Continuum Soliton Chain Analog to Heisenberg Spin Chain System. Modulation Stability and Spectral Characteristics

H. I. Abdel-Gawad[1] 🔵

Abstract

The one-dimensional (1D) Heisenberg spin chain system (HSCS) allows to investigate anomalous features originating from strong quantum fluctuations, which become more significant than those in higher dimensions. A continuum model equation of the HSCS, based on the discrete model, was constructed in the literature. It is nonlinear Schrodinger equation (NLSE) with biquadratic dispersion and fifth degree nonlinearity. Rare research works were done in this area. Notably for deriving the exact solutions and investigate the physical phenomena produced. Our objective, here, is to obtain these solutions, which we think they are new. Further, an analog of the different geometric solutions structures to the known characteristics of HSCS is performed. The unified method is implemented to find the exact solutions of the continuum model equation. A variety of solutions are obtained where they are evaluated numerically and represented in graphs. In these graphs, it is remarked that the solutions exhibit soliton chain (or dense soliton chain).in an analog to spin chain. In the contour plots, they show different shapes of super lattices. Furthermore, complex chirped waves are observed. A significant result is that these solutions are bounded by $-1/4$ and $1/4$, which can be relevant to the spins$-1/2$ and $1/2$. The analysis of modulation stability is carried and it is found that there is a critical value for the dominant parameters, where below this value, modulation instability holds otherwise modulation stability occurs.For the spectral characteristics, it is shown that the wave number increases abruptly and decreases to an asymptotic state, while the frequency is monotonic increasing. The spectrum is periodic wave away from the origin, but near the origin it is soliton.

Keywords Heisenberg · Spin chain · Soliton chain · Super lattices · Unified method

1 Introduction

In the study of quantum phase and phase transition, and quantum computations are paradigmatic.In quantum spin systems, lower energy state for a nuclear spin in an external field is

---

✉ H. I. Abdel-Gawad
    hamdyig@yahoo.com

[1]   Mathematics Department Faculty of Science Cairo University, Giza, Egypt

spin $(+1/2)$ while the higher energy corresponds to spin $(-1/2)$. The quantum spin chains in a magnetic field was studied in [1], by using Bethe-ansatz solutions which arise from string solutions that continuously connect the mode of the lowest-energy excitation. In [2], It was reported that antiferromagnetic HSCs, in a 1D trap, are stabilized by strong repulsive interactions between the two spin components in the absence of an external potential. A scheme for conditional state transfer in a HSC, produced in a transistor, was proposed and analyzed in [3, 4]. Exact solutions of the generalized HSCS were found and the thermodynamic features were studied on the basis of the exact solutions [5]. Analytical and numerical studies of spin transport in a 1D Heisenberg model, in linear-response regime, were carried [6]. In [8], the integrability of 1D classical continuum inhomogeneous biquadratic HSC and the effect of nonlinear inhomogeneity on the soliton of completely integrable spin model are studied.The integrability aspects of a classical one-dimensional continuum isotropic biquadratic HSC in its continuum limit was studied. This was achieved via a differential geometric approach, the dynamical equation for the spin chain is expressed in the form of a higher-order NLSE [7]. The conserved quantities are expressed in terms of a sum over simple polynomials in spin variables. Very recently, It was shown that for a NLSE, whatever its formulation, is integrable (or compeletly integrable) when the real and imaginary parts are linearly dependent [9–12]. A direct construction of explicit expressions for all the quantum integrals of motion for the isotropic Heisenberg s=1/2 spin chain was presented [13]. Although the 1D Heisenberg ferromagnetic spin chain (HFSC) was rarely studied in the literature, the (2+1) dimensional HFSC was remarkably considered. This may be argued to that in the first case the model equation (ME) is taken a NLSE with biquadratic dispersion and fifth degree nonlinearity. While in the second case, the ME is taken a NLSE with quadratic dispersion and Kerr nonlinearituy but in (2+1) dimensions. In [14], the (2+1)-dimensional HFSC that describes the nonlinear dynamics of magnet was studied. Two mathematical approaches for showing dark, bright, kink-type and singular soliton solutions to the HFSCS were presented. The NLSE in (2 + 1) dimensions, with beta derivative evolution, was considered to study nonlinear coherent structures for HFSC with magnetic exchanges [15]. In [16], the NLSE in (2+1)-dimensions for the HFSC, with anisotropic and bilinear interactions in the semi classical limit,. where two integrating schemes were used, was studied. The (2+1)-dimensions HFSCS was considered for the objective of finding the exact solutions via a specific transformation and adopting a modified version of the Jacobi elliptic expansion method [17]. An ansatz method,to solve The HFSC equation was used to get bright and dark 1-soliton solutions.Some conditions of integrability were given which guarantee the existence of solitons [18]. In [19], constructio0n of further exact soliton solutions of the (2 + 1)-dimensional HFSCE and investigating the nonlinear dynamics of magnets and explains their ordering in ferromagnetic materials were carried.. The collision dynamics of soliton in discrete classical ferromagnetic spin chain with Dzyaloshinskii-Moriya (DM) interaction in the classical limit are analyzed [20]. In [21], The conformable fractional derivative HFSC was considered via the complete discrimination system for polynomial method. The rational combined multi-wave solutions were obtained for HFSCE by using the logarithmic transformation and symbolic computation with ansatz functions [22]. The NLSE that describes the spin dynamics of (2 + 1)-dimensional inhomogeneous IHFSC with bilinear and anisotropic interactions in the semi classical limit was investigated [23]. n [24], Hirota bilinear method with appropriate polynomial functions in bilinear forms, the one-order rogue waves solution and its existence condition were obtained. Different methods and techniques were used to solve nonlinear evolution equations; $Tanh$ and Exp-function [25, 26], $\frac{G'}{G}$ expansion [27], Darboux transformation [28], Kyrdiashov

method, [29], Hirota-bilinear transformation [30], Lie symmetries of NLPDEs [31].Very recently different methods and techniques were introduced, among them, the first integral method, the improved q- homotopy perturbation method and the unified algebraic and auxiliary equation expansion methods [32–41]. Here, the unified method (UM) [42–47] is used in this paper.

The method used here is compared with the known methods, known in the in the literature.

Indeed, the UM [36] prevails the all known methods such as, the tanh, modified, and extended versions, the F-expansion, the exponential, the G'/G expansion method, the Kerdyashov methods, as it is of low time cost in symbolic computation.Further, It provides solutions which cannot be obtained by the other methods.

The outlines of the paper is in what follows. Section 2 is devoted to mathematical equations and outlines of the UM and GUM. the solutions in the polynomial form are presented in Section 3. While the solutions in the rational forms are carried in Section 4. Section 5 is devoted to modulation stability analysis. Conclusions are given in Section 6.

2 The Model equation and outlines of the UM

2.1 The Model equation

For the 1D classical HFSCS with biquadratic exchange and a bilinear exchange interaction, the Hamiltonian is,

$$H = -J \sum_n (S_n..S_{n+1}) - \alpha J \sum_n (S_n..S_{n+1})^2, \quad Sn = (S_n^x, S_n^y, S_n^z), S_n^2 = 1, \quad (J > 0), \tag{1}$$

where $S_n := S_n(t)$ and α is the biquadratic exchange parameter, which is considered in a dominant parameter. By assuming that the lattice side, a, is small, in the continuum analog, we write $x = na$ and $S_{n+1} = S(x, t) + aS_x(x, t) + a^2 S_{xx}(x,t)/2!+...$ Up to the order of a^4, the equation of motion takes the form,

$$S_t = S \times [S_{xx} + \nu S_{xxx} + \beta((S..S_{xx})S_x + \frac{2}{3}(S..S_{xxx})S_x)], \tag{2}$$

where$\nu = a^2/12$ and $\beta = \alpha a^2/(1 + 2\alpha)$. The continuum model equation, based on (2) was constructed [47]

$$\begin{aligned} &iw_t + w_{xx} + 2w \mid w \mid^2 + \nu w_{xxxx} - 4\delta w_{xx}^* w^2 - 4 \mid w \mid^2 w_{xx} \\ &-4\alpha \mid w \mid^2 w_{xx} - 4\nu_0 w^* (w_x)^2 - 24\sigma \mid w \mid^4 w = 0, \end{aligned} \tag{3}$$

where $\alpha := 4\beta + 9\nu$; $\mu = 2\beta + 3\nu$; $\nu_0 = 2\beta + \frac{7\mu}{2}$, $\sigma = \frac{\beta}{2} + \nu$; $\delta = \beta + 2\nu$, and $w := w(x, t)$. Equation (1) is a NLSE with the highest biquadratic dispersion and highest nonlinearity of fifth degree.

