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D-Dimensional Metrics with $D-3$ Symmetries

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Abstract Symmetry transformations in a space of D-dimensional vacuum metrics with $D-3$ commuting Killing vectors are studied. We solve directly the Einstein equations in the Maison formulation under additional assumptions. We show that the Reissner-Nordström solution is related by the symmetry transformation to a particular case of the 5-dimensional Gross-Perry metric and the 5-dimensional plane wave solution is related to the Gross-Perry-Sorkin metric.

Keywords Higher dimensional gravity · Maison symmetries · Exact solutions

1 Introduction

Stationary vacuum Einstein equations admit the symmetry group $SL(2,\mathbb{R})$ [2, 3]. This symmetry was generalized to an action of the group $SL(D-2,\mathbb{R})$ in the class of D-dimensional vacuum metrics with $D-3$ commuting Killing vectors [4]. In the case $D=5$ this group contains the group $SO(1,2)$ which preserves asymptotical flatness of a metric [5]. For instance, using this action one can reproduce the Myers-Perry solution from the Schwarzschild-Tangherlini metric [5].

In this paper we investigate in detail the action of the $SL(D-2,\mathbb{R})$ symmetry group when integral submanifolds of the Killing vectors are not null. We identify relevant parameters of this action and we discuss corresponding changes of the metric signature. We solve directly the Einstein equations in $D=5$ assuming two commuting Killing vectors and additional symmetries which are not isometries. We give an example of the $SL(3,\mathbb{R})$ symmetry transformation in $D=5$ which generates the Reissner-Nordström 4-dimensional solution, with a dyonic electromagnetic field, from the 5-dimensional Gross-Perry metric [1, 7] of the Euclidean signature.

In the considered class of solutions there are near horizon metrics of extremal black holes [6] and metrics obtained by Clément (see [8] and references therein).

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2 Generation Method for Reduced Vacuum Einstein Equations

Let M be $(n+2)$-dimensional manifold with metric g admitting $n-1$ commuting Killing vectors (thus, $D = n+2$) which define a non-null integrable distribution. In special coordinates x^i, $i = 1, \dots, n-1$, and x^a, $a = n, n+1, n+2$, the Killing vectors are ∂_i and the metric takes the form

$$g = g_{ij}(dx^i + A^i)(dx^j + A^j) + \tau^{-1}\tilde{g}_{ab}dx^a dx^b, \tag{1}$$

where $\tau = |\det g_{ij}|$, $A^i = A^i_a dx^a$ and functions g_{ij}, A^i_a, $\tilde{g}_{ab}$ do not depend on coordinates x^i. The vacuum Einstein equations for metrics (1) are equivalent to the following equations [4]

$$d(\chi^{-1} * d\chi) = 0, \tag{2}$$

$$\tilde{R}_{ab} = \frac{1}{4}\mathrm{Tr}(\chi^{-1}\chi_{,a}\chi^{-1}\chi_{,b}) \tag{3}$$

for $\tilde{g}_{ab}$ and $n \times n$ symmetric matrix χ constrained by the conditions

$$\det\chi = \pm 1, \qquad \varepsilon\chi_{nn} < 0, \tag{4}$$

where $\varepsilon = \mathrm{sgn}(\det\tilde{g}_{ab})$. Here $\tilde{R}_{ab}$ is the Ricci tensor of the metric $\tilde{g} = \tilde{g}_{ab}dx^a dx^b$ and $*$ denotes the Hodge dualization with respect to this metric. The matrix χ is related to components of (1) via the equations

$$\chi = \begin{pmatrix} g_{ij} - \frac{\varepsilon}{\tau}V_i V_j & \frac{1}{\tau}V_i \\ \frac{1}{\tau}V_j & -\frac{\varepsilon}{\tau} \end{pmatrix}, \tag{5}$$

$$dV_i = \tau g_{ij} * dA^j. \tag{6}$$

Note that (6) is integrable by virtue of (2) and that

$$\mathrm{sgn}(\det g_{ij}) = -\varepsilon\det\chi. \tag{7}$$

Equations (2)–(4) are preserved by the transformation

$$\chi \mapsto \chi' = \varepsilon' S^T \chi S \tag{8}$$

where $S \in SL(n, \mathbb{R})$ is a constant matrix and value of $\varepsilon' = \pm 1$ is fixed by the condition

$$\varepsilon'\varepsilon(S^T\chi S)_{nn} < 0. \tag{9}$$

Transformations given by (8)–(9) can be used to obtain new vacuum metrics from known ones. They generalize the Ehlers transformation [2] for stationary 4-dimensional metrics. In dimension 5 they contain the group $SO(1,2)$ preserving asymptotical flatness of metrics [5].

