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RESEARCH

The covariant approach to static spacetimes in Einstein and extended gravity theories

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Abstract

We present a covariant study of static space-times, as such and as solutions of gravity theories. By expressing the relevant tensors through the velocity and the acceleration vectors that characterise static space-times, the field equations provide a natural non-redundant set of scalar equations. The same vectors suggest the form of a Faraday tensor, that is studied in itself and in (non)-linear electrodynamics. In spherical symmetry, we evaluate the explicit expressions of the Ricci, the Weyl, the Cotton and the Bach tensors. Simple restrictions on the coefficients yield well known and new solutions in Einstein, $f(R)$, Cotton and Conformal gravity, with or without charges, in vacuo or with fluid source.

Keywords Static spacetime · Faraday tensor · Bach tensor · Electrodynamics · F(R) theories · Conformal gravity · Cotton gravity

Mathematics Subject Classification 83C20 (Primary) · 83C55 · 83D05 (Secondary)

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Contents

1 Introduction

The field equations of gravitational theories are covariant, and equate a geometric tensor (e.g. Einstein, Cotton, Bach tensor) to a tensor describing matter. Solutions are usually found in coordinates that exploit the symmetries. However, there are advantages in keeping the coordinate-free tensor description as far as possible. Besides the formal elegance, it naturally addresses scalar identities. We do so in this study of static space-times, beginning with local geometry and then discussing gravity.

A covariant characterization of a static space-time involves an equation for a time-like velocity u_k with a closed space-like acceleration $\dot u_k = u^j \nabla_j u_k$. The two defining vectors are the natural start for the expansion of the relevant tensors. In absence of symmetries they are complemented by two others. This is basically the spirit of the 1+1+2 formalism introduced by Clarkson and Barrett [15, 16]. Here the second vector is fixed by the context. Carloni used the formalism to specify the stress tensor, and the Ricci tensor ensuing from the Einstein equation, in spherically symmetric metrics [14]. In this work the two unspecified orthogonal vectors are not explicitly required, as the Ricci tensor is constructed via the integrability conditions and linked to the electric part of the Weyl tensor.

With the vectors u_j, $\dot u_j$ and an orthogonal space-like pair, we consider the antisymmetric tensor ($\eta = \dot u^k \dot u_k$)

$$F_{jk} = \mathbb{E}\frac{1}{\sqrt{\eta}}(u_j \dot u_k - \dot u_j u_k) + \mathbb{B}(y_j z_k - y_k z_j)$$

and obtain the conditions on $\mathbb{E}$ and $\mathbb{B}$ to yield a Faraday tensor. The terms correspond to the electric and magnetic fields. In the Einstein theory, the equations of linear and non-linear electrodynamics constrain the Ricci tensor to a simple structure, with coefficients R and $R^\star$. The weak energy condition imposes a non-negative spatial curvature scalar, $R^\star \geq 0$, while the scalar curvature R is zero if and only if the electrodynamics is linear.

After the general setting, we turn to the much studied spherically symmetric static space-times, with line element

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2).$$

The early solutions in General Relativity, named after Schwarzschild, de Sitter, Reissner and Nordstrøm, belong to this class. In the years many others were found as black hole (BH) or compact star solutions, that fit in the present covariant description.

Beginning with geometry, we obtain the relevant static spherical tensors (Ricci, Weyl, Cotton, Bach) as combinations of $u_i u_j$, g_{ij} and $\dot{u}_i \dot{u}_j$. Remarkably, also the energy–momentum tensor of linear and non-linear electrodynamics is a combination of the same elementary tensors, with the magnetic term $\mathbb{B}$ being forced to be the field of a magnetic monopole. By constraining the tensor coefficients to simple forms, we obtain scalar equations that recover the early metrics and others.

By the equality inherent gravity theories mentioned in the beginning, the geometric Ricci, Cotton and Bach tensors fix the form of the energy–momentum tensor respectively in Einstein, Cotton and Conformal gravity. A similar construction is made in $f(R)$ theory, where the Ricci tensor and the Hessian provide the form of the matter tensor. In these four theories, it has the form of energy–momentum tensor of an anisotropic fluid or of linear or nonlinear electrodynamics.
The identification of the geometric and physical coefficients of the tensors in the left and right sides of the field equations, provides scalar equations. The functions $B(r)$ found on geometric grounds specify as solutions of field equations.

In a century of gravity theories, many static spherical solutions were found. This is a very short and partial recount.
In 1968 James Bardeen (son of John B. of BCS theory) obtained the first singularity-free BH solution of the Einstein equation [2, 4]:

$$B(r) = 1 - \frac{2Mr^2}{(r^2 + g^2)^{3/2}}$$

Ayon-Beato and Garcia reinterpreted it as a magnetic monopole solution in Einstein-non-linear electrodynamics [1].
In 2003 Kiselev published a new exact solution of the Einstein equation for quintessential matter surrounding a BH [32]

$$B(r) = 1 - \frac{2M}{r} - \frac{K}{r^{1+3w}}$$

It raised a debate, until Visser showed in 2020 that the Kiselev BH is neither a perfect fluid nor a quintessence [57]. Generalisations of Kiselev space-times were recently used in the framework of gravitational lensing [49].
In 1996 Hayward [27] discovered a line element describing the local formation of a BH out of vacuum, its Bardeen-like static quiescence and final evaporation.

Bronnikov [9] showed that Einstein gravity coupled to nonlinear electrodynamics has nontrivial spherical solutions with global regular metric if and only if the electric charge is zero and the Lagrangian $\mathscr{L}(F)$ has a finite limit as $F \to \infty$. In the same context, Dymnikowa [20] studied the existence of regular spherically symmetric electrically charged solutions. The effects of torsion were considered by Cotton [18].

Gravastars (gravitational vacuum stars) were introduced in 2001 [46] as an alternative to BH that avoid the problems associated with horizons and singularities. Models in nonlinear electrodynamics were constructed by Lobo and Arellano [37].

Among a great variety of spherical metrics in Einstein gravity, we quote the Yukawa BH [45], the Van der Waals BH [50], the global monopole [5], the Rindler–Grumiller metric [23], the logotropic BH [13], quantum corrections to Reissner–Nordstrøm BH [58]. The study of non-linear electrodynamics coupled to $f(R)$ gravity was started by Hollenstein and Lobo [29, 52].

In 1989 Mannheim and Kazanas [39] obtained an exact vacuum solution of Conformal gravity, and applied it to describe the rotation curve of galaxies without dark matter

$$B(r) = -\frac{1}{r}\beta(2-3\beta\gamma) + (1-3\beta\gamma) + \gamma r - \kappa r^2$$

Topological black holes in conformal gravity were studied by Klemm [31].

Based on the conformal action, but with variation in the connection, a theory named Cotton gravity was recently proposed by Harada [25]. We showed that the field equation can be recast as Einstein equations, with the freedom of a Codazzi tensor. As such, they are second order in the derivatives of the metric tensor [44].

This is the ouline of the paper. In Sect. 2 we discuss static space-times in general, the Ricci and the Weyl tensors. It is partly based on our study of the larger family of doubly-warped space-times [43]. Useful equations are collected in "Appendix 1".

In Sect. 3 we introduce the Faraday tensor and prove necessary and sufficient conditions on the scalars $\mathbb{E}$ and $\mathbb{B}$ (the proof is in "Appendix 2"). The general discussion of Einstein gravity coupled to linear (LE) and non-linear electrodynamics (NLE) is in Sect. 4. An interesting form of the Ricci tensor is obtained, with conclusions about the curvature space-time and space scalars R and $R^\star$. Section 5 discusses the anisotropic fluid source, concluding that the energy density is proportional to $R^\star \geq 0$.
In Sect. 6 we discuss spherical symmetry, where the full form of the Ricci, Weyl, Cotton and Bach tensors are obtained, as combinations of the basic tensors g_{jk}, $u_j u_k$ and $\dot{u}_j \dot{u}_k$ with coefficients that are linear or at most quadratic in $B(r)$, $B'(r)$ and $B''(r)$. Simple conditions yield the early static metrics. In Sect. 7 we consider the Einstein gravity, where the metrics are solutions of field equations. Pure dust or perfect fluid solutions are not possible, unless $p = -\mu$. We then discuss LE and NLE, with some

identities that allow for reconstructing the Lagrangian $\mathscr{L}(F)$ from $B(r)$ (actually from a coefficient of the Ricci tensor).

NLE coupled to $f(R)$ gravity is presented in Sect. 8, with immediate recognition of the known property that $f_R(r)$ is constrained to be linear in r. This allows the integration of one field equation in presence of point charges. Since only the solution appears in previous papers, we offer its deduction in "Appendix 3".

In Cotton gravity (Sect. 9), after a brief presentation of our interpretation as an Einstein equation, we show that the vacuum solution by Harada is also a solution of the Einstein theory with an anisotropic energy–momentum. We then present two solutions: with perfect fluid and LE. Finally, in Sect. 10 we turn to Conformal gravity. We recall the vacuum solution by Mannheim and Kazanas, and present a solution in LE. We end with the conclusions.

In this paper the static four-dimensional Lorentzian spacetimes have signature $(-,+++)$. A dot denotes the action of $u^k\nabla_k$.

2 Static space-times

The velocity is eigenvector of the Ricci tensor, the Electric tensor evaluates the Weyl tensor, the form of the Ricci tensor is obtained.

There are various characterisations of static spacetimes:

• The existence of a time-like vector field u_k (named velocity) that is normalized, $u^k u_k = -1$, with gradient

$$\nabla_j u_k = -u_j \dot u_k \tag{1}$$

such that the 'acceleration' $\dot u_k = u^j \nabla_j u_k$ is a closed vector field [54]:

$$\nabla_j \dot u_k = \nabla_k \dot u_j \tag{2}$$

The acceleration is spacelike ($u^k \dot u_k = 0$) with normalization $\eta = \dot u^p \dot u_p > 0$. Contraction of (2) with u^j shows that $\ddot u_k = u^j \nabla_k \dot u_j = -\dot u^j \nabla_k u_j = \dot u^j u_k \dot u_j = \eta u_k$.

• The existence of coordinates $(t, \mathbf{x})$ where the metric tensor has the static form

$$ds^2 = -B(\mathbf{x})dt^2 + g^\star_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu$$

In this frame: $u_k = (-\sqrt{B}, \mathbf{0})$, $\dot u_k = (0, \partial_\mu \log\sqrt{B})$.

• The existence of a time-like hypersurface orthogonal Killing vector: $\nabla_i \xi_j + \nabla_j \xi_i = 0$. The vector is $\xi_j = u_j \sqrt{B}$, and $\dot u_j = \nabla_j \log\sqrt{B}$.

