



Codazzi tensors and their space-times and Cotton gravity

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Abstract

We study the geometric properties of certain Codazzi tensors for their own sake, and for their appearance in the recent theory of Cotton gravity. We prove that a perfect-fluid tensor is Codazzi if and only if the metric is a generalized Stephani universe. A trace condition restricts it to a warped space-time, as proven by Merton and Derdziński. We also give necessary and sufficient conditions for a space-time to host a current-flow Codazzi tensor. In particular, we study the static and spherically symmetric cases, which include the Nariai and Bertotti-Robinson metrics. The latter are a special case of Yang Pure space-times, together with spatially flat FRW space-times with constant curvature scalar. We apply these results to the recent Cotton gravity by Harada. We show that the equation of Cotton gravity is Einstein's equation modified by the presence of a Codazzi tensor, which can be chosen freely and constrains the space-time where the theory is staged. In doing so, the tensor (chosen in forms appropriate for physics) implies the form of the Ricci tensor. The two tensors specify the energy-momentum tensor, which is the source in the equation of Cotton gravity for the metric implied by the Codazzi tensor. For example, we show that the Stephani, Nariai and Bertotti-Robinson space-times are characterized by a “current flow” Codazzi tensor. Because of it, they solve Cotton gravity with physically sensible energy-momentum tensors. Finally, we discuss Cotton gravity in constant curvature space-times.

Keywords Codazzi tensor · Cotton gravity · Stephani universe · Yang Pure space · Alternative gravity theories · Anisotropic fluid

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1 Introduction

In Ref. [1] Junpei Harada proposed an extension named ‘‘Cotton gravity’’ of the Einstein equations, where the geometric term (the Einstein tensor) is replaced by the Cotton tensor, and the source (the energy-momentum tensor) is replaced by gradients of the energy-momentum. In a space-time of dimension n :

$$C_{jkl} = \nabla_j T_{kl} - \nabla_k T_{jl} - \frac{g_{kl} \nabla_j T - g_{jl} \nabla_k T}{n - 1} \tag{1}$$

T is the trace T^k_k and Newton’s constant is absorbed in T_{jk} . The Cotton tensor is defined as

$$C_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} - \frac{g_{kl} \nabla_j R - g_{jl} \nabla_k R}{2(n - 1)} \tag{2}$$

It is related to the Weyl tensor, $C_{jkl} = -\frac{n-2}{n-3} \nabla_m C_{jkl}{}^m$, and contains third-order derivatives of the metric tensor.

Harada showed that his gravity Eq. (1) descend from a variational principle with action, in $n = 4$, $S = \int d^4x \sqrt{-g} (C_{jklm} C^{jklm} - R_{jklm} T^{jklm})$. The variation is taken in the connection. T^{jklm} is a curvature tensor built with the energy-momentum and the metric tensors (Eq. 14 in [1]). Conformal gravity has the same action, but the variation in the metric tensor gives the equations $-4(2\nabla^j \nabla^m C_{jklm} + R^{jm} C_{jklm}) = T_{kl}$, that are fourth-order in the derivatives of the metric tensor.

While solving (1) for a vacuum ($T_{kl} = 0$) static spherically symmetric space-time, Harada obtained a generalization of the Schwarzschild solution:

$$ds^2 = -b^2(r)dt^2 + \frac{1}{b^2(r)}dr^2 + r^2d\Omega_2^2 \tag{3}$$

with $b^2(r) = 1 - 2M/r + \gamma r - \frac{1}{3}\Lambda r^2$. He remarked the similarity with a vacuum solution of conformal gravity, where $b^2(r)$ is replaced by $\hat{b}^2(r) = [1 - 3\beta\gamma - \beta(2 - 3\beta\gamma)/r + \gamma r - Kr^2]$ (see [2, 3]).

In ref. [4], Harada applied his theory to describe the rotation curves of several galaxies, where the effect of the possible dark-matter halo is supplanted by the modified gravitational potential.

After its appearance, Harada's paper was criticized by objecting that it adds nothing to standard General Relativity [5, 6]. This paper shows that it is not the case: there is a rich structure in the theory, that includes the standard one.

The difference of Eqs. (1) and (2) shows that

$$\mathcal{C}_{kl} = R_{kl} - T_{kl} - g_{kl} \frac{R - 2T}{2(n-1)} \quad (4)$$

is a Codazzi tensor:

$$\nabla_j \mathcal{C}_{ik} = \nabla_i \mathcal{C}_{jk} \quad (5)$$

Equations (4) and (5) are equivalent to the Harada equation (1) for Cotton gravity. In fact, with $R_{kl} = \mathcal{C}_{kl} + T_{kl} + g_{kl} \frac{R-2T}{2(n-1)}$ the Cotton tensor (2) is constructed, and the Codazzi condition ensures that (1) is obtained.

The third order character of (1) is reduced to second order in (4), with the appearance of a supplemental term, the Codazzi tensor. The latter may be thought of as a modification of the Ricci tensor, or the energy-momentum tensor, or both.

The case $\mathcal{C}_{kl} = 0$ in (4) restores the Einstein equations, and Eq. (1) is identically true. The "trivial" case $\mathcal{C}_{jk} = Bg_{jk}$, adds a cosmological constant.

If $\mathcal{C}_{kl} \neq 0$, Eq. (4) can still be interpreted as the Einstein equation, with a modified energy-momentum tensor:

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + \mathcal{C}_{kl} - g_{kl}\mathcal{C}^r_r \quad (6)$$

Let us mention that Codazzi tensors appear in the geometry of hypersurfaces [7]. A Lorentzian hypersurface in a Minkowski space-time has Riemann tensor $R_{jklm} = \Omega_{jl}\Omega_{km} - \Omega_{jm}\Omega_{kl}$ where Ω_{jk} is a Codazzi tensor. The trivial case $\Omega_{jk} = \frac{R}{n(n-1)}g_{jk}$ corresponds to a constant-curvature hypersurface, and the tensor has a single, constant eigenvalue.

A non-trivial Codazzi tensor poses important limitations on the geometry of the hosting space-time.

Among the possible tensors, we choose to investigate two simple and physically relevant ones, that often appear in the expressions of the Ricci or of the energy-momentum tensors. They involve the basic kinematic quantities u_i and \dot{u}_i .

We begin with the "perfect fluid" tensor $\mathcal{C}_{jk} = Au_ju_k + Bg_{jk}$ with the Codazzi property. Andrzej Derdziński [8] proved that if \mathcal{C}^k_k is a constant, then the space-time is warped (GRW, generalized Robertson-Walker space-time), i.e. there are coordinates

such that

$$ds^2 = -dt^2 + a^2(t)g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu \quad (7)$$

with Riemannian metric $g_{\mu\nu}^*$. The hypothesis was weakened by Gabe Merton [9], who showed that a necessary and sufficient condition for the GRW space-time is $v^j \nabla_j \mathcal{C}^k_k = 0$ for all vectors $v^j u_j = 0$ (the result was proven in Riemannian signature, but it also holds in Lorentzian).

In Theorem 2.1 we prove that a perfect fluid tensor is Codazzi if and only if the space-time is “doubly twisted”, i.e. there are coordinates such that

$$ds^2 = -b^2(t, \mathbf{x})dt^2 + a^2(t, \mathbf{x})g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu \quad (8)$$

with the special condition that $(\partial_t \log a)/b$ only depends on time t . Remarkably, this metric with the constraint happens to be a generalization of the well known Stephani Universes.

We discuss special cases, including Merton’s result, and obtain the general form of the Ricci tensor.

Next we study the “current flow” tensor $\mathcal{C}_{jk} = \lambda(u_j \dot{u}_k + \dot{u}_j u_k)$ with the Codazzi condition and closed vector field \dot{u}_j . The field u_j turns out to be vorticity-free but not shear-free. This makes the metric more general than doubly-twisted, Eq. (28). If it is constrained to be static, a useful form of the Ricci tensor is obtained. We list some of the several examples that can be found in the literature.

Finally, we consider Yang Pure space-times. They are characterised by a Ricci tensor that is a Codazzi tensor. Among the examples, we show that a Friedmann-Robertson-Walker metric is Yang Pure if and only if $\nabla_j R = 0$.