3 Relevant Parameters and Change of Signature

Most of parameters in S (symmetry) do not change the seed metric in a nontrivial way. Any matrix S with $S^n_n \neq 0$ can be uniquely decomposed into a product of three matrices $S = S_0 H T$, where

$$S_0 = \begin{pmatrix} \delta^i_j & \alpha^i \\ 0 & 1 \end{pmatrix}, \tag{10}$$

$$H = \begin{pmatrix} \beta^i_j & 0 \\ 0 & (\det \beta^i_j)^{-1} \end{pmatrix}, \qquad T = \begin{pmatrix} \delta^i_j & 0 \\ \gamma_j & 1 \end{pmatrix}. \tag{11}$$

The matrix T (translation) yields translations of V_i by constants $-\varepsilon\gamma_i$. It does not change the seed metric. The matrix H (homothety) corresponds to a linear transformation of x^i and A^i combined with a multiplication of the full metric g by $(\det \beta^i_j)^{-2}$. Thus, modulo coordinate transformations, H is a homothety and can be replaced by

$$H_0 = \begin{pmatrix} \beta\delta^i_j & 0 \\ 0 & \beta^{1-n} \end{pmatrix} \tag{12}$$

with an appropriate constant β. The only nontrivial action is that of the matrix S_0 (true symmetry),

$$S_0^T \chi S_0 = \begin{pmatrix} g_{ij} - \frac{\varepsilon}{\tau} V_i V_j & \alpha_i - \frac{\varepsilon}{\tau} V_i (\alpha^k V_k - \varepsilon) \\ \alpha_j - \frac{\varepsilon}{\tau} V_j (\alpha^k V_k - \varepsilon) & V^k V_k - \frac{\varepsilon}{\tau} (\alpha^k V_k - \varepsilon)^2 \end{pmatrix}, \tag{13}$$

where $\alpha_i = g_{ij}\alpha^j$ and $V^i = g^{ij} V_j$.

If $S^n_n = 0$ then S is equivalent, modulo H and T, to one of the matrices $S_l, l = 1, \ldots, n-1$ with entries given by

$$S^i_{ln} = \begin{cases} 1 & \text{if } i = n-l, \\ 0 & \text{if } i > n-l, \end{cases} \qquad S^n_{lj} = \begin{cases} -1 & \text{if } j = n-l, \\ 0 & \text{if } j \neq n-l, \end{cases}$$
$$S^i_{lj} = \begin{cases} 1 & \text{if } i = j \neq n-l, \\ 0 & \text{if } i \neq j \text{ or } i = j = n-l. \end{cases} \tag{14}$$

Note that S_l contains $n-l-1$ free parameters (components S^i_{ln} for $i < n-l$).

Summarizing, without loss of generality, symmetry transformations (8) can be reduced to the action of one of matrices S_0, S_l composed with H_0, the latter equivalent to a homothety of g.

Transformations (8) can change signature of g_{ij}, and hence the signature of (1). Let the initial signature of g_{ij} be (p, q), where p denotes the multiplicity of the value $+1$. If $\varepsilon' > 0$ then transformation (8) preserves this signature. If $\varepsilon' < 0$ then the signature of the final metric g_{ij} is $(q + \varepsilon, p - \varepsilon)$. (Note that if $(p, \varepsilon) = (0, 1)$ or $(q, \varepsilon) = (0, -1)$ then $\varepsilon' > 0$.) For instance, if we start with a 5-dimensional metric of the Lorentz signature $(- + + + +)$ then the transformed metric has the same signature or the Euclidean one. In this case transformation (13) adds two parameters to the seed metric. (They have to be dependent, $\alpha^2 = \frac{1}{2}(\alpha^1)^2$, if the asymptotical flatness is to be preserved [5].) Transformations S_l take the form

$$S_1 = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{15}$$

$$S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{16}$$

4 Solutions with 2-Dimensional Space of Constant Curvature

Let us assume that $\chi = \chi(z)$ depends only on one coordinate z and metric $\tilde{g}$ has the following form

$$\tilde{g} = dz^2 + f(z) g^{(k)}, \tag{17}$$

where $g^{(k)}$ is a 2-dimensional metric of constant curvature $k = 0, \pm 1$ and signature $++$ or $+-$,

$$g^{(k)} = \frac{4(d\tilde{x}^2 + \varepsilon d\tilde{y}^2)}{(1 + k(\tilde{x}^2 + \varepsilon \tilde{y}^2))^2}. \tag{18}$$