Proposition 2.1 *The velocity is an eigenvector of the Ricci tensor,*

$$R_{jk} u^k = -(\nabla_p \dot u^p) u_j \tag{3}$$

and it is Riemann compatible:

$$(u_i R_{jklm} + u_j R_{kilm} + u_k R_{ijlm})u^m = 0. \tag{4}$$

Proof $R_{jklm}u^m = (\nabla_j\nabla_k - \nabla_k\nabla_j)u_l = (u_j\dot u_k - \dot u_j u_k)\dot u_l + u_j\nabla_k\dot u_l - u_k\nabla_j\dot u_l$. Contraction with g^{jl} gives property (3). Multiplication by u_i and cyclic sum gives compatibility. □

As expected, the "time derivative" of geometric invariants is zero:

Proposition 2.2 *Let $\eta = \dot u^k\dot u_k$ and $R = g^{jk}R_{jk}$,*

$$\dot\eta = 0, \quad \dot R = 0, \quad u^k\nabla_k(\nabla_p\dot u^p) = 0 \tag{5}$$

Proof 1) $\dot\eta = u^k\nabla_k(\dot u^p\dot u_p) = 2\ddot u^p u_p = 2\eta u^p\dot u_p = 0$. 2) $u^k\nabla_k\nabla_p\dot u^p = u^k R_{kp}{}^p{}_m\dot u^m + u^k\nabla_p\nabla_k\dot u^p = u^k R_{kp}\dot u^p + \nabla_p(\ddot u^p) - (\nabla_p u^k)(\nabla_k\dot u^p) = \nabla_p(\eta u^p) + u^p\dot u^k\nabla_p\dot u_k = \dot\eta + \frac{1}{2}\dot\eta = 0$. 3) Eq. (3) and $\nabla_j R^j{}_k = \frac{1}{2}\nabla_k R$ give $\dot R = -2u^j\nabla_j(\nabla_p\dot u^p) = 0$. □

Being Riemann compatible, the velocity is also "Weyl compatible" (Theorem 2.1 in [42]): $(u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm})u^m = 0$. The Weyl tensor is

$$C_{jklm}=R_{jklm}+\frac{1}{2}(g_{jm}R_{kl}-g_{km}R_{jl}+g_{kl}R_{jm} - g_{jl}R_{km})-\frac{R}{6}(g_{jm}g_{kl}-g_{km}g_{jl}).$$

The contraction $E_{kl} = u^j C_{jklm}u^m$ is the Electric tensor; it is symmetric, traceless and $E_{jk}u^k = 0$. Weyl compatibility is equivalent to the relation

$$C_{jklm}u^m = u_k E_{jl} - u_j E_{kl}$$

The explicit evaluation of the Electric tensor gives an identity with the Ricci tensor:

$$E_{kl} = -\frac{1}{2}R_{kl} + \frac{1}{2}(\nabla_p\dot u^p)(g_{kl} + 2u_l u_k) + \frac{1}{6}R(g_{kl} + u_k u_l) + u^j R_{jklm}u^m \tag{6}$$

where $u^j R_{jklm}u^m = -\dot u_k\dot u_l - \nabla_k\dot u_l - \eta u_k u_l$ (see Prop. 2.1).

Proposition 2.3 (Weyl tensor) *In a four-dimensional static spacetime the Weyl tensor is solely determined by the electric tensor:*

$$\begin{aligned} C_{jklm} = {} & (g_{kl} + 2u_k u_l)E_{jm} - (g_{jl} + 2u_j u_l)E_{km} \\ & + (g_{jm} + 2u_j u_m)E_{kl} - (g_{km} + 2u_k u_m)E_{jl} \end{aligned} \tag{7}$$

and $C_{jklm}C^{jklm} = 8E^{kl}E_{kl}$.

Proof In $n = 4$ the following identity by Lovelock holds [38]: $0 = g_{ir}C_{jklm} + g_{jr}C_{kilm} + g_{kr}C_{ijlm} + g_{im}C_{jkrl} + g_{jm}C_{kirl} + g_{km}C_{ijrl} + g_{il}C_{jkmr} + g_{jl}C_{kimr} + g_{kl}C_{ijmr}$. The contraction with $u^i u^r$ and Weyl compatibility give the Weyl tensor. □

At each point we choose a basis of vectors formed by u_i and three orthonormal space-like vectors $\frac{1}{\sqrt{\eta}}\dot u_i$, y_i, z_i:

$$g_{ij} = -u_i u_j + \frac{\dot u_i \dot u_j}{\eta} + y_i y_j + z_i z_j \tag{8}$$

Lemma 2.4 *In a* $n=4$ *static space-time:*

$$\nabla_k y_j = -Y_k \dot u_j - \Omega_k z_j \tag{9}$$
$$\nabla_k z_j = -Z_k \dot u_j + \Omega_k y_j \tag{10}$$
$$\nabla_k \dot u_j = -\eta u_k u_j + \tfrac{1}{2\eta}\dot u_j \nabla_k \eta + \eta(y_j Y_k + z_j Z_k) \tag{11}$$

where $Y_k = \frac{1}{\eta} y^p \nabla_k \dot u_p$, $Z_k = \frac{1}{\eta} z^m \nabla_k \dot u_m$ *and* $\Omega_k = y^p \nabla_k z_p$. *It is also:*

$$u^k Y_k = 0, \quad u^k Z_k = 0, \quad y^k Z_k = z^k Y_k$$
$$\eta(y^k Y_k + z^k Z_k) = -\eta + \nabla^p \dot u_p - \frac{\dot u^p \nabla_p \eta}{2\eta} \tag{12}$$

Proof The gradient of (8) is: $0 = u_k(u_i \dot u_j + \dot u_i u_j) + \frac{1}{\eta}(\dot u_i \nabla_k \dot u_j + \dot u_j \nabla_k \dot u_i) - \dot u_i \dot u_j \frac{\nabla_k \eta}{\eta^2} + y_i \nabla_k y_j + y_j \nabla_k y_i + z_i \nabla_k z_j + z_j \nabla_k z_i$. The contractions with y^i or z^i give the first two relations. While contracting with $\dot u^i$ note that $\dot u^i \nabla_k y_i = -y^i \nabla_k \dot u_i = -\eta Y_k$, and $\dot u^i \nabla_k z_i = -\eta Z_k$.
Similarly: $u^k Y_k = \frac{1}{\eta} y^j \ddot u_j = y^j u_j = 0$ and $u^j Z_j = 0$, and $y^k Z_k = z^k Y_k$. Finally, (12) results from the contraction g^{jk} of (11). □

With this choice of basis vectors, the Ricci tensor (6) is:

$$R_{kl} = u_k u_l \left[\frac{R}{3} + 2\nabla_p \dot u^p\right] + g_{kl}\left[\frac{R}{3} + \nabla_p \dot u^p\right] - 2\dot u_k \dot u_l - 2E_{kl}$$
$$- \frac{1}{\eta}\dot u_l \nabla_k \eta - 2\eta(Y_k y_l + Z_k z_l). \tag{13}$$

In static space-times the curvature scalar R and the space curvature scalar $R^\star$ are related by the identity (see [43] Eq. 34):

$$R = R^\star - 2\nabla_p \dot u^p. \tag{14}$$

3 The Faraday tensor in static space-times

The conditions for a Faraday tensor and the conserved current are given.

The antisymmetric tensors $u_i \dot u_j - u_j \dot u_i$ and $y_i z_j - y_j z_i$ are "time independent": $u^k \nabla_k (u_i \dot u_j - u_j \dot u_i) = 0$ and $u^k \nabla_k (y_i z_j - y_j z_i) = 0$. The other antisymmetric combinations of the basis vectors do not share this property.

Therefore, we consider the following antisymmetric tensor

$$F_{jk} = \frac{\mathbb{E}}{\sqrt{\eta}}(u_j\dot{u}_k - \dot{u}_j u_k) + \mathbb{B}(y_j z_k - y_k z_j) \tag{15}$$

where $\mathbb{E}$ and $\mathbb{B}$ are scalar fields with $\dot{\mathbb{E}} = \dot{\mathbb{B}} = 0$. Since $\dot{\eta} = 0$, it is $\dot{F}_{jk} = 0$. F_{jk} is a Faraday tensor if:

$$\nabla_j F_{kl} + \nabla_k F_{lj} + \nabla_l F_{jk} = 0. \tag{16}$$

Theorem 3.1 (The Faraday tensor) *In a static space-time, the tensor* (15) *with* $\dot{\mathbb{E}} = \dot{\mathbb{B}} = 0$ *is Faraday if and only if*

$$\nabla_k \frac{\mathbb{E}}{\sqrt{\eta}} = \varkappa \dot{u}_k, \tag{17}$$

$$\dot{u}^k \nabla_k \mathbb{B} = \mathbb{B}\left[\eta - \nabla^k \dot{u}_k + \frac{\dot{u}^k \nabla_k \eta}{2\eta}\right]. \tag{18}$$

Then it is $\dot{\varkappa} = 0$.

Proof See "Appendix 2". □

The vector field $J_k = \nabla_j F^j{}_k$ is a conserved current $\nabla_k J^k = 0$.

Proposition 3.2 (The current)

$$J^k = J u^k + \left(z^k y^m - y^k z^m\right)\left(\nabla_m \mathbb{B} - \mathbb{B}\frac{\nabla_m \eta}{2\eta}\right) \tag{19}$$

$$J = -\eta\varkappa + \mathbb{E}\sqrt{\eta} - \frac{\mathbb{E}}{\sqrt{\eta}}\nabla_j \dot{u}^j \tag{20}$$

Proof Equation (92) gives $J_k = -\eta\varkappa u_k + \frac{\mathbb{E}}{\sqrt{\eta}}(\ddot{u}_k - u_k \nabla_i \dot{u}^i) + (y^j \nabla_j \mathbb{B} - \mathbb{B}\dot{u}^j Y_j) z_k - (z^j \nabla_j \mathbb{B} - \mathbb{B}\dot{u}^j Z_j) y_k + \mathbb{B}(Y_j z^j - Z_j y^j)\dot{u}_k$. The last term is zero and $\ddot{u}_k = \eta u_k$. It is also $\dot{u}^j Y_j = \frac{1}{\eta} y^m \dot{u}^j \nabla_j \dot{u}_m = \frac{1}{\eta} y^m \dot{u}^j \nabla_m \dot{u}_j = \frac{1}{2\eta} y^m \nabla_m \eta$, and $\dot{u}^j Z_k = \frac{1}{2\eta} z^m \nabla_m \eta$. The current is obtained, and is orthogonal to $\dot{u}_k$. □

4 Linear/non-linear electrodynamics in static Einstein gravity

The equations of the Einstein—LE and NLE theory are discussed.