This concludes Sect. 2 of the paper.

In Sect. 3 we show that these results are interesting for the Cotton gravity by Harada. If nontrivial, the Codazzi tensor introduces geometric or unconventional matter content in the Einstein equation, depending on the point of the view, in a way different than other extended theories of gravity.

This suggests a solution to the Harada equations which goes as follows: given the form of a Codazzi tensor, this determines a class of space-times that host the tensor. The space-time in turn determines the Ricci tensor. Finally, the Codazzi and the Ricci tensors in Eq. (4) determine the energy-momentum tensor of the Harada equation.

The two Codazzi tensors that are here studied, modify the energy-momentum in its perfect-fluid component or in the current component.

We end with a discussion of constant curvature space-times, for which Ferus [10] identified the general form of Codazzi tensors.

We employ the Lorentzian signature $(- + \dots +)$, latin letters for space-time components and greek letters for space components. A dot on a quantity X is the operator $\dot{X} = u^k \nabla_k X$. The symbols η, ϵ are the scalar functions $\eta = \dot{u}^k \dot{u}_k$ and $\epsilon = \dot{u}^k \nabla_k \eta$.

2 Codazzi tensors and their space-times

In refs. [11, 12] we showed that a Codazzi tensor always satisfies an algebraic identity with the Riemann tensor (it is ‘‘Riemann compatible’’):

$$\mathcal{C}_{im}R_{jkl}{}^m + \mathcal{C}_{jm}R_{kil}{}^m + \mathcal{C}_{km}R_{ijl}{}^m = 0. \quad (9)$$

This property implies that a Codazzi tensor is also Weyl compatible, with the Weyl tensor C_{jklm} replacing R_{jklm} . The contraction with the metric tensor g^{il} gives $\mathcal{C}_{jm}R_k{}^m = \mathcal{C}_{km}R_j{}^m$, i.e. a Codazzi tensor commutes with the Ricci tensor.

As anticipated, we investigate two forms of Codazzi tensor. We name them in analogy with terms in an energy-momentum tensor: $\mathcal{C}_{jk} = Au_ju_k + Bg_{jk}$ (perfect fluid) and $\mathcal{C}_{jk} = \lambda(u_j\dot{u}_k + \dot{u}_ju_k)$ (current flow). $A \neq 0$, B , λ are scalar fields. The vector field u_j is time-like unit, $u^ju_j = -1$, and is named velocity. The vector field $\dot{u}_j = u^k\nabla_k u_j$ is spacelike, orthogonal to the velocity, and is named acceleration.

We show that the Codazzi property of such tensors strongly restricts the space-times they live in.

2.1 Perfect fluid Codazzi tensors and Stephani universes.

Theorem 2.1 *The perfect fluid tensor $\mathcal{C}_{jk} = Au_ju_k + Bg_{jk}$ with $u^ju_j = -1$ is Codazzi if and only if*

$$\nabla_i u_j = \varphi(g_{ij} + u_i u_j) - u_i \dot{u}_j \quad (10)$$

where $(n - 1)\varphi$ is the usual expansion parameter,

$$\nabla_i A = -u_i \dot{A} - \dot{u}_i A \quad (11)$$

$$\nabla_i B = -u_i \dot{B} \quad (12)$$

$$\varphi = -\dot{B}/A \quad (13)$$

This relation is useful:

$$\nabla_i \varphi = -u_i \dot{\varphi} \quad (14)$$

Proof See Appendix 1. □

Proposition 2.2 *If \mathcal{C}_{jk} is a perfect-fluid Codazzi tensor, the velocity u_i is Riemann compatible, $u_i R_{jklm}u^m + u_j R_{kilm}u^m + u_k R_{ijlm}u^m = 0$, and it is an eigenvector of the Ricci tensor, $R_{jk}u^k = \gamma u_j$, with eigenvalue*

$$\gamma = (n - 1)(\dot{\varphi} + \varphi^2) - \nabla_k \dot{u}^k \quad (15)$$

The following identity for the acceleration holds:

$$(\varphi \dot{u}_k + \ddot{u}_k)u_l - u_k(\varphi \dot{u}_l + \ddot{u}_l) = \nabla_k \dot{u}_l - \nabla_l \dot{u}_k. \quad (16)$$

Proof The first statement is an obvious consequence of (9) and of the first Bianchi identity. For the eigenvalue we evaluate:

$$\begin{aligned}
 R_{jklm}u^m &= \nabla_j \nabla_k u_l - \nabla_k \nabla_j u_l \\
 &= \nabla_j [\varphi(g_{kl} + u_k u_l) - u_k \dot{u}_l] - \nabla_k [\varphi(g_{jl} + u_j u_l) - u_j \dot{u}_l] \\
 &= - (g_{kl} u_j - g_{jl} u_k) \dot{\varphi} + (\nabla_j u_k - \nabla_k u_j) (\varphi u_l - \dot{u}_l) \\
 &\quad + u_k (\varphi \nabla_j u_l - \nabla_j \dot{u}_l) - u_j (\varphi \nabla_k u_l - \nabla_k \dot{u}_l) \\
 &= - (g_{kl} u_j - g_{jl} u_k) (\dot{\varphi} + \varphi^2) - (u_j \dot{u}_k - u_k \dot{u}_j) (\varphi u_l - \dot{u}_l) - u_k \nabla_j \dot{u}_l + u_j \nabla_k \dot{u}_l
 \end{aligned}$$

The contraction with g^{jl} gives: $R_{km}u^m = (n - 1)(\dot{\varphi} + \varphi^2)u_k + \varphi \dot{u}_k - u_k \eta - u_k \nabla_j \dot{u}^j + u^j \nabla_k \dot{u}_j$. Since $\dot{u}^j u_j = 0$, the last term is: $-\dot{u}^j \nabla_k u_j = -\varphi \dot{u}_k + u_k \eta$ by Eq. (10), and cancels three terms. The eigenvalue γ is read.

The contraction with u^j gives the symmetric tensor

$$u^j R_{jklm}u^m = (g_{kl} + u_l u_k) (\dot{\varphi} + \varphi^2) + \dot{u}_k (\varphi u_l - \dot{u}_l) - u_k \ddot{u}_l - \nabla_k \dot{u}_l \tag{17}$$

Subtraction with indices k, l exchanged gives the identity for the acceleration. \square

With the aid of the Weyl tensor, we obtain the expression of the Ricci tensor on a space-time with a perfect fluid Codazzi tensor.

Proposition 2.3 (The Ricci tensor)

$$\begin{aligned}
 R_{kl} &= \frac{R - n\gamma}{n - 1} u_k u_l + \frac{R - \gamma}{n - 1} g_{kl} + \Pi_{kl} \\
 \Pi_{kl} &= \frac{1}{2} (n - 2) [u_k (\varphi \dot{u}_l - \ddot{u}_l) + u_l (\varphi \dot{u}_k - \ddot{u}_k) - (\nabla_k \dot{u}_l + \nabla_l \dot{u}_k)] \\
 &\quad - (n - 2) [\dot{u}_k \dot{u}_l + E_{kl}] + \frac{n - 2}{n - 1} (g_{kl} + u_k u_l) \nabla_p \dot{u}^p \tag{18}
 \end{aligned}$$

where γ is the eigenvalue (15), Π_{kl} is symmetric traceless with $\Pi_{kl}u^l = 0$, and $E_{kl} = u^j u^m C_{jklm}$ is the electric tensor. It is symmetric, traceless, with $E_{jku}^k = 0$.

