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Mutually Unbiasedness between Maximally Entangled Bases and Unextendible Maximally Entangled Systems in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$

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Abstract The mutually unbiasedness between a maximally entangled basis (MEB) and an unextendible maximally entangled system (UMES) in the bipartite system $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ $(k > 1)$ are introduced and discussed first in this paper. Then two mutually unbiased pairs of a maximally entangled basis and an unextendible maximally entangled system are constructed; lastly, explicit constructions are obtained for mutually unbiased MEB and UMES in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^8$, respectively.

Keywords Mutually unbiased bases · Maximally entangled state · Unextendible maximally entangled basis

Mutually unbiased bases (MUBs) have been extensively investigated due to their important roles played in quantum kinematics [1], quantum state tomography [2, 3], quantum key distribution [4], cryptographic protocols [5, 6], mean king problem [7], quantum teleportation and superdense coding [8–10], and in quantifying wave-particle duality in multipath interferometers [4]. Two orthogonal bases $\mathcal{B}_1 = \{|\phi_i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|\psi_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle\phi_i|\psi_j\rangle| = \frac{1}{\sqrt{d}}(i, j = 1, 2, \ldots, d).$$

A set of orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ in $\mathbb{C}^d$ is said to be a set of mutually unbiased bases if every pair of the bases in the set is mutually unbiased. The maximum number of

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MUBs in $\mathbb{C}^d$ is shown to be no more than $d+1$ [3]. For prime power dimensional case and qubits systems, different constructions of MUBs have been presented in [11–18].

For bipartite systems, there are many different kinds of bases such as product bases (PB) [19], unextendible product basis (UPB) [20], unextendible maximally entangled system (UMES) [21–26] and maximally entangled basis (MEB) [27] etc., according to the quantum entanglement of the related basic vectors in the bases.

The maximally entangled states play a vital role in quantum information processing tasks such as perfect teleportation [28–36]. A pure state $|\psi\rangle$ is said to be a $d \otimes d'$ ($d' > d$) maximally entangled state if and only if for an arbitrary given orthonormal basis $\{|i_A\rangle\}$ of subsystem A, there exists an orthonormal basis $\{|i_B\rangle\}$ of subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$ [35]. A maximally entangled basis (MEB) is an orthonormal basis consisting of maximally entangled states. In [27], the authors provided a systematic way of constructing MEBs in arbitrary bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$). Then necessary and sufficient conditions of constructing two mutually unbiased maximally entangled systems (MUMEBs) are derived, and explicit constructions of MUMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^6$ are presented.

An unextendible maximally entangled system (UMES) in $\mathbb{C}^d \otimes \mathbb{C}^d$ is a set of less than d^2 orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ such that whose complementary space has no maximally entangled vectors that are mutually orthogonal. It has been proved that UMESs do not exist for $d = 2$, and explicit examples are presented for a 6-member UMES for $d = 3$ and a 12-member UMES for $d = 4$ [21]. In [22], a systematic way of constructing a set of d^2 orthonormal maximally entangled states in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ ($\frac{d'}{2} < d < d'$) was established. In [23, 24], UMESs in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ with $d' = dq + r$ ($0 < r < d$) have been constructed. Also, UMESs in $\mathbb{C}^d \bigotimes \mathbb{C}^{qd}$ ($q \geq 2$) have been constructed in [24]. UMESs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ have been investigated in [25]. In [22], the authors first considered the mutually unbiased bases in which all the bases are unextendible maximally entangled ones, and presented two mutually unbiased unextendible maximally entangled bases(MUUMESs) in $\mathbb{C}^2 \otimes \mathbb{C}^3$. Necessary conditions of constructing a pair of MUUMESs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ are derived in [26]. In [23], two MUUMESs in $\mathbb{C}^2 \otimes \mathbb{C}^5$ and $\mathbb{C}^3 \otimes \mathbb{C}^4$ are established.