The Einstein equations of gravity coupled to an electromagnetic field descend from the action (see [1])

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\,[R - 4\mathscr{L}(F)]$$

where $\mathscr{L}$ is a scalar function of the squared Faraday tensor $F = \frac{1}{4}F_{jk}F^{jk}$. In linear electrodynamics $\mathscr{L}(F) = F$.
The vanishing of the variations of the action in the metric tensor and in the vector potential ($F_{jk} = \nabla_j A_k - \nabla_k A_j$) respectively give:

$$R_{jk} - \tfrac{1}{2}g_{jk}R = 2\mathscr{L}_F(F)F_{jm}F_k{}^m - 2g_{jk}\mathscr{L}(F) \tag{21}$$

$$\nabla_j(\mathscr{L}_F(F)F^{jk}) = 0 \tag{22}$$

where $\mathscr{L}_F = d\mathscr{L}/dF$. The right-hand-side of Eq. (21) is the energy–momentum tensor T^{nlin}_{jk} of non-linear electrodynamics.
In the static setting with F_{jk} given by (15), it is $F = \frac{1}{2}(\mathbb{B}^2 - \mathbb{E}^2)$ and

$$T^{nlin}_{jk} = 2(\mathbb{E}^2 + \mathbb{B}^2)\left[u_j u_k - \frac{\dot{u}_j \dot{u}_k}{\eta}\right]\mathscr{L}_F(F) + 2g_{jk}[\mathbb{B}^2\mathscr{L}_F(F) - \mathscr{L}(F)]. \tag{23}$$

Note the disappearance of the space-like vectors y_j and z_j.
In the linear case the tensor is traceless:

$$T^{lin}_{jk} = 2(\mathbb{E}^2 + \mathbb{B}^2)\left[u_j u_k + \frac{1}{2}g_{jk} - \frac{\dot{u}_j \dot{u}_k}{\eta}\right]. \tag{24}$$

The contractions of the Einstein equation (21) with u^j and g^{jk} give two interesting relations between the geometry and the scalars of electrodynamics:

$$\mathbb{E}^2\mathscr{L}_F(F) + \mathscr{L}(F) = \tfrac{1}{2}\nabla_p \dot{u}^p + \tfrac{1}{4}R \tag{25}$$

$$F\mathscr{L}_F(F) - \mathscr{L}(F) = -\tfrac{1}{8}R \tag{26}$$

These are immediate consequences:

Proposition 4.1 *(1) $R = 0$ if and only if $\mathscr{L}(F) = cF$ (we take $c = 1$).*
(2) In linear electrodynamics: $\mathbb{B}^2 + \mathbb{E}^2 = \nabla_p \dot{u}^p$.

The sum of (25) and (26) is: $(\mathbb{B}^2 + \mathbb{E}^2)\mathscr{L}_F = \nabla_p \dot{u}^p + \frac{1}{4}R$. The Einstein equation of non-linear electrodynamics becomes a geometric prescription for the Ricci tensor, which acquires a form much simpler than the general one in static space-times (13):

$$R_{jk} = \left(R^\star - \frac{1}{2}R\right)\left[u_j u_k + \frac{1}{2}g_{jk} - \frac{\dot{u}_j \dot{u}_k}{\eta}\right] + \frac{1}{4}g_{jk}R \tag{27}$$

u^k and $\dot{u}^k$ are eigenvectors of the Ricci tensor with eigenvalue $-(\nabla_p \dot{u}^p)$, while y^k and z^k are eigenvectors with eigenvalue $\frac{R}{2} + \nabla_p \dot{u}^p$ (in the linear case: $R = 0$).
The second field equation (22) describes the current. It is equivalent to the following three equations

$$\nabla_j\left[\frac{\dot{u}^j}{\sqrt{\eta}}\mathbb{E}\mathscr{L}_F\right] = \sqrt{\eta}\,[\mathbb{E}\mathscr{L}_F] \tag{28}$$

$$\nabla_j [y^j \mathbb{B}\mathscr{L}_F] = -(z^j \Omega_j)[\mathbb{B}\mathscr{L}_F] \tag{29}$$

$$\nabla_j [z^j \mathbb{B}\mathscr{L}_F] = (y^j \Omega_j)[\mathbb{B}\mathscr{L}_F] \tag{30}$$

Proof The expression (15) of the Faraday tensor is placed in (22):

$$\begin{aligned}
0 &= \nabla_j \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F \left(u^j \dot u_k - u_k \dot u^j\right)\right] + \nabla_j \left[\mathbb{B}\mathscr{L}_F (y^j z_k - y_k z^j)\right] \\
&= \dot u_k u^j \nabla_j \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right] - u_k \dot u^j \nabla_j \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right] + \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right]\left(\ddot u_k - u_k \nabla_j \dot u^j\right) \\
&\quad + z_k y^j \nabla_j [\mathbb{B}\mathscr{L}_F] - y_k z^j \nabla_j [\mathbb{B}\mathscr{L}_F] + [\mathbb{B}\mathscr{L}_F]\Big(z_k \nabla_j y^j + y^j \nabla_j z_k \\
&\quad - y_k \nabla_j z^j - z^j \nabla_j y_k\Big) = -u_k \dot u^j \nabla_j \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right] + u_k \left[\tfrac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right]\left(\eta - \nabla_j \dot u^j\right) \\
&\quad + z_k y^j \nabla_j [\mathbb{B}\mathscr{L}_F] - y_k z^j \nabla_j [\mathbb{B}\mathscr{L}_F] \\
&\quad + [\mathbb{B}\mathscr{L}_F]\left(z_k \nabla_j y^j - y^j Z_j \dot u_k + y^j \Omega_j y_k - y_k \nabla_j z^j + z^j Y_j \dot u_k + z^j \Omega_j z_k\right)
\end{aligned}$$

The coefficient of $\dot u_k$ is proportional to $(z^j Y_j - y^j Z_j) = 0$. The vector equation gives three conditions for the coefficients of the components along u_k, z_k and y_k. □

An extension with $\mathscr{L}(F, {}^*F)$, where the invariant scalar ${}^*F$ is built with the dual Faraday tensor, is studied by Bokulić et al. [7].

5 Anisotropic perfect fluid in static Einstein gravity

Absence of convective term. Positive energy means positive space-curvature scalar.

The Einstein equation for a static anisotropic fluid with velocity u_i is:

$$R_{jk} - \frac{1}{2} g_{jk} R = (p + \mu) u_j u_k + p g_{jk} + \Pi_{jk}$$

where Π_{jk} is the stress-tensor (traceless and $\Pi_{jk} u^k = 0$), p is the effective pressure and μ is the energy density. A convective term $(u_j q_k + u_k q_j)$ is forbidden in static space-times as it would violate Eq. (3).
By the general property $R_{jk} u^k = (-\nabla_p \dot u^p) u_j$ the contraction of the Einstein equation with u^k gives $\nabla_p \dot u^p + \frac{R}{2} = \mu$. Now use (14) and obtain the simple relation:

$$\mu = \frac{1}{2} R^\star \tag{31}$$

The trace and the previous equation give the pressure:

$$3p = \frac{1}{2} R^\star - R \tag{32}$$

Remark 5.1 In general, in a static space-time the Einstein equations relate the positive energy constraint to the space curvature scalar:

$$T_{ij}u^i u^j = R_{jk}u^j u^k + \tfrac{1}{2}R = \nabla_p \dot{u}^p + \tfrac{1}{2}R = \tfrac{1}{2}R^\star \geq 0$$

In spherical symmetry (see "Appendix 1"): $R^\star = 2(\frac{1-B}{r^2} + \frac{B'}{r})$. The condition becomes

$$\frac{d}{dr}\frac{B(r)-1}{r} \geq 0 \tag{33}$$

6 Spherical symmetry

Expressions of the Ricci, Weyl, Cotton and Bach tensors in terms of $u_j u_k$, g_{jk} *and* $\dot{u}_j \dot{u}_k$.
Natural constraints give notorious metrics.
The magnetic part of the Faraday tensor is a monopole.

The majority of static spherical metrics discussed in the literature depend on a single scale function $B(r) > 0$:

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{34}$$

In coordinates (t, r, θ, ϕ), the acceleration is the radial vector $\dot{u}_k = (0, \frac{B'}{2B}, 0, 0)$, where a prime is a derivative in r.
If $X(r)$ is a scalar function, its gradient is parallel to $\dot{u}_j$:

$$\nabla_j X = \frac{\dot{u}_j \dot{u}^k}{\eta}\nabla_k X = \dot{u}_j \frac{2B}{B'} X' \tag{35}$$

The following scalars are obtained from expressions valid for the broader class of spherical doubly-warped space-times (see "Appendix 1" or equations 51, 40 and 53 in [43]):

$$\eta = \frac{1}{4}\frac{B'^2}{B}, \qquad \frac{\dot{u}^p \nabla_p \eta}{\eta} = B'' - \frac{B'^2}{2B}, \qquad \nabla_p \dot{u}^p = \frac{B''}{2} + \frac{B'}{r}. \tag{36}$$

A key quantity is the following:

Proposition 6.1

$$\nabla_j \dot{u}_k = \left[-\eta + \frac{B'}{2r}\right] u_j u_k + \frac{B'}{2r} g_{jk} + \left[\frac{B''}{2} - \frac{B'}{2r} - \eta\right]\frac{\dot{u}_j \dot{u}_k}{\eta}. \tag{37}$$

Proof In spherical coordinates, with $y^j = (0,0,r,0)$ and $z^j = (0,0,0,r\sin\theta)$ one evaluates

$$Y_j = \frac{2B}{rB'}y_j, \qquad Z_j = \frac{2B}{rB'}z_j$$

Then $Y_j y_k + Z_j z_k = \frac{2B}{rB'}(g_{jk} + u_j u_k - \frac{\dot u_j \dot u_k}{\eta})$. The static expression (11) becomes

$$\begin{aligned}\nabla_j \dot u_k &= -\eta u_j u_k + \frac{\dot u_j \dot u_k}{2\eta^2}\dot u^p \nabla_p \eta + \eta \frac{2B}{rB'}\left(g_{jk} + u_j u_k - \frac{\dot u_j \dot u_k}{\eta}\right)\\ &= u_j u_k \left[-\eta + \frac{B'}{2r}\right] + g_{jk}\frac{B'}{2r} + \frac{\dot u_j \dot u_k}{\eta}\left[\frac{1}{2\eta}\dot u^p \nabla_p \eta - \frac{B'}{2r}\right]\end{aligned}$$

With insertions of the scalars (36), the result is obtained. □

We now obtain the covariant expressions of the Ricci, the Weyl, the Cotton and the Bach tensors. They will appear as combinations of the tensors $u_j u_k$, g_{jk} and $\dot u_j \dot u_k$, with coefficients that are scalar functions of r.