The spectral characteristics are, here, introduced. To this issue, we write,

$$w(x, t) = \mid w(x, t) \mid e^{i(Kx - \Omega t)}, \tag{4}$$

where K is the w2ave number and Ω is the frequency;

$$K = \frac{\int_0^\infty (\int_{-\infty}^\infty \mid w_x(x, t) \mid dx)dt}{\int_0^\infty (\int_{-\infty}^\infty \mid w(x, t) \mid dx)dt}, \quad \Omega = \frac{\int_0^\infty (\int_{-\infty}^\infty \mid w_t(x, t) \mid dx)dt}{\int_0^\infty (\int_{-\infty}^\infty \mid w(x, t) \mid dx)dt}, \tag{5}$$

and the spectrum content is,

$$W(k_0, t\} = \int\limits_{-\infty}^{\infty} e^{-i\,k_0 x} w(x, t)dx. \tag{6}$$

For the objective of finding the solutions of (3), we introduce a transformation for $w(x, t)$ with complex amplitude,

$$w(x, t) = (u(x, t) + iv(x, t))e^{i(kx - \omega t)}, \quad w(x, t)^* = (u(x, t) - iv(x, t))e^{-i(kx - \omega t)}. \tag{7}$$

This transformation allows to inspect the effect of soliton- periodic wave collision, which is elastic or inelastic depending on the waves solutions if they are smooth or not smooth. When inserting (7) into (3) gives rise, for the real and imaginary parts respectively, to,

$$-4\alpha k^2 u^3 + 4\delta k^2 u^3 + 4k^2 \mu u^3 + 4k^2 \nu o u^3 + k^4 \nu u - k^2 u - 24\sigma u^5 + 2u^3 + u\omega$$

$$-v_t - 4\alpha k^2 u v^2 + 4\delta k^2 u v^2 + 4k^2 \mu u v^2 + 4k^2 \nu o u v^2 - 48\sigma u^3 v^2 + +2u v^2$$

$$-24\sigma \mathrm{uv}^4 - 4k^3 \nu v_x + +8\alpha k u v u_x - 16\delta k u v u_x - 2k v_x - 4\alpha u (u_x)^2 - 4\nu o u (u_x)^2$$

$$-8\alpha k u^2 v_x + 8\delta k u^2 v_x + 8k\mu u^2 v_x + 8k\nu o u^2 v_x - 8\delta k v^2 v_x + 8k\mu v^2 v_x^) \tag{8}$$

$$-6k^2 \nu u_{xx} + +8k\nu o v^2 v_x - 4\delta u^2 u_{xx} - 4\mu u^2 u_{xx} - 8\nu o v u_x v_x + u_{xx} - 4\alpha u (v_x)^2$$

$$+4\nu o u (v_x)^2 - 4k\nu v_{xxx} + \nu u_{xxxx} + +4\delta v^2 u_{xx} - 4\mu v^2 u_{xx} - 8\delta u v v_{xx} = 0,$$

$$-4\alpha k^2 u^2 v + 4\delta k^2 u^2 v + 4k^2 \mu u^2 v + 4k^2 \nu o u^2 v + k^4 \nu v - k^2 v + 2u^2 v + v\omega$$

$$2k u_x + u_t - 4\alpha k^2 v^3 + 4\delta k^2 v^3 + 4k^2 \mu v^3 + 4k^2 \nu o v^3 - 48\sigma u^2 v^3 - 24\sigma u^4 v - 24\sigma v^5$$

$$+2v^3 + [-4k^3 \nu + 8\delta k u^2 - 8k\mu u^2 - 8k\nu o u^2 + 8\alpha k v^2 - 8\delta k v^2 - 8k\mu v^2 - 8k\nu o v^2]u_x$$

$$-4\alpha v (u_x)^2 + 4\nu o v (u_x)^2 - 8\alpha k u v v_x + 16\delta k u v v_x - 8\,\nu o u u_x v_x - 4\alpha v (v_x)^2$$

$$-4\nu o v (v_x)^2 - 6k^2 \nu v_{xx} + 4k\nu u_{xxx} + 4\delta v_{xx} u(x, t)^2 - 4\mu u^2 v_{xx} - 8\delta u v u_{xx}$$
$$-4\delta v^2 v_{xx} - 4\mu v^2 v_{xx} + \nu v_{xxxx} + v_{xx} = 0. \tag{9}$$

We search for traveling waves solutions. To this end, we use the transformations $u(x, t) = U(z)$, $v(x, t) = V(z)$ and $z = ax + bt$. Under these transformations (8) and (9) become, respectively,

$$-V' \left(8a^2 \nu o V U' - 4ak^3 v + 8akV^2(\delta - \mu - \nu o) + 2ak + b\right) + U^3$$

$$\left(4k^2(-\alpha + \delta + \mu + \nu o) - 48\sigma V^2 + 2\right) + 4aU^2 \left(a(\delta + \mu)U'' + 2kV'(\alpha - \delta - \mu - \nu o)\right)$$

$$-24\sigma U^5 + U \left(k^4 v + V^2 \left(4k^2(-\alpha + \delta + \mu + \nu o) + 2\right) - k^2 + \omega\right)$$

$$-4a^2(\alpha + \nu o) \left(U'\right)^2 - 4a^2 \alpha \left(V'\right)^2 + 4a^2 \nu o \left(V'\right)^2 - 8aV \left(a\delta V'' - k(\alpha - 2\delta)U'\right)$$
$$-24\sigma V^4) + a^2 \left(av \left(aU^{(4)} - 4kV^{(3)}\right) + U'' \left(-6k^2 v + 4V^2(\delta - \mu) + 1\right)\right) = 0, \tag{10}$$

$$V^3 \left(4k^2(-\alpha + \delta + \mu + vo) - 48\sigma U^2 + 2\right) - 24\sigma V^5 - k^2 - 24\sigma U^4 + \omega$$
$$+U' \left(-4ak^3 v + 2ak + b\right.$$
$$+8akU^2(\delta - \mu - vo) - 8a^2 voUV') + V \left(k^4 v\right.$$
$$+U^2 \left(4k^2(-\alpha + \delta + \mu + vo) + 2\right)$$
$$+4a^2(vo - \alpha) \left(U'\right)^2 - 4a^2\alpha \left(V'\right)^2 - 4a^2 vo \left(V'\right)^2 - 8aU \left(a\delta U'' + k(\alpha - 2\delta)V'\right)\Big)$$

$$a^2 \left(av \left(aV^{(4)} + 4kU^{(3)}\right) + V'' \left(-6k^2 v + 4U^2(\delta - \mu) + 1\right)\right)$$
$$-4aV^2 \left(a(\delta + \mu)V'' + 2kU'(-\alpha + \delta + \mu + vo)\right) == 0.$$
$$\tag{11}$$

Here, the solutions of (10) and (11) are found by using the UM and GUM. The Um asserts that the solutions of NLPDEs (NLODEs) are formulated in polynomial and rational forms, in auxiliary functions that satisfy appropriate auxiliary equations AE).

