General Relativity and Gravitation (2023) 55:80
https://doi.org/10.1007/s10714-023-03131-6

RESEARCH

Quantum atmosphere effective radii for different spin fields from quantum gravity inspired black holes

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Received: 22 April 2023 / Accepted: 26 June 2023 / Published online: 3 July 2023

Abstract

Quantum atmosphere effective radii for the emission of spin-0, 1/2, 1, and 2 massless fields from Schwarzschild, Tangherlini, non-commutative geometry inspired, and polymeric black holes are calculated. The power observed from the black hole at spatial infinity taking greybody factors into account is compared to an equal-power black-body radiator of the same temperature but different effective radius. A large range of different radii are obtained for different spin fields and black holes. The equal-power black-body effective radius is not, in general, a good proxy for the location of the quantum atmosphere.

Keywords Black holes · Quantum gravity · Greybody factors · Extra dimensions · Non-commutative geometry

Contents

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1 Introduction

The Hawking radiation from evaporating black holes is thought to originate from quantum excitations near the horizon [1]. Giddings [2] has argued that the radiation originates from an effective radius r_A outside the horizon radius r_H call the quantum atmosphere: $r_A - r_H \sim r_H$. It is of interest to test the validity of Giddings' claim.

The quantum atmosphere is the location where most of the Hawking radiation comes from. A few different arguments have been given for the location of the quantum atmosphere. The thermal wavelength of typical Hawking radiation is much larger than the horizon size. Heuristic arguments using a gravitational version of the Schwinger effect for particle production by tidal forces outside the horizon have been made [3, 4]. Another reasoning uses the $(1 + 1)$-dimensional renormalized stress-energy tensor [2, 3, 5]. In addition, the radius can be given by an effective black-body emission surface [2, 6]. In this paper, we examine the later of these definitions.

Ref. [3, 5] have corroborated Giddings' conclusion by obtaining $r_A - r_H \approx r_H$ for the Schwarzschild black hole using gravitational Schwinger effect arguments and a more precise calculation using the stress-energy tensor. While the different arguments agree that the location of the quantum atmosphere is some distance from the horizon, they do not all give a common estimate for the numerical value.

It is of interest to examine if Giddings' arguments are applicable to other types of black holes. Hod [6] showed that the quantum atmosphere radius for a massless scalar field from a Tangherlini black hole emitting radiation in the bulk is a decreasing function of the number of space dimensions; Hod finds $r_A - r_H \ll r_H$ for high number of extra dimensions. The Reissner-Nordström black hole has also been considered in Ref. [4]. These metrics give contradicting conclusions to Ref. [2, 3, 5].

In this paper, we calculate exact greybody factors numerically for all spin fields. The results are used to calculate the double-differential frequency spectrum which is then integrated over all frequencies to obtain the power. By equating the power to that of a black body, we determine an effective emission surface of the quantum atmosphere. The potentials seen by different spin fields are different so we could expect the quantum atmosphere to depend on the emitted field's spin. We find that the apparent radius should not be used, in general, as a proxy for the location of the quantum atmosphere. For example, r_A can not be used as a definition for the location of the quantum atmosphere for gravitons for most black hole metrics we consider.

2 Effective radius calculation

The effective potential barrier around a black hole is commonly encoded in a set of transmission coefficients, greybody factors, that depend on the properties of the black hole, the properties of the emitted radiation, frequency and modes of the emitted radiation. A physical observable that can be formed from the transmission coefficients is the absorption cross section which is a sum of the transmission coefficients over all radiation modes divided by the frequency squared. Weighting the absorption cross section by a temperature-dependent statistical factor corresponding to the spin-statistics of the emitted radiation gives the radiation flux or power per unit frequency. By integrating over all energies, the total radiated power or luminosity is obtained. In the absence of absorption—step-function transmission coefficients—the Stefan-Boltzmann law is obtained. For black holes, one can convert the temperature dependence into a dependence on the horizon radius, and in principle a dependence on the black hole parameters. The power thus allows a determination of the effects of the transmission coefficients integrated over all frequencies. By comparing the power generated by a black hole with the equivalent power from a black body seen at spatial infinity, one obtains and effective area for the black hole, or in the case of spherically symmetric black holes, an effective radius. The method of equal-power infers the size of the radiating body.