Proposition 6.2 *(The Ricci tensor)*

$$\begin{aligned}R_{jk} &= g_{jk}\frac{R(r)}{4} + \left[u_j u_k + \frac{1}{2}g_{jk} - \frac{\dot u_j \dot u_k}{\eta}\right]A(r)\\ A(r) &= \frac{1-B}{r^2} + \frac{B''}{2}\end{aligned} \tag{38}$$

Proof Insert the expression for $Y_k y_l + Z_k z_l$ evaluated in Prop. 6.1 in Eq. (13):

$$\begin{aligned}R_{kl} &= u_k u_l \left[\tfrac{R}{3} + 2\nabla_p \dot u^p\right] + g_{kl}\left[\tfrac{R}{3} + \nabla_p \dot u^p\right] - 2\dot u_k \dot u_l - 2E_{kl}\\ &\quad - \tfrac{\dot u_k \dot u_l}{\eta}\tfrac{\dot u^p \nabla_p \eta}{\eta} - 2\tfrac{B'^2}{4B}\tfrac{2B}{rB'}(g_{kl} + u_k u_l - \tfrac{\dot u_k \dot u_l}{\eta})\\ &= u_k u_l \left[\tfrac{R}{3} + 2\nabla_p \dot u^p - \tfrac{B'}{r}\right] + g_{kl}\left[\tfrac{R}{3} + \nabla_p \dot u^p - \tfrac{B'}{r}\right]\\ &\quad - \tfrac{\dot u_k \dot u_l}{\eta}\left[2\eta + \tfrac{\dot u^l \nabla_l \eta}{\eta} - \tfrac{B'}{r}\right] - 2E_{kl}\end{aligned}$$

The spherical scalars η, $\dot u^p \nabla_p \eta$ and $\nabla_p \dot u^p$ are given in Eq. (36). The electric tensor and the curvature scalar are obtained from equations (86) and (90) in "Appendix 1":

$$\begin{aligned}E_{kl} &= E(r)\left[\frac{\dot u_k \dot u_l}{\eta} - \frac{u_k u_l + g_{kl}}{3}\right]\\ E(r) &= \frac{1}{2}\left[\frac{1-B}{r^2} + \frac{B'}{r} - \frac{B''}{2}\right]\end{aligned} \tag{39}$$

$$R(r) = 2\frac{1-B}{r^2} - 4\frac{B'}{r} - B'' \tag{40}$$

The Ricci tensor is then written as a sum with a trace-less term. □

Proposition 6.3 *(*The Weyl tensor) *With the expression of the electric tensor, the static Weyl tensor* (7) *becomes*:

$$C_{jklm} = E(r)\Big[(g_{kl} + 2u_k u_l)\frac{\dot u_j \dot u_m}{\eta} - (g_{jl} + 2u_j u_l)\frac{\dot u_k \dot u_m}{\eta} + (g_{jm} + 2u_j u_m)\frac{\dot u_k \dot u_l}{\eta} \\ - (g_{km} + 2u_k u_m)\frac{\dot u_j \dot u_l}{\eta} - u_k u_l g_{jm} + u_j u_l g_{km} - u_j u_m g_{kl} + u_k u_m g_{jl} \\ - \frac{2}{3}(g_{kl} g_{jm} - g_{jl} g_{km})\Big]. \tag{41}$$

The Riemann tensor can then be obtained.

Proposition 6.4 The Cotton tensor (Cotton [17]) *The Cotton tensor* $C_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} - \frac{1}{6}(g_{kl}\nabla_j R - g_{jl}\nabla_k R)$ *is proportional to* $\nabla_m C_{jkl}{}^m$. *Here it is*

$$C_{jkl} = \frac{B'}{2\eta}\left(A' + \frac{A}{r}\right)\left[\dot u_j (u_k u_l + \frac{1}{3} g_{kl}) - (u_j u_l + \frac{1}{3} g_{jl})\dot u_k\right]. \tag{42}$$

Proof The evaluation is rather long. Let us specify some building steps. With (37) we obtain:

$$\nabla_j \frac{\dot u_k \dot u_l}{\eta} = -u_j(u_k \dot u_l + u_l \dot u_k) + \frac{B'}{2r\eta}\Big(h_{jk}\dot u_l + h_{jl}\dot u_k - \frac{2}{\eta}\dot u_j \dot u_k \dot u_l\Big) \tag{43}$$

Next, with the spherical static Ricci tensor (38):

$$\nabla_j (R_{kl} - \frac{R}{6} g_{kl}) = -A(r)\frac{B'}{2r\eta}(h_{jk}\dot u_l + h_{jl}\dot u_k - \frac{2}{\eta}\dot u_j \dot u_k \dot u_l) \\ + \frac{B'}{2\eta}\dot u_j \left[\left(u_k u_l + \frac{1}{2} g_{kl} - \frac{\dot u_k \dot u_l}{\eta}\right) A' + g_{kl}\frac{R'(r)}{12}\right].$$

The subtraction with j, k exchanged gives

$$C_{jkl} = \frac{B'}{2\eta}\left[(\dot u_j u_k u_l - \dot u_k u_j u_l)\left(A' + \frac{A}{r}\right) + (\dot u_j g_{kl} - \dot u_k g_{jl})\left(\frac{A'}{2} + \frac{R'}{12} + \frac{A}{r}\right)\right].$$

With the identity $R' = -8\frac{A}{r} - 2A'$ the Cotton tensor gains the useful form (42). □

Proposition 6.5 The Bach tensor (Bach [3]) *With the Weyl tensor* $C_{jkl}{}^m$, *the Bach tensor is the only algebraically independent one that is invariant for a conformal transformation* $g'_{ij}(x) = e^{2\phi(x)} g_{ij}(x)$ *in* $n = 4$ [56]*:*

$$\mathscr{B}_{kl} = 2\nabla^j \nabla^m C_{jklm} + R^{jm} C_{jklm} = -\nabla^j C_{jkl} + R^{jm} C_{jklm}$$

where C_{jkl} is the Cotton tensor. It is symmetric, traceless and divergence-free. In the metric (34) we find:

$$\mathscr{B}_{kl} = B_1 u_k u_l + \frac{1}{4}(B_1 - B_2) g_{kl} + B_2 \frac{\dot{u}_k \dot{u}_l}{\eta} \tag{44}$$

$$B_1 + B_2 = -\frac{2B}{r}\left(A' + \frac{A}{r}\right) - \frac{2B}{3}\left(A' + \frac{A}{r}\right)' \tag{45}$$

$$B_1 - B_2 = -\frac{8}{3} A E - \frac{4}{3}\left(A' + \frac{A}{r}\right)\left(B' + \frac{B}{r}\right) - \frac{4B}{3}\left(A' + \frac{A}{r}\right)' \tag{46}$$

Proof With Eq. (7) and using $E_{kl} u^l = 0$, it is:

$$R^{jm} C_{jklm} = A(r)\left[-(g_{kl} + 2u_k u_l) E_{jm} \frac{\dot{u}^j \dot{u}^m}{\eta} + \frac{\dot{u}_l E_{km} \dot{u}^m}{\eta} + \frac{\dot{u}_k E_{jl} \dot{u}^j}{\eta} \right]$$

Now use $E_{jk} \dot{u}^k = \frac{2}{3} E(r) \dot{u}_l$. Then:

$$R^{jm} C_{jklm} = -\frac{4}{3} A(r) E(r) \left[u_k u_l + \frac{1}{2} g_{kl} - \frac{\dot{u}_k \dot{u}_l}{\eta} \right] \tag{47}$$

The divergence of the Cotton tensor is

$$\begin{aligned}
\nabla^j C_{jkl} &= \frac{B'}{2\eta} \frac{d}{dr}\left[\frac{B'}{2\eta}\left(A' + \frac{A}{r}\right)\right] \dot{u}^j \left[\dot{u}_j \left(u_k u_l + \frac{1}{3} g_{kl}\right) - \left(u_j u_l + \frac{1}{3} g_{jl}\right) \dot{u}_k \right] \\
&\quad + \frac{B'}{2\eta}\left(A' + \frac{A}{r}\right) \nabla^j \left[\dot{u}_j \left(u_k u_l + \frac{1}{3} g_{kl}\right) - \left(u_j u_l + \frac{1}{3} g_{jl}\right) \dot{u}_k \right] \\
&= \frac{1}{3}\left(A' + \frac{A}{r}\right)\left[2B'\left(u_k u_l + \frac{1}{2} g_{kl} - \frac{\dot{u}_k \dot{u}_l}{\eta}\right) + \frac{B}{r}\left(5 u_k u_l + g_{kl} + \frac{\dot{u}_k \dot{u}_l}{\eta}\right)\right] \\
&\quad + B\left(A' + \frac{A}{r}\right)' \left[u_k u_l + \frac{1}{3} g_{kl} - \frac{1}{3} \frac{\dot{u}_k \dot{u}_l}{\eta} \right].
\end{aligned} \tag{48}$$

It turns out that $\mathscr{B}_{kl}$ is a traceless linear combination of $u_k u_l$, g_{kl} and $\dot{u}_k \dot{u}_l$. The expressions $B_1 \pm B_2$ result from the scalars $\mathscr{B}_{jk} u^j u^k$ and $\mathscr{B}_{jk} \dot{u}^j \dot{u}^k$ evaluated with (47) and (48). □

Now we pin down some space-times that solve special geometric constraints.

Proposition 6.6 *A spherically symmetric static space-time*
(a) has zero scalar curvature *($R = 0$) if $2\frac{1-B}{r^2} - 4\frac{B'}{r} - B'' = 0$ i.e.*

$$B(r) = \frac{b_{-2}}{r^2} + \frac{b_{-1}}{r} + 1 \tag{49}$$

The Ricci tensor (38) *is traceless, with* $A(r) = 2b_{-2}/r^4$. *(b) is* conformally flat ($C_{jklm} = 0$) *if* $E(r) = 0$ *i.e.*

$$B(r) = 1 + b_1 r + b_2 r^2 \tag{50}$$

(c) is harmonic ($\nabla^m C_{jklm} = 0$) *if*

$$B(r) = \frac{b_{-1}}{r} + 1 + b_1 r + b_2 r^2 \tag{51}$$

Proof: Equation (42) *gives* $A' + A/r = 0$ *i.e.* $A = -b_1/r$. *Then:* $B'' + 2\frac{1-B}{r^2} + \frac{2b_1}{r} = 0$, *with the above solution.* □
(d) is bi − harmonic ($\nabla^j C_{jkl} = 0$) *if it is harmonic* (51), *or if*

$$B(r) = \kappa r^2 \tag{52}$$

with arbitrary constant (note that it is not a special case of harmonic).
Proof: The expression (48) *for* $\nabla^j C_{jkl}$ *is zero if the components* g_{kl} *and* $\dot{u}_k \dot{u}_l$ *vanish (the component* $u_k u_l$ *vanishes because of the trace condition). The difference gives:*

$$\left(A' + \frac{A}{r}\right)\left[-\frac{B'}{3} + \frac{2}{3}\frac{B}{r}\right] = 0$$

The first factor vanishes for harmonic space-times, the other gives $B = \kappa r^2$, *that also solves the other constraint and sets* $A(r) = 1/r^2$, $E(r) = 1/(2r^2)$. □
(e) is Einstein *if* $A(r) = 0$ *i.e.*

$$B(r) = \frac{b_{-1}}{r} + 1 + b_2 r^2 \tag{53}$$

(f) is Constant Curvature *if* $A(r) = 0$ *and* $C_{jklm} = 0$.

$$B(r) = 1 + b_2 r^2 \tag{54}$$

The Riemann tensor has the form $R_{jklm} = \frac{R}{12}(g_{jl}g_{km} - g_{jm}g_{kl})$ *with* $R = -12b_2$.
(g) has zero Bach tensor ($\mathscr{B}_{kl} = 0$) *if*

$$B(r) = -\frac{\beta(2 - 3b_1\beta)}{r} + (1 - 3b_1\beta) + b_1 r + b_2 r^2 \tag{55}$$

Proof: the Bach tensor (44) *is zero if* $B_1 \pm B_2 = 0$. *With* $B \neq 0$ *the sum gives* $3\left(A' + \frac{A}{r}\right) + r\left(A' + \frac{A}{r}\right)' = 0$ *i.e* $[r^2(Ar)']' = 0$, $A(r) = \frac{a_{-2}}{r^2} + \frac{a_{-1}}{r}$. *Then* $B(r) = \frac{b_{-1}}{r} + b_0 + b_1 r + b_2 r^2$ *with* $1 - b_0 = a_{-2}$ *and* $b_1 = -a_{-1}$.