Proof The general expression of the Weyl tensor is:

$$C_{jklm} = R_{jklm} + \frac{g_{jm}R_{kl} - g_{km}R_{jl} + g_{kl}R_{jm} - g_{jl}R_{km}}{n - 2} - R \frac{g_{jm}g_{kl} - g_{km}g_{jl}}{(n - 1)(n - 2)}$$

The contraction with $u^j u^m$ and (17) give:

$$\begin{aligned}
 E_{kl} &= (g_{kl} + u_l u_k) (\dot{\varphi} + \varphi^2) + \dot{u}_k (\varphi u_l - \dot{u}_l) - u_k \ddot{u}_l - \nabla_k \dot{u}_l \\
 &\quad - \frac{R_{kl} + 2\gamma u_k u_l + \gamma g_{kl}}{n - 2} + R \frac{g_{kl} + u_k u_l}{(n - 1)(n - 2)}.
 \end{aligned}$$

The Ricci tensor is obtained:

$$R_{kl} = \left[\frac{R - n\gamma + (n-2)\nabla_p \dot{u}^p}{n-1} \right] u_k u_l + \left[\frac{R - \gamma + (n-2)\nabla_p \dot{u}^p}{n-1} \right] g_{kl} - (n-2)[\dot{u}_k \dot{u}_l - \varphi \dot{u}_k u_l + u_k \ddot{u}_l + \nabla_k \dot{u}_l + E_{kl}].$$

The expression is symmetrized with the identity (16) and the correction to the perfect fluid part is made traceless by subtraction. \square

We discuss the geometric restrictions posed by a perfect-fluid Codazzi tensor. The presence of a shear-free and vorticity-free velocity field, Eq. (10), classifies the space-time as *doubly-twisted* [13], i.e. there is a coordinate frame such that the metric has the form (8).

In this frame, with the Christoffel symbols

$$\Gamma_{00}^0 = \frac{\partial_t b}{b}, \quad \Gamma_{\mu 0}^0 = \frac{\partial_\mu b}{b}, \quad \Gamma_{\mu\nu}^0 = \frac{\partial_t a}{ab^2} g_{\mu\nu}^*, \quad \Gamma_{0\mu}^\nu = \frac{\partial_t a}{a} \delta_\mu^\nu$$

Eq. (10) for u_j and $\dot{u}_j = u^k \nabla_k u_j$ give: $u_0 = -b(t, \mathbf{x})$, $u_\mu = 0$, and

$$\dot{u}_0 = 0, \quad \dot{u}_\mu = \frac{\partial_\mu b(t, \mathbf{x})}{b(t, \mathbf{x})}; \quad \varphi = \frac{1}{b(t, \mathbf{x})} \frac{\partial_t a(t, \mathbf{x})}{a(t, \mathbf{x})}. \quad (19)$$

By Eq. (14), the doubly twisted metric has the constraint that φ only depends on time. With $a = 1/V(\mathbf{x}, t)$, the metric (8) with the constraint becomes:

$$ds^2 = - \left[\frac{1}{\varphi(t)} \frac{\partial_t V}{V} \right]^2 dt^2 + \frac{g_{\mu\nu}^*(\mathbf{x}) dx^\mu dx^\nu}{V^2(\mathbf{x}, t)} \quad (20)$$

This metric generalizes the well known Stephani metrics, presented in the following example.

Example 2.4 Remarkably, Eqs. (10)–(13) in $n = 4$ coincide with eqs. 37.32–37.34 in the book by Stephani et al. [15]. They were derived for a Riemann tensor of the form $R_{jklm} = \mathcal{C}_{jl} \mathcal{C}_{km} - \mathcal{C}_{jm} \mathcal{C}_{kl}$, with $\mathcal{C}_{jk} = Au_k u_l + Bg_{jk}$ (note that if \mathcal{C}_{jk} is invertible then the Bianchi identity implies that it is a Codazzi tensor [14]). Such space-times are conformally flat and are named Stephani universes [15, 16]. They are solutions of the Einstein equations with a perfect fluid source T_{jk} .

The Stephani metric is

$$ds^2 = - \left[\frac{1}{\varphi(t)} \frac{\partial_t V}{V} \right]^2 dt^2 + \frac{dx^2 + dy^2 + dz^2}{V^2(\mathbf{x}, t)}$$

with $V(\mathbf{x}, t) = V_0(t) + \frac{B^2(t) - \varphi^2(t)}{4V_0(t)} \|\mathbf{x} - \mathbf{x}_0(t)\|^2$, where V_0 , φ and \mathbf{x}_0 are arbitrary functions of time.

We now consider some special conditions of the perfect fluid Codazzi tensor.

Lemma 2.5 *If the acceleration is closed, $\nabla_j \dot{u}_k = \nabla_k \dot{u}_j$, then $b(t, \mathbf{x}) = \hat{b}(t)b(\mathbf{x})$, and $\ddot{u}_k = \eta u_k - \varphi \dot{u}_k$.*

Proof The condition that matters is $\nabla_0 \dot{u}_\mu = \nabla_\mu \dot{u}_0$ i.e. $\partial_t \dot{u}_\mu - \Gamma_{0\mu}^\nu \dot{u}_\nu = -\Gamma_{\mu 0}^\nu \dot{u}_\nu$. By the symmetry of the Christoffel symbols, we remain with $0 = \partial_t \dot{u}_\mu$ i.e. $\dot{u}_\mu = \partial_\mu \log b$ is independent of t . Then $b(t, \mathbf{x}) = b_1(t)b_2(\mathbf{x})$.

Equation(16) now is: $(\varphi \dot{u}_k + \ddot{u}_k)u_l - u_k(\varphi \dot{u}_l + \ddot{u}_l) = 0$. Contraction with u^l is: $\ddot{u}_k = -\varphi \dot{u}_k - u_k(u^l \ddot{u}_l)$. The identity $u^l \dot{u}_l = 0$ gives $u^l \ddot{u}_l = -\dot{u}^l \dot{u}_l \equiv -\eta$. \square

- If $\nabla_k A = -u_k \dot{A}$ i.e. $\dot{u}_k = 0$, then $b(t, \mathbf{x})$ is only a function of time. It is $b = 1$ after a rescaling of time. The equations $\partial_\mu \varphi = 0$ show that a only depends on time. Therefore, the space-time is a *generalised Robertson Walker* (GRW) space-time, Eq. (7) [17, 18].

This agrees with Theorem 1.2 in [9], stating that (in a Riemannian setting) a perfect fluid Codazzi tensor such that $h^{jk} \nabla_k \mathcal{C}^i{}_i = 0$ implies a warped metric.

With $\xi \equiv (n - 1)(\dot{\varphi} + \varphi^2)$, the Ricci tensor now is:

$$R_{jk} = \frac{R - n\xi}{n - 1} u_j u_k + \frac{R - \xi}{n - 1} g_{jk} - (n - 2)E_{jk}. \tag{21}$$

- If $B = 0$, i.e. $\mathcal{C}_{jk} = Au_j u_k$, then $\nabla_i u_j = -u_i \dot{u}_j$ and A solves (11). The equation $\varphi = 0$ gives that $a(t, \mathbf{x})$ is independent of time, and can be absorbed in the space metric to give

$$ds^2 = -b^2(t, \mathbf{x})dt^2 + g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu$$

Its conformally flat and spherically symmetric version generalises the Schwarzschild interior solution, Eq. 37.39 in [15]. If moreover \dot{u}_i is closed, then the metric is *static* ([15], page 283):

$$ds^2 = -b^2(\mathbf{x})dt^2 + g_{\mu\nu}^*(\mathbf{x})dx^\mu dx^\nu. \tag{22}$$

- In General Relativity the vanishing of the Cotton tensor $C_{jkl} = 0$ means that $R_{kl} - g_{kl} \frac{R}{2(n-1)}$ is a Codazzi tensor. The Einstein equations then imply that also $\mathcal{C}_{kl} = T_{kl} - \frac{T}{n-1} g_{kl}$ is a Codazzi tensor.

2.2 Current-flow Codazzi tensors

We investigate Codazzi tensors with the form of a current-flow tensor $\mathcal{C}_{jk} = \lambda(u_j \dot{u}_k + \dot{u}_j u_k)$, with closed \dot{u}_i .

