n \times \mathbb{R}^n$ — by a quantum characteristic function, is a function of the latter type; see [37, 38]. Furthermore, for every pair of density states $\rho, \upsilon \in \mathcal{D}(\mathcal{H})$, $\breve{\rho}\overline{\breve{\upsilon}}$ is a *classical* characteristic function; namely, the (symplectic) Fourier-Stieltjes transform of the probability measure $\nu_{\rho,\upsilon}$ on $\mathbb{R}^n \times \mathbb{R}^n$, with

$$\mathrm{d}\nu_{\rho,\upsilon}(q,p) = (2\pi)^{-n}\mathrm{tr}\big(\rho(\mathsf{S}_U(q,p)\upsilon)\big)\mathrm{d}^n q\,\mathrm{d}^n p. \tag{58}$$

Indeed, considering the convolution $\mathscr{W}_\rho \circledast \widehat{\mathscr{W}_\upsilon}$ of $\mathscr{W}_\rho$ with $\widehat{\mathscr{W}_\upsilon}$ — $\widehat{\mathscr{W}_\upsilon}(q,p) := \mathscr{W}_\upsilon(-q,-p)$ — we have that

$$\begin{aligned}\big(\breve{\rho}\overline{\breve{\upsilon}}\big)(q,p) &= \int \frac{\mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p}}{(2\pi)^n}\; \mathrm{tr}\big(\rho(\mathrm{e}^{\mathrm{i}\tilde{p}\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}\tilde{q}\cdot\hat{p}}\upsilon\mathrm{e}^{\mathrm{i}\tilde{q}\cdot\hat{p}}\mathrm{e}^{-\mathrm{i}\tilde{p}\cdot\hat{q}})\big)\exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q}))\\ &= \int \mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p}\;\big(\mathscr{W}_\rho \circledast \widehat{\mathscr{W}_\upsilon}\big)(\tilde{q},\tilde{p})\exp(\mathrm{i}(q\cdot\tilde{p} - p\cdot\tilde{q})).\end{aligned} \tag{59}$$

In conclusion, the weighted product $(\breve{\rho}, \breve{\sigma}) \mapsto \breve{\varpi}\,\breve{\rho}\overline{\breve{\upsilon}}\breve{\sigma}$ may be thought of as the pointwise product of two *classical* characteristic functions — $\breve{\varpi}$ and $\breve{\rho}\overline{\breve{\upsilon}}$ — which is still a function of this type (i.e., the Fourier-Stieltjes transform of the convolution of two probability measures), multiplied (again pointwise) by the *quantum* characteristic function $\breve{\sigma}$, which eventually provides a function of the latter kind. We stress that exploiting the quantum characteristic function $\breve{\upsilon}$ of the probe υ, as a suitable ‘weight’, in the pointwise product *cannot be dispensed with* if one wants to achieve a state-preserving binary operation.

6.5 Quantizing the Weighted Pointwise Product

We close our analysis of the phase-space stochastic product with the observation that it can be obtained by quantizing the weighted pointwise product. In addition, we show that the *purity* of the output state of the product cannot exceed the purity of the input, the probe and the whirligig; precisely, we have:

Proposition 5 *The phase-space stochastic product admits the expression*

$$\tau \equiv \rho \underset{\varpi}{\overset{\upsilon}{\odot}} \sigma = \int \frac{\mathrm{d}^n q \mathrm{d}^n p}{(2\pi)^n} \; \breve{\varpi}(q, p)\breve{\rho}(q, p)\overline{\breve{\upsilon}(q, p)}\breve{\sigma}(q, p)\, \mathrm{e}^{-\mathrm{i}q\cdot p/2}\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}}, \tag{60}$$

where a weak integral is understood. Moreover, for the purity of the states τ, ρ, υ and σ the following inequality holds:

$$\mathrm{tr}(\tau^2) \le \min\left\{\mathrm{tr}(\rho^2), \mathrm{tr}(\upsilon^2), \mathrm{tr}(\sigma^2)\right\}. \tag{61}$$

Proof The map $\mathfrak{Q} := \mathfrak{D}^* \colon \mathrm{L}^2(G) \to \mathcal{S}(\mathcal{H})$, where $\mathrm{L}^2(G) = \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}\mathrm{d}^n q \mathrm{d}^n p; \mathbb{C})$ and $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$, is of the form [24]

$$f \mapsto \int \frac{\mathrm{d}^n q \mathrm{d}^n p}{(2\pi)^n} \; f(q, p)\, \mathrm{e}^{-\mathrm{i}q\cdot p/2}\mathrm{e}^{\mathrm{i}p\cdot\hat{q}}\mathrm{e}^{-\mathrm{i}q\cdot\hat{p}} \quad \text{(weak integral)}. \tag{62}$$

Hence, formula (60) follows directly from the expression (56). Observe, moreover, that, since the dequantization map $\mathfrak{D} \colon \mathcal{S}(\mathcal{H}) \to \mathrm{L}^2(G)$ is an isometry, we have:

$$\mathrm{tr}(\tau^2) = \langle \tau, \tau \rangle_{\mathrm{HS}} = \int \frac{\mathrm{d}^n q \mathrm{d}^n p}{(2\pi)^n} \; |\breve{\varpi}(q, p)|^2 |\breve{\rho}(q, p)|^2 |\breve{\upsilon}(q, p)|^2 |\breve{\sigma}(q, p)|^2. \tag{63}$$

Here, the characteristic function $\breve{\varpi}$ of the probability measure ϖ is such that $|\breve{\varpi}(q, p)| \le |\breve{\varpi}(0)| = 1$ and, analogously, we have that $|\breve{\rho}(q, p)| \le 1$, $|\breve{\upsilon}(q, p)| \le 1$, $|\breve{\sigma}(q, p)| \le 1$. Then, relation (61) follows immediately. □

7 Conclusions and Prospects

We have presented a notion of *stochastic product* of two quantum states as a binary operation on the convex set of density operators that preserves the convex structure. We have also shown that, by a group-theoretical construction, it is possible to achieve a class of associative stochastic products, the so-called *twirled products*. The twirled products exist for every Hilbert space dimension and admit a remarkable physical interpretation in the framework of quantum measurement theory and quantum information. In the case where the relevant group involved in the construction is the group of phase-space translations, one obtains a commutative stochastic product that may be regarded as a quantum counterpart of the 'classical' convolution product on the same group.

Interestingly, a quantum stochastic product, together with the standard operator product, gives rise, in a natural way, to an abstract notion of *stochastic* H*-*algebra* [10], which is now under study. The extension of the group-theoretical construction underlying the twirled products to the case where the relevant group is, in general, *not* unimodular, and the expression of these generalized twirled products in terms of the covariant symbols of quantum states is also work in progress.

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Declarations

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