This expression in Eq. (46) *gives:* $a_{-2}(1 + b_0) = 3a_{-1}b_{-1}$ *and* $a_{-2}b_1 = -a_{-1}(1 - b_0)$. *The first one is the constraint* $1 - b_0^2 = -3b_1 b_{-1}$, *the other equation is trivial. A possible parameterization of* $B(r)$ *is* (55). □

Proposition 6.7 (The Faraday tensor) *In a static spherical-symmetric space-time the magnetic coefficient* $\mathbb{B}(r)$ *of the Faraday tensor* (15) *is*

$$\mathbb{B}(r) = \frac{q_m}{r^2} \tag{56}$$

where q_m *is a magnetic charge. The current is time-like and independent of* $\mathbb{B}$*:*

$$J^k = \left(-\eta\varkappa + \mathbb{E}\sqrt{\eta} - \frac{\mathbb{E}}{\sqrt{\eta}}\nabla_j \dot{u}^j\right) u^k \tag{57}$$

Proof The Eq. (17) for $\mathbb{E}/\sqrt{\eta}$ is satisfied by any function of r. The Eq. (18) for $\mathbb{B}$ becomes

$$\frac{1}{2}\mathbb{B}' = -\mathbb{B}\frac{1}{r}.$$

with solution (56). The expression of the current (19) simplifies as directional derivatives other than $\dot{u}^k\nabla_k$ are zero for scalars that only depend on r. □

The geometric cases presented in Prop. 6.6 correspond to well known static spherically symmetric solutions of gravitational theories.
We consider the Einstein, the Cotton, the $f(R)$ and the Conformal Gravity theories.

7 Static solutions in Einstein gravity

The imperfect fluid cannot be perfect. Early solutions. Properties of LE and NLE

The Einstein tensor $G_{jk} = R_{jk} - \frac{R}{2}g_{jk}$ for the static spherical metric (34) is

$$G_{jk} = \frac{2}{3}A(r)u_j u_k + \left[\frac{B''}{3} + \frac{B'}{r} - \frac{1-B}{3r^2}\right]g_{jk} - A(r)\left[\frac{\dot{u}_j\dot{u}_k}{\eta} - \frac{u_j u_k + g_{jk}}{3}\right]. \tag{58}$$

Its tensor structure and the Einstein equation $G_{jk} = T_{jk}$ dictate that of the energy–momentum density T_{jk}. In the picture of a fluid it is:

$$T_{jk} = (p+\mu)u_j u_k + p g_{jk} + (p_r - p_\perp)\left[\frac{\dot{u}_j\dot{u}_k}{\eta} - \frac{u_j u_k + g_{jk}}{3}\right] \tag{59}$$

The structure of the stress tensor is fully specified. The energy density μ, the effective pressure $p = \frac{1}{3}(p_r + 2p_\perp)$, the radial and transverse pressures p_r, $p_\perp$ are functions r:

$$\mu = -p_r = \frac{1-B}{r^2} - \frac{B'}{r}, \quad p_\perp = \frac{B''}{2} + \frac{B'}{r} \tag{60}$$

Pressure isotropy ($p_r = p_\perp$) imposes $A(r) = \frac{B''}{2} + \frac{1-B}{r^2} = 0$, i.e. the space-time is Einstein with fluid equation of state $\mu = -p$, and

$$B(r) = 1 + \frac{b_{-1}}{r} + b_2 r^2 \tag{61}$$

The field equation with a dust source, $G_{jk} = \mu u_j u_k$ does not admit a static solution (34) (the source term must contain a pressure anisotropy to compensate the equality).

Remark 7.1 There has been a discussion whether a static spacetime with spherical metric (34) may host a perfect fluid. Faraoni et al. [21] and Visser [57] showed the inconsistency of the Kiselev metric with a perfect fluid source. A definite negative answer has been given by Lake and Bisson [6, 35]. Here, again, we have shown that it does not occur unless $p = -\mu$.

For the static anisotropic fluid tensor (59), the equation $\nabla_j T^j{}_k = 0$ is:

$$\begin{aligned}0 &= (p+\mu)\dot u_k + \nabla_k p + (p_r - p_\perp)\left[\nabla_j \frac{\dot u_j \dot u_k}{\eta} - \frac{\dot u_k}{3}\right] + \left[\frac{\dot u_j \dot u_k}{\eta} - \frac{g_{jk}}{3}\right]\nabla_j (p_r - p_\perp)\\ &= (\mu + p_\perp)\dot u_k + \nabla_k p_\perp + (p_r - p_\perp)\nabla_j \frac{\dot u^j \dot u_k}{\eta} + \frac{\dot u_k}{\eta}\dot u^j \nabla_j (p_r - p_\perp)\end{aligned}$$

A gradient is evaluated in (43): $\nabla_j \frac{\dot u^j \dot u_k}{\eta} = \dot u_k + \frac{B'}{r\eta}\dot u_k$. In spherical symmetry: $\nabla_k p_\perp = \frac{1}{\eta}\dot u_k \dot u^j \nabla_j p_\perp$. The derivative of the radial pressure is obtained:

$$0 = p_r' + \frac{B'}{2B}(\mu + p_r) + \frac{2}{r}(p_r - p_\perp) \tag{62}$$

7.1 Some static solutions of the Einstein equations

Simple conditions provide the classical static solutions

• *Schwarzschild* space-time.

$R_{jk} = 0$ gives the famous vacuum spherical solution (61), with $b_2 = 0$:

$$B(r) = 1 - \frac{2M}{r}$$

• *Schwarzschild - de Sitter* (SdS) space-time. The equation $G_{jk} + \Lambda g_{jk} = 0$ is solved by (61) with free parameter b_{-1}. The coefficient of g_{jk} fixes $b_2 = -\frac{1}{3}\Lambda$.

• *Reissner-Nordstrøm* space-time [19].

If $R = 0$ the Einstein equation $R_{kl} = T_{kl}$ implies the energy–momentum tensor $T_{jk} = \frac{b_{-2}}{r^4}\left[u_j u_k - \frac{1}{2}g_{jk} - \frac{\dot u_j \dot u_k}{\eta}\right]$. In comoving coordinates the non-zero Faraday component

$$F_{tr} = \frac{\mathbb{E}}{\sqrt{\eta}} u_0 \dot u_r = \mathbb{E}\big(-\sqrt{B}\big)\frac{1}{2}\frac{B'}{B} = -\frac{\sqrt{b_{-2}}}{r^2}$$

corresponds to the radial electric field of a point charge $q_e = \sqrt{b_{-2}}$.
The metric function is (Reissner 1916, Nordström 1913):

$$B(r) = 1 - \frac{2M}{r} + \frac{q_e^2}{r^2} \tag{63}$$

• *Reissner-Nordstrøm-(anti) de Sitter* space-time [36].
It is a variant of the previous metric where a cosmological term $-\Lambda g_{jk}$ is added to the traceless T^{em}_{jk}. The scale function is:

$$B(r) = 1 - \frac{2M}{r} + \frac{q_e^2}{r^2} - \frac{1}{3}\Lambda r^2$$

7.2 Linear and non-linear electrodynamics in Einstein gravity

While $\mathbb{B}(r)$ is fixed and equal to q_m/r^2 by the Faraday conditon in spherical symmetry, $\mathbb{E}(r)$ is model dependent. In linear ($\mathscr{L}_F = 1$) or non-linear electrodynamics:

Proposition 7.2 $\mathbb{E}(r)$ *solves the implicit equation (see* [10, 24]*)*

$$\mathbb{E}(r) = \frac{q_e}{r^2 \mathscr{L}_F(F)} \tag{64}$$

Proof In spherical symmetry the Eq. (28) for $\mathbb{E}$ is

$$\frac{B'}{2}\frac{d}{dr}\log\left[\frac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F\right] = \frac{B'^2}{4B} - \frac{B''}{2} - \frac{B'}{r}$$

Then $\frac{d}{dr}\log[\frac{\mathbb{E}}{\sqrt{\eta}}\mathscr{L}_F] = \frac{d}{dr}\log\frac{\sqrt{B}}{B' r^2}$. The integration yields a constant q_e. □

In linear electrodynamics Eq. (64) is solved by a Coulomb field

$$\mathbb{E}^{lin}(r) = \frac{q_e}{r^2} \tag{65}$$

and the electromagnetic energy–momentum density tensor is

$$T^{lin}_{jk} = 2\,\frac{q_e^2 + q_m^2}{r^4}\left[u_j u_k + \frac{1}{2}g_{jk} - \frac{\dot{u}_j \dot{u}_k}{\eta}\right]. \tag{66}$$

This result recovers a generalization of Birkhoff's theorem, stating that a spherical symmetric solution of the Einstein–Maxwell equations is necessarily a piece of the Reissner–Nordstrøm geometry with monopole charges (see [47], p. 844).

The expression T^{nlin}_{jk} in the Einstein equation (21) with the Einstein tensor (58) gives Eq. (26) and

$$2(\mathbb{B}^2 + \mathbb{E}^2)\mathscr{L}_F(F) = A(r) \tag{67}$$

The case $R = 0$ i.e. $\mathscr{L}_F = 1$ (linear electrodynamics) is the Reissner-Nördstrom solution with "dyonic charge", i.e. $b_{-2} = q_m^2 + q_e^2$.

Equation (67) has been exploited to infer the Lagrangian $\mathscr{L}$ from the metric function $B(r)$ (through $A(r)$), or the opposite, in two situations: $\mathbb{E} = 0$ or $\mathbb{B} = 0$. The feasibility of the correspondence has been investigated by Bronnikov [11].