The calculated power emitted from a black hole seen by an observe at spatial infinity is compared to the equivalent power P_{B} from an idealize black-body radiator in flat space using the generalized Stefan-Boltzmann relation (see for example Ref. [7])

$$P_{\mathrm{B}} = \sigma A_{n+2}(r_{\mathrm{A}}) T_{\mathrm{B}}^{n+4} , \tag{1}$$

where T_{B} is the black-body temperature, $A_{n+2}(r_{\mathrm{A}})$ is the surface area of a $(n+4)$-dimensional emitting body, and σ is the appropriate Stefan-Boltzmann constant for bosons or fermions in n extra dimensions. We use units of $G = c = \hbar = k_{\mathrm{B}} = 1$. Although Eq. (1) is written in the general form to allow comparison with higher-dimensional black holes, it reduces to the more familiar form of the Stefan-Boltzmann law when $n = 0$.

For a black hole, once the greybody factors $\Gamma_{s,\ell}(\omega)$ for massless spin field s emitted with spheroidal harmonic mode ℓ with frequency ω have been calculated, the absorption cross section in four spacetime dimensions is obtained:

$$\sigma_s(\omega) = \frac{\pi}{\omega^2} \sum_{\ell \geq s} (2\ell + 1) \Gamma_{s,\ell}(\omega) . \tag{2}$$

The $(2\ell + 1)$ factor is the degeneracy of the axial quantum number or angular momentum m modes.

The total power in four spacetime dimensions is then given by

$$P = \frac{1}{2\pi^2} \int_0^\infty \frac{\omega^3 \sigma_s(\omega)}{\exp(\omega/T) - (-1)^{2s}} \mathrm{d}\omega , \tag{3}$$

where T is the Hawking temperature as measured at spatial infinity.

We define the effective radius r_{A} of the black hole quantum atmospheres by equating the Hawking radiation power from the black hole Eq. (3) with the corresponding Stefan-Boltzmann radiation power of a flat space perfect black-body emitter Eq. (1):

$$P(r_{\mathrm{H}}, T) = P_{\mathrm{B}}(r_A, T_{\mathrm{B}}) . \tag{4}$$

This equation determines the effective radius assuming equal temperature: $T = T_B$. One could likewise determine the effective temperature of the filtered radiation by assuming equal radii [8].

Using $A_{n+2} \propto R^{n+2}$, we obtain the effective radius using

$$\frac{r_A}{r_H} = \left[\frac{P(r_H, T)}{P_B(r_H, T)}\right]^{\frac{1}{n+2}} , \tag{5}$$

where P depends on the emitted field's spin and P_B is different for bosons and fermions. The dimensionless radii r_A/r_H characterizes the black hole quantum atmospheres.

As in Ref. [6], it is beneficial to characterize the effective quantum atmosphere using

$$\bar{r}_A = \frac{r_A - r_H}{r_H} . \tag{6}$$

Values of $\bar{r}_A \gtrsim 1$ validate Giddings' argument and negative values imply the quantum atmosphere is behind the horizon.

3 Black holes thermodynamics

In this section, we write down the black-body power for the different metrics consider. The formula contain only a single polarization for each spin field. We make no claim about the validity of the two quantum inspired black hole metrics considered here. They are partly chosen for their different black-body features and the ease of greybody calculation.