2.2 Outlines of the UM

2.2.1 Polynomial forms

By considering (10) and (11), the polynomial forms are,

$$U(z) = \sum_{j=0}^{m_1} a_j g(z)^j, \quad V(z) = \sum_{j=0}^{m_2} b_j g(z)^j, \, (g'(z))^p = \sum_{j=0}^{rp} c_j g(z)^j, \, p = 1, 2. \quad (12)$$

To determine m_i, $i = 1, 2$ and r, we use the balance and compatibility conditions. First, we consider the case when $p = 1$. The balance condition is determined by balancing the highest order derivative and highest nonlinearity terms. In the present case, the balance condition reads $m_1 = m_2 = r - 1$. To determine the consistency condition, we require the following

(a) the number of equations that result from inserting (12) into (10) and (11) and by setting the coefficients of $g(z)^i$., $i = 0, 1, 2, ...,$ say h(k).

(b) the number of arbitrary parameters a_i, b_i, c_i, say f(k). For integrable equations the condition is $h(k) - f(k) \leq s$, where s is the highest order derivative (here $s = 4$). In the present case the consistency condition reads $1 \leq r \leq 3$.

The case case when $p = 2$ can be analyzed by the same way. We mention that when $p = 1$, the solutions of the AE are elementary functions, while they are periodic or ellptic when $p = 2$.

2.2.2 Rational forms

In the UM the rational solutions are written,

$$U(z) = \frac{a_1 g(z) + a_0}{s_1 g(z) + s_0}, \quad V(z) = \frac{b_1 g(z) + b_0}{s_1 g(z) + s_0},$$

$$(g'(z))^p = \sum_{j=0}^{rp} c_j g(z)^j, \, p = 1, 2.$$
$$\tag{13}$$

3 Polynomial solutions of (10) and (11)

Here, we consider the following cases.

3.1 When $p = 1$ and $r = 2$

In this case (12) reduces to

$$U(z) = a_1 g(z) + a_0, \quad V(z) = b_1 g(z) + b_0, \ g'(z) = c_2 g(z)^2 + c_1 g(z) + c_0. \quad (14)$$

When inserting (14) into (10) and (11), and by setting the coefficients of $g(z)^i = 0$, $i = 0, 1, ..,$ we have, (for linearity dependent solutions, $b_0 = a_0 b_1 / a_1$),

$$v = \frac{(a_1^2 + b_1^2)(a^2 c_2^2(\alpha + 2\delta + 2\mu + v o) + 6\sigma(a_1^2 + b_1^2))}{6a^4 c 2^4}, c_2 = -r^2,$$

$$v_0 = \frac{6\sigma(a_1^2 + b_1^2)}{a^2 c_2^2} + \alpha + 4\delta, \quad \delta := \frac{1}{12c_2 a_1^2 k^2 (a_1^2 + b_1^2)} \left(4a_0^2 b_1^2 c2 \left(12 b_1^2 \sigma - a^2 c2^2(\alpha - 2\mu)\right)\right.$$

$$- 48 a_1^6 c_0 \sigma + a_1^4 \left(4a^2 c_0 c_2^2(\alpha - 2\mu) + c_2 \left(48 a_0^2 \sigma + 6\alpha k^2 - 3\right) - 96 c_0 b_1^2 \sigma\right)$$

$$+ a_1^2 \left(b_1^2 c2 \left(4a^2 c_2 c_0(\alpha - 2\mu) + 96 a_0^2 \sigma + 6\alpha k^2 - 3\right)\right)\right)$$

$$+ a_1^2 (+a^2 c2^3 \left(-4a_0^2(\alpha - 2\mu) - 3\right) - 48 b_1^4 c o\sigma))$$

$$c_1 = \frac{2a_0 c_2}{a_1}, b = \frac{P}{H}, \ P = 2ak(-\left(\alpha \left(12 k^2 v - 1\right) + 2\mu - 3v\right)$$

$$+ 3a^4 v r^8 \left(4k^2 v + 1\right) a^2 r^4 \left(a_1^2 + b_1^2\right), \ H = \left(a_1^2 + b_1^2\right) \left(12\sigma \left(a_1^2 + b_1^2\right) - a^2 r^4(\alpha - 2\mu)\right),$$

$$c_0 = \frac{H_1}{H_2}, \ H_1 = -a^4 r^8 \left(a1^2 \left(4 a o^2(\alpha - 2\mu) + 12 k^2 v + 3\right) + 4 a_0^2 b_1^2(\alpha - 2\mu)\right)$$

$$+ 24 a1^2 k^2 \sigma \left(a_1^2 + b_1^2\right)^2 + a^2 r^4 \left(a_1^2 + b_1^2\right), \left(a_1^2 \left(48 a_0^2 \sigma + 2k^2 (5\alpha + 2\mu) - 3\right) + 48 a_0^2 b_1^2 \sigma\right)$$

$$H_2 = 4a^2 a_1^2 r^2 \left(a_1^2 + b_1^2\right) \left(a^2 r^4(\alpha - 2\mu) - 12\sigma \left(a_1^2 + b_1^2\right)\right),$$

$$\omega = \frac{H_3}{H_4}, \quad H_3 = 90 a^{10} k^2 v r^{20}(\alpha - 2\mu) \left(4k^2 v + 1\right) - 432 a^2 k^2 r^4 \left(a_1^2 + b_1^2\right)^4 \sigma^2 \left(2k^2(\alpha + 2\mu) - 1\right)$$

$$- 1728 k^4 \sigma^3 \left(a_1^2 + b_1^2\right)^5 - 3a^8 \ a_1^2 + b_1^2 \ r^{16} \left(-8k^2 \mu^2 \left(13 k^2 v + 2\right)\right.$$

$$+ 2\alpha^2 k^2 \left(71 k^2 v + 4\right) - \alpha \left(232 k^4 \mu v + k^2(8\mu + 42v) + 3\right)$$

$$+ \mu \left(84 k^2 v + 6\right) 9\sigma \left(112 k^4 v^2 + 16 k^2 v - 3\right) - \alpha \left(232 k^4 \mu v + k^2(8\mu + 42v) + 3\right))$$

$$+ 9a^4 r^8 \sigma \left(a_1^2 + b_1^2\right)^3 \left(4k^4 \left(5\alpha^2 - 28\alpha\mu - 12\mu^2 + 168 v\sigma\right)\right.$$

$$+ 4k^2(\alpha + 10\mu + 12\sigma) - 3) + \alpha \left(-144 k^4 \left(\mu^2 - 14 v\sigma\right) + 12 k^2(8\mu - 33\sigma) + 9\right)),$$

$$H_4 = 6a^4 r^8 \left(a_1^2 + b_1^2\right) \left(a^2 r^4(\alpha - 2\mu) - 12\sigma \left(a_1^2 + b_1^2\right)\right)^2.$$

$$(15)$$

By solving the AE in (14), the solutions of (8) and (9) are,

$$u(x, t) = a_1 \frac{K \tanh \left(\frac{(z + A_0)K}{K_0}\right)}{K_0}$$

$$K = \sqrt{3a^4 r^8 \left(4k^2 v + 1\right) - \left(24 k^2 \sigma \left(a_1^2 + b_1^2\right) + M},$$

$$M = a^2 r^4 \left(2k^2(5\alpha + 2\mu) - 3)\right) \left(a_1^2 + b_1^2\right)$$

$$K_0 = 2ar^2 \sqrt{a_1^2 + b_1^2} \sqrt{12\sigma \left(a_1^2 + b_1^2\right) - a^2 r^4 (\alpha - 2\mu)}, \tag{16}$$

$$v(x,t) = \frac{b_1}{a_1} u(x,t), \ z = ax + bt.$$

The solutions in (16) are evaluated numerical and the results are used to display $Re w$ in Fig. 1(i)-(v).

Figures 4 and 5. In (iv), the contour plots is displayed, while in (v) the variation of $Re w$ against x for different values of t is done.

Figure 1(i) shows "continuum" soliton chain with trapping, while Fig. 1(ii) and (iii) show complex soliton chain. Fig. 1(iv) shows super lattices and Fig. 1(v) shows "continuum" solton chain.

The Spectral characteristics which are given in (5) and (6) are shown in Fig. 2(i)-(iii), for the wave number, the frequency and spectrum respectively.