The eigenvalues are 0 and $\pm i\lambda\sqrt{\eta}$, the latter being non-degenerate with complex eigenvectors $V_k^\pm = \pm\sqrt{\eta}u_k + i\dot{u}_k$, $g^{jk}V_j^+ V_k^- = 0$. Since the Codazzi tensor commutes with the Ricci tensor, V_k^\pm are also eigenvectors of the Ricci tensor. From $0 = V_j^+ R^{jk} V_k^-$ one obtains

$$\dot{u}^j R_{jk} \dot{u}^k = -\eta u^j R_{jk} u^k \tag{23}$$

Theorem 2.6 *The tensor $\mathcal{C}_{jk} = \lambda(u_j \dot{u}_k + \dot{u}_j u_k)$ with closed acceleration is Codazzi if and only if:*

$$\nabla_j u_k = -\frac{\dot{\lambda}}{\lambda} \frac{\dot{u}_j \dot{u}_k}{\eta} - u_j \dot{u}_k \tag{24}$$

$$\nabla_j \lambda = -u_j \dot{\lambda} - \lambda \dot{u}_j \left(2 + \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \right) \tag{25}$$

$$\nabla_j \dot{u}_k = -\eta u_j u_k - \frac{\dot{\lambda}}{\lambda} (\dot{u}_j u_k + u_j \dot{u}_k) + \dot{u}_j \dot{u}_k \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \tag{26}$$

Proof See Appendix 2. □

A useful relation found in the proof is

$$\nabla_k \eta = -2 \frac{\dot{\lambda}}{\lambda} \eta u_k + \dot{u}_k \frac{\dot{u}^p \nabla_p \eta}{\eta}. \tag{27}$$

We discuss the geometric restrictions posed by a current-flow Codazzi tensor with closed acceleration.

Since the velocity has non-zero shear tensor

$$\sigma_{jk} = \frac{\dot{\lambda}}{\lambda} \left[\frac{g_{jk} + u_j u_k}{n-1} - \frac{\dot{u}_j \dot{u}_k}{\eta} \right]$$

there are coordinates such that the metric has the structure [19]:

$$ds^2 = -b^2(t, \mathbf{x}) dt^2 + G_{\mu\nu}^*(t, \mathbf{x}) dx^\mu dx^\nu \tag{28}$$

with Christoffel symbols $\Gamma_{00}^0 = \frac{\partial_t b}{b}$, $\Gamma_{\mu 0}^0 = \frac{\partial_\mu b}{b}$, $\Gamma_{00}^\mu = G^{*\mu\nu} b \partial_\nu b$, $\Gamma_{\mu\nu}^0 = \frac{\partial_t G_{\mu\nu}^*}{2b^2}$, $\Gamma_{0\nu}^\mu = \frac{1}{2} G^{*\mu\rho} \partial_t G_{\nu\rho}^*$ and $\Gamma_{\rho\sigma}^\mu = G^{*\mu}_\rho \sigma_\sigma$. The equations for u, \dot{u} give:

$$u_0 = -b(t, \mathbf{x}), \quad u_\mu = 0, \quad \dot{u}_0 = 0, \quad \dot{u}_\mu = \frac{\partial_\mu b(t, \mathbf{x})}{b(t, \mathbf{x})}$$

In this frame, the equations $\nabla_\mu u_\nu = -\frac{\dot{\lambda}}{\lambda} \frac{\dot{u}_\mu \dot{u}_\nu}{\eta}$ and $\nabla_0 \dot{u}_\mu = -\frac{\dot{\lambda}}{\lambda} u_0 \dot{u}_\mu$ are:

$$-\frac{1}{2} b \frac{\partial G_{\mu\nu}^*}{\partial t} = \frac{\dot{\lambda}}{\lambda} \frac{\partial_\mu b}{\eta} \partial_\nu b, \quad \frac{\partial \dot{u}_\mu}{\partial t} - \frac{1}{2} \dot{u}_\nu G^{*\nu\rho} \frac{\partial G_{\mu\rho}^*}{\partial t} = \frac{\dot{\lambda}}{\lambda} b \dot{u}_\mu \tag{29}$$

We now specialize to static space-times.

2.2.1 Static space-times

If $\dot{\lambda} = 0$, Eq. (29) shows that $G_{\mu\nu}^*$ is independent of time t , as well as \dot{u}_μ . Then $b(t, \mathbf{x}) = \beta(t)b(\mathbf{x})$. The product $\beta^2(t)dt^2$ in ds^2 redefines the time, and the metric is static, Eq. (22).

Theorem 2.6 becomes: the current-flow tensor with $\dot{\lambda} = 0$ and closed acceleration is Codazzi if and only if:

$$\nabla_j u_k = -u_j \dot{u}_k \tag{30}$$

$$\nabla_j \lambda = -\lambda \dot{u}_j \left(2 + \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \right) \tag{31}$$

$$\nabla_j \dot{u}_k = -\eta u_j u_k + \dot{u}_j \dot{u}_k \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \tag{32}$$

Eq. (30) and closedness of \dot{u}_i covariantly confirm the space-time as static.

Proposition 2.7 *In a static space-time with Eqs.(30)–(32) with closed \dot{u}_i , the vectors u_i and \dot{u}_i are eigenvectors of the Ricci tensor with the same eigenvalue.*

Proof (1) For brevity, put $\epsilon = \dot{u}^p \nabla_p \eta$.

Equation(27) with $\dot{\lambda} = 0$ is: $\nabla_j \eta^2 = 2\epsilon \dot{u}_j$ Now it is $\nabla_k \nabla_j \eta^2 = 2\dot{u}_j \nabla_k \epsilon + 2\epsilon \nabla_k \dot{u}_j$. Antisymmetrization, with the property that $\nabla_k \dot{u}_j = \nabla_j \dot{u}_k$ gives: $\dot{u}_j \nabla_k \epsilon = \dot{u}_k \nabla_j \epsilon$ i.e.

$$\nabla_j \epsilon = \dot{u}_j \frac{\dot{u}^p \nabla_p \epsilon}{\eta}$$

(2) $R_{jklm} u^m = \nabla_j \nabla_k u_l - \nabla_k \nabla_j u_l = -\nabla_j (u_k \dot{u}_l) + \nabla_k (u_j \dot{u}_l) = (u_j \dot{u}_k - u_k \dot{u}_j) \dot{u}_l - u_k \nabla_j \dot{u}_l + u_j \nabla_k \dot{u}_l = (u_j \dot{u}_k - u_k \dot{u}_j) \dot{u}_l (1 + \epsilon/2\eta^2)$. Contraction with g^{jl} :

$$R_{km} u^m = -\left(\eta + \frac{\epsilon}{2\eta}\right) u_k \tag{33}$$

(3) $R_{jklm} \dot{u}^m = \nabla_j \nabla_k \dot{u}_l - \nabla_k \nabla_j \dot{u}_l = \nabla_j (-\eta u_k u_l + \dot{u}_k \dot{u}_l \frac{\epsilon}{2\eta^2}) - \nabla_k (-\eta u_j u_l + \dot{u}_j \dot{u}_l \frac{\epsilon}{2\eta^2}) = -(u_k \nabla_j \eta - u_j \nabla_k \eta) u_l + \eta (u_j \dot{u}_k - \dot{u}_j u_k) u_l + (\dot{u}_k \nabla_j \dot{u}_l - \dot{u}_j \nabla_k \dot{u}_l) \frac{\epsilon}{2\eta^2} + (\dot{u}_k \nabla_j \frac{\epsilon}{2\eta^2} - \dot{u}_j \nabla_k \frac{\epsilon}{2\eta^2}) \dot{u}_l$. The last parenthesis is zero because $\nabla_j \frac{\epsilon}{2\eta^2}$ is proportional to \dot{u}_j . Then: $R_{jklm} \dot{u}^m = (u_j \dot{u}_k - \dot{u}_j u_k) u_l (\eta + \frac{\epsilon}{\eta}) + (\dot{u}_k \nabla_j \dot{u}_l - \dot{u}_j \nabla_k \dot{u}_l) \frac{\epsilon}{2\eta^2}$. Contraction with g^{jl} :

$$R_{km} \dot{u}^m = -\left(\eta + \frac{\epsilon}{2\eta}\right) \dot{u}_k \tag{34}$$

□

The Ricci tensor is now obtained.