Note that z can be shifted by a constant and metric (17) can be multiplied by another constant since this transformation can be compensated by a change of coordinates x^i. Thus, we admit the following transformations

$$z \mapsto cz + c_0, \qquad f \mapsto c^{-2} f, \quad c, c_0 = \text{const}. \tag{19}$$

The Ricci tensor of $\tilde{g}$ reads

$$\text{Ricci}(\tilde{g}) = \frac{(f_{,z})^2 - 2 f f_{,zz}}{2 f^2} dz^2 + \left(k - \frac{f_{,zz}}{2}\right) g^{(k)}. \tag{20}$$

It follows from (3) that

$$f = kz^2 + az + b, \quad a, b = \text{const} \tag{21}$$

and

$$\text{Tr}\left(\chi^{-1}\chi_{,z}\right)^2 = 2a^2 - 8kb. \tag{22}$$

A double integration of (2) gives

$$\chi = \chi_0 \exp\big((w + w_0) C\big), \tag{23}$$

where χ_0, C are constant matrices such that

$$\chi_0 = \chi_0^T, \qquad \chi_0 C = C^T \chi_0, \qquad \text{Tr}\, C = 0, \qquad \text{Tr}\, C^2 = 2a^2 - 8kb \tag{24}$$

and $w(z)$ is a particular solution of

$$w_{,z} = \pm f^{-1}. \tag{25}$$

The constant w_0 and the sign in (25) can be arbitrarily chosen.

One can classify functions f and w by putting them into a canonical form. First, let us note that by virtue of (19) one can transform f into one of the following expressions labelled by k and a new index $k' = 0, \pm 1$,

$$f(z) = \begin{cases} z & \text{if } k = k' = 0, \\ kz^2 + k' & \text{if } k^2 + k'^2 \neq 0. \end{cases} \tag{26}$$

Using (26) we find particular solutions of (25). They are presented in Table 1.

The symmetry (8) induces transformations $C \mapsto S^{-1} C S$ and $\chi_0 \mapsto S^T \chi_0 S$ which allow us to reduce matrices C and χ_0 to simpler forms. First, we put matrix C into a canonical

Table 1 $\operatorname{Tr} C^2$ and solutions of (25)–(26)

	$w(z)$	Conditions	$\operatorname{Tr} C^2$
(i)	$\log\|z\|$	$k = k' = 0$	2
(ii)	$\operatorname{arctanh} z$	$k' = -k \neq 0,\ z^2 < 1$	8
(iii)	$\operatorname{arccoth} z$	$k' = -k \neq 0,\ z^2 > 1$	8
(iv)	z	$k = 0,\ k' \neq 0$	0
(v)	$1/z$	$k \neq 0,\ k' = 0$	0
(vi)	$\operatorname{arccot} z$	$k' = k \neq 0$	−8

Jordan form. Then, we find χ_0 satisfying (24) and we simplify it by means of matrices which commute with C. Below we present results of this procedure in the case of 5-dimensional metrics. Then $n = 2$ and there are four canonical forms of C and χ, in which α, β are constants and $\varepsilon, \varepsilon_i = \pm 1$,

$$C = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}, \qquad \chi = \begin{pmatrix} -e^{\alpha w}\sin(\beta w) & e^{\alpha w}\cos(\beta w) & 0 \\ e^{\alpha w}\cos(\beta w) & e^{\alpha w}\sin(\beta w) & 0 \\ 0 & 0 & -\varepsilon e^{-2\alpha w} \end{pmatrix},$$
$$\operatorname{Tr} C^2 = 6\alpha^2 - 2\beta^2, \quad \beta \neq 0, \tag{27}$$

