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Bell Inequalities and Group Symmetry

Katarzyna Bolonek-Lasoń[1]

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Abstract Recently the method based on irreducible representations of finite groups has been proposed as a tool for investigating the more sophisticated versions of Bell inequalities (V. Ugŭr Gűney, M. Hillery, Phys. Rev. **A90**, 062121 (2014) and Phys. Rev. **A91**, 052110 (2015)). In the present paper an example based on the symmetry group S_4 is considered. The Bell inequality violation due to the symmetry properties of regular tetrahedron is described. A nonlocal game based on the inequalities derived is described and it is shown that the violation of Bell inequality implies that the quantum strategies outperform their classical counterparts.

Keywords Bell inequalities · Group theoretical methods · Nonlocal games

1 Introduction

We present here a short review of the results obtained in our papers [1, 2] devoted to group-theoretical aspects of Bell inequalities. The Bell inequalities [3] describe a fundamental difference between classical and quantum correlations. These inequalities must be satisfied by the classical theory, whereas the quantum theory does not have such restrictions. Since the time when Bell published his paper numerous authors derived various Bell inequalities [4–10] (for a review, see [11, 12]).

A common scenario for a Bell inequality is that a bipartite quantum system is prepared, and one part is sent to each of two parties, Alice and Bob. Each party then performs a measurement on their part. This procedure is repeated a number of times with different measurements choices. An event is a choice of measurement observables by each party and

✉ Katarzyna Bolonek-Lasoń
kbolonek@uni.lodz.pl

1 Department of Statistical Methods, Faculty of Economics and Sociology, University of Lodz, 41/43 Rewolucji 1905 St., Lodz 90-214, Poland

the results of their measurements. Bell inequality involves the corresponding correlation functions. It can be, however, expressed directly in terms of sums of probabilities of the events.

A significant contribution which allowed the transparent interpretation of Bell inequality has been made by Fine [13, 14] (see also [15, 16]). He assumed that a number of random variables possess the joint probability distribution and the relevant probabilities entering the Bell inequalities are obtained as marginals from joint probability distribution. The Bell ineqaulities form necessary and sufficient conditions for the existence of such a joint probability distribution returning the initial probabilities as marginals.

In order to derive the Bell inequalities we express the relevant correlation functions in terms of probabilities of the events (described above) and write the resulting combination of probabilities as marginals of joint probability distribution arriving at the expression of the form $\sum_{\alpha} c(\alpha)p(\alpha)$. The coefficient $c(\alpha)$ determines how many times $p(\alpha)$ appears in this sum. Due to $0 \leq p(\alpha) \leq 1$, $\sum_{\alpha} p(\alpha) = 1$ one obtains

$$\min_{\alpha} c(\alpha) \leq \sum_{\alpha} c(\alpha)p(\alpha) \leq \max_{\alpha} c(\alpha) \tag{1}$$

which is a Bell inequality.

One of the methods which allow to find examples of Bell inequality violation was proposed by Gűney and Hillery [17, 18]. It is based on group theory and can be described as follows. We take some finite group G and select its irreducible representation D. Then the space carrying the representation D becames the space of states of one party. Next, we take some state $|\varphi\rangle \otimes |\psi\rangle$ and construct the operator [17, 18]

$$X(\varphi, \psi) \equiv \sum_{g \in G} (D(g)|\varphi\rangle \otimes D(g)|\psi\rangle)(\langle\varphi|D^{+}(g) \otimes \langle\psi|D^{+}(g)). \tag{2}$$

Defining

$$\begin{aligned} &|g, \varphi\rangle \equiv D(g)|\varphi\rangle, \qquad |g, \psi\rangle \equiv D(g)|\psi\rangle \\ &|g, \varphi, \psi\rangle \equiv |g, \varphi\rangle \otimes |g, \psi\rangle \end{aligned} \tag{3}$$

we find for arbitrary bipartite state $|\chi\rangle$

$$\langle\chi|X|\chi\rangle = \sum_{g \in G} |\langle g, \varphi, \psi|\chi\rangle|^2. \tag{4}$$

The states $|g, \varphi\rangle$ and $|g, \psi\rangle$ can be viewed as eigenstates of observables, the first of an observable on the first system (Alice) and the second of an observable on the second system (Bob). Each term in the above sum then represents the probability of an event, and the whole sum is just the sum of probabilities of events, exactly the kind of expression that appears in a Bell inequality. The set $\{\mathcal{D}(g)|\alpha\rangle | g \in G\}$ is called an orbit of G corresponding to the representation $\mathcal{D}$ of a group G and passing through $|\alpha\rangle$. In this terminology the sum (4) involves the orbit corresponding to $\mathcal{D} = D \otimes D$ and $|\alpha\rangle = |\varphi\rangle \otimes |\psi\rangle$.