• Purely magnetic ($\mathbb{E} = 0$). Then $4F\mathscr{L}_F(F) = A(r)$ with $F = q_m^2/(2r^4)$.
Ayón-Beato and Garcia [1] started with the metric of the Bardeen black-hole, and deduced the Lagrangian:

$$B(r) = 1 - \frac{2mr^2}{(r^2 + q_m^2)^{3/2}} \quad \to \quad \mathscr{L}(F) = \frac{3m}{q_m^3}\left[\frac{\sqrt{2q_m^2 F}}{1 + \sqrt{2q_m^2 F}}\right]^{5/2}$$

Kruglov [34] obtained the Lagrangian of the Hayward black-hole [27]:

$$B(r) = 1 - \frac{2mr^2}{r^3 + q_m^3} \quad \to \quad \mathscr{L}(F) = \frac{3}{2^{3/4}} \frac{(2q_m^2 F)^{3/2}}{(1 + (2q_m^2 F)^{3/4})^2}$$

• Purely electric ($q_m = 0$). Then $A(r) = -4F\mathscr{L}_F(F)$ with $F = -\mathbb{E}^2/2$.
With the aid of Eq. (64) Halilsoy et al. [24] obtained the metric from the Lagrangian:

$$\mathscr{L}(F) = \frac{a}{b\sqrt{2} - \sqrt{-4F}} \quad \to \quad A(r) = \frac{2bq_e}{r^2} - \sqrt{\frac{|a|q_e}{\sqrt{2}}\frac{2q_e}{r}}$$

The function $A(r)$ has the same form as the Mannheim–Kazanas solution (55) Prop. 6.6. The same metric function $B(r)$ is also a vacuum solution of Conformal Gravity. An interesting application has been the forecast of the shadow of the black hole in M87 [33].

8 Linear and non-linear electrodynamics in f(R) gravity

The equations and the charged solution by Hollenstein and Lobo

$f(R)$ gravity is an extension of Einstein gravity, where a function $f(R)$ replaces R in the Einstein-Hilbert action. The equations in spherical symmetry are studied by Capozziello et al. in [12]. With coupling to non-linear electrodynamics, the equations of motion, with $f_R = df/dR$ are [29]:

$$R_{jk} f_R(R) - \frac{1}{2} g_{jk} f(R) + [g_{jk}\Box - \nabla_j \nabla_k] f_R(R) = T_{jk}^{nlin} \tag{68}$$

$$\nabla_j (F^{jk} \mathscr{L}_F(F)) = 0 \tag{69}$$

The second one is the same as in the Einstein theory. The equations are studied in the static metric (34). For any function $g(r)$: $\nabla_k g = \dot u_k \frac{2B}{B'} g'$ (Eq. (35)), and

$$\begin{aligned}\nabla_j\nabla_k g &= (\nabla_j \dot u_k)\Big(\frac{2B}{B'}g'\Big) + \dot u_j \dot u_k \frac{2B}{B'}\frac{d}{dr}\Big(\frac{2B}{B'}g'\Big)\\ &= u_j u_k\left[\frac{B}{r}-\frac{B'}{2}\right]g' + g_{jk}\frac{B}{r}g' + \frac{\dot u_j \dot u_k}{\eta}\left[Bg''+\frac{1}{2}B'g' - \frac{Bg'}{r}\right]\end{aligned}$$

where $\nabla_j \dot u_k$ is (37). In particular: $\Box g = 2\frac{B}{r}g' + B'g' + Bg''$.
Given the expressions of the Ricci tensor (38) and of T^{nlin}_{jk} (23), the first field equation corresponds to three scalar equations:

$$-\frac{1}{2}f + f_R\left[\frac{R}{4}+\frac{1}{2}A(r)\right] + \Box f_R - \frac{B}{r}f_R' = 2[\mathbb{B}^2\mathscr{L}_F - \mathscr{L}] \tag{70}$$

$$f_R A(r) + \left[\frac{B'}{2}-\frac{B}{r}\right]f_R' = 2(\mathbb{E}^2+\mathbb{B}^2)\mathscr{L}_F \tag{71}$$

$$f_R A(r) + \left[\frac{B'}{2}-\frac{B}{r}\right]f_R' + Bf_R'' = 2(\mathbb{E}^2+\mathbb{B}^2)\mathscr{L}_F \tag{72}$$

The difference of equations (71) and (72) is $f_R'' = 0$. Thus, we reobtain a simple general result by Hollenstein and Lobo [29]:

Proposition 8.1 *In $f(R)$-nonlinear electrodynamics with static metric (34), it is $f_R(R(r)) = cr + d$, where c and d are constants.*

The case $c = 0$, $d = 1$ is Einstein's gravity ($f = R$).
The result greatly simplifies Eqs. (70) and (71). With $\frac{1}{4}R + \frac{1}{2}A = \frac{1-B}{r^2} - \frac{B'}{r}$ and $\Box f_R = 2\frac{B}{r}c + B'c$, they become:

$$\frac{1}{2}f(R) = (cr+d)\left[\frac{1-B}{r^2}-\frac{B'}{r}\right] + c\left[\frac{B}{r}+B'\right] - 2[\mathbb{B}^2\mathscr{L}_F - \mathscr{L}] \tag{73}$$

$$(cr+d)\left[\frac{1-B}{r^2}+\frac{B''}{2}\right] + c\left[\frac{B'}{2}-\frac{B}{r}\right] = 2(\mathbb{E}^2+\mathbb{B}^2)\mathscr{L}_F \tag{74}$$

The spherical symmetry always forces $\mathbb{B} = q_m/r^2$. To go further, we consider linear electrodynamics $\mathscr{L}_F = 1$, $\mathbb{E} = q_e/r^2$. Equation (74) can now be solved and is Eq. 34 in [29] (since it is without explanation, we offer a derivation in "Appendix 3").

$$\begin{aligned}B(r) = 1 + \frac{cK}{2d^2} - \frac{K}{3dr} - \left[1 + \frac{cK}{d^2} + 4\frac{q_e^2+q_m^2}{d}(\frac{c}{d})^2\right]\left[\frac{c}{d}r - (\frac{c}{d})^2 r^2 \log\frac{cr+d}{r}\right]\\ + \frac{q_e^2+q_m^2}{d}\left[\frac{1}{r^2} - (\frac{c}{d})\frac{4}{3r} + 2(\frac{c}{d})^2\right] + K_0 r^2.\end{aligned} \tag{75}$$

The solution $B(r)$ has to produce in (73) a function $f(R)$, and be compatible with $f_R(R) = cr + d$.

The thermodynamics of a $f(R) = R - 2\alpha\sqrt{R}$ black hole with metric (34) are studied in [48], in power law electrodynamics.

9 Static solutions in Cotton gravity

Cotton gravity is Einstein gravity with a free Codazzi tensor.
Two new solutions: perfect fluid and LE.

Cotton gravity was introduced by Harada [25], as an extension of Einstein's gravity. In the Harada equation, the Einstein tensor is replaced by the Cotton tensor, and the energy–momentum tensor is replaced by gradients of it:

$$C_{jkl} = \nabla_j T_{kl} - \nabla_k T_{jl} - \frac{1}{3}(g_{kl}\nabla_j T - g_{jl}\nabla_k T) \tag{76}$$

where $T = T^k{}_k$. As we showed in [44] the Harada equation is equivalent to the Einstein equation with an energy momentum modified by an arbitrary Codazzi tensor

$$\begin{aligned} &R_{jk} - \tfrac{1}{2}g_{jk}R = T_{jk} + \mathscr{C}_{jk} - g_{jk}\mathscr{C}^k{}_k \\ &\nabla_i \mathscr{C}_{jk} = \nabla_j \mathscr{C}_{ik} \end{aligned} \tag{77}$$

9.1 The Harada solution

Harada found a static spherical solution of his equation with $C_{jkl} = 0$. It is (51) in Prop. 6.6:

$$B(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \gamma r \tag{78}$$

It generalizes the Schwarzschild solution by a cosmological term, and corresponds to solving the Einstein equation with the energy momentum

$$T_{jk} = -\Lambda g_{jk} + \frac{\gamma}{r}\left[-\frac{2}{3}u_j u_k + \frac{4}{3}g_{jk} + \left(\frac{\dot{u}_j\dot{u}_k}{\eta} - \frac{u_j u_k + g_{jk}}{3}\right)\right]$$

Therefore, the Harada vacuum solution is a solution of the Einstein equation for an exotic anisotropic fluid, with velocity u_k, energy density $\mu = -p_r = -\frac{2\gamma}{r} + \Lambda$ and transverse pressure $p_\perp = \frac{\gamma}{r} - \Lambda$.
The same function $B(r)$ appears as solution of a model for gravity at large distances studied by Grumiller [23], with an analogous energy–momentum tensor.

Harada numerically solved the equations for Cotton-gravity to describe the rotation curves of galaxies [26], where a linear term γr provides the observed gravitational potential without the need of dark matter.

9.2 Perfect fluid solution

While in Einstein gravity there are no perfect fluid solutions with the static spherical metric (34), this is no longer true in Cotton gravity because of the freedom of choosing the Codazzi tensor.
The following one, with constants K and κ,

$$\mathscr{C}_{jk} = \frac{K}{\sqrt{B(r)}} u_j u_k + \kappa g_{jk} \tag{79}$$

is a Codazzi tensor in the metric (34) (see [44]). By choosing $B(r) = \frac{b_{-1}}{r} + 1 + b_2 r^2$ (as (53), i.e. $A(r) = 0$), the Ricci tensor is Einstein, $R_{jk} = -3b_2 g_{jk}$. The energy–momentum tensor

$$\begin{aligned} T_{jk} &= R_{jk} - \tfrac{1}{2} R g_{jk} - \mathscr{C}_{jk} + g_{jk} \mathscr{C}^k{}_k \\ &= -\tfrac{K}{\sqrt{B(r)}} (u_j u_k + g_{jk}) - \left[\tfrac{R}{4} + 4\kappa \right] g_{jk} \end{aligned}$$

is perfect fluid, and the Harada equation (76) is solved by the metric (34) with the function $B(r)$. The perfect fluid has constant energy density $\mu = 4\kappa + R/4$, while $p + \mu = -\frac{K}{\sqrt{B(r)}}$ is a function of r because of the Codazzi term.

- If $\mathscr{C}_{jk} = 0$ we recover GR with the cosmological law $p = -\mu$.
- If $K = 0$, $\kappa = -\frac{R}{16}$ we get the empty solution $p = \mu = 0$. Thus in Cotton gravity the same metric (SdS or SadS) is compatible with different energy–momentum tensors.
- The electric function is $E(r) = -\frac{3}{2}\frac{b_{-1}}{r^3}$. If $b_{-1} = 0$ then $C_{jklm} = 0$. $B(r) = 1 + b_2 r^2$ gives the metric of a constant curvature space-time $R_{jklm} = \frac{1}{12} R(g_{jl} g_{km} - g_{jm} g_{kl})$ in presence of a perfect fluid in Cotton gravity.

Remark 9.1 Apparently, the statement that (79) is a Codazzi tensor in a constant curvature space-time comes at odds with the theorem by Ferus [22] stating that the only Codazzi tensors is such spacetimes are $\nabla_j \nabla_k \varphi + \frac{1}{12} R \varphi g_{jk}$, where φ is an arbitrary scalar field. Actually, it can be shown that the tensor is in this class with $\varphi(r) = K\sqrt{B(r)}$. The term κg_{jk} is the trivial Codazzi tensor.

9.3 Linear electrodynamics

We obtain a new solution for Cotton gravity in presence of the linear tensor of electrodynamics.