$$C = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -\alpha-\beta \end{pmatrix}, \qquad \chi = \begin{pmatrix} \varepsilon_1 e^{\alpha w} & 0 & 0 \\ 0 & \varepsilon_2 e^{\beta w} & 0 \\ 0 & 0 & -\varepsilon e^{-(\alpha+\beta) w} \end{pmatrix},$$
$$\operatorname{Tr} C^2 = 2(\alpha^2 + \alpha\beta + \beta^2), \tag{28}$$

$$C = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix}, \qquad \chi = \begin{pmatrix} 0 & \varepsilon_1 e^{\alpha w} & 0 \\ \varepsilon_1 e^{\alpha w} & \varepsilon_1 w e^{\alpha w} & 0 \\ 0 & 0 & -\varepsilon e^{-2\alpha w} \end{pmatrix},$$
$$\operatorname{Tr} C^2 = 6\alpha^2, \tag{29}$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \chi = -\varepsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & w \\ 1 & w & \frac{1}{2}w^2 \end{pmatrix}, \qquad \operatorname{Tr} C^2 = 0. \tag{30}$$

Note that case (27), with parameters in an appropriate range, is compatible with any function from Table 1, cases (28) and (29) admit functions (i)–(v) while (30) is only compatible with functions (iv)–(v).

5 Canonical Metrics

Below, we present metrics corresponding to cases (27)–(30). In what follows, the coordinates x^1, x^2 will be denoted by x, y.

5.1 Case (27)

The following metric

$$g = e^{\alpha w}\left[\sin(\beta w)(dy^2 - dx^2) + 2\cos(\beta w)dx\,dy\right] + e^{-2\alpha w}\left(dz^2 + f(z) g^{(k)}\right) \tag{31}$$

corresponds to solution (27) where $f(z)$ is defined by (26) and $w(z)$ is any function from Table 1. In case when $\varepsilon = 1$ metrics (31) are Lorentzian. In the subcase (v) with $k = \varepsilon = 1$ solution (31) reads

$$g = e^{\alpha/z}\left[\sin(\beta/z)(dy^2 - dx^2) + 2\cos(\beta/z)dx\,dy\right] + e^{-2\alpha/z}\left(dz^2 + z^2 d\Omega^2\right), \tag{32}$$

where $\beta = \pm\sqrt{3}\,\alpha$ and $d\Omega^2$ is the standard metric of the 2-dimensional sphere. The Kaluza-Klein reduction of the latter solution with respect to x or y leads to 4-dimensional asymptotically flat Lorentzian metric. Coordinate z plays a role the a 3-dimensional length.

It is noted that for $k = 1$ and subcase (ii) solution (31) reduces to one of metrics found in [9].

5.2 Case (28)

Metric which corresponds to the subcase (i) of solution (28) is given by

$$g = \varepsilon_1 z^{\alpha} dx^2 + \varepsilon_2 z^{\beta} dy^2 + z^{-\alpha-\beta}\left(dz^2 + 4z(d\tilde{x}^2 + \varepsilon d\tilde{y}^2)\right), \tag{33}$$

where $\alpha^2 + \alpha\beta + \beta^2 = 1$. Metrics related to (ii)–(iii) are given by

$$\begin{aligned} g = {} & \varepsilon_1 \left|\frac{z+1}{z-1}\right|^{\alpha/2} dx^2 + \varepsilon_2 \left|\frac{z+1}{z-1}\right|^{\beta/2} dy^2 \\ & + \left|\frac{z+1}{z-1}\right|^{-(\alpha+\beta)/2} \left(dz^2 + k(z^2-1)g^{(k)}\right), \end{aligned} \tag{34}$$

where $\alpha^2 + \alpha\beta + \beta^2 = 4$. We postpone metrics corresponding to subcases (iv) and (v) because they are flat as well as metrics obtained from them by the symmetry transformation (8).

