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New Approach in Analysis of Sensitivity of Temperature Response to Selected Parameters of Two-Layer Structure

Piotr Majchrzak · Zbigniew Suszyński

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Abstract The analysis of the temperature response versus frequency of excitation provides information about the depth profile of the thermal parameters of the examined objects. The main parameters that determine the amplitude and the phase of the temperature response are: the frequency of thermal excitation, the thermal effusivities, and the relative thermal thicknesses of particular layers. The sensitivity of the amplitude and the phase of the thermal wave signal to changes of particular thermal parameters has been defined by partial derivatives of amplitude or phase with respect to those parameters. On the other hand, the method proposed in this article consists of the analysis of the sensitivity of the thermal response to relative changes of thermal parameters of object layers. The advantage of such a solution is the possibility of determining the optimal measurement conditions and assessing relative errors of measured values.

Keywords Layer structures · Sensitivity · Thermal parameters · Thermal waves

1 Introduction

The analysis of the thermal wave response of a layer structure to harmonic power excitation may provide information about the thermal properties of materials of which the object is fabricated [1–3]. In this case, the variable component of the thermal wave response depends on excitation and on the thermal impedance of the structure. The parameters which influence the impact of the value of the thermal impedance of layer structures are the ratios of thermal effusivities of adjacent layers,

P. Majchrzak (✉) · Z. Suszyński
Faculty of Electronics and Computer Science, Koszalin University of Technology,
Śniadeckich Street 2, 75-453 Koszalin, Poland
e-mail: piotr.majchrzak@tu.koszalin.pl

$$\varepsilon_{\mathrm{rel}}^{(i,i+1)} = \frac{\varepsilon^{(i)}}{\varepsilon^{(i+1)}} \tag{1}$$

and the relative thicknesses of layers [2],

$$L_{\mathrm{rel}}^{(i)} = d^{(i)}\sqrt{\frac{\pi f}{\alpha^{(i)}}} = \frac{d^{(i)}}{\mu^{(i)}} \tag{2}$$

where i is the number of layers, ε is the thermal effusivity, α is the thermal diffusivity (in $\mathrm{m}^2 \cdot s^{-1}$), d is the thickness of a layer (in m), f is the frequency of the thermal excitation (in Hz), and μ is the depth of thermal wave diffusion (in m).

The main problems in depth profiling of thermal parameters are: the selection of the proper frequency of excitation and the assessment of accuracy of the obtained results. The analysis of the sensitivity of the thermal response (its amplitude and phase) may provide the solutions for both the problems. The accuracy of determination of the thermal parameters of the analyzed structure may be estimated based on the known accuracy of measurements of the amplitude and the phase, and information about the influence of the thermal parameters on the amplitude and the phase of the thermal response.

Moreover, the estimation of optimum measurement conditions (frequency), for which the accuracy estimate of the thermal parameters is the highest, can be carried out based on information obtained from the analysis of the sensitivity of the thermal response to changes of parameters which depend on frequency.

The sensitivity of the amplitude and phase of the thermal wave signal on selected thermal parameters changes may be described by partial derivatives [4–8]. In this way, information about the amplitude or phase increase, corresponding to changes of independent variables, may be obtained. Unfortunately, derivatives show an increment of the function value corresponding to the absolute (constant, unitary) increment of the argument. In practice, it means that the same value of the derivative, for arguments differing by a few orders of magnitude, may have different relative weights.

The accuracy of a measurement is usually given by a relative measure. Therefore, calculating the sensitivity of the amplitude and phase of the thermal response, as changes of those values that correspond to relative changes of parameters of the thermal impedance is profitable.

The sensitivity of the amplitude of the thermal response to changes of the value of the chosen independent variable (iv) is defined as a relative difference of amplitude caused by a 1 % change of this parameter

$$SA_{\mathrm{nl}}^{(\mathrm{iv})} = \frac{\left|\dot{\Theta}'^{(\mathrm{iv})}\right| - \left|\dot{\Theta}\right|}{\left|\dot{\Theta}\right|} 100\ \% \tag{3}$$

where “nl” is the number of layers, “iv” represents an independent variable, $\dot{\Theta}$ is the value of the temperature harmonic component (in K), and $\dot{\Theta}'^{(\mathrm{iv})}$ is the value of the temperature harmonic component calculated for the value of the chosen independent variable increased by 1 % (K).