3.1 Schwarzschild-Tangherlini black holes

For the Schwarzschild-Tangherlini [9] black hole radiating into the bulk, the higher dimensional $(n + 4)$ black-body power is [6, 7]

$$P_B = \sigma A_{n+2}(R) T^{n+4} , \tag{7}$$

where the higher-dimensional Stefan-Boltzmann constant is

$$\sigma = \frac{(n+3)\Gamma((n+3)/2)\zeta(n+4)}{2\pi^{(n+3)/2+1}} , \tag{8}$$

and Γ is the gamma function and ζ is the Riemann zeta function. The higher-dimensional surface area of the emitting body of radius R is

$$A_{n+2}(R) = \frac{2\pi^{(n+3)/2}}{\Gamma((n+3)/2)} R^{n+2} . \tag{9}$$

We will also need the black hole temperature

$$T = \frac{n+1}{4\pi r_{\rm H}}\,, \tag{10}$$

where

$$r_{\rm H} = \frac{1}{\sqrt{\pi}M_*}\left(\frac{M}{M_*}\right)^{1/(n+1)}\left[\frac{8\Gamma((n+3)/2)}{n+2}\right]^{1/(n+1)}\,. \tag{11}$$

The above equations reduce to the familiar Stefan-Boltzmann law and Schwarzschild black hole when $n = 0$, and $M_* = \sqrt{\hbar c/G}$ is the Planck mass.

3.2 Non-commutative geometry inspired black holes

Non-commutative geometry inspired black holes are interesting in that the form of the black-body area of the Schwarzschild-Tangherlini remains unchanged but the temperature dependence is different [10, 11]. The temperature is given by

$$T = \frac{n+1}{4\pi r_{\rm H}}\left[1 - \frac{2}{n+1}\left(\frac{r_{\rm H}}{2\sqrt{\theta}}\right)^{n+3}\frac{\mathrm{e}^{-r_{\rm H}/(4\theta)}}{\gamma\left(\frac{n+3}{2}, \frac{r_{\rm H}^2}{4\theta}\right)}\right]\,, \tag{12}$$

where γ is the upper incomplete gamma function. The horizon radius is obtained by solving

$$\frac{M}{M_*} = \frac{n+2}{8\gamma\left(\frac{n+3}{2}, \frac{r^2}{4\theta}\right)}(\sqrt{\pi}M_* r_{\rm H})^{n+1}\,. \tag{13}$$

The minimum length parameter $\sqrt{\theta}$ is take to be a free parameter and could be well above the Planck length. As $\theta \to 0$, the radius and temperature approach the Tangherlini values. The metric gives one, two, or no horizon. For a single horizon the temperature vanishes and a black hole remnant is expected to form. The temperature has a maximum but vanishes at the remnant radius. The non-commutative black hole is similar to the Tangherlini black hole for large masses.

To model the effects of an effective ultra-violet cut-off in the frequency ω of the emitted quanta an additional factor [12] of $\exp(-\theta\omega^2/2)$ should multiply Eq. (3). Although we have included this factor, it has a small effect.

3.3 Polymeric black holes

In loop quantum gravity, semi-classical corrections due to the effects of quantum gravity have been derived to give a so-called polymer Schwarzschild black hole [13, 14]. The model has two free parameters ϵ and a_0. The parameter $a_0 = 8\pi A_{\rm min}$ is related to the minimum area of loop quantum gravity and is expected to be of the Planck scale.

A positive deformation parameter ϵ represents the typical scale of the geometry fluctuations in the Hamiltonian constraints of the theory as they get renormalized from the Planck scale to the astrophysical scales. It's thought that $\epsilon \ll 1$, and values of $\epsilon \lesssim 0.8$ will have little effect on what follows. For large ϵ, deviations from the Schwarzschild metric are apparent for astronomical size black holes.

The horizon area is not the usual form but is given by

$$A = 4\pi(2m)^2 \left[1 + \left(\frac{\sqrt{a_0}}{2m}\right)^4\right]. \tag{14}$$

The temperature is given by

$$T = \frac{1}{4\pi(2m)}(1 - P(\epsilon)^2) \left[1 + \left(\frac{\sqrt{a_0}}{2m}\right)^4\right]^{-1}, \tag{15}$$

where the polymerization function is

$$P(\epsilon) = \frac{\sqrt{1+\epsilon^2} - 1}{\sqrt{1+\epsilon^2} + 1}. \tag{16}$$

The total integrated power given by the Stefan-Boltzmann law is

$$P = \frac{\sigma}{256\pi^3} m^{-2}(1 - P(\epsilon)^2)^4 \left[1 + \left(\frac{\sqrt{a_0}}{2m}\right)^4\right]^{-3}, \tag{17}$$

where $\sigma = \pi^2/120$ for bosons and $\sigma = 7\pi^2/960$ for fermions. In the above equations m is a parameter that is related to the ADM mass M by $M = m(1 + P)^2$.