In Prop.2.7 we evaluated $R_{jklm}u^m = (u_j\dot{u}_k - u_k\dot{u}_j)\dot{u}_l(1 + \epsilon/2\eta^2)$. Contraction with u^j is $u^j R_{jklm}u^m = -\dot{u}_k\dot{u}_l(1 + \epsilon/2\eta^2)$. The contraction of the Weyl tensor and (33) give

$$E_{kl} = -\dot{u}_k\dot{u}_l \left(1 + \frac{\epsilon}{2\eta^2} \right) - \frac{R_{kl} - (g_{kl} + 2u_k u_l)(\eta + \epsilon/2\eta)}{n - 2} + R \frac{g_{kl} + u_k u_l}{(n - 1)(n - 2)}$$

We then find:

$$R_{kl} = \left[\frac{R}{n - 1} + 2\eta + \frac{\epsilon}{\eta} \right] u_k u_l + \left[\frac{R}{n - 1} + \eta + \frac{\epsilon}{2\eta} \right] g_{kl} - (n - 2) \left[E_{kl} + \dot{u}_k \dot{u}_l \left(1 + \frac{\epsilon}{2\eta^2} \right) \right]. \tag{35}$$

In particular, by Eq. (34), one has the eigenvalue equation

$$(n - 2)E_{kl}\dot{u}^l = \left[\frac{R}{n - 1} - (n - 4) \left(\eta + \frac{\epsilon}{2\eta} \right) \right] \dot{u}_k$$

Eq. (32) with $\epsilon = 0$ (then η is a constant) was obtained by Rao and Rao [20] in a static metric to characterize the relativistic generalisation of the uniform Newton force at a spatial hypersurface.

2.2.2. We restrict the static space-time to be *spherically symmetric*, and give some examples in the end:

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + f_2^2(r)d\Omega_{n-2}^2 \tag{36}$$

In spherical symmetry \dot{u} is radial and $\dot{u}_r = b'(r)/b(r)$ (a prime is a derivative in r). The definition $\eta = \dot{u}^k \dot{u}_k$ gives:

$$\eta(r) = \frac{1}{f_1^2(r)} \frac{b'^2(r)}{b^2(r)} \tag{37}$$

In such coordinates the solution of Eq. (31) is

$$\lambda(r) = \kappa \frac{f_1(r)}{b(r)b'(r)} \tag{38}$$

with a constant κ .

Since \dot{u} is a radial vector, the angular components of Eq. (32) are $\nabla_a \dot{u}_{a'} = 0$ (where $a, a' = 1, \dots, n - 2$ enumerate the angles). It implies $\Gamma_{a,a'}^r \dot{u}_r = 0$ i.e. $\Gamma_{a,a'}^r = 0$. With the expression in [21] Appendix 9.6, one gets the condition on the metric:

$$\frac{df_2}{dr} = 0.$$

In conclusion, a static spherically symmetric space-time with Codazzi tensor $\mathcal{C}_{jk} = \lambda(u_j \dot{u}_k + \dot{u}_j u_k)$ with closed acceleration has the form:

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + L^2 d\Omega_{n-2}^2 \tag{39}$$

where L is a positive constant.

The electric tensor and the scalar curvature of the space manifold are obtained from Eq. (33) in [21] with $a = 1$, $f_2 = L$, and the relations (37) and (38):

$$E_{jk}(r) = \frac{n-3}{n-2} \frac{1}{f_1^2} \left(\frac{f_1^2}{L^2} + \frac{b'}{b} \frac{f_1'}{f_1} - \frac{b''}{b} \right) \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{n-1} \right] \tag{40}$$

$$R^* = \frac{1}{L^2} (n-2)(n-3) \tag{41}$$

where $h_{jk} = g_{jk} + u_k u_j$.

The Ricci tensor (35) in spherical coordinates is sum of three tensors, proportional to $u_j u_k$, g_{jk} and $\dot{u}_j \dot{u}_k$.

We list some examples of the metric (39). They share the same form of Ricci tensor (35), with electric tensor (40). Moreover, they are endowed with a current-flow Codazzi tensor with non-zero components $\mathcal{C}_{0r} = \mathcal{C}_{r0} = \kappa f_1(r)/b(r)$ in the coordinates of each below-listed metric.

Example 2.8 Nariai space-times solve Einstein’s equations in vacuo [22–25]:

$$ds^2 = \frac{1}{\Lambda} \left[-a \cos \log \left(\frac{r}{r_0} \right) dt^2 + \frac{dr^2}{r^2} + d\Omega_2^2 \right].$$

where Λ is the cosmological constant. A coordinate change (see [22]) brings the metric to the direct product of de Sitter $(dS)_2$ with the sphere S_2 of radius $\sqrt{3/\Lambda}$: $ds^2 = -b d\tau^2 + b^{-1} d\rho^2 + (3/\Lambda) d\Omega_2^2$, with $b(\rho) = 1 - \frac{1}{3} \Lambda \rho^2$.

Example 2.9 Bertotti-Robinson space-times are conformally flat solutions of the source-free Einstein-Maxwell equations with non-null e.m. field [26–28]:

$$ds^2 = \frac{r_0^2}{r^2} \left[-dt^2 + dr^2 + r^2 d\Omega_{n-2}^2 \right]. \tag{42}$$

The Ricci tensor is (35) with $R = 0$, $\epsilon = 0$, $\eta = \frac{1}{r_0^2} \lambda = -\kappa \frac{r^2}{r_0^2}$:

$$R_{kl} = \frac{1}{r_0^2} (2u_k u_l + g_{kl}) - 2\dot{u}_k \dot{u}_l. \tag{43}$$

Eqs. (30) and (32), that reads $\nabla_j \dot{u}_k = \frac{1}{r_0^2} u_j u_k$, imply $\nabla_i R_{jk} = \nabla_j R_{ik}$. Therefore, Bertotti-Robinson space-times have two Codazzi tensors: the Ricci tensor and

$$\mathcal{C}_{jk} = -\kappa \frac{r^2}{r_0^2} (u_j \dot{u}_k + \dot{u}_j u_k)$$

A coordinate change [29] brings the metric to: $ds^2 = -b d\tau^2 + b^{-1} d\rho^2 + r_0^2 d\Omega_{n-2}^2$ with $b(\rho) = 1 + \rho^2/r_0^2$. It is the direct product of $(\text{AdS})_2$ with the sphere $S_{n-2}(r_0)$.

Example 2.10 In [30] black holes are studied in string-corrected Einstein-Maxwell theory coupled to a dilaton field. The solution displayed in Eq. 35 is

$$ds^2 = -(ar^2 + br + c)dt^2 + \frac{dr^2}{ar^2 + br + c} + L^2 d\Omega_2^2.$$

Depending on the constants, it may reduce to $(\text{AdS})_2 \times S_2(L)$ or $(\text{dS})_2 \times S_2(L)$.

Example 2.11 In [31] the Bertotti-Robinson-type black hole solutions of string theory are obtained, by CFT methods. This one (Eq. 38) is an example:

$$ds^2 = -\left[\frac{r^2}{\ell^2} + \frac{J^2}{r^2} - M\right] dt^2 + \frac{dr^2}{\left[\frac{r^2}{\ell^2} + \frac{J^2}{r^2} - M\right]} + L^2 d\Omega_2^2$$

where M is the mass, J is the angular momentum, ℓ^2 is proportional to the cosmological constant.

Example 2.12 In [32] spherical black hole solutions of the Einstein-Maxwell-scalar equations are found, where the scalar field is non-minimally coupled to the Maxwell invariant. Among others the following metric is given (Eq. 4.11), where a is a constant:

$$ds^2 = a \left(-r^2 dt^2 + \frac{1}{r^2} dr^2 \right) + L^2 d\Omega_2^2.$$

The examples describe “near horizon geometries of extremal black-holes” [33]. They are direct products of a Lorentzian 2D spacetime with the 2-sphere, whose general properties were investigated by Ficken [34].