Proposition 9.2 *The metric function $B(r)$ solving the Cotton gravity equation in linear electrodynamics is:*

$$B(r) = \left[1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 + \gamma r \right] + \frac{q_e^2 + q_m^2}{r^2} \tag{80}$$

It is the sum of the solution of $C_{jkl} = 0$ (in square brackets) and a dyonic charge term.

Proof The traceless energy–momentum tensor T^{lin}_{jk} in Eq. (24), is entered in the Cotton gravity equation: $C_{jkl} = \nabla_j T^{lin}_{kl} - \nabla_k T^{lin}_{jl}$:

$$C_{jkl} = \left[K' + \frac{K}{r}\right](u_k\dot u_j - u_j\dot u_k)u_l + \left[\frac{K'}{2} + \frac{K}{r}\right](\dot u_j g_{kl} - \dot u_k g_{jl}) \tag{81}$$

where for brevity $K = 2[\frac{q_m^2}{r^4} + \mathbb{E}^2(r)]$. The static spherical Cotton tensor is (42). The contraction with g^{kl} gives $0 = \frac{K'}{2} + \frac{2K}{r}$, with solution

$$\mathbb{E}(r) = \frac{q_e}{r^2}$$

The contraction with $u^k u^l$ is $A' + \frac{A}{r} = \frac{3}{4}K'$ with solution $A(r) = -\frac{\gamma}{r} + 2\frac{q_e^2+q_m^2}{r^4}$, with a constant γ. The corresponding metric function is obtained. □

10 Static solutions in conformal gravity

The field equations, the Mannheim-Kazanas and LE solutions.

The action of conformal gravity is $S = -\alpha_G \int d^4x\sqrt{(-g)}C_{jklm}C^{jklm} + S_{matter}$ In $n = 4$ the Weyl term, that accounts for geometry, is invariant for the conformal transformation[1] $g'_{jk}(x) = e^{2\phi(x)}g_{jk}(x)$.
The variation in the metric tensor, neglecting boundary terms, is:

$$\delta S = 2\alpha_G \int d^4x\sqrt{(-g)}\mathscr{B}_{kl}\,\delta g^{kl} - \frac{1}{2}\int d^4x\sqrt{(-g)}\,T_{kl}\,\delta g^{kl}$$

where $\mathscr{B}_{kl} = 2\nabla^j\nabla^m C_{jklm} + R^{jm}C_{jklm} = -\nabla^j C_{jkl} + R^{jm}C_{jklm}$ is the Bach tensor and T_{kl} is the energy–momentum density tensor. The field equation of Conformal gravity is:

$$4\alpha_G\,\mathscr{B}_{kl} = T_{kl} \tag{82}$$

The property $\nabla_j T^j{}_k = 0$ is mantained by the identity $\nabla_j \mathscr{B}^j{}_k = 0$.

Equation (44) for the static spherical Bach tensor fixes the form of the energy–momentum tensor as an anisotropic fluid (59):

$$\begin{aligned}4\alpha_G\left[B_1 u_j u_k + \frac{B_1 - B_2}{4}g_{jk} + B_2\frac{\dot u_j \dot u_k}{\eta}\right]\\ = (\mu + p_\perp)u_j u_k + p_\perp g_{jk} + (p_r - p_\perp)\frac{\dot u_j\dot u_k}{\eta}\end{aligned}$$

with $\mu = \alpha_G(3B_1 + B_2)$, $p_r = \alpha_G(B_1 + 3B_2)$ and $p_\perp = \alpha_G(B_1 - B_2)$.

[1] It is $C'_{jklm} = e^{2\phi}C_{jklm}$, $C'^{jklm} = e^{-6\phi}C_{jklm}$ and $\sqrt{-g'} = e^{4\phi}\sqrt{-g}$.

Since the Bach tensor is traceless, it is $T^k{}_k = 0$, i.e. in static conformal gravity the fluid always satisfies

$$p_r + 2p_\perp = \mu \tag{83}$$

The continuity equation for the energy momentum is Eq. (62).

Let us view some special cases:

10.1 Vacuum solution

Mannheim and Kazanas [39] obtained the vacuum spherical static solution for conformal gravity, $\mathscr{B}_{jk} = 0$. It is the metric function $B(r)$ in Eq. (55) Prop. 6.6. The solution arose much interest for the description of the rotation curves of galaxies, where the linear term γr accounts for the plateau without need of dark matter [28, 30, 39, 41]. Constraints on the value of the constant γ were obtained by Sultana et al. [55], using data for perihelion shift.

Bach showed that every static spherically symmetric space-time that is conformally related to the Schwarzschild-de Sitter (SdS) metric solves $\mathscr{B}_{jk} = 0$ [40]. The converse was later proved by Buchdahl (see [28]). In some papers it is actually proven that (55) is conformally equivalent to the SdS metric.

10.2 Perfect fluid

The anisotropic term is zero if $B_2 = 0$, and $p_r = p_\perp = \mu/3$.
The condition on B_2 is a fourth order non-linear differential equation for $B(r)$.

10.3 The anisotropic EoS $\mu = -p_r$

This occurs for $B_1 + B_2 = 0$. The trace-less energy momentum tensor takes the same form of T^{lin}_{jk} of linear electrodynamics, Eq. (24):

$$T_{kl} = \mu\left[u_k u_l + \frac{1}{2} g_{kl} - \frac{\dot u_k \dot u_l}{\eta}\right]$$

For this reason, the case is by far the most studied in the literature. Remarkably, Riegert [51] proved that Birkhoff's theorem holds in conformal gravity and implies that a spherical symmetric solution of the Bach-Maxwell equations is necessarily static, with $B(r)$ given below.

Equation (45) gives: $0 = \frac{1}{r}\left(A' + \frac{A}{r}\right) + \frac{1}{3}\left(A' + \frac{A}{r}\right)'$, with solution $A(r) = \frac{1}{r^2}(1 - b_0) - \frac{1}{r} b_1$. The equation has solution

$$B(r) = \frac{b_{-1}}{r} + b_0 + b_1 r + b_2 r^2$$

It follows that $E(r) = \frac{1}{2r^2}(1-b_0^2) - \frac{3}{2}\frac{b_{-1}}{r^3}$. This, in Eq. (46) gives $2B_1 = -\frac{4}{3r^4}[(1-b_0^2) + 3b_1 b_{-1}]$. The energy density is $\mu = p_\perp = -p_r = 2\alpha_G B_1$:

$$\mu(r) = -\alpha_G \frac{8}{3r^4}[(1-b_0^2) + 3b_1 b_{-1}]$$

The dependence r^{-4} agrees with the monopole field in linear electrodynamics. In this picture:

$$2(q_e^2 + q_m^2) = -\alpha_G \frac{8}{3}[(1-b_0^2) + 3b_1 b_{-1}]$$

The parameter b_2 is free while $b_{\pm 1}$ and b_0 are constrained. The metric function B can be cast as follows [40]

$$B(r) = -\frac{1}{r}\left[\beta(2-3\beta\gamma) + \frac{q_e^2 + q_m^2}{4\gamma\alpha_G}\right] + (1-3\beta\gamma) + \gamma r - \kappa r^2$$

Let's look at two subcases:

10.3.1 The harmonic solution ($\nabla_m C_{jkl}{}^m = 0$)

The function $B(r)$ by Harada Eq. (78) solves $C_{jkl} = 0$. It implies $A(r) = -\gamma/r$ and $E(r) = 3M/r^3$, i.e. $b_0 = 1$, $b_1 = \gamma$ and $b_{-1} = -2M$. Therefore: the Harada metric (78) solves the field equation of conformal gravity in presence of electric and magnetic monopoles, with charge $q_e^2 + q_m^2 = 8M\gamma\,\alpha_G$.

10.3.2 The bi-harmonic solution ($\nabla_j \nabla_m C^j{}_{kl}{}^m = 0$)

Besides the harmonic solution, the function $B(r) = \kappa r^2$, see (52), cancels the term $\nabla^j C_{jkl}$. With $b_0 = 0$, $b_1 = b_{-1} = 0$, $B(r) = \kappa r^2$ solves the field equation of conformal gravity coupled to monopole charges $q_e^2 + q_m^2 = -\frac{4}{3}\alpha_G$, independent of κ, with $\alpha_G < 0$.

Some equations of state $p_r = p_r(\mu)$ have been numerically studied by Brihaye and Verbin [8].

11 Conclusions

The initial effort of writing tensors with the vectors u_j, $\dot{u}_j$ that define static spacetimes, and two other orthogonal vectors, is rewarded by the simplicity of the study of the field equations in gravitation theories. In spherical symmetry the first two vectors suffice, the others being projected away with entrance of the metric tensor. In the field equations, a geometric tensor equals a matter tensor; the tensor form of the first determines that of the latter, and the equality of the coefficients are scalar field equations.

With this plan we obtain a list of solutions in Einstein, Cotton, $f(R)$ and conformal gravity, with results on the Faraday tensor and (non)linear—electrodynamics, and new solutions in Cotton gravity.

New and old results are here obtained in the natural and simple covariant formalism. This strategy may be applied to other extended theories, as Gauss–Bonnet gravity.

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Appendix 1

We report some useful formulas valid for static spherical space-times. They result from the equations for doubly-warped spherical space-times with $a(t) = 1$ presented in Ref. [43]. In this paper $b^2 = B$, $f_1^2 = 1/B$, $f_2^2 = r^2$ and $n = 4$.