In order to find an example of violation of Bell inequality we look for the state $|\chi\rangle$ which maximizes the sum of probabilities on the right hand side of the last equation. To this end we have to find the maximal eigenvalue of operator $X(\varphi, \psi)$. To this end it is assumed that in the decomposition of $D \otimes D$ into irreducible pieces,

$$D \otimes D = \bigoplus_{s} D^{(s)} \tag{5}$$

each $D^{(s)}$ appears only once. This assumption simplifies the form of $X(\varphi, \psi)$ which becomes diagonal and reduces, according to the Schur lemma, to a multiple of unity on each irreducible piece. Using the orthogonality relations it is easy to see that the relevant eigenvalues of $X(\varphi, \psi)$ are [18]

$$\frac{|G|}{d_s} \parallel (|\varphi\rangle \otimes |\psi\rangle)_s \parallel^2 \tag{6}$$

where $|G|$ is the order of G, d_s - the dimension of $D^{(s)}$ and $(|\varphi\rangle \otimes |\psi\rangle)_s$ is the projection of $|\varphi\rangle \otimes |\psi\rangle$ on the carrier space of $D^{(s)}$. In general, in order to violate the Bell inequality it is not sufficient to consider only one orbit. Therefore, we look for the sum of eigenvalues of all $X(\varphi_n, \psi_n)$, where the pairs of vectors (φ_n, ψ_n) generate the orbits. In this way we can maximize the sum of probabilities

$$\sum_{n=1}^{N} \sum_{g \in G} |\langle g, \varphi_n, \psi_n | \chi \rangle|^2. \tag{7}$$

which is a candidate for the example of Bell inequality violation.

The paper is organized as follows. In Section 2 we give some details concerning the group S_4. The method of analyzing the violation of Bell inequality described by Gűney and Hillery is applied in Section 3. The results are rewritten in term of nonlocal game in Section 4. The last section contains some conclusions.

2 The S_4 Group

We consider the symmetric group S_4 consisting of 24 elements. This group has five irreducible representations: trivial representation D_0, the alternating representation D_1, the twodimensional one D_2 and two threedimensional representations D and $\widetilde{D}$. We select threedimensional representation D because it can be interpreted as the set of symmetry transformations of the regular tetrahedron [1]. The matrices representing transpositions generate D and their explicit form reads:

$$D(12) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad D(13) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{8}$$

$$D(14) = \begin{bmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{5}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} & \frac{1}{2} \end{bmatrix}, \qquad D(23) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{9}$$

$$D(24) = \begin{bmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{5}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{bmatrix}, \qquad D(34) = \begin{bmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

According to method proposed by Gűney and Hillery, we construct the operator $X(\varphi, \psi)$. The condition that in the Clebsh-Gordan decomposition each representation apears only once is in our case satisfied. The product $D \otimes D$ decomposes into

$$D \otimes D = D \oplus \widetilde{D} \oplus D_2 \oplus D_0. \tag{11}$$

The transformation leading from the product basis to the one in which decomposition (11) is explicit given by

$$C = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{12}$$

We look for the eigenvalues defined by (6). To this end with the help of (12) we decompose $|\varphi\rangle \otimes |\psi\rangle$ into irreducible pieces (explicit calulations are presented in [1]) and use (6). Finally, we have to select the orbits of S_4 in the space of states. The orbit consisting of eight triples of mutually ortogonal vectors $|x^i_\alpha\rangle$, $\alpha = 0, 1, 2$, $i = 1, \ldots, 8$, appears to be the most optimal choice. It can be found using only elementary Euclidean geometry [1, 2]. There are eight observables a_i for Alice and eight observables b_i for Bob

$$a_i = \sum_{\alpha=0}^{2} \alpha \left|x^i_\alpha\right\rangle\left\langle x^i_\alpha\right|, \quad b_i = \sum_{\beta=0}^{2} \beta \left|x^i_\beta\right\rangle\left\langle x^i_\beta\right|. \tag{13}$$

The vectors $|x^i_\alpha\rangle$ are written out explicitly in [1, 2].