2.3 Yang Pure space-times

A Yang Pure space-time is defined by a Ricci tensor that is a Codazzi tensor:

$$\nabla_j R_{kl} = \nabla_k R_{jl} \tag{44}$$

equivalent to $\nabla^m R_{jklm} = 0$. Contraction with g^{jl} gives $\nabla_k R = 0$. They were introduced by Chen Ning Yang in 1974 in the geometry of Yang-Mills theories [35, 36]. These are examples of solutions of Yang’s equation (44):

- Vacuum solutions of Einstein’s equations: $R_{kl} = \Lambda g_{kl}$.
- Wei-Tou Ni obtained the conformally-flat non-static solution [37]

$$ds^2 = \left[C + \frac{f(r-t)}{r} + \frac{g(r+t)}{r} \right] (-dt^2 + dr^2 + r^2 d\Omega_2^2)$$

where C is a constant, f and g are arbitrary functions, and also the solution:

$$ds^2 = -dt^2 + \left[1 + \frac{a}{r} + br^2 \right]^{-1} dr^2 + r^2 d\Omega_2^2$$

- In 1975 A. H. Thompson [38] found geometrically degenerate solutions of Yang’s gravitational equations. In particular, he showed that the Bertotti-Robinson metric Eq. (42) is Yang Pure.
- Friedmann-Robertson-Walker (FRW) space-times

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right] \tag{45}$$

may be also characterized by a “perfect fluid” Ricci tensor

$$R_{jk} = \frac{1}{3}(R - 4\xi)u_j u_k + \frac{1}{3}(R - \xi)g_{jk} \tag{46}$$

and zero Weyl tensor. Here (see [39]): $u^k u_k = -1$, $\nabla_i u_j = H(u_i u_j + g_{ij})$ where $H = \dot{a}/a$ is Hubble’s parameter and $\xi = 3(H^2 + \dot{H}) = 3\ddot{a}/a$. The Cotton tensor being zero, a FRW space-time is Yang Pure if and only if $\nabla_j R = 0$.

The flat case $k = 0$ was solved by the authors [39]. While the two geometric constraints fix the Ricci tensor, the Einstein equations provide a source which is a perfect fluid with equation of state $p = w(t)\mu$ that evolves from $w = 1/3$ (pure radiation) to $w = -1$ (accelerated expansion, without a cosmological constant Λ).

3 Harada-Cotton gravity

The results of the previous section are interesting for Harada’s Cotton gravity. The symmetries of the Weyl tensor imply two important facts [1]:

- (1) $g^{kl}C_{jkl} = 0$, then (1) maintains the law $0 = \nabla_k T^{jk}$.
- (2) $\nabla^l C_{jkl} = 0$ implies that R_{jk} and T_{jk} commute:

$$0 = \nabla_l (\nabla_j T_k^l - \nabla_k T_j^l) = [\nabla_l, \nabla_j] T_k^l - [\nabla_l, \nabla_k] T_j^l + \nabla_j (\nabla_l T_k^l) - \nabla_k (\nabla_l T_j^l) = R_{ljkm} T^{ml} + R_{jm} T_k^m - R_{lkjm} T^{ml} - R_{km} T_j^m.$$
 The first term cancels the third one by the first Bianchi identity: $0 = (R_{ljkm} + R_{jklm} + R_{kljm}) T^{ml}$ (the second term vanishes).

As stated in the introduction, Eq. (1) naturally provides the Codazzi tensor in Eq. (4). Depending on its form, there are different levels of Cotton gravity, that are extensions of the Einstein gravity.

The choice of the Codazzi tensor restricts the space-time which, in turn, provides the structure of the Ricci tensor. Together, the Ricci and the Codazzi tensors determine the energy-momentum tensor:

$$T_{kl} = R_{kl} - \frac{1}{2}Rg_{kl} - \mathcal{C}_{kl} + g_{kl}\mathcal{C}^j{}_j \quad (47)$$

By construction, the metric of the space-time solves the Cotton-gravity equation (1) with the energy-momentum tensor (47). This approach reverses the standard one, where the matter tensor is the input. Of course it is simpler, but not alternative, than solving high order differential equations for the metric. Here a form of the metric is given a priori through the Codazzi tensor.

Equation (47) is Einstein's equation corrected by a Codazzi tensor, Eq. (6), in analogy with other theories of extended gravity (the H -term of Eq. 26 in [40]).

3.1 Yang Pure spaces.

Since the Ricci tensor is Codazzi, the definition (2) of Cotton tensor shows that Yang Pure spaces are solutions of the vacuum Harada equations $\mathcal{C}_{jkl} = 0$.

Harada's vacuum solution (3) is not a Yang Pure space.

Now we present the simplest Codazzi tensors, with examples that only aim at illustrating the procedure.

3.2 The trivial Codazzi tensors $\mathcal{C}_{jk} = 0$ and $\mathcal{C}_{jk} = Bg_{jk}$ (with B constant by the Codazzi condition) give the Einstein equations without or with a cosmological constant.

3.3 Case $\mathcal{C}_{jk} = Au_ju_k + Bg_{jk}$, $u^k u_k = -1$

The generalized Stephani Universes are solutions of the Harada equation with energy-momentum tensor (47) built with the Ricci tensor (18) and the Codazzi tensor.

Such inhomogeneous cosmological models may provide an explanation of the observed accelerated expansion of the universe and bypass the dark energy problem (see for example [41, 42] and references therein).

We here give the Ricci tensor for the simpler Stephani Universe in $n = 4$:

$$R_{kl} = 2AB u_k u_l + g_{kl}(3B^2 - AB) \quad (48)$$

Its perfect fluid form implies a perfect fluid source in the Einstein equations, as well as in the Harada equations (with different density and pressure).

3.4 Case $\mathcal{C}_{jk} = Au_ju_k$, $u_k u^k = -1$, \dot{u} closed

The Codazzi condition is equivalent to $\nabla_i u_j = -u_i \dot{u}_j$ and (11). The space-time is static and the velocity is eigenvector of the Ricci tensor: $R_{jk} u^k = -u_j (\nabla_k \dot{u}^k)$.

This example in $n = 4$ is static and spherically symmetric:

$$ds^2 = -b^2(r)dt^2 + f(r)^2dr^2 + r^2d\Omega_2^2 \tag{49}$$

The function $A(r)$ solves (11), where the time component is an identity and $A' = -Ab'/b$ (a prime is a derivative in r). The equation is solved by

$$A(r) = \frac{k}{b(r)}$$

where k is a constant. The covariant form of the Ricci tensor on static isotropic space-times was obtained in [21] (Eq. 49 with $\varphi = 0$):

$$\begin{aligned} R_{jk} &= u_j u_k \frac{R + 4\nabla_p \dot{u}^p}{3} + g_{jk} \frac{R + \nabla_p \dot{u}^p}{3} + \Pi_{jk} \\ \Pi_{jk} &= \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{3} \right] \left[\nabla_p \dot{u}^p - 3 \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) - 2E(r) \right] \end{aligned} \tag{50}$$

where $\eta = \dot{u}^j \dot{u}_j = b'^2/(b^2 f^2)$. $E(r)$ is the amplitude of the electric tensor

$$\begin{aligned} E(r) &= \frac{1}{2f^2} \left[\frac{f^2}{r^2} - \frac{1}{r^2} - \frac{f'}{fr} + \frac{b'}{b} \frac{d}{dr} \log(fr) - \frac{b''}{b} \right] \\ R &= \frac{2}{r^2} \left(1 - \frac{1}{f^2} \right) + \frac{4}{r} \frac{f'}{f^3} - \frac{2}{bf^2} \left(b'' - b' \frac{f'}{f} + 2 \frac{b'}{r} \right) \\ \nabla_p \dot{u}^p &= \frac{1}{bf^2} \left(b'' - b' \frac{f'}{f} + 2 \frac{b'}{r} \right) \end{aligned} \tag{51}$$

The traceless tensor Π_{jk} modifies the perfect fluid term. It is $\Pi_{jk} u^k = 0$ and $\Pi_{jk} \dot{u}^k \propto \dot{u}_j$.