The sensitivity of the phase of the thermal response to changes of the value of the chosen independent variable is defined as the relative difference of phases caused by a 1 % change of this parameter:

$$\mathrm{SPh}_{\mathrm{nl}}^{(\mathrm{iv})} = \mathrm{Arg}\left(\dot{\Theta}'^{(\mathrm{iv})}\right) - \mathrm{Arg}\left(\dot{\Theta}\right) = \mathrm{Arg}\left(\frac{\dot{\Theta}'^{(\mathrm{iv})}}{\dot{\Theta}}\right) \tag{4}$$

This article presents an analysis of the sensitivity of a thermal wave response to changes of the thermal parameters of a two-layer structure. A theoretical temperature response of the layer structure for harmonic excitation was evaluated with the help of the TLM model [2]. In all cases the assumption that the last layer (the second one) is thermally thick (for all analyzed frequencies) was made.

2 Analysis

Furthermore, the analysis of the sensitivity of the amplitude and phase of the temperature response of a two-layer structure (Table 1) is presented. The harmonic component of the temperature on the structure surface, when the surface of the first layer is excited, is described by the following:

$$\Theta = \tilde{P} Z_{\mathrm{in}} \tag{5}$$

where $\tilde{P}$ is the variable component of excitation power (in W) and Z_{in} is the thermal input impedance of the structure (in $\mathrm{K \cdot W^{-1}}$).

The thermal wave input impedance of a two-layer structure ($Z_{\mathrm{in}(2l)}$) is described by

$$Z_{\mathrm{in}(2l)} = z_{11}^{(1)} - \frac{\left(z_{12}^{(1)}\right)^2}{z_{11}^{(1)} + Z_{\mathrm{c}}^{(2)}} \tag{6}$$

where
$z_{11}^{(i)} = Z_{\mathrm{c}}^{(i)} \frac{\cosh\left(\Gamma^{(i)}\right)}{\sinh\left(\Gamma^{(i)}\right)}$;
$z_{12}^{(i)} = Z_{\mathrm{c}}^{(i)} \frac{1}{\sinh\left(\Gamma^{(i)}\right)}$;
Γ is the operator of propagation: $\Gamma = (1+j)\sqrt{\frac{\omega}{2\alpha}}d$;
$\mathrm{Z_c}$ is the characteristic impedance: $Z_{\mathrm{c}} = (1-j)\frac{1}{S\varepsilon\sqrt{2\omega}}$;
S is the area of excitation (in $\mathrm{m^2}$); and
ω is the angular frequency of excitation (in $\mathrm{rad \cdot s^{-1}}$).

Table 1 Two-layered structure

$\varepsilon_{(1)}$	$L_{\mathrm{rel}}^{(1)}$
$\varepsilon_{(2)}$	$L_{\mathrm{rel}}^{(2)}$

After rearranging these equations, the input impedance of a two-layer structure can be described by

$$Z_{\mathrm{in}(2l)} = Z_{\mathrm{c}}^{(1)} \left(\frac{\sinh(\Gamma^{(1)}) + \varepsilon_{\mathrm{rel}}^{(1,2)} \cosh(\Gamma^{(1)})}{\cosh(\Gamma^{(1)}) + \varepsilon_{\mathrm{rel}}^{(1,2)} \sinh(\Gamma^{(1)})} \right) = Z_{\mathrm{c}}^{(1)} H_{2l} \tag{7}$$

Based on Eqs. 3, 5, and 7, an equation for the amplitude sensitivity of the thermal wave response of an n-layer structure to relative changes of the independent variables and for the harmonic excitation can be derived:

$$\mathrm{SA}_{\mathrm{nl}}^{(\mathrm{iv})} = \frac{\left| H_{\mathrm{nl}}^{\prime(\mathrm{iv})} \right| - |H_{\mathrm{nl}}|}{|H_{\mathrm{nl}}|} 100\ \% \tag{8}$$

Similarly, from Eqs. 4, 5, and 7, the sensitivity of the phase of the thermal wave response of an n-layer structure to relative changes of independent variables and for harmonic excitation may be derived:

$$\mathrm{SPh}_{\mathrm{nl}}^{(\mathrm{iv})} = \mathrm{Arg} \left(\frac{H_{\mathrm{nl}}^{\prime(\mathrm{iv})}}{H_{\mathrm{nl}}} \right) \tag{9}$$