3 Examples of Bell Inequality

In order to find the example of violation of Bell inequalities we have to choose at least two orbits [1]. Starting from the vector $|\varphi_1\rangle \otimes |\psi_1\rangle = |x^1_0\rangle \otimes |x^8_1\rangle$ we find the following orbit:

$$\begin{array}{llllll} |1,0;8,1\rangle, & |1,1;8,2\rangle, & |1,2;8,0\rangle, & |2,0;7,2\rangle, & |2,1;7,0\rangle, & |2,2;7,1\rangle \\ |3,0;5,0\rangle, & |3,1;5,2\rangle, & |3,2;5,1\rangle, & |4,0;6,2\rangle, & |4,1;6,1\rangle, & |4,2;6,0\rangle \\ |5,0;3,0\rangle, & |5,1;3,2\rangle, & |5,2;3,1\rangle, & |6,0;4,2\rangle, & |6,1;4,1\rangle, & |6,2;4,0\rangle \\ |7,0;2,1\rangle, & |7,1;2,2\rangle, & |7,2;2,0\rangle, & |8,0;1,2\rangle, & |8,1;1,0\rangle, & |8,2;1,1\rangle \end{array} \tag{14}$$

where $|i, \alpha; j, \beta\rangle = |x^i_\alpha\rangle \otimes |x^j_\beta\rangle$, $i, j = 1, \ldots, 8$, $\alpha, \beta = 0, 1, 2$. The second orbit is obtained by acting elements of S_4 on the vector $|\varphi_2\rangle \otimes |\psi_2\rangle = |x^1_0\rangle \otimes |x^4_1\rangle$:

$$\begin{array}{llllll} |1,0;4,1\rangle, & |1,1;5,0\rangle, & |1,2;7,1\rangle, & |2,0;4,2\rangle, & |2,1;8,1\rangle, & |2,2;5,2\rangle \\ |3,0;4,0\rangle, & |3,1;8,0\rangle, & |3,2;7,2\rangle, & |4,0;3,0\rangle, & |4,1;1,0\rangle, & |4,2;2,0\rangle \\ |5,0;1,1\rangle, & |5,1;6,0\rangle, & |5,2;2,2\rangle, & |6,0;5,1\rangle, & |6,1;7,0\rangle, & |6,2;8,2\rangle \\ |7,0;6,1\rangle, & |7,1;1,2\rangle, & |7,2;3,2\rangle, & |8,0;3,1\rangle, & |8,1;2,1\rangle, & |8,2;6,2\rangle \end{array} \tag{15}$$

The states belonging to both orbits are eigenstates of observables entering the inequalities we are going to derive. Therefore, the sum of 48 probabilities corresponding to the states of orbits has form

$$S = \sum_{(i,\alpha;j,\beta)} p(a_i = \alpha; b_j = \beta). \tag{16}$$

The joint probabilities for a_i and b_j make sense both in classical and quantum physics because the observables a_i and b_j commute. On the quantum level the maximal value of the sum (16) equals the maximal eigenvalue of the operator $X(\varphi, \psi)$ which in our case is the sum of two eigenvalues of $X(\varphi_n, \psi_n)$, $n = 1, 2$, and approximately equals 14,036. Now, we want to find the classical upper bound of the sum (16). According to Fine's theorem the probabilities entering the Bell inequality are obtained as marginals from the joint probability distribution. Therefore the probabilities appearing on the right hand side of (16) can be expressed by the joint probabilities $p(\sigma) \equiv p(a_1 = \alpha_1, \ldots, a_8 = \alpha_8; b_1 = \alpha'_1, \ldots, b_8 = \alpha'_8)$. In this way we obtain the following classical sum of probabilities

$$S = \sum_{\sigma} c(\sigma) p(\sigma). \tag{17}$$

The maximal value of S is equal to the largest coefficient $c(\sigma)$ (cf. (1)). It can be shown that the maximal value of $c(\sigma)$ is 14 (see [1]). Summarizing, we have shown that the Bell inequality is violated with our choice of two orbits defining the quantum states.

One can select more than two orbits. Then the difference between classical and quantum upper bound of Bell expression is larger. For instance, if we add third orbit to the orbits defined by (14) and (15) we obtain the following sum of probabilities appearing on the right hand side of (4) [2]