Figure 2(i) shows the there a critical value of v where it incre4ases and decreases abruptly. Figure 2(ii) shows that the frequency increases with v. Figure 2(iii) shows soliton chain with small amplitude apart near $x = 0$.
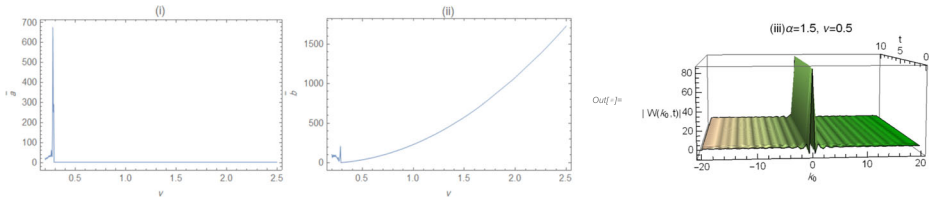
3.2 When $p = 2$ and $r = 2$

In this case consider the solution in (14), but the AE is,

$$g'(z) = g(z)\sqrt{c_2 g(z)^2 + c_1 g(z) + c_0}. \tag{17}$$
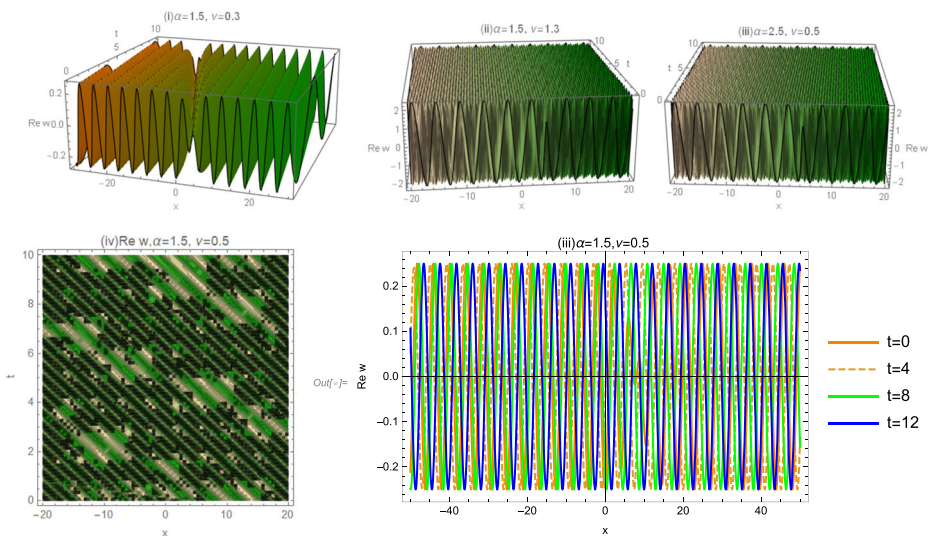
Fig. 1 The 3D plot is displayed for $Re w$ against x and t by varying the values of α and v when $r := 1.2, k = 1.5, a := 1.2, a_1 = 0.7, b_1 = 0.5, \sigma = 0.6, \mu = 0.5, A_0 = -5$. In (i) $\alpha = 1.5, v = 0.3$, (ii) $\alpha = 1.5, v = 1.3$ (iii) $\alpha = 2.5, v = 0.3$

Fig. 2 The wave number, the frequency and the spectrum are shown for the same caption as in Fig. 1 (i)-(v), but in (iii)$v = 0.5$

By substituting from (14) and (17) into (10) and (11), by the same way as in Section 3.1, we get,

$$a_1 = \frac{\sqrt{3a^2 c_2 v - b_1^2(-\delta + \mu + vo)}}{\sqrt{-\delta + \mu + vo}}, \quad a_0 = \frac{c_1}{4c_2}a_1, \quad c_0 = \frac{3a^3 c_1^2 kv + ac_1 \, k(8k^2 v - 4) - 2bc_2}{8a^3 c_2 kv},$$

$$\sigma = \frac{(\alpha + 4\delta - vo)(\delta - \mu - vo)}{18v}, \quad b = \frac{2ak(3\alpha(4k^2 v - 1) + \delta(4k^2 v - 7) - 4k^2 \mu v - 4k^2 vvo + \mu - 3v + vo)}{3\alpha + 8\delta - 2(\mu + vo)},$$

$$c_2 = -\frac{a^2 v(3\alpha + 8\delta - 2(\mu + vo))c_1^2}{4(-6\alpha k^2 v + \delta(12k^2 v - 1) + \mu + 3v + vo)},$$ (18)

$$\omega = \frac{3(\alpha + 4\delta - vo)(-\delta - 6\alpha k^2 v + 12\delta k^2 v + \mu + 3v + vo)^2}{4v(\delta - \mu - vo)(3\alpha + 8\delta - 2(\mu + vo))^2} + k^4(-v) + k^2$$

$$- \frac{3(2k^2(-\alpha + \delta + \mu + vo) + 1)(-6\alpha k^2 v + \delta(12k^2 v - 1) + \mu + 3v + vo)}{2(\delta - \mu - vo)(3\alpha + 8\delta - 2(\mu + vo))}.$$

Finally, the solutions of (8) and (9) are,.

$$u(x, t) = \frac{K_1}{K_2}, \quad K_1 = -6\alpha k^2 v + \delta(12k^2 v - 1) + \mu + 3v$$
$$+ vo\left(\exp\left((A_0 + z)\left(3v\left(4\delta k^2 + 1\right) + \mu + vo\right)K\right)\right.$$
$$+ a^2 c_1 \, \exp\left((Ao + z)\left(\delta + 6\alpha k^2 v\right)K\right)v(3\alpha + 8\delta - 2(\mu + vo)))$$

$$\sqrt{-\frac{3a^4 c_1^2 v^2(3\alpha + 8\delta - 2(\mu + vo))}{4(-6\alpha k^2 v + \delta(12k^2 v - 1) + \mu + 3v + vo)}} + b_1^2(\delta - \mu - vo),$$ (19)

$$K_2 = a^2 c_1 \, v\sqrt{-\delta + \mu + vo}3\alpha + 8\delta - 2(\mu + vo)$$
$$\left(-\exp\left((A_0 + z)\left(3v\left(4\delta k^2 + 1\right) + \mu + vo\right)K\right)\right.$$
$$+ a^2 c1v(3\alpha + 8\delta - 2(\mu + vo))\exp\left((A_0 + z)\left(\delta + 6\alpha k^2 v\right)K\right)),$$

$$K = \sqrt{\frac{1}{a^2 v(2(-4\delta + \mu + vo) - 3\alpha)(-6\alpha k^2 v + \delta(12k^2 v - 1) + \mu + 3v + vo)}},$$
$$v(x, t) = \frac{b_1}{a_1}u(x, t), z = ax + \frac{2ak(3\alpha(4k^2 v - 1) + \delta(4k^2 v - 7) - 4k^2 \mu v - 4k^2 vvo + \mu - 3v + vo)}{3\alpha + 8\delta - 2(\mu + vo)}t.$$

The solutions in (20) are used to display Rew in Fig. 3 (i)-(iii)

When $v_0 = 3.2, k := 1.5, a := 1.2, a_1 = 0.7, b_1 = 0.1, \sigma = \backslash 0.6, \alpha = 0.5, \mu := 2.5, v = 1.3, A_0 = -5, \delta = 0.1, c_1 = 0.6$. Theses figures show "continuum" soliton chain.
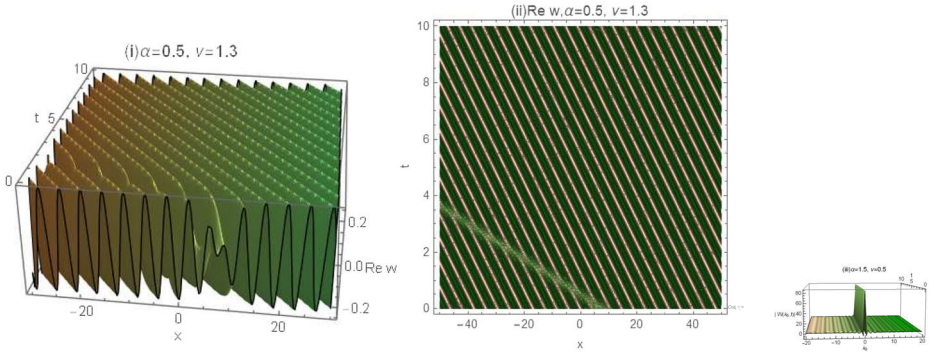
Fig. 3 In Fig. 3(i) and , the 3D plot , contour plots are displayed for $Re\ w$, while the variation against$x$ for different values of$t$ is displayed in Fig. 3(iii)