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + f_2^2(r)d\Omega_{n-2}^2 \tag{84}$$

The Ricci tensor is Eq. 49 in [43]. Here $\xi = 0$. It is the sum of a perfect fluid term and a traceless tensor:

$$\begin{aligned} R_{kl} &= \frac{R + n\nabla_p \dot{u}^p}{n-1} u_k u_l + \frac{R + \nabla_p \dot{u}^p}{n-1} g_{kl} + \Sigma(r) \left[\frac{\dot{u}_k \dot{u}_l}{\eta} - \frac{u_k u_l + g_{kl}}{n-1} \right] \\ \Sigma(r) &= \nabla_p \dot{u}^p - (n-1)\left(\eta + \frac{\dot{u}^p \nabla_p \eta}{2\eta} \right) - (n-2)E(r). \end{aligned} \tag{85}$$

While in general u^l is an eigenvector, with spherical symmetry also $\dot{u}^l$ is an eigenvector. The electric tensor (Eqs. 48 and 44 in [43]) is

$$\begin{aligned} E_{kl} &= E(r) \left[\frac{\dot{u}_k \dot{u}_l}{\eta} - \frac{u_k u_l + g_{kl}}{n-1} \right] \\ E(r) &= \frac{n-3}{n-2} \frac{1}{f_1^2} \left[\frac{f_1^2}{f_2^2} + \frac{f_2''}{f_2} - \frac{f_2'^2}{f_2^2} - \frac{f_1' f_2'}{f_1 f_2} + \frac{b' f_1'}{b f_1} + \frac{b' f_2'}{b f_2} - \frac{b''}{b} \right] \end{aligned} \tag{86}$$

$$\nabla_p \dot{u}^p = \frac{1}{b f_1^2} \left[b'' - b' \frac{d}{dr} \log(f_1 f_2) + (n-1) b' \frac{f_2'}{f_2} \right] \tag{87}$$

$$\frac{\dot{u}^p \nabla_p \eta}{2\eta} = \frac{1}{f_1}\frac{d}{dr}\frac{b'}{f_1 b} \tag{88}$$

The scalar $\Sigma(r)$ is evaluated with the aid of Eq. 51 in [43]:

$$\Sigma(r) = -\frac{1}{f_1^2}\left[\frac{b''}{b} - \frac{b' f_1'}{b f_1} - \frac{b' f_2'}{b f_2}\right] - \frac{n-3}{f_1^2}\left[\frac{f_1^2}{f_2^2} + \frac{f_2''}{f_2} - \frac{f_2'^2}{f_2^2} - \frac{f_1' f_2'}{f_1 f_2}\right]. \tag{89}$$

The curvature scalar of space-time and of the space submanifold are:

$$R = R^\star - 2\nabla_p \dot{u}^p \tag{90}$$

$$R^\star = \frac{(n-2)(n-3)}{f_2^2} - \frac{n-2}{f_1^2}\left[2\frac{f_2''}{f_2} - 2\frac{f_1' f_2'}{f_1 f_2} + (n-3)\frac{f_2'^2}{f_2^2}\right] \tag{91}$$

Appendix 2: Proof of Theorem 3.1

With (15) and Lemma 2.4:

$$\nabla_i F_{jk} = \left(\nabla_i \frac{\mathbb{E}}{\sqrt{\eta}}\right)(u_j \dot{u}_k - \dot{u}_j u_k) + (\nabla_i \mathbb{B})(y_j z_k - y_k z_j) + \frac{\mathbb{E}}{\sqrt{\eta}}(u_j \nabla_i \dot{u}_k - u_k \nabla_j \dot{u}_i)$$
$$+ \mathbb{B}(-Y_i \dot{u}_j z_k - Z_i y_j \dot{u}_k + Y_i z_j \dot{u}_k + Z_i \dot{u}_j y_k) \tag{92}$$

The cyclic sum is:

$$\begin{aligned}
&(\nabla_i \frac{\mathbb{E}}{\sqrt{\eta}})(u_j \dot{u}_k - \dot{u}_j u_k) + (\nabla_j \frac{\mathbb{E}}{\sqrt{\eta}})(u_k \dot{u}_i - \dot{u}_k u_i) + (\nabla_k \frac{\mathbb{E}}{\sqrt{\eta}})(u_i \dot{u}_j - \dot{u}_i u_j) \\
&+ (\nabla_i \mathbb{B})(y_j z_k - y_k z_j) + (\nabla_j \mathbb{B})(y_k z_i - y_i z_k) + (\nabla_k \mathbb{B})(y_i z_j - y_j z_i) \\
&+ \frac{\mathbb{E}}{\sqrt{\eta}}(u_j \nabla_i \dot{u}_k - u_k \nabla_i \dot{u}_j + u_k \nabla_j \dot{u}_i - u_i \nabla_j \dot{u}_k + u_i \nabla_k \dot{u}_j - u_j \nabla_k \dot{u}_i) \\
&+ \mathbb{B} Y_i (z_j \dot{u}_k - z_k \dot{u}_j) - \mathbb{B} Z_i (y_j \dot{u}_k - y_k \dot{u}_j) + \mathbb{B} Y_j (z_k \dot{u}_i - z_i \dot{u}_k) \\
&- \mathbb{B} Z_j (y_k \dot{u}_i - y_i \dot{u}_k) + \mathbb{B} Y_k (z_i \dot{u}_j - z_j \dot{u}_i) - \mathbb{B} Z_k (y_i \dot{u}_j - y_j \dot{u}_i)
\end{aligned}$$

The third line is zero because the acceleration is closed. For the cyclic sum to be zero, all contractions with vectors must be zero, and give conditions. Contraction with u^i gives: $(\nabla_j \frac{\mathbb{E}}{\sqrt{\eta}})\dot{u}_k - (\nabla_k \frac{\mathbb{E}}{\sqrt{\eta}})\dot{u}_j = 0$ with solution

$$\nabla_j \left(\frac{\mathbb{E}}{\sqrt{\eta}}\right) = \varkappa \dot{u}_j$$

With this result the ciclic condition simplifies:

$$(\nabla_i \mathbb{B})(y_j z_k - y_k z_j) + (\nabla_j \mathbb{B})(y_k z_i - y_i z_k) + (\nabla_k \mathbb{B})(y_i z_j - y_j z_i)$$

$$+ \mathbb{B}Y_i(z_j\dot u_k - z_k\dot u_j) - \mathbb{B}Z_i(y_j\dot u_k - y_k\dot u_j) + \mathbb{B}Y_j(z_k\dot u_i - z_i\dot u_k)$$
$$- \mathbb{B}Z_j(y_k\dot u_i - y_i\dot u_k) + \mathbb{B}Y_k(z_i\dot u_j - z_j\dot u_i) - \mathbb{B}Z_k(y_i\dot u_j - y_j\dot u_i) = 0.$$

Contraction with y^i:

$$(y^i\nabla_i\mathbb{B})(y_jz_k - y_kz_j) - (\nabla_j\mathbb{B})z_k + (\nabla_k\mathbb{B})z_j$$
$$+ \mathbb{B}y^iY_i(z_j\dot u_k - z_k\dot u_j) - \mathbb{B}y^iZ_i(y_j\dot u_k - y_k\dot u_j) + \mathbb{B}(Z_j\dot u_k - Z_k\dot u_j) = 0.$$

A further contraction with z^j gives: $\nabla_k\mathbb{B} = (y^i\nabla_i\mathbb{B})y_k + (z^j\nabla_j\mathbb{B})z_k - \mathbb{B}(y^jY_j + z^jZ_j)\dot u_k$ i.e. $\dot u^k\nabla_k\mathbb{B} = -\eta\mathbb{B}(y^rY_r + z^rZ_r)$. The right-hand-side is evaluated in Lemma 2.4 and gives the second condition.

Using the form of $\nabla_j\mathbb{B}$, the cyclic condition becomes

$$- (y^rY_r + z^rZ_r)[\dot u_i(y_jz_k - y_kz_j) + \dot u_j(y_kz_i - y_iz_k) + \dot u_k(y_iz_j - y_jz_i)]$$
$$+ Y_i(z_j\dot u_k - z_k\dot u_j) - Z_i(y_j\dot u_k - y_k\dot u_j) + Y_j(z_k\dot u_i - z_i\dot u_k) - Z_j(y_k\dot u_i - y_i\dot u_k)$$
$$+ Y_k(z_i\dot u_j - z_j\dot u_i) - Z_k(y_i\dot u_j - y_j\dot u_i) = 0.$$

The contractions with $\dot u^i$, y^i or z^i or with the metric tensor are trivial. Indeed it is satisfied by the generic expansions $Y_i = ay_i + bz_i + c\dot u_i$ and $Z_i = a'y_i + b'z_i + c'\dot u_i$.

$\dot\varkappa = u^k\nabla_k(\frac{1}{\eta}\dot u^j\nabla_j\frac{\mathbb{E}}{\sqrt\eta})$. Use $\dot\eta = 0$, $\ddot u^j = \eta u^j$ and $u^j\nabla_j\frac{\mathbb{E}}{\sqrt\eta} = 0$. Then $\dot\varkappa = \frac{1}{\eta}\dot u^ju^k\nabla_k\nabla_j\frac{\mathbb{E}}{\sqrt\eta} = \frac{1}{\eta}\dot u^ju^k\nabla_j\nabla_k\frac{\mathbb{E}}{\sqrt\eta}$. Now:

$$u^k\nabla_j\nabla_k\frac{\mathbb{E}}{\sqrt\eta} = \nabla_j\left(u^k\nabla_k\frac{\mathbb{E}}{\sqrt\eta}\right) - (\nabla_ju^k)\nabla_k\frac{\mathbb{E}}{\sqrt\eta} = \nabla_j(\varkappa u^k\dot u_k) + u_j\dot u^k(\varkappa\dot u_k) = u_j\eta\varkappa$$

Then: $\dot\varkappa = \dot u^ju_j\varkappa = 0$.

Appendix 3: Solution of Eq. (74) with point charges

After multiplication of (74) by $2r^2$, the equation takes the form $PB'' + QB' + TB = S$ with $P(r) = cr^3 + dr^2$, $Q(r) = cr^2$, $T(r) = -4cr - 2d$. The circumstance $P'' - Q' + T = 0$ makes the equation integrable (see [59] Eq. 67.5). Indeed it can be written as

$$[(cr^3 + dr^2)B]'' - [B(5cr^2 + 4rd)]' = 4r^2\frac{q_e^2 + q_m^2}{r^4} - 2(cr + d)$$

An integration gives a constant K:

$$B' - \frac{2}{r}B = -\frac{cr + 2d}{r(cr + d)} + \frac{1}{r^2(cr + d)}\left[K + 4\int^r dr'\,\frac{q_e^2 + q_m^2}{r'^2}\right]$$

Define $B(r) = r^2 H(r)$. The equation now is:

$$\begin{aligned}\frac{dH}{dr} &= -\frac{cr+2d}{r^3(cr+d)} + \frac{1}{r^4(cr+d)}\left[K - 4\frac{q_e^2+q_m^2}{r}\right] \\ &= -\frac{1}{r^3} + \frac{K}{dr^4} - \frac{d^2+cK}{d}\frac{1}{r^3(cr+d)} - 4\frac{q_e^2+q_m^2}{r^5(cr+d)}\end{aligned}$$

Note that: $\frac{1}{r^3(cr+d)} = \frac{1}{d}[\frac{1}{r^3} - \frac{c}{d}\frac{1}{r^2} + (\frac{c}{d})^2\frac{1}{r}] - (\frac{c}{d})^3\frac{1}{cr+d}$, and similar with power 5. The integral gives another constant K_0:

$$\begin{aligned}H(r) = \frac{1}{2r^2} - \frac{K}{3dr^3} - \frac{d^2+cK}{d^2}\left[-\frac{1}{2r^2} + \frac{c}{d}\frac{1}{r} - (\frac{c}{d})^2 \log\frac{cr+d}{r}\right] + K_0 \\ -4\frac{q_e^2+q_m^2}{d}\left[-\frac{1}{4r^4} + \left(\frac{c}{d}\right)\frac{1}{3r^3} - \left(\frac{c}{d}\right)^2\frac{1}{2r^2} + \left(\frac{c}{d}\right)^3\frac{1}{r} - \left(\frac{c}{d}\right)^4\log\frac{cr+d}{r}\right]\end{aligned}$$

Multiplication by r^2 gives the solution $B(r)$ in (75). It coincides with Eq. 34 in [29]. With zero charge it is Eq. 22 in [53].

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