$$\begin{aligned}
S \equiv\ & P(a_1 = 0, b_5 = 2) + P(a_1 = 1, b_7 = 0) + P(a_1 = 2, b_4 = 0) + P(a_2 = 0, b_8 = 0) + \\
& + P(a_2 = 1, b_5 = 1) + P(a_2 = 2, b_4 = 1) + P(a_3 = 0, b_7 = 1) + P(a_3 = 1, b_4 = 2) + \\
& + P(a_3 = 2, b_8 = 2) + P(a_4 = 0, b_2 = 1) + P(a_4 = 1, b_3 = 2) + P(a_4 = 2, b_1 = 1) + \\
& + P(a_5 = 0, b_2 = 0) + P(a_5 = 1, b_1 = 2) + P(a_5 = 2, b_6 = 2) + P(a_6 = 0, b_8 = 1) + \\
& + P(a_6 = 1, b_5 = 0) + P(a_6 = 2, b_7 = 2) + P(a_7 = 0, b_3 = 1) + P(a_7 = 1, b_6 = 0) + \\
& + P(a_7 = 2, b_1 = 0) + P(a_8 = 0, b_6 = 1) + P(a_8 = 1, b_3 = 0) + P(a_8 = 2, b_2 = 2) + \\
& + P(a_1 = 0, b_4 = 1) + P(a_1 = 1, b_5 = 0) + P(a_1 = 2, b_7 = 1) + P(a_2 = 0, b_4 = 2) + \\
& + P(a_2 = 1, b_8 = 1) + P(a_2 = 2, b_5 = 2) + P(a_3 = 0, b_4 = 0) + P(a_3 = 1, b_8 = 0) + \\
& + P(a_3 = 2, b_7 = 2) + P(a_4 = 0, b_3 = 0) + P(a_4 = 1, b_1 = 0) + P(a_4 = 2, b_2 = 0) + \\
& + P(a_5 = 0, b_1 = 1) + P(a_5 = 1, b_6 = 0) + P(a_5 = 2, b_2 = 2) + P(a_6 = 0, b_5 = 1) + \\
& + P(a_6 = 1, b_7 = 0) + P(a_6 = 2, b_8 = 2) + P(a_7 = 0, b_6 = 1) + P(a_7 = 1, b_1 = 2) + \\
& + P(a_7 = 2, b_3 = 2) + P(a_8 = 0, b_3 = 1) + P(a_8 = 1, b_2 = 1) + P(a_8 = 2, b_6 = 2) + \\
& + P(a_1 = 0, b_8 = 1) + P(a_1 = 1, b_8 = 2) + P(a_1 = 2, b_8 = 0) + P(a_2 = 0, b_7 = 2) + \\
& + P(a_2 = 1, b_7 = 0) + P(a_2 = 2, b_7 = 1) + P(a_3 = 0, b_5 = 0) + P(a_3 = 1, b_5 = 2) + \\
& + P(a_3 = 2, b_5 = 1) + P(a_4 = 0, b_6 = 2) + P(a_4 = 1, b_6 = 1) + P(a_4 = 2, b_6 = 0) + \\
& + P(a_5 = 0, b_3 = 0) + P(a_5 = 1, b_3 = 2) + P(a_5 = 2, b_3 = 1) + P(a_6 = 0, b_4 = 2) + \\
& + P(a_6 = 1, b_4 = 1) + P(a_6 = 2, b_4 = 0) + P(a_7 = 0, b_2 = 1) + P(a_7 = 1, b_2 = 2) + \\
& + P(a_7 = 2, b_2 = 0) + P(a_8 = 0, b_1 = 2) + P(a_8 = 1, b_1 = 0) + P(a_8 = 2, b_1 = 1)
\end{aligned} \tag{18}$$

where the third orbit is obtained from the vector $|\varphi_3\rangle \otimes |\psi_3\rangle = |x_0^1\rangle \otimes |x_2^5\rangle$. In this case, the classical upper bound in Bell inequality equals 16 whereas the quantum one is 17,38. More examples of violation of Bell inequalities are presented in paper [2].

4 Bell Inequality as Nonlocal Game

It is shown in the paper [17, 18] that the Bell inequalities can be rewritten in terms of nonlocal game. In such a game two players Alice and Bob receive values s and t, respectively, from an arbitrator, where $t, s = 1, 2, \dots, 8$. Then both players send back the numbers a and

Table 1 Winning values for nonlocal game defined by two orbits of S_4

s, t	Alice, Bob
14	01
15	10
17	21
18	01, 12, 20
24	02
25	22
27	02, 10, 21
28	11
34	00
35	00, 12, 21
37	22
38	10
41	10
42	20
43	00
46	02, 11, 20
51	01
52	22
53	00, 12, 21
56	10
64	02, 11, 20
65	01
67	10
68	22
71	12
72	01, 12, 20
73	22
76	01
81	02, 10, 21
82	11
83	01
86	22

b to the arbitrator, where $a, b = 0, 1, 2$. All configurations $(a_s = a, b_t = b)$ appearing on the right hand side of (4) are winning. In the case of two orbits the set of winning values is collected in Table 1 [1].

Assuming that all configuration of (s, t) are equally likely, the probability of winning the game is $\frac{7}{32}$ in the classical case. On the quantum level this probability is slightly higher and equals $\frac{7.018}{32}$. This implies that the quantum strategy outperforms its classical counterpart. The similar reasoning can be made for the Bell inequalities obtained for three orbits [2].

5 Conclusions

We gave here examples of violation of Bell inequalities. To this end, using the method presented in Refs. [17, 18], we selected two and three orbits of S_4 group. In view of the fact that the symmetric group S_4 can be considered as the symmetry of regular tetrahedron we could construct the relevant states and observables using Euclidean geometry in three dimensions. In the case of two orbits the Bell inequality is only slightly violated but when third oribt is added the classical upper bound of the sum of probabilities is exceed by $8,625\%$.

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