4 Results

We calculate the quantum atmosphere effective radius for spin-0, 1/2, 1, and 2 massless fields from two quantum inspired black holes. Our calculations are numerical and follow the procedures used in Ref [15] which are based on the general potentials in Ref [16] and the path-ordered matrix exponentials in Ref [17]. The procedure enables previously rather difficult calculations.

4.1 Schwarzschild black hole

We consider the Schwarzschild black holes as a warm-up. Table 1shows dimensionless effective radii for all spin fields from a Schwarzschild black hole. Our numerical calculations reproduce the results of Page [18] for spin-1, 1/2, 2, and Elster [19] for scalars. In terms of the quantum atmosphere, the case of spin-1 was first discussed in

Table 1 Dimensionless radii $\bar{r}_A$ for massless fields of spin s from a Schwarzschild black hole

s	0	1/2	1	2
$\bar{r}_A$	1.68	1.13	0.27	−0.57

Table 2 Dimensionless radii $\bar{r}_A$ for a massless scalar field from a $(n+4)$-dimensional Tangherlini black hole radiating in the bulk

n	1	2	3	4	5	6	7
$\bar{r}_A$	0.99	0.71	0.59	0.50	0.44	0.39	0.33

Table 3 Dimensionless radii $\bar{r}_A$ for massless fields of spin s from a $(n+4)$-dimensional Tangherlini black hole radiating on the brane

n							
s	1	2	3	4	5	6	7
0	3.44	4.98	5.83	5.86	5.21	4.16	2.78
1/2	3.44	5.10	5.90	5.84	5.13	4.04	2.71
1	2.68	4.70	5.82	5.99	5.39	4.32	2.95
2	0.96	2.67	3.89	4.37	4.14	3.43	2.34

Ref. [2] and the spin-0 in Ref. [6]. The case of spin-2 shows a breakdown of Gidding's principle (Ref. [2] restricted the discussion to $s \leq 1$).

For the Schwarzschild black hole, the black-body power is well known to have a $P \sim M^{-2}$ dependence. We find that including graybody factors, this mass dependence is maintained, i.e. Γ does not introduce any additional M dependence.

4.2 Tangherlini black hole

To help validate our procedure, we reproduce a previous result in Ref. [6]. Table 2 shows dimensionless effective radii for scalars from a Tangherlini black hole radiating in the bulk. We have taken $M = M_* = 1$. To obtain these results, we have calculated the emission on the brane and used the bulk-to-brane emission ratios obtained in Ref. [20]. Our results agree with Ref. [6] to within the numerical accuracy of the calculations.

We are now equipped to calculate something new. Table 3 shows dimensionless effective radii for all spin fields from a Tangherlini black hole radiating on the brane. Looking at the large values of $\bar{r}_A$ for brane emission, we reach a different conclusion from bulk emission, and support Giddings' argument much better.

4.3 Non-commutative geometry inspired black hole

The non-commutative geometry inspired black hole we consider has a minimum horizon radius at a finite mass (a black hole remnant), and a temperature that has a maximum before the temperature vanishes. Thus the power does not follow the M^{-2} dependence near the end of the black hole's lifetime and $\bar{r}_A$ depends on the black hole mass. For high $M\sqrt{\theta}$, we reproduce the Schwarzschild results. For the

Table 4 Dimensionless radii $\bar{r}_A$ for massless fields of spin s from a $(n+4)$-dimensional non-commutative geometry inspired black hole radiating on the brane with the maximum temperature and $\sqrt{\theta} = 1$

n								
s	0	1	2	3	4	5	6	7
0	1.70	−0.98	5.85	7.35	7.93	7.62	6.60	5.21
1/2	1.03	−0.98	6.03	7.51	7.99	7.57	6.47	5.04
1	0.12	−0.99	5.33	7.22	8.04	7.83	6.83	5.42
2	−0.68	−0.99	2.75	4.50	5.55	5.77	5.25	4.28