The Ricci tensor has three eigenvalues and builds a Cotton tensor C_{jkl} that, by construction, solves Harada’s equation (1) for the following energy-momentum tensor:

$$\begin{aligned} T_{jk} &= u_j u_k \frac{R + 4\nabla_p \dot{u}^p}{3} + g_{jk} \frac{R + \nabla_p \dot{u}^p}{3} + g_{kl} \frac{T}{3} - g_{kl} \frac{R}{6} \\ &\quad - \mathcal{C}_{kl} + \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{3} \right] \left[\nabla_p \dot{u}^p - 3 \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) - 2E(r) \right]. \end{aligned}$$

A simplification is done with the expression of the trace T , and with the following identity (Lemma 3.4 in [21]):

$$\nabla_p \dot{u}^p - 3 \left(\eta + \frac{\dot{u}^i \nabla_i \eta}{2\eta} \right) = -\frac{2}{f^2} \left[\frac{b''}{b} - \frac{b'}{b} \frac{d}{dr} \log(rf) \right].$$

The result is:

$$T_{jk} = u_j u_k \left[\frac{R}{3} + \frac{4}{3} \nabla_p \dot{u}^p - \frac{k}{b} \right] + g_{jk} \left[-\frac{R}{6} + \frac{1}{3} \nabla_p \dot{u}^p - \frac{k}{b} \right] + \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{3} \right] \frac{1}{f^2} \left[-\frac{b''}{b} + \frac{b'}{b} \left(\frac{1}{r} + \frac{f'}{f} \right) - \frac{f^2 - 1}{r^2} + \frac{f'}{fr} \right].$$

The tensor specifies the parameters of an anisotropic fluid

$$T_{jk} = (P + \mu) u_j u_k + P g_{jk} + \left[\frac{\dot{u}_j \dot{u}_k}{\eta} - \frac{h_{jk}}{3} \right] (p_r - p_\perp)$$

with $P = \frac{1}{3} p_r + \frac{2}{3} p_\perp$ (effective pressure), density μ , radial pressure p_r , transverse pressure p_\perp , constructed with the free parameters $b(r)$, $f(r)$, k . Note the pressure anisotropy despite the spherical symmetry of the metric.

3.5 Case $\mathcal{C}_{jk} = \lambda(u_j \dot{u}_k + \dot{u}_j u_k)$ with closed \dot{u}_j

The metrics in examples 2.8–2.12 are static spherically symmetric solutions of equations of various gravity theories, Einstein, Einstein-Maxwell, low energy string, with their own matter or radiation content. However, since they all contain a current-flow Codazzi tensor, they all solve the Harada equation (1) with a proper energy-momentum tensor that is obtained below, characterized by a current-flow term.

The metrics determine the Ricci tensor (35) with radial symmetry. The energy-momentum tensor is (47) with $R = R^* - 2\eta - \epsilon/\eta$ and $R^* = 2/L^2$:

$$T_{kl} = u_k u_l \frac{R^*}{2} + h_{kl} \left[-\frac{R^*}{6} + \frac{2\eta}{3} + \frac{\epsilon}{3\eta} \right] - \mathcal{C}_{kl} - 2 \left[\frac{\dot{u}_k \dot{u}_l}{\eta} - \frac{h_{kl}}{3} \right] \left[\eta + \frac{\epsilon}{2\eta} + E(r) \right]$$

$$E(r) = \frac{1}{2} \frac{1}{f_1^2} \left(\frac{f_1^2}{L^2} + \frac{b'}{b} \frac{f_1'}{f_1} - \frac{b''}{b} \right).$$

It is the energy-momentum tensor of a fluid with velocity u_j , acceleration \dot{u}_j , energy density $\mu = \frac{1}{2} R^*$, pressure anisotropy $p_r - p_\perp = -2\eta - \epsilon/\eta - 2E(r)$ and effective pressure $3P = p_r + 2p_\perp = -\frac{R^*}{2} + 2\eta + \frac{\epsilon}{\eta}$.

Example 3.1 Consider the Bertotti-Robinson metric in 2.9: $b(r) = f_1(r) = r_0/r$, $L = r_0$. It is $\eta = 1/r_0^2$, $\epsilon = 0$ and $\lambda = -\kappa r^2/r_0$.

The Ricci tensor for this metric is: $R_{kl} = \frac{1}{r_0^2} (2u_k u_l + g_{kl}) - 2\dot{u}_k \dot{u}_l$ and $R = 0$. The metric (42) solves the Harada equation with the traceless energy-momentum tensor

$$T_{kl} = \frac{1}{r_0^2} (2u_k u_l + g_{kl}) - 2\dot{u}_k \dot{u}_l + \frac{\kappa}{r_0} r^2 (u_k \dot{u}_l + \dot{u}_k u_l).$$

3.6 Cotton gravity in constant curvature space-times

Harada made the remark that a de Sitter metric is a vacuum solution ($T_{jk} = 0$) of the Cotton gravity equation (1). We extend his remark.

Constant curvature space-times are defined by the Riemann tensor

$$R_{jklm} = \frac{R}{n(n-1)}(g_{jl}g_{km} - g_{jm}g_{kl})$$

They are conformally flat ($C_{jklm} = 0$) and Einstein ($R_{jk} = g_{jk}R/n$). In the Lorentzian signature there are exactly three cases: Minkowski ($R = 0$), de Sitter ($R > 0$) and anti-de Sitter ($R < 0$) (see [43], pages 124, 131).

Ferus [10] proved that the only non-trivial Codazzi tensor in a constant curvature space-time has the form

$$\mathcal{C}_{jk} = \nabla_j \nabla_k \phi + \frac{R\phi}{n(n-1)}g_{jk} \quad (52)$$

where ϕ is a smooth scalar field (see also [44] p.436). Then

$$\begin{aligned} T_{kl} &= \frac{R}{n}g_{kl} - \frac{R}{2}g_{kl} - \mathcal{C}_{kl} + g_{kl}\mathcal{C}^j_j \\ &= g_{kl} \left[\nabla_j \nabla^j \phi + \frac{1}{n}R\phi - \frac{n-2}{2n}R \right] - \nabla_k \nabla_l \phi \end{aligned} \quad (53)$$

is the most general energy-momentum tensor for Cotton gravity in constant curvature space-times. Therefore, Minkowski, de Sitter, anti-de Sitter space-times solve the Cotton gravity equation. The Codazzi tensor introduces a coupling of gravity with a scalar field.

4 Conclusion

Codazzi tensors have an intrinsic geometric importance. They also naturally enter in the recently proposed Cotton gravity by Harada. The specific form of a Codazzi tensor restricts the space-time it lives in.

We presented a strategy to find solutions of Cotton gravity. In essence, we showed that the equation for Cotton gravity is the Einstein equation modified by the presence of a Codazzi tensor. We investigated two specific forms of Codazzi tensors: the perfect fluid and the current flow.

In the first case the hosting metric turns out to be a generalization of Stephani Universes. Stephani Universes are conformally flat cosmological solutions of the Einstein equations with perfect fluid source.

In the second case, a static current flow Codazzi tensor generates metrics that embrace Nariai and Bertotti-Robinson space-times, and extensions. In the literature they are solutions of various gravity theories, such as Einstein, low energy string,

Einstein-Maxwell and so on. By construction, all these metrics solve the Harada-Cotton gravity in geometries selected by the Codazzi tensor, with stress-energy tensors different from the original theory.

An interesting question is whether other forms of Codazzi tensors may give rise to new solutions of Cotton gravity of physical interest, using the same strategy.

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Appendix 1: Proof of theorem 2.1

Proof For a perfect fluid tensor, the Codazzi condition $0 = \nabla_i \mathcal{C}_{jk} - \nabla_j \mathcal{C}_{ik}$ is:

$$0 = u_k(u_j \nabla_i A - u_i \nabla_j A) + (g_{jk} \nabla_i B - g_{ik} \nabla_j B) + Au_k(\nabla_i u_j - \nabla_j u_i) + A(u_j \nabla_i u_k - u_i \nabla_j u_k) \quad (54)$$

Contraction with u^k , and $u^k \nabla_j u_k = 0$ give: $0 = -u_j \nabla_i (A - B) + u_i \nabla_j (A - B) - A(\nabla_i u_j - \nabla_j u_i)$. Another contraction with u^j

$$0 = \nabla_i (A - B) + u_i (\dot{A} - \dot{B}) + A \dot{u}_i \quad (55)$$

simplifies the previous equation to: $0 = A(u_j \dot{u}_i - u_i \dot{u}_j - \nabla_i u_j + \nabla_j u_i)$. Since $A \neq 0$: $0 = u_j \dot{u}_i - u_i \dot{u}_j - \nabla_i u_j + \nabla_j u_i$. By inserting the standard decomposition

$$\nabla_i u_j = \varphi(g_{ij} + u_i u_j) + \sigma_{ij} + \omega_{ij} - u_i \dot{u}_j$$

where φ is the expansion parameter, σ_{ij} is the shear and ω_{ij} is the vorticity, the result is: $\omega_{ij} = 0$ (the perfect fluid velocity is vorticity-free).