The operator of propagation Γ is described by

$$\Gamma = (1 + j)\, L_{\mathrm{rel}} \tag{10}$$

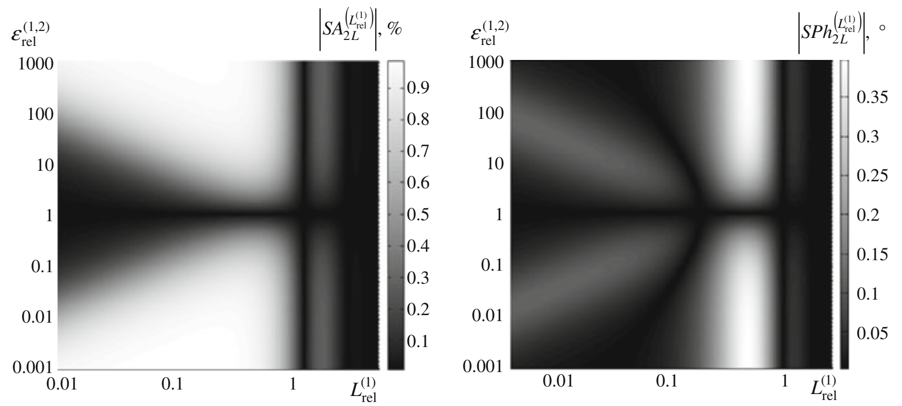
So, it follows from Eq. 7 that in a two-layer structure (assuming that the second layer is thermally thick), two parameters, which determine the signal of the harmonic temperature response, may be indicated. Those parameters are the relative thickness of the first layer ($L_{\mathrm{rel}}^{(1)}$) and the ratio of the first and second layer effusivities ($\varepsilon_{\mathrm{rel}}^{(1,2)}$).

The graphs in Figs. 1 and 2 present the amplitude and phase changes which correspond to a 1 % change of values of the independent variables ($L_{\mathrm{rel}}^{(1)}$ and $\varepsilon_{\mathrm{rel}}^{(1,2)}$). To determine these characteristics, the following ranges of arguments were taken:

$$L_{\mathrm{rel}}^{(1)} \in \left\langle 5 \times 10^{-3}, 5 \times 10^{0} \right\rangle \tag{11}$$

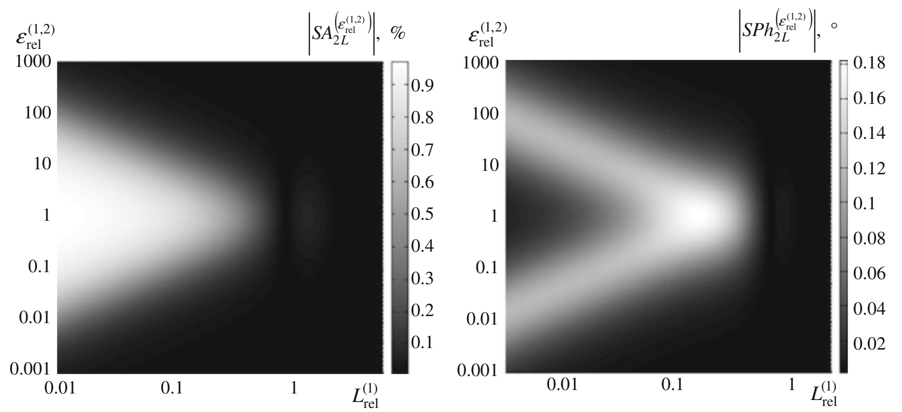
$$\varepsilon_{\mathrm{rel}}^{(1,2)} \in \left\langle 10^{-3}, 10^{3} \right\rangle \tag{12}$$

These ranges of parameters assure the analysis for a very wide spectrum of the materials and their connections. The amplitude damping on the L_{rel} depth is equal to multiplication of $\mathrm{e}^{L_{\mathrm{rel}}}$ [2]. Therefore, for $L_{\mathrm{rel}}^{(1)} = 5$, the damping of the signal is 148. Since the signal is registered on the surface of the analyzed object, one should take into consideration that the temperature wave, after reflecting from medium boundaries, travels the same distance to the top of this surface.