3.3 When $p = 2$ and $r = 2$

Here, we consider the AE,

$$g'(z) = \sqrt{c_4 g(z)^4 + c_2 g(z)^2 + c_0}. \tag{20}$$

By inserting (14) and (20) into (10) and (11), we have,

$$a_1 = \frac{\sqrt{3a^2 c_4 v - b_1^2(-\delta+\mu+v o)}}{\sqrt{-\delta+\mu+v o}}, \; a_0 = 0, \; b = -2ak\left(2a^2 c_2 v - 2k^2 v + 1\right),$$

$$\sigma = \frac{(\alpha+4\delta-v o)(\delta-\mu-v o)}{18 v}, \; k = \frac{\sqrt{-6\alpha a^2 c_2 v - \delta(16 a^2 c_2 v+1)+4a^2 c_2 \mu v+4a^2 c_2 v v o+\mu+3v+v o}}{\sqrt{6}\sqrt{v(\alpha-2\delta)}},$$

$$\omega = a^4 v\left(-c_2^2 - \frac{12 c_4 c o(\alpha+\delta-\mu)}{\delta-\mu-v o}\right) - \frac{\left(-6\alpha a^2 c_2 v - 16 a^2 c_2 \delta v + 4a^2 c_2 \mu v + 4a^2 c_2 v v o - \delta+\mu+3v+v o\right)^2}{36 v(\alpha-2\delta)^2}$$

$$+ \frac{a^2 c_c\left(-\alpha\left(6a^2 c_2 v+1\right)-16 a^2 c_2 \delta v+4a^2 c_2 \mu v+4a^2 c_2 v v o+\delta+\mu+3v+v o\right)}{\alpha-2\delta}$$

$$+ \frac{-6\alpha a^2 c_2 v - \delta\left(16 a^2 c_2 v+1\right)+4a^2 c_2 \mu v+4a^2 c_2 v v o+\mu+3v+v o}{6 v(\alpha-2\delta)},$$

$$c_4 = m^2, \; c_0 = r^2, \; c_2 = -n^2, \tag{21}$$

The solutions of (8) and (9) are,

$$u(x,t) = \frac{\sqrt{3a^2 m^2 v - b_1^2(-\delta+\mu+v o)}}{\sqrt{-\delta+\mu+v o}}$$

$$\frac{\sqrt{2}\left(n^2\sqrt{n^4-4m^2 r^2}+2m^2 r^2-n^4\right)\mathrm{sn}\left(\sqrt{2}r\sqrt{\frac{m^2}{n^2-\sqrt{n^4-4m^2 r^2}}}(Ao+z),|\frac{n^2-\sqrt{n^4-4m^2 r^2}}{n^2+\sqrt{n^4-4m^2 r^2}}\right)}{\left(n^2-\sqrt{n^4-4m^2 r^2}\right)^2\sqrt{\frac{m^2}{n^2-\sqrt{n^4-4m^2 r^2}}}} \tag{22}$$

$$v(x,t) = \frac{b_1}{a_1}u(x,t), z = ax - 2ak\left(2a^2 c_2 v - 2k^2 v + 1\right)t.$$

The solutions in (22) are used to display $\mid w \mid$ in Fig. 4 (i)-(iii).

When $v_0 = 3.2$; $k := 1.5$, $a = 1.2$, $a_1 = 0.7$; $b_1 = 0.1$, $\sigma = 0.6$, $\alpha = 0.5$, $\mu = 2.5$, $v = 1.3$, $A_0 = -5$, $\delta = 0.3$, $m := 0.6$, $r = 1.2$, $n = 3$.

Figure 4(i) shows complex chirped waves, while Fig. 4(ii) shows super lattices. Figure 4(iii) shows dense "continuum" solion chain.
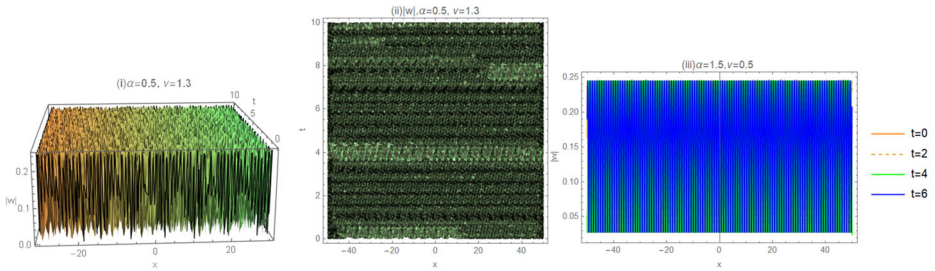
4 Rational solutions of (10) and (11)

We consider the solution in (13), together with AE,

$$g'(z) = c_1 g(z) + c_0. \tag{23}$$

By using (13) and (23) into (10) and (11), we get,

$$
\begin{aligned}
\omega = {} & \tfrac{1}{s_1^4}\left(2b_1^2 s_1^2\left(2k^2(\alpha - \delta - \mu - v_0) - 1\right) + k^2 s_1^4\left(1 - k^2 v\right)\right.\\
& + 24a_1^4\sigma + 24b_1^4\sigma + a_1^2\left(s_1^2\left(4k^2(\alpha - \delta - \mu - v_0) - 2\right)\right) + 48b_1^2\sigma)),
\end{aligned}
$$

$$
\begin{aligned}
b = {} & \tfrac{1}{a_1 c_1 s_1^4}\left(a^2 b_1 c_1^2 s_1^4\left(a^2 c_1^2 v - 6k^2 v + 1\right) - 4b_1^3 s_1^2\left(a^2 c_1^2(\delta + \mu)\right.\right.\\
& + 2k^2(\alpha - \delta - \mu - v_0) - 1)\\
& - 2aa_1 c_1 k s_1^2\left(s_1^2\left(2a^2 c1^2 v - 2k^2 v + 1\right) + 4b_1^2(\delta - \mu - v_0)\right)\\
& + 8aa_1^3 c_1 k s_1^2(-\delta + \mu + v_0) - 96a_1^4 b_1\sigma - 96b_1^5\sigma\\
& - 4a_{21}\left(b_1\, s_1^2\left(a^2 c_1^2(\delta + \mu) + 2k^2(\alpha - \delta - \mu - v_0) - 1\right) + 48b_1^3\sigma\right)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
c_0 = {} & -\tfrac{3c_1 s_0}{2s_1},\quad \mu = \tfrac{P_1}{Q},\quad P_1 = -5a^5 b_1^2 c_1^5 v s_1^2 + 242a^4 a_1 b_1 c_1^4 k v s_1^2\\
& + 4a_1 b_1 k\left(a_1^2\left(11 - 38\alpha k^2\right) + b_1^2\left(11 - 38\alpha k^2\right) + 3k^2 s_1^2\left(6k^2 v - 1\right)\right)\\
& + a^3 c_1^2 s_1^2\left(456a_1^2 k^2 v + b_1^2\left(99k^2 v + 1\right)\right) + 2a^2\, a_1 b_1 c_1^2 k\left(-4\alpha\left(a_1^2 + b_1^2\right)\right.\\
& + s_1^2\left(114k^2 v + 13\right)) + a_1 c_1\left(b_1^4\left(4 - 16\alpha k^2\right) + 3b_1^2 k^2 s1^2\left(1 - 6k^2 v\right)\right.\\
& - 4a_1^2\left(b_1^2\left(4\alpha k^2 - 1\right) + 12k^4 v s_1^2\right)),\quad Q = 16a_1 b_1 k\left(a_1^2 + b_1^2\right)\left(3k^2 - a^2 c1^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\delta = {} & \tfrac{P_2}{Q},\quad P_2 = -102a^4 a_1 b_1 c_1^4 k v s_1^2 + 5a^5 b_1^2 c_1^5 v s1^2\\
& + 12a_1 b_1 k\left(a_1^2 + b_1^2\right)\left(2\alpha k^2 - 1\right)\\
& + ab_1 c_1^2\left(4a_1^2\left(4\alpha k^2 - 1\right) + 4b_1^2\left(4\alpha k^2 - 1\right) + 3k^2 s1^2\left(6k^2 v - 1\right)\right)\\
& + 2a^2 a_1 b_1 c_1^2 k\left(4\alpha\left(a_1^2 + b_1^2\right) - 3s_1^2 s1^2\left(8k^2 v + 1\right)\right)\\
& - a^3 c_1^2 s_1^2\left(120a_1^2 k^2 v + b_1^2\left(99k^2 v + 1\right)\right),\quad a = \tfrac{k}{\sqrt{2}c_1},\quad k = \tfrac{1}{2\sqrt{\alpha}},
\end{aligned}
$$