Instead of the investigation of mutually unbiased bases from two MEBs or from two UMESs, in this paper we study the mutually unbiasedness between one MEB and one UMES in the bipartite system $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ ($k > 1$). We first present an approach of constructing a mutually unbiased pair of a maximally entangled basis and unextendible maximally entangled system and then present explicit constructions of mutually unbiased MEB and UMES in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^8$.

Definition 1 Two orthonomal systems $\{|\phi_i\rangle\}_{i=1}^m$ and $\{|\psi_j\rangle\}_{j=1}^n$ in $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle\phi_i|\psi_j\rangle| = \frac{1}{\sqrt{d}} (1 \leq i \leq m, 1 \leq j \leq n).$$

We first begin with examples of MEB and UMES in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ ($k > 1$). Let $\{|0\rangle, |1\rangle\}$ and $\{|i'\rangle\}_{i=0}^{2^k-1}$ denote the orthonormal bases of $\mathbb{C}^2$ and $\mathbb{C}^{2^k}$ ($k > 1$), respectively. An MEB for $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ ($k > 1$) has been constructed in [27],

$$|\phi_{n,m}^{(\alpha)}\rangle = \frac{1}{\sqrt{2}} \sum_{p=0}^{1} (-1)^{np} |p \oplus_2 m\rangle |(p + 2\alpha)'\rangle (\alpha = 0, 1, \ldots, 2^{k-1} - 1; n, m = 0, 1), \quad (1)$$

where $p\oplus_2 m$ denotes $(p+m) \mod 2$. And a $(2^{k+1}-2)$-member UMES in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}$ $(k>1)$ has been constructed in [24],

$$|\eta_{i,j}\rangle = \frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|(q\oplus_{(2^k-1)} j)'\rangle (i=0,1;\ j=0,1,\ldots,2^k-2)$$

where $q\oplus_{(2^k-1)} j$ denotes $(q+j) \mod (2^k-1)$.

Let $\{|a_i'\rangle\}_{i=0}^{2^k-1}$ be another orthonormal basis in $\mathbb{C}^{2^k}$ that is different from $\{|i'\rangle\}_{i=0}^{2^k-1}$. New $(2^{k+1}-2)$-member UMES in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}$ can be obtained in the following way,

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|a'_{q\oplus_{(2^k-1)} j}\rangle (i=0,1;\ j=0,1,\ldots,2^k-2) \tag{2}$$

By Definition 1, the MEB (1) and the UMES (2) are mutually unbiased if and only if the following relations are satisfied.

$$|\langle\phi_{n,m}^{(\alpha)}|\psi_{i,j}\rangle| = \frac{1}{\sqrt{2^{k+1}}}(\alpha=0,1,\ldots,2^{k-1}-1;\ n,m,i=0,1;\ j=0,1,\ldots,2^k-2). \tag{3}$$

Let T denote the transition matrix from the basis $\{|i'\rangle\}_{i=0}^{2^k-1}$ to the basis $\{|a_i'\rangle\}_{i=0}^{2^k-1}$ for $\mathbb{C}^{2^k}$, i.e.

$$\begin{pmatrix} |a_0'\rangle \\ |a_1'\rangle \\ \vdots \\ |a'_{(2^k-1)}\rangle \end{pmatrix} = T\begin{pmatrix} |0'\rangle \\ |1'\rangle \\ \vdots \\ |(2^k-1)'\rangle \end{pmatrix}, \tag{4}$$

equivalently, $|a_i'\rangle = \sum_{j=0}^{2^k-1} t_{ij}|j'\rangle$, where t_{ij} are (i,j)-entries of the matrix T.