(a) **(b)**

Fig. 1 Sensitivity of (a) amplitude and (b) phase of the temperature response from two-layer structure, for an increase in parameter $L_{\mathrm{rel}}^{(1)}$ of 1 %



(a) **(b)**

Fig. 2 Sensitivity of (a) amplitude and (b) phase of the temperature response from two-layer structure, for an increase in parameter $\varepsilon_{\mathrm{rel}}^{(1,2)}$ of 1 %

So the registered amplitude of the thermal response on the surface of the analyzed structure is damped 22 026 times, which is such a huge value that it allows the assumption that the layer is thermally thick. In the same way, the $L_{\mathrm{rel}}^{(1)} = 5 \times 10^{-3}$ value results in a damping value equal to 1.01. In this case, the layer may be treated as thermally thin.

The range of $\varepsilon_{\mathrm{rel}}^{(1,2)}$ is also very wide. The effusivity of thermal insulators is usually on the order of several hundreds of $\mathrm{W} \cdot \mathrm{s}^{1/2} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-1}$, while the effusivity of materials that conducts heat very well (metals) is on the order of several ten thousands of $\mathrm{W} \cdot \mathrm{s}^{1/2} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-1}$. The range of effusivity ratios was taken to assure that the analysis of joints of materials cover a wide range of effusivities.

In order to determine the accuracy of the values obtained from thermal profiling, the direction of changes (sign) of the amplitude and phase sensitivity is not significant. Therefore, all the analyzed values of sensitivity are presented as their absolute values.

Based on the ranges in Eqs. 11 and 12, the vectors containing values of independent variables $L_{\text{rel}}^{(1)}$ and $\varepsilon_{\text{rel}}^{(1,2)}$ and also vectors containing values increased by 1 % ($L_{\text{rel}}^{\prime(1)}$ and $\varepsilon_{\text{rel}}^{\prime(1,2)}$) were evaluated. In Fig. 1, the graphs were evaluated based on Eqs. 8 and 9, and vectors $L_{\text{rel}}^{(1)}$, $\varepsilon_{\text{rel}}^{(1,2)}$, and $L_{\text{rel}}^{\prime(1)}$ values are presented.

From the graphs (Fig. 1) it follows that the maximum of the sensitivity that can be achieved in the case of changing the relative depth by 1 % is equal to about 1 % for the amplitude and about 0.4° for the phase of the thermal response from the two-layer structure. It may also be observed that the maxima of both sensitivities correspond to different ranges of relative depth ($L_{\text{rel}}^{(1)}$) values. In practice, it means that the optimal conditions for the best accuracy of measurements occur in different ranges of frequency for the amplitude and the phase.

In Fig. 2, the characteristics of the sensitivity of the amplitude and the phase of thermal wave response of a two-layer structure to the changes of the ratio of the first and the second layer effusivities are presented. These characteristics were calculated based on $L_{\text{rel}}^{(1)}$, $\varepsilon_{\text{rel}}^{(1,2)}$, and $\varepsilon_{\text{rel}}^{\prime(1,2)}$ vectors.

From the graphs (Fig. 2) it follows that the highest accuracy of measurements for both the amplitude and the phase registered on the surface of the two-layer structure, may be received when the effusivities of the first and the second layers are almost equal ($\varepsilon_{\text{rel}}^{(1,2)} \approx 1$). It may be explained by analyzing the values of the reflection coefficient of the heat flux (r) from the planar boundary between the two layers:

$$r\left(\varepsilon_{\text{rel}}\right) = \frac{|1 - \varepsilon_{\text{rel}}|}{|1 + \varepsilon_{\text{rel}}|} \tag{13}$$

The magnitude of the signal registered on the object surface depends on the value of the reflection coefficient of the heat flux [2]. In Fig. 3, the dependence of the reflection coefficient (r) on the ratio (ε_{rel}) of the effusivities is presented.

As may be observed, the highest sensitivity of the reflection coefficient for relative changes of ε_{rel} appears near the value of the argument of $\varepsilon_{\text{rel}} = 1$. In practice, it means that the more similar the adjacent layers effusivities are, the more accurate determination of one of the effusivities is based on a known value of the second effusivity and the measured value of the ε_{rel} ratio.

Based on known values of the amplitude and the phase measuring accuracy and sensitivity graphs presented in Figs. 1 and 2, one can estimate the accuracy of the determined values of L_{rel} or ε_{rel}.