5.3 Case (29)

The subcases (i), (iv) and (v) correspond, respectively, to the following metrics

$$g = \varepsilon_1 z^{\pm 1/\sqrt{3}} dy\,(\log|z|dy + 2dx) + z^{\mp 2/\sqrt{3}}(dz^2 + 4z(d\tilde{x}^2 + \varepsilon d\tilde{y}^2)), \tag{35}$$

$$g = \varepsilon_1 dy\,(z dy + 2dx) + dz^2 + 4k'(d\tilde{x}^2 + \varepsilon d\tilde{y}^2), \tag{36}$$

$$g = \varepsilon_1 dy\left(2dx + \frac{1}{z}dy\right) + dz^2 + 4kz^2 g^{(k)}, \tag{37}$$

while the metric

$$\begin{aligned} g = {} & \frac{\varepsilon_1}{2}\left|\frac{z+1}{z-1}\right|^{\pm 1/\sqrt{3}} dy\left(\log\left|\frac{z+1}{z-1}\right| dy + 4dx\right) \\ & + \left|\frac{z+1}{z-1}\right|^{\mp 2/\sqrt{3}} \left(dz^2 + k(z^2-1)g^{(k)}\right) \end{aligned} \tag{38}$$

follows in subcases (ii) and (iii). Although (36) is flat, it can be transformed to non-flat solutions (see Sect. 6).

5.4 Case (30)

The subcases (iv)–(v) correspond respectively to the following metrics

$$g = \frac{2\varepsilon}{z^2}dx^2 + \frac{4\varepsilon}{z}dx\,(dy - 2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})) + \varepsilon\,(dy - 2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y}))^2 + \frac{z^2}{2}\left(dz^2 + 4k'(d\tilde{x}^2 + \varepsilon d\tilde{y}^2)\right), \tag{39}$$

$$g = g_{ij}\theta^i\theta^j + \frac{1}{2z^2}\left(dz^2 + kz^2 g^{(k)}\right), \tag{40}$$

where

$$g_{ij} = \varepsilon\begin{pmatrix} 2z^2 & 2z \\ 2z & 1\end{pmatrix}, \qquad \theta^1 = dx, \qquad \theta^2 = dy - \frac{2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})}{1 + k(\tilde{x}^2 + \varepsilon\tilde{y}^2)}. \tag{41}$$

Both solutions are not flat. Metric (40) for $k = \varepsilon = 1$ is Lorentzian and can be slightly simplified by introducing spherical angles θ, ϕ instead of $\tilde{x}$, $\tilde{y}$. Then (40)–(41) yields

$$g = -2z^2dx^2 + (2zdx + dy' + \cos\theta d\phi)^2 + (2z^2)^{-1}\left(dz^2 + z^2d\Omega^2\right), \tag{42}$$

where $y' = y + \phi$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

6 Examples of Transformed Metrics

In this section, new metrics are constructed by application of the symmetry transformation (8) to some canonical solutions found in the previous section. Among them there is the Reissner-Nordström solution lifted to 5 dimensions, the Gross-Perry-Sorkin monopole solution and the near horizon metrics. It is noted that this transformation cannot always be used. For instance, transformation (8) with S given by (16) is not applicable to (39)–(41) or (35)–(37) because then $\chi'_{33} = 0$.

Below we present all metrics which can be obtained from (39)–(41). Solutions corresponding to matrix S_0 become

$$g = g_{ij}\theta^i\theta^j + \tau^{-1}\left(dz^2 + f(z)g^{(k)}\right), \tag{43}$$

where

$$g_{ij} = \varepsilon\tau\begin{pmatrix} 1 & w + \alpha^2 \\ w + \alpha^2 & \frac{1}{2}w^2 - 2\alpha^1\end{pmatrix}, \qquad \tau = \frac{2}{|(w + 2\alpha^2)^2 + 4\alpha^1 - 2(\alpha^2)^2|},$$
$$\theta^1 = dx - \frac{2\varepsilon\alpha^2(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})}{1 + k(\tilde{x}^2 + \varepsilon\tilde{y}^2)}, \qquad \theta^2 = dy - \frac{2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})}{1 + k(\tilde{x}^2 + \varepsilon\tilde{y}^2)}. \tag{44}$$