$$
b_1 = \tfrac{1}{4}\sqrt{\frac{\sqrt{\frac{2}{15}}s_1^2\sqrt{\sigma}(8\alpha - 17v)}{\alpha\sigma}} - 16a_1^2,\quad a_0 = -\tfrac{4a_1 s_0}{s_1}. \tag{24}
$$

Fig. 4 In Fig. [4](i) and , the 3D plot , contour plots are displayed for $| w |$, while the variation against x for different values of t is displayed in Fig. [4](iii)

The solutions of ([8](8)) and ([9](9)) are,

$$
u(x,t) = \frac{2A_0 a_1 s_1 e^{c_1 z} - 5a_1 s_0}{2A_0 s_1^2 e^{c_1 z} + 5a_1 s_0}, \qquad v(x,t) = = \frac{1}{4}\sqrt{s1^2 \frac{\sqrt{\frac{2}{15}}\sqrt{\sigma(8\alpha - 17v)}}{\alpha\sigma} - 16a_1^2 u(x,t)},
$$

$$
z = \frac{H_3}{H_4}, \qquad H_3 = -3\sigma s_1^4 t\left(448\alpha^2 - 8384\alpha v + 15793v^2\right)
$$
$$
+ 24\, a_1 \alpha \sigma s_1^2 \sqrt{\sigma(8\alpha - 17v)}\left(\sqrt{30}\, a_1 t(56\alpha - 449v)\right.
$$
$$
+ 4\left(-6\alpha t + 99vt + 10\alpha^{3/2}x\right)\sqrt{\frac{s1^2\sqrt{30}\sqrt{\sigma(8\alpha - 17v)} - 240\alpha a1^2\sigma}{\alpha\sigma}}\right)
$$
$$
+ 8\, s1^2\, t(8\alpha - 17v)(\alpha + vo)\left(s_1^2\sqrt{30}\sqrt{\sigma(8\alpha - 17v)} - 240\alpha a_1^2\sigma\right.
$$
$$
+ 2\sqrt{30}\alpha a1\sigma\sqrt{\frac{s_1^2\sqrt{30}\sqrt{\sigma(8\alpha - 17v)} - 240\alpha a1^2\sigma}{\alpha\sigma}}\right),
$$
(25)

$$
H_4 = 1920\sqrt{2}\alpha^3 a1 c1\sigma s_1^2\sqrt{\sigma(8\alpha - 17v)}\sqrt{\frac{s_1^2\sqrt{30}\sqrt{\sigma(8\alpha - 17v)} - 240\alpha a1^2\sigma}{\alpha\sigma}}.
$$

The results in ([25](25)) are used to display Rew in Fig. [5](5) (i)-(iii).

In Fig. [5](i) and (ii0, the 3D and contour plots of Rew are displayed, while the variation of Rew against x fo0r different values of t is displayed.

Figure [5](i) shows "—continuum" soliton chain with trap, while)ii) shows mixed lattice-solitons.
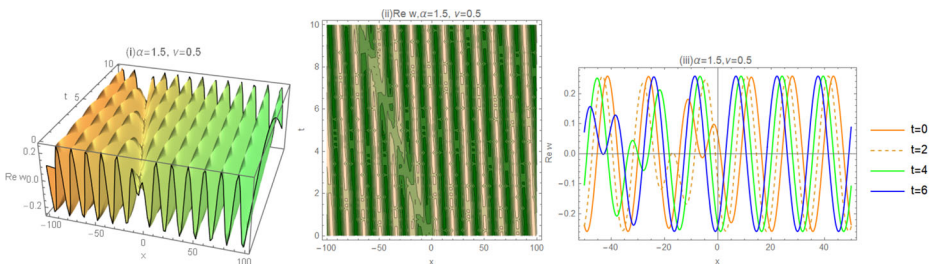
Fig. 5 When $a_1 = 0.5, v_0 = 3.2, k = 1.5, a := 1.2, \sigma = 0.6, \alpha = 1.5, \mu := 2.5, v = 0.5, A_0 = 5, s_1 = 3, s_0 = 1.5, c_1 = 2.5$

5 Modulation stability analysis

To study the modulation instability of a system, it should exhibit a normal mode. That is a periodic standing waves. Here, (3) has a solution of the form.

$$w_m(x,t) = Q e^{i(rx-qt)}, \qquad w_m(x,t)^* = Q e^{-i(rx-qt)}. \tag{26}$$

By inserting (24) into (3), we get

$$q = 4\alpha Q^2 r^2 - 4\delta Q^2 r^2 - 4\mu Q^2 r^2 - 4v o Q^2 r^2 + 24 Q^4 \sigma - 2Q^2 - vr^4 + r^2. \tag{27}$$

We write the solution expansion near w_m,

$$\begin{aligned}
w(x,t) &= Q e^{i(rx-qt)} \left(1 + \varepsilon_1 e^{\lambda t}(U(x) + i V(x)) + O(\varepsilon^2)\right) \\
w(x,t)^* &= Q e^{-i(rx-qt)} \left(1 + \epsilon 2 e^{\lambda t}(U(x) - i V(x)) + O(\varepsilon^2)\right),
\end{aligned} \tag{28}$$

in (3) and Calculations give rise to,

$$H \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = 0, \ \ H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$$\begin{aligned}
h_{11} &= U\left(q + Q^2\left(8r^2(-\alpha + \delta + \mu + vo) + 4\right) - 72 Q^4 \sigma + r^2\left(vr^2 - 1\right)\right) \\
&\quad + 2rV'\left(Q^2(4(\mu + vo) - 2\alpha) + 2vr^2 - 1\right) - 4\mu Q^2 U'' - 6vr^2 U'' \\
&\quad - 4vr V^{(3)} + v U^{(4)} + U'', \\
h_{21} &= V\left(q + Q^2\left(8r^2(-\alpha + \delta + \mu + vo) + 4\right) - 72 Q^4 \sigma + r^2\left(vr^2 - 1\right)\right)' \\
&\quad + 4\alpha Q^2 rU - 8\mu Q^2 rU' - 8vo Q^2 rU' - 4\mu Q^2 V'' - 4vr^3 U' \\
&\quad - 6vr^2 V'' + 4vr U^{(3)} + 2rU' + \lambda U + v V^{(4)} + V'', \\
h_{12} &= 2Q^2 U\left(-24 Q^2 \sigma + 2r^2(-\alpha + \delta + \mu + vo) + 1\right) \\
&\quad - 4Q^2\left(r(\alpha - 2\delta)V' + \delta U''\right), \\
h_{22} &= 2Q^2\left(V\left(24 Q^2 \sigma + 2r^2(\alpha - \delta - \mu - vo) - 1\right) - 2r(\alpha - 2\delta)U' + 2\delta V''\right).
\end{aligned} \tag{29}$$
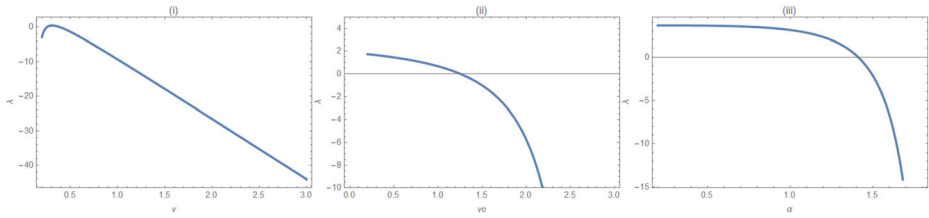
The solution of (26) is $det H = 0$, which yields,