Theorem 1 *The MEB* (1) *and the UMES* (2) *are mutually unbiased if and only if T satisfies the following conditions:*

$$\left|\sum_{p=0}^{1}\xi^p t_{p\oplus_{(2^k-1)}j,\ p+2\alpha}\right| = \frac{1}{\sqrt{2^{k-1}}}, \tag{5}$$

$$|\sum_{p=0}^{1}\xi^p t_{p\oplus_{(2^k-1)}j,\ 1-p+2\alpha}| = \frac{1}{\sqrt{2^{k-1}}}, \tag{6}$$

where $\xi=1,-1;\ j=0,1,\ldots,2^k-2;\ \alpha=0,1,\ldots,2^{k-1}-1.$

Proof From (1), (2) we have

$$\begin{aligned}|\langle\phi_{n,m}^{(\alpha)}|\psi_{i,j}\rangle| &= \frac{1}{2}|\sum_{p=0}^{1}\sum_{q=0}^{1}(-1)^{np+qi}\langle p\oplus_2 m|q\rangle\langle(p+2\alpha)'|a'_{q\oplus_{(2^k-1)}j}\rangle| \\ &= \frac{1}{2}|\sum_{p,q=0}^{1}\sum_{q=p\oplus_2 m}(-1)^{np+qi}\langle(p+2\alpha)'|a'_{q\oplus_{(2^k-1)}j}\rangle|\end{aligned}$$

where $\alpha=0,1,\ldots,2^{k-1}-1;\ n,m,i=0,1;\ j=0,1,\ldots,2^k-2.$

so the conditions (3) are equivalent to the following conditions:

$$|\sum_{p,q=0}^{1}\sum_{q=p\oplus_2 m}\langle(p+2\alpha)'|a'_{q\oplus_{(2^k-1)}j}\rangle| = \frac{1}{\sqrt{2^{k-1}}}$$
$$(\alpha=0,1,\ldots,2^{k-1}-1;\ n,m,i=0,1;\ j=0,1,\ldots,2^k-2). \quad (7)$$

Since $q=p$ if $m=0$ and $q=p\oplus_2 1=1-p$ if $m=1$, then the above conditions (7) are equivalent to the following two conditions:

$$|\sum_{p=0}^{1}(-1)^{p(n+i)}\langle(p+2\alpha)'|a'_{p\oplus_{(2^k-1)}j}\rangle| = \frac{1}{\sqrt{2^{k-1}}}$$
$$(\alpha=0,1,\ldots,2^{k-1}-1;\ n,i=0,1;\ j=0,1,\ldots,2^k-2). \quad (8)$$

$$|\sum_{p=0}^{1}(-1)^{np+(1-p)i}\langle(p+2\alpha)'|a'_{(1-p)\oplus_{(2^k-1)}j}\rangle| = \frac{1}{\sqrt{2^{k-1}}}$$
$$(\alpha=0,1,\ldots,2^{k-1}-1;\ n,i=0,1;\ j=0,1,\ldots,2^k-2). \quad (9)$$

After taking over all possible values of n and i, we can simplify the above conditions (8) and (9) as follows:

$$\left|\sum_{p=0}^{1}\xi^p t_{p\oplus_{(2^k-1)}j,\ p+2\alpha}\right| = \frac{1}{\sqrt{2^{k-1}}}(\alpha=0,1,\ldots,2^{k-1}-1;\ j=0,1,\ldots,2^k-2.)$$

$$|\sum_{p=0}^{1}\xi^p t_{p\oplus_{(2^k-1)}j,\ 1-p+2\alpha}| = \frac{1}{\sqrt{2^{k-1}}}(\alpha=0,1,\ldots,2^{k-1}-1;\ j=0,1,\ldots,2^k-2.)$$

where $\xi=1,-1$ and thus the theorem is proved.

For a detailed construction of a pair of mutually unbiased MEB and UMES, we first consider the case of $\mathbb{C}^2\otimes\mathbb{C}^4$. Let us take the second basis $\{|a'_i\rangle\}_{i=0}^3$ in $\mathbb{C}^4$ as $(|a'_0\rangle,|a'_1\rangle,|a'_2\rangle,|a'_3\rangle)^t = T\,(|0'\rangle,|1'\rangle,|2'\rangle,|3'\rangle)^t$, where t denotes transposition,