Applying transformation (15) to metrics (39)–(40) gives the following solutions

$$g = 2\varepsilon dy\,(dx - 2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})) + \varepsilon\left((z + \alpha_0)^2 - \frac{z^2}{2}\right)dy^2 + dz^2 + 4k'(d\tilde{x}^2 + \varepsilon d\tilde{y}^2), \tag{45}$$

$$g = g_{ij}\theta^i\theta^j + dz^2 + kz^2g^{(k)}, \tag{46}$$

where

$$g_{ij} = \varepsilon \begin{pmatrix} 0 & 1 \\ 1 & (\frac{1}{z}+\alpha_0)^2 - \frac{1}{2z^2} \end{pmatrix}, \qquad \theta^1 = dx - \frac{2\varepsilon(\tilde{y}d\tilde{x} - \tilde{x}d\tilde{y})}{1+k(\tilde{x}^2+\varepsilon\tilde{y}^2)}, \qquad \theta^2 = dy. \tag{47}$$

Transformation (15) applied to (32) leads to

$$g = \eta e^{-2\alpha/z} dy^2 + \frac{e^{2\alpha/z}\cos^2\gamma}{|\sin(\beta/z+2\gamma)|}\left(dx' + \frac{\beta}{\cos^2\gamma}\cos\theta d\phi\right)^2 + \frac{e^{\alpha/z}|\sin(\beta/z+2\gamma)|}{\cos^2\gamma}\left(dz^2 + z^2 d\Omega^2\right), \tag{48}$$

where $x' = x + \beta\phi/\cos^2\gamma$, $\tan\gamma = \alpha_0$, $\beta = \pm\sqrt{3}\alpha$ and $\eta = \mathrm{sgn}(\sin(\beta/z+2\gamma))$. For $\gamma = 0$, (48) reduces to metric equivalent to a transform of (32) by means of (16).

It is noted that 5-dimensional near horizon geometries [6] may be associated with (28) by a suitable symmetry transformation. The 3-dimensional metric $2dvdr - C^2r^2dv^2 + C^{-2}d\theta^2$ used in [6] is transformed to $C^{-2}(1-z^2)^{-1}(dz^2 + (1-z^2)g^{(-1)})$ by the following coordinate transformation

$$v = \frac{\tilde{x}+\tilde{y}-1}{\tilde{x}+\tilde{y}+1}, \qquad r = C^{-2}\left(\frac{2(\tilde{x}+1)}{1-\tilde{x}^2+\tilde{y}^2} - 1\right), \qquad \theta = \arccos z \tag{49}$$

and setting $\varepsilon = -1$. It can be shown that the symmetry transformation $S = S_0HT$ ($\varepsilon' = 1$), where

$$S_0 = \begin{pmatrix} 1 & 0 & -C/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad H = \frac{1}{2BC^{3/2}}\begin{pmatrix} 8AB^3 & C^2 & 0 \\ -8B^3 & 0 & 0 \\ 0 & 0 & C \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \tag{50}$$

relates the near horizon geometry with horizon topology $\mathcal{H} \cong S^1 \times S^2$ (see [6]) with solution (34) with $\varepsilon = k = -1$, $\varepsilon_1 = \varepsilon_2 = 1$, $\alpha = 0$, $\beta = 2$, and (x, y) replaced by $(x/\sqrt{C}, y/\sqrt{C})$.

Let us consider metric (34) with $k = -k' \neq 0$. If $\varepsilon_1 = \varepsilon_2 = \varepsilon = k = 1$, $\alpha = -\beta = -2$ and we admit a constant conformal factor in g we obtain the (Euclidean) Gross-Perry solution [1, 7]

$$g = \left(\frac{\rho-q}{\rho+q}\right)^2 dx^2 + \left(\frac{\rho+q}{\rho-q}\right)^2 dy^2 + \frac{1}{4}\left(1-\frac{q^2}{\rho^2}\right)^2\left(d\rho^2 + \rho^2 d\Omega^2\right), \tag{51}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $q = \mathrm{const}$ and $\rho = q(z+\sqrt{z^2-1})$. Let us transform metric (51) by means of (8) with

$$S = -\frac{1}{2q}\begin{pmatrix} \sqrt{2(m^2-q^2)} & m+q & m-q \\ -\sqrt{2(m^2-q^2)} & -m+q & -m-q \\ 2m & \sqrt{2(m^2-q^2)} & \sqrt{2(m^2-q^2)} \end{pmatrix}. \tag{52}$$