$$\begin{aligned}
&\left(V\left(24 Q^2 \sigma + 2r^2(\alpha - \delta - \mu - vo) - 1\right) - 2r(\alpha - 2\delta)U' + 2\delta V''\right) \\
&\left(U\left(q + Q^2\left(8r^2(-\alpha + \delta + \mu + vo) + 4\right) - 72 Q^4 \sigma + r^2\left(vr^2 - 1\right)\right) + 2rV'\right. \\
&\left(Q^2(4(\mu + vo) - 2\alpha) + 2vr^2 - 1\right) - 4\mu Q^2 U'' - 6vr^2 U'' - 4vr V^{(3)} + v U^{(4)} + U'') \\
&- \left(U\left(-24 Q^2 \sigma + 2r^2(-\alpha + \delta + \mu + vo) + 1\right) - 2\left(r(\alpha - 2\delta)V' + \delta U''\right)\right) \\
&\left(V\left(q + Q^2\left(8r^2(-\alpha + \delta + \mu + vo) + 4\right) - 72 Q^4 \sigma + r^2\left(vr^2 - 1\right)\right) + \lambda U' \right. \\
&+ 4\alpha Q^2 rU - 8\mu Q^2 rU' - 8vo Q^2 rU' - 4\mu Q^2 V'' \\
&\left. -4vr^3 U' - 6vr^2 V'' + 4vr U^{(3)} + +2rU' + v V^{(4)} + V''\right) = 0.
\end{aligned} \tag{30}$$

We solve the eigenvalue problem in (27) subjected to the boundary conditions $| U(\pm\infty) | \leq U_0$ and $| V(\pm\infty) | \leq V_0$. Thus the eigenfunctions take the form.

$$U(x) = U0 e^{i(hx)}, \qquad V(x) = V0 e^{i(hx)}. \tag{31}$$

By substituting from (28) into (27), we have,

$$\begin{aligned}
\lambda = \tfrac{1}{Q} 2V_0 &\left(2\delta h^6 v + 2Q^2\left(24 Q^2 \sigma + 2r^2(\alpha - \delta - \mu - vo) - 1\right)^2\right. \\
&+ h^4\left(v + \delta\left(8\mu Q^2 + 30 vr^2 - 2\right) + 2vr^2(-5\alpha + \mu + vo) - 24v Q^2 \sigma\right) \\
&+ h^2\left(-\left(4Q^2(\delta + \mu) - 1\right) - 4vr^4(5\alpha - 7\delta - 3(\mu + vo))\right) \\
&\left(24 Q^2 \sigma - 1\right) + 2r^2\left(-\mu + 3v - vo + (4\alpha^2 + 4\delta^2 + 4\mu^2 + 4\mu vo)Q^2\right. \\[4pt]
&\left.\left. + \alpha\left(3 - 4Q^2(3\delta + 3\mu + 2vo)\right) + \delta\left(4Q^2(6\mu + 5vo) - 5\right) - 72v Q^2 \sigma\right)\right),
\end{aligned} \tag{32}$$

Fig. 6 When $r=1.2, k:=1.5, \delta = 0.2, h=0.5, \mu=0.5, U_0 = 5, V_0 = 3$. In (i) $\alpha=2.5, v_0 = 2.3$, (ii) , $\alpha=2.5, v=1.3$

$$Q := \frac{\sqrt{U_0^2 - V_0^2}\sqrt{2h^2v + 2vr^2 - 1}}{2\sqrt{U_0^2(\alpha - \delta - \mu - v\text{o}) + \text{Vo}^2(-\delta + \mu + v\text{o})}}, \tag{33}$$

together with as lengthy equation for σ, which will not produced here. The eigenvalue λ given in (29) is displayed against the dominant parameters v, α and v_0 in Fig. 6 (i)-(iii)

and (iii); $\alpha=2.5, v=1.7, v_0 = 2.3$.

Figure 6(i) shows that modulation stability holds against v. While, in Fig. 6(ii) and (iii) there are the critical values $v_{ocr} = 1.25$ and $\alpha_{cr} = 1.4$, where below these values instability hols otherwise stability occurs.

6 Conclusions

The 1D Heisenberg spin chain system is considered. In continuum analog a o equation wad derived in the literature. Which is a nonlinear Schrodinger equation with bi-quadratic dispersion and fifth degree nonlinearity. Here, this equation is studied for the objectives of finding the exact solutions and investigating the relevant phenomena vis-a-vis with spin chain. The exact solutions are obtained by using the unified method in polynomial and rational forms. A transformation that enables us to inspect the effects of soliton- periodic wave collision is proposed. The collision can be elastic or inelastic according to the waves solutions are smooth or not smooth. Numerical evaluation of the solutions are carried and displayed in figures. It is found that the solutions exhibit "continuum" soliton chain, while the contour plots show super lattices or lattices with trapping. It is remarked the the solutions are bounded by $-1/4$ and $1/4$ which may be relevant with the spin $-1/2$ and $1/2$. The modulation stability analysis is carried and it is shown that there is a critical value of the dominant parameters that separates stability and instability. It is worth noticing that the waves solutions found here are smooth, so waves collision is elastic.

Declarations

Conflict of Interests The author declares that there is no conflict of interest.