$$T=\frac{1}{2}\begin{pmatrix}1 & i & 1 & -i\\ -1 & i & 1 & i\\ 1 & i & -1 & i\\ 1 & -i & 1 & i\end{pmatrix}$$

with $i=\sqrt{-1}$. It is direct to verify that the matrix T satisfies the mutually unbiased conditions (5)–(6). From (1) and (2) we have the mutually unbiased MEB and the 6-member UMES in $\mathbb{C}^2\otimes\mathbb{C}^4$, respectively,

$$|\phi^{(\alpha)}_{n,m}\rangle = \frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{np}|p\oplus_2 m\rangle|(p+2\alpha)'\rangle,\quad \alpha=0,1;\quad n,m=0,1, \quad (10)$$

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|a'_{q\oplus_3 j}\rangle,\quad i=0,1;\ j=0,1,2. \quad (11)$$

As another example, we present a detailed construction of mutually unbiased MEB and 14-member UMES in $\mathbb{C}^2 \otimes \mathbb{C}^8$. According to (1) we have a MEB for $\mathbb{C}^2 \otimes \mathbb{C}^8$,

$$|\phi_{n,m}^{(\alpha)}\rangle = \frac{1}{\sqrt{2}} \sum_{p=0}^{1} (-1)^{np} |p \oplus_2 m\rangle |(p + 2\alpha)'\rangle, \quad \alpha = 0, 1, 2, 3; \quad n, m = 0, 1. \tag{12}$$

To construct a UMES in $\mathbb{C}^2 \otimes \mathbb{C}^8$ that is mutually unbiased with the MEB above, we take a basis $\{|b'_j\rangle\}_{j=0}^{7}$ for $\mathbb{C}^8$ as

$$(|b'_0\rangle, |b'_1\rangle, |b'_2\rangle, |b'_3\rangle, |b'_4\rangle, |b'_5\rangle, |b'_6\rangle, |b'_7\rangle)^t = T\,(|0'\rangle, |1'\rangle, |2'\rangle, |3'\rangle, |4'\rangle, |5'\rangle, |6'\rangle, |7'\rangle)^t,$$

where

$$T = \frac{1}{\sqrt{8}} \begin{pmatrix} -i & -1 & 1 & -i & -i & -1 & 1 & -i \\ -i & -1 & -1 & i & -i & -1 & -1 & i \\ -i & 1 & -1 & -i & -i & 1 & -1 & -i \\ -i & 1 & 1 & i & -i & 1 & 1 & i \\ -i & -1 & 1 & -i & i & 1 & -1 & i \\ -i & -1 & -1 & i & i & 1 & 1 & -i \\ -i & 1 & -1 & -i & i & -1 & 1 & i \\ -i & 1 & 1 & i & i & -1 & -1 & -i \end{pmatrix}.$$

Then the corresponding complete 14-member UMES in $\mathbb{C}^2 \otimes \mathbb{C}^8$ has the form,

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{2}} \sum_{q=0}^{1} (-1)^{qi} |q\rangle |b'_{q \oplus_7 j}\rangle, \; i = 0, 1; \; j = 0, 1, \ldots, 6. \tag{13}$$

It is direct to verify that the transformation matrix B satisfies the relation (5)–(6) and so the MEB (12) and the completed 14-member UMES (13) in $\mathbb{C}^2 \otimes \mathbb{C}^8$ are mutually unbiased.

We have constructed a maximally entangled basis and an unextendible maximally entangled system in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ $(k > 1)$ and derived sufficient and necessary conditions of mutually unbiasedness between them. As detailed applications, we have constructed mutually unbiased pairs of a maximally entangled basis and an unextendible maximally entangled system in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and in $\mathbb{C}^2 \otimes \mathbb{C}^8$, respectively.

There are still many open problems related to maximally entangled bases and unextendible maximally entangledsystems which are mutually unbiased, such as the case in $\mathbb{C}^d \otimes \mathbb{C}^d$ or $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ $(k \in \mathbb{Z}^+)$ for $d > 2$, as well as to the roles played by such bases in quantum information processing. □

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