For $\varepsilon' = -1$ this transformation yields surprisingly the following 5-dimensional vacuum solution

$$g = \left(dx - \sqrt{2}Q\cos\theta d\phi + \frac{\sqrt{2}Q}{r}dt\right)^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2$$

$$+\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2, \tag{53}$$

where $t=y$, $\rho=r-m+\sqrt{r^2-2mr+Q^2}$ and $q=\sqrt{m^2-Q^2}$. Within the Kaluza-Klein approach the latter metric decomposes into the 4-dimensional Reissner-Nordström solution on the manifold $x=$ constant

$$g'=-\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)dt^2+\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2 \tag{54}$$

and dyonic electromagnetic field given by

$$A=-\sqrt{2}Q\cos\theta d\phi+\frac{\sqrt{2}Q}{r}dt. \tag{55}$$

This field represents a magnetic and electric monopoles of the same strength placed at the same point.

Transformation (15) applied to (29) leads to the following class of solutions

$$g=\frac{\varepsilon e^{\alpha w}}{|w+2\alpha_0|}(\theta^1)^2+\eta\varepsilon_1 e^{-2\alpha w}(\theta^2)^2+e^{\alpha w}|w+2\alpha_0|\left(dz^2+f(z)g^{(k)}\right), \tag{56}$$

where $\eta=\mathrm{sgn}(w+2\alpha_0)$, $\theta^1=dx-\frac{2\varepsilon(\tilde{y}d\tilde{x}-\tilde{x}d\tilde{y})}{1+k(\tilde{x}^2+\varepsilon\tilde{y}^2)}$, $\theta^2=dy$ and functions $f(z)$, w are given by (26) and Table 1 for subcases (i)–(v). For instance, in the subcase (iv) the latter metric can be written in the form

$$g=\varepsilon_1 dy^2+\frac{1}{z}\left(dx+\frac{\rho^2}{2}d\phi\right)^2+z\left(dz^2+d\rho^2+\rho^2d\phi^2\right), \tag{57}$$

where $\varepsilon=k_1=1$ and cylindrical coordinates ρ, ϕ ($\tilde{x}=\frac{\rho}{2}\cos\phi$, $\tilde{y}=\frac{\rho}{2}\sin\phi$) are introduced for simplicity. Metric (57) provides an example of a non flat metric obtained from a flat one by means of a symmetry.

It turns out that in the subcase (v) of (29) the matrix S_1 transforms 5-dimensional plane wave solution into Gross-Perry-Sorkin metric [7, 10, 11]. To show that we set $\varepsilon=\varepsilon_1=k=1$ in (37) and introduce spherical coordinates with $z=-r$. Then, applying coordinate transformation $x=\frac{2u+t}{2\sqrt{m}}$, $y=\sqrt{m}\,t$ we obtain the plane wave solution

$$g=\left(1-\frac{m}{r}\right)dt^2+2dtdu+dr^2+r^2d\Omega^2. \tag{58}$$

Transforming the latter metric by virtue of matrix S_1 with $\alpha_0<-1/2$ we get the Gross-Perry-Sorkin monopole solution

$$g=-dt^2+\frac{1}{1+\frac{M}{\tilde{r}}}(d\tilde{u}-M\cos\theta d\phi)^2+\left(1+\frac{M}{\tilde{r}}\right)(d\tilde{r}^2+\tilde{r}^2d\Omega^2), \tag{59}$$

where $\tilde{u}=u/\sqrt{\alpha_0'}$, $\tilde{r}=\sqrt{\alpha_0'}\,r$, $M=m/\sqrt{\alpha_0'}$ and $\alpha_0'=-2\alpha_0-1$.

7 Summary

We showed that symmetry transformation (8) of the D-dimensional vacuum Einstein equations with $D-3$ commuting Killing vectors may lead to at most $(D-3)$-parameter family of new solutions. The symmetry transformations can be reduced to (10), (12) or (14). We outlined a method of solving (2)–(3) under the assumption that the 3-dimensional metric $\tilde{g}$ is given by (17) and the matrix χ (see (5)) depends only on the coordinate z. 5-dimensional metrics obtained in this way are given in Sects. 5.1–5.4. Some of these metrics were used as seed solutions for the symmetry transformation (Sect. 6). Unexpectedly, it turns out that the Kaluza-Klein version of the Reissner-Nordström solution is the symmetry transform of a Gross-Perry metric and that the 5-dimensional plane wave metric (58) is related to the Gross-Perry-Sorkin monopole solution (59).

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