References

1. Kohno, M.: Dynamically Dominant Excitation of String Solutions in the Spin-1/2 Antiferromagnetic Heisenberg Chain in a Magnetic Field. Phy. Rev. Lett. **102**, 037203 (2009)
2. Murmann, S., Deuretzbacher, F., Zürn, G., Bjerlin, J., Reimann, S.M., Santos, L., Lompe, T., Jochim, S.: Antiferromagnetic Heisenberg Spin Chain of a Few Cold Atoms in a One-Dimensional Trap. Phys. Rev. Lett. **115**, 215301 (2015)
3. Marchukov, O.V., Volosniev, A.G., Valiente, M., Petrosyan, D., Zinner, N.T.: Quantum spin transistor with a Heisenberg spin chain. Nat Commun **7**, 13070 (2016)
4. Devoret, M.H., Glattli, C.: Single-electron transistors. Phys. World **11**, 29 (1998)
5. Babujian, H.M.: Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model. Nuclear Physics B **215**(3), 317–336 (1983)
6. Žnidar, M.: Spin transport in a one-dimensional Anisotropic Heisenberg model, Phys. Rev. Lett. **106**, 22060 (2011)
7. Porsezian, K., Daniel, M., Lakshmanan, M.: On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain. J. of Math.l Phys. **33**, 1807 (1992)
8. Kavitha1, L., Daniel, M.: Integrability and soliton in a classical one-dimensional site-dependent biquadratic Heisenberg spin chain and the effect of nonlinear inhomogeneity. J. Phys. A:, Math. Gen. **36**, 10471 (2003)
9. Tantawy, M., Abdel-Gawad, H.I.: On multi-geometric structures optical waves propagation in self-phase modulation medium: Sasa– Satsuma equation. Eur. Phys. J. Plus **135**, 928 (2020)
10. Wazwaz, A.M.: A study on linear and nonlinear Schrodinger equations by the variational iteration method. Chaos Solitons Fractals **37**, 1136–1142 (2008)
11. Ablowitz, M.J., Solitons: Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, New York (1991)
12. Abdel-Gawad, H.I.: Chirped, breathers, diamond and W-shaped optical waves propagation in non self-phase modulation medium. Biswas–Arshed equation, I. J.of Mod. Phys. B 35(07) (2021)
13. Grabwoski, M.P., Mathe, P.: Quantum integrals of motion for the Heisenberg spin chain. Mod. Phys. Lett. A **09**(24), 2197–2206 (1994)
14. Sulaiman, T.A., Aktürk, T., Bulut, H., Baskonus, H.I.M.: Investigation of various soliton solutions to the Heisenberg ferromagnetic spin chain equation, J. of Electromag. Waves and Appl. 32 (9) (2018)
15. Uddin, M.F., Hafez, M.G., Hammouch, Z., Baleanu, D.: Periodic and rogue waves for Heisenberg models of ferromagnetic spin chains with fractional beta derivative evolution and obliqueness, Waves in Rand. and Complex Med. 1-15. https://doi.org/10.1080/17455030.2020.1722331 (2022)
16. Inc, M., Aliyu, A.I., Yusuf, A., Baleanu, D.: Optical solitons and modulation instability analysis of an integrable model of (2+1)-Dimensional Heisenberg ferromagnetic spin chain equation. Superlattices and Microstruct. **112**, 628–638 (2017)
17. Hosseini, K., Salahshour, S., Mirzazadeh, M., Ahmadian, A., Baleanu, D., Khoshrang, A.: The (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation: its solitons and Jacobi elliptic function solutions. Eur. Phys. J. Plus **136**, 206 (2021)
18. Tang, G., Wang, S., Wang, G.: Solitons and complexitons solutions of an integrable model of (2+1)-dimensional Heisenberg ferromagnetic spin chain. Nonlinear Dyn 88 (2017) 2319– (2327)
19. Osman, M.S., Tariq, K.U., Bekir, A., Elmoasry, A., Elazab1, N.S., Younis7, M., Abdel-Aty, M.: Investigation of soliton solutions with different wave structures to the (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation. Commun. Theor. Phys. **72**, 035002 (2020)
20. Parasuraman, E.: Dynamics of soliton collision phenomena on classical discrete Heisenberg weak ferromagnetic spin chain. J. of Magnetism and Magnetic Mat. **489**, 165403 (2019)
21. Guana, B., Chena, S., Liu, Y., Wang, X., Zhao, J.: Wave patterns of (2+1)-dimensional nonlinear Heisenberg ferromagnetic spin chains in the semi classical limit. Results in Phys.. **16**, 102834 (2020)
22. Yusuf, A., Tchier, F., Inc, M.: New interaction and combined multi-wave solutions for the Heisenberg ferromagnetic spin chain equation. Eur. Phys. J. Plus **135**, 416 (2020)

23. Douvagai, Y., Amadou, G., Betchewe, A., Houwe, M., Inc, S.Y.D., Almohsen, B.: Dynamic behaviors for a (2 + 1)-dimensional inhomogenous Heisenberg ferromagnetic spin chain system. Mod.Phys. Lett. B. **35**(15), 2150251 (2021)
24. Li, B.-Q., Ma, Y.-L.: Characteristics of rogue waves for a (2,+,1)-dimensional Heisenberg ferromagnetic spin chain system. J. of Magnetism and Magnetic Mat. **474**, 537–543 (2019)
25. Wazwaz, A.M.: The tanh method for traveling wave solutions of nonlinear equations. Appl. Math. and Comput. **154**(3), 713–723 (2004)
26. Ji-Huan, H., Xu-Hong, W.: Exp-function method for nonlinear wave equations. Chaos. Solitons & Fractals **30**(3), 700–708 (2006)
27. Bekir, A.: Application of the -expansion method for nonlinear evolution equations. Phys. Lett. A **372**(19), 3400–3406 (2008)
28. Bueno, M.I., Marcellán, F.: Darboux transformation and perturbation of linear functionals. Linear Algebra Applications **384**(1), 215–242 (2004)
29. Hosseini, K., Ansari, R.R.: New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method, Waves in Rand. and Compl Media 27 (4) (2017)
30. Belmonte-Beitia, J., Pérez-García, V.M., Vekslerchik, V., Torres, P.J.: Lie Symmetries and solitons in nonlinear Systems with spatially inhomogeneous nonlinearities. Phys. Rev. Lett. **98**, 064102 (2007)
31. Akinyemi, L., Rezazadeh, H., Shic, Q.-H., Inc, M., M.A.Khater, M., Ahmad, H., Jhangeer, A.J., Ali Akbar, M.: New optical solitons of perturbed nonlinear Schrödinger–Hirota equation with spatio-temporal dispersion. Res.in Phys. **29**, 104656 (2021)
32. Akinyemi, L., Nisar, K.S., Saleel, C.A., Rezazadeh, H., Veeresha, P., M.A.Khater, M., Inc, M.: Novel approach to the analysis of fifth-order weakly nonlocal fractional Schrödinger equation with Caputo derivative. Res. in Phys. **31**, 104958 (2021)
33. Kilic, B., Inc, M.: On optical solitons of the resonant schrödinger's equation in optical fibers with dual-power law nonlinearity and time-dependent coefficients, Waves in Rand. Complex Media **25**(3), 334–341 (2015)
34. Inc, M., Kilic, B., Baleanu, D.: Optical soliton solutions of the pulse propagation generalized equation in parabolic-law media with space-modulated coefficients. Optik **127**(3), 1056–1058 (2016)
35. Kilic, B., Inc, M.: Soliton solutions for the Kundu–Eckhaus equation with the aid of unified algebraic and auxiliary equation expansion method. J. of Elect. Waves Appli. **30**(7), 371–379 (2016)
36. Silva, S.L.L.: Thermal Entanglement in 2 × 3 Heisenberg Chains via Distance Between States. Int J Theor Phys **60**, 3861–3867 (2021)
37. Zhou, Q., Liu, S.: Dark optical solitons in quadratic nonlinear media with spatio-temporal dispersion. Nonlinear Dyn. **81**, 733–738 (2015)
38. Abdel-Gawad, H.I., Tantawy, M., Mani Rajan, M.S.: Similariton regularized waves solutions of the (1+2)-dimensional non-autonomous BBME in shallow water and stability. J. Ocean Eng. and Sci. https://doi.org/10.1016/j.joes.2021.09.002 (2021)
39. Liu, W.: Parallel line rogue waves of a -dimensional nonlinear Schrödinger equation describing the (2+1) Heisenberg ferromagnetic spin chain Romanian. J Phys **62**(118), 1–16 (2017)
40. Alvarez, J.V., Gros, C.: Low-Temperature Transport in Heisenberg Chains. Phys. Rev. Lett. **88**, 077203 (2002)
41. Abdel-Gawad H.I.: Towards a unified method for exact Solutions of evolution Equations. An application to reaction diffusion equations with finite memory transport. J. Stat. Phys. **147**, 506–521 (2012)
42. Alderremy, A.A., Abdel-Gawad, H.I., Saad, K.M., Aly, S.H.: New exact solutions of time conformable fractional Klein Kramer equation. Opt. and Quantum Elect. **53**, 693 (2021)
43. Abdel-Gawad, H.I.: Zig-zag, bright, short and long solitons formation in inhomogeneous ferromagnetic materials. Kraenkel-Manna-Merle equation with space dependent coefficients (2021)
44. Abdel-Gawad, H.I.: Study of modulation instability and geometric structures of multisolitons in a medium with high dispersivity and nonlinearity, Pramana 95, Article number: 146 (2021)
45. Tantawy, M., Abdel-Gawad, H.I.: On continuum model analog to zig-zag optical lattice in quantum optics. Appl Phys. B **127**, 120 (2021)
46. Abdel-Gawad, H.I.: A generalized Kundu--Eckhaus equation with an extra-dispersion: pulses configuration, optical Quant. Elect. 53, Article number: 705 (2021)
47. Lakshamanan, M., Porsezian, K., Daniel, M.: Effect of discreteness on the continuum limit of the Heisenberg spin chain. Phys. Lett. A **133**, 9 (1988)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.