To do this, one should find (on adequate axes of charts) selected values of independent variables and read the values of the corresponded sensitivities. The error of determining the parameter value may be evaluated from the ratio of the measurement error and selected value of the amplitude or phase sensitivity:

$$\delta^{(\text{iv})} = \frac{\delta_{\text{A}}}{\text{SA}_{\text{nw}}^{(\text{iv})}} 1\ \% \tag{14}$$

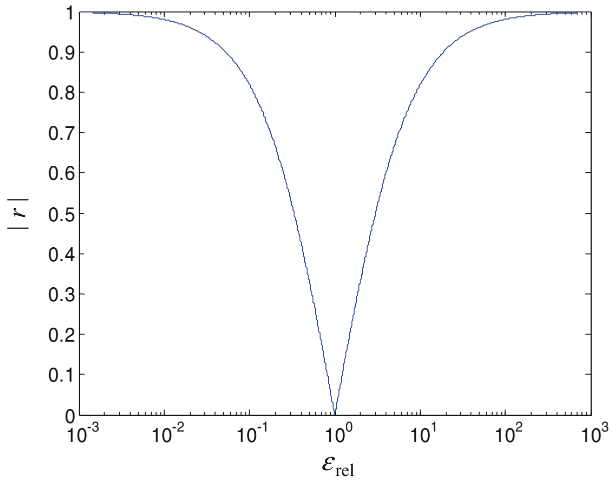


Fig. 3 Reflection coefficient of heat flux versus relative effusivity

or

$$\delta^{(\mathrm{iv})} = \frac{\delta_{\mathrm{Ph}}}{\mathrm{SPh}_{\mathrm{nw}}^{(\mathrm{iv})}} 1\ \% \tag{15}$$

where $\delta^{(\mathrm{iv})}$ is the accuracy of determining the independent variable (in %), δ_{A} is the accuracy of the amplitude measurement (in %), δ_{Ph} is the accuracy of the phase measurement (in °), $SA_{\mathrm{nw}}^{(\mathrm{iv})}$ is the sensitivity of the amplitude to changes of the independent variable (in %), and $\mathrm{SPh}_{\mathrm{nw}}^{(\mathrm{iv})}$ is the sensitivity of the phase to changes of the independent variable (in °).

The presented graphs in Figs. 1 and 2 may be also used to determine the excitation frequency, for which the response signal will be characterized by the highest sensitivity for relative changes of chosen parameters. To do this, one should use values of L_{rel} (for which the sensitivity has the maximum value), and from the expression below, determine the adequate frequency. Rearranging Eq. 2, one obtains

$$f = \frac{\alpha L_{\mathrm{rel}}^2}{\pi d^2} \tag{16}$$

To determine the frequency, the thermal diffusivity (α) and thickness of layer (d) are also necessary. Usually these parameters are unknown, but most of them can be estimated based on preliminary information about the object, its structure, and the technology of the fabrication process.

Based on the determined L_{rel} value and the preliminary specified ranges of the layer thickness variation $\langle d_{\min}, d_{\max}\rangle$ and thermal-diffusivity variation $\langle \alpha_{\min}, \alpha_{\max}\rangle$, the range of the excitation frequency may be determined:

$$f_{\min} = \frac{\alpha_{\min} L_{\text{rel}}^2}{\pi d_{\max}^2} \tag{17}$$

$$f_{\max} = \frac{\alpha_{\max} L_{\text{rel}}^2}{\pi d_{\min}^2} \tag{18}$$

In this way, the range of frequency for which the measurement of the amplitude and (or) phase frequency characteristic is determined.

3 Conclusions

A method of analyzing the sensitivity of the amplitude and the phase of the thermal response of layered structures for independent variables of thermal wave model changes is presented in this article. Based on the analysis of sensitivity graphs obtained, one may come to the conclusion that the excitation frequency for which the maximum value of the sensitivity of the amplitude of the thermal response appears, differs from the frequency for which the maximum of the sensitivity of the phase of this signal occurs. In practice, it means that for different excitation frequencies, better or worse (in consideration of measurement accuracy) measurement conditions, for both amplitudes and phases, may be obtained.

The accuracy of the measurement of different thermal wave parameters for the same excitation frequency will also vary. Based on presented analyses, the selection of thermal wave measurement conditions which assure the best conditions for experimental determination of thermal parameters of examined structures is possible. Moreover, these analyses allow an estimate the accuracy of the thermal parameter profile of layered structures.

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