Table 5 Dimensionless radii $\bar{r}_A$ for massless fields of spin s from a polymeric black hole with the maximum temperature, and $\epsilon = 0.01$ and $a_0 = 1$

s	0	1/2	1	2
$\bar{r}_A$	1.57	0.62	−0.22	−0.90

black-body case, below about $M\sqrt{\theta} < 6$, the power dependence deviates from a pure M^{-2} dependence and vanishes as $M \to 1.9/\sqrt{\theta}$. The black hole power falls faster than the black-body power with M except for the spin-0 field.

Table 4 shows dimensionless effective radii for all spin fields from a non-commutative geometry inspired black hole in higher dimensions radiating on the brane at the maximum temperature; we have taken $\sqrt{\theta} = 1$.

4.4 Polymeric black hole

The polymeric black hole also has a maximum temperature but the temperature vanishes at zero mass. Combined with the non-trivial area dependence, the power does not follow a M^{-2} dependence and $\bar{r}_A$ depends on the mass of the black hole. We have taken $\epsilon = 0.01$ and $a_0 = 1$. For this value of ϵ, $P = 2.5 \times 10^{-5}$ and gives a negligible contribution to the power, and causes $m \approx M$. For high $2M/\sqrt{a_0}$, we reproduce the Schwarzschild results. For the black-body case, below about $2M/\sqrt{a_0} < 2$, the power dependence deviates from a pure M^{-2} dependence and vanishes as $M \to 0$. The black hole power falls faster than the black-body power with M except for the spin-0 field.

Table 5 shows dimensionless effective radii for all spin fields from a polymeric black hole at the maximum temperature.

5 Discussion

We have calculated the quantum atmosphere for all massless spin fields for the first time. Two quantum gravity inspired metrics posing different black-body power formula have been compared with exact numerical calculations of the total power from the black hole including greybody factors.

Giddings' argument of $\bar{r}_A \sim 1$ clearly depends on the spin of the emitted radiation, decreasing by a factor of about six when going from scalars to vectors, and in general does not apply to gravitons.

Hod's [6] result $\bar{r}_{\mathrm{A}} < 1$ for higher-dimensional black holes is reproduced, but if the radiation is confined to our brane, the conclusion is very different. Values of $\bar{r}_{\mathrm{A}} \sim 5$ for most spins and extra dimensions are obtained. The higher-dimensional form of the black-body formula plays a significant role beyond just the greybody factors.

We have examined two quantum gravity inspired black holes in the regime were quantum effects are important and the radiation will have its maximum intensity. The quantum atmosphere for scalar fields in four space-time dimensions appears similar regardless of the quantum inspired metric and is similar to Schwarzschild black holes.

The power in the spin-0 field always has a quantum atmosphere radius of about 1.7 times the horizon radius in four space-time dimensions. We can see that, in general, the effective radius of an equivalent black-body radiator is not a good proxy for the quantum atmosphere. On the other hand, the effective radius $\bar{r}_{\mathrm{A}}$ could be considered an intuitive measure of greybody effects on the total power received by an observer.

The greybody factors themselves are of little interest until they are used to calculate physical observables. It is common to calculate the absorption cross section and compare the high-frequency limit against the geometric cross section and the low-frequency limit against the surface area. These limits allow an easily quantifiable measure of the greybody effects of different metrics. Perhaps a more measurable observable, someday, will be the total particle fluxes and energy spectra measured by a distant observer. First measurements of these quantities are likely to be integrated over the detecting instrument's acceptance and resolution to obtain single numbers for the number of particles per unit time and energy per unit time (or power), before full spectra are measured. Expressing these measurements in terms of an effective black-body radius could prove to be a useful mnemonic for elucidating quantum gravity effects.

Acknowledgements We acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Nous remercions le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG) de son soutien.

Author Contributions The single author of this paper is responsible for all the research contained herein and the entire manuscript preparation

Declarations

Conflict of interest The authors declare no competing interests.

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