The contraction of (54) with g^{jk} is:

$$0 = -\nabla_i A - u_i \dot{A} + (n - 1)\nabla_i B - A \dot{u}_i - Au_i \nabla_k u^k \quad (56)$$

Contraction with u^i : $(n - 1)\dot{B} + A\nabla_k u^k = 0$. Then the equation becomes:

$$0 = -\nabla_i[A - (n - 1)B] - u_i[\dot{A} - (n - 1)\dot{B}] - A\dot{u}_i$$

Together with (55) the equations give:

$$0 = -\nabla_i[A - (n - 1)B] - u_i[\dot{A} - (n - 1)\dot{B}] + \nabla_i(A - B) + u_i(\dot{A} - \dot{B})$$

i.e. $\nabla_i B = -u_i \dot{B}$. Then Eq. (55) gives: $\nabla_i A = -u_i \dot{A} - A\dot{u}_i$.

Contraction with u^i of (56) gives $(n - 1)\dot{B} + A\nabla_k u^k = 0$ i.e. $\varphi = -\frac{\dot{B}}{A}$.

Let us introduce the expansion of $\nabla_i u_j$ with $\omega_{ij} = 0$ in (54). Several terms simplify to give: $u_j \sigma_{ik} = u_i \sigma_{jk}$ i.e. $\sigma_{jk} = 0$. Therefore: $\nabla_i u_j = \varphi(g_{ij} + u_i u_j) - u_i \dot{u}_j$. Finally, we evaluate the gradient of the expansion parameter:

$$\begin{aligned} \nabla_i \varphi &= -\frac{1}{A} \nabla_i \dot{B} + \dot{B} \frac{\nabla_i A}{A^2} \\ &= -\frac{1}{A} \nabla_i (u^k \nabla_k B) + \frac{\dot{B}}{A^2} (-u_i \dot{A} - \dot{u}_i A) \end{aligned}$$

The first term contains $(\nabla_i u^k) \nabla_k B + u^k \nabla_i \nabla_k B = [\varphi(\delta_i^k + u_i u^k) - u_i \dot{u}^k](-u_k \dot{B}) + u^k \nabla_k (-u_i \dot{B}) = -\dot{u}_i \dot{B} - u_i \ddot{B}$. Then: $\nabla_i \varphi = -u_i (\frac{\dot{B}\dot{A}}{A^2} - \frac{\ddot{B}}{A}) = -u_i \dot{\varphi}$.

The opposite statement holds: if a perfect fluid tensor solves eqs.(10)-(13) then it is Codazzi. The right hand side of Eq. (54) is evaluated with the conditions:

$$\begin{aligned} &u_k(u_j \nabla_i A - u_i \nabla_j A) + (g_{jk} \nabla_i B - g_{ik} \nabla_j B) \\ &\quad + Au_k(\nabla_i u_j - \nabla_j u_i) + A(u_j \nabla_i u_k - u_i \nabla_j u_k) \\ &= -Au_k(u_j \dot{u}_i - u_i \dot{u}_j) - \dot{B}(g_{jk} u_i - g_{ik} u_j) \\ &\quad - Au_k(u_i \dot{u}_j - u_j \dot{u}_i) + A[u_j(\varphi g_{ik} - u_i \dot{u}_k) - u_i(\varphi g_{jk} - u_j \dot{u}_k)] \\ &= -(\dot{B} + \varphi A)(g_{jk} u_i - g_{ik} u_j) = 0 \end{aligned}$$

with use of the expression (13) for φ . □

Appendix 2: Proof of theorem 2.6

Proof Suppose that the tensor $\nabla_i \mathcal{C}_{jk} - \nabla_j \mathcal{C}_{ik} = 0$ is Codazzi, with $\nabla_j \dot{u}_k = \nabla_k \dot{u}_j$.

- (1) Since $\dot{u}^k u_k = 0$, it is $\dot{u}^k \nabla_j u_k = -u^k \nabla_j \dot{u}_k = -u^k \nabla_k \dot{u}_j = -\dot{u}_j$.
- (2) Since $\dot{u}^k u_k = 0$, it is $\ddot{u}^k u_k = -\dot{u}^k \dot{u}_k = -\eta$.
- (3) Contraction of the Codazzi condition with g^{jk} gives:

$$\begin{aligned} 0 &= \nabla^k [\lambda(u_i \dot{u}_k + \dot{u}_i u_k)] \\ &= u_i (\dot{u}^p \nabla_p \lambda + \lambda \nabla_k \dot{u}^k) + \dot{u}_i (\dot{\lambda} + \lambda \nabla_k u^k) + \lambda \ddot{u}_i + \lambda \dot{u}^k \nabla_k u_i \end{aligned}$$

Contraction with \dot{u}^i and contraction with u^i , with properties (1), (2) give:

$$\begin{aligned} \dot{\lambda} + \lambda \nabla_k u^k &= 0 \\ \dot{u}^p \nabla_p \lambda + \lambda \nabla_k \dot{u}^k + \lambda \eta &= 0 \end{aligned}$$

What remains of the equation is $0 = -\eta u_i + \ddot{u}_i + \dot{u}^k \nabla_k u_i$.

(4) Contraction of the Codazzi condition with u^k is:

$$\begin{aligned} 0 &= \nabla_i (u^k \mathcal{E}_{jk}) - \mathcal{E}_{jk} \nabla_i u^k - \nabla_j (u^k \mathcal{E}_{ik}) + \mathcal{E}_{ik} \nabla_j u^k \\ &= \nabla_i (-\lambda \dot{u}_j) - \lambda u_j \dot{u}_k \nabla_i u^k \nabla_j (-\lambda \dot{u}_i) - \lambda u_i \dot{u}_k \nabla_j u^k \\ &= -\nabla_i (\lambda \dot{u}_j) + \lambda u_j \ddot{u}_i + \nabla_j (\lambda \dot{u}_i) - \lambda u_i \ddot{u}_j \\ &= -(\nabla_i \lambda) \dot{u}_j + \lambda u_j \ddot{u}_i + (\nabla_j \lambda) \dot{u}_i - \lambda u_i \ddot{u}_j \end{aligned} \tag{57}$$

Contraction with u^i is: $0 = -\dot{\lambda} \dot{u}_j + \lambda u_j u^i \ddot{u}_i + \lambda \ddot{u}_j$ i.e.

$$\lambda \ddot{u}_j = \lambda \eta u_j + \dot{\lambda} \dot{u}_j$$

Using this in Eq. (57) gives: $0 = -(\nabla_i \lambda + \dot{\lambda} u_i) \dot{u}_j + (\nabla_j \lambda + \dot{\lambda} u_j) \dot{u}_i$ with solution

$$\nabla_i \lambda = -\dot{\lambda} u_i + \dot{u}_i \frac{\dot{u}^p \nabla_p \lambda}{\eta} \tag{58}$$

(5) Let us rewrite in full the Codazzi condition, using the results found so far.

To manage it, begin with:

$$\nabla_i \mathcal{E}_{jk} = [-\dot{\lambda} u_i + \dot{u}_i \frac{\dot{u}^p \nabla_p \lambda}{\eta}] (u_j \dot{u}_k + \dot{u}_j u_k) + \lambda \nabla_i (u_j \dot{u}_k + \dot{u}_j u_k)$$

Now subtract the expression with first two indices exchanged, and use closedness:

$$\begin{aligned} 0 &= (\dot{u}_i u_j - u_i \dot{u}_j) (\dot{\lambda} u_k + \frac{\dot{u}^p \nabla_p \lambda}{\eta} \dot{u}_k) + \lambda (\nabla_i u_j - \nabla_j u_i) \dot{u}_k \\ &\quad + \lambda (u_j \nabla_i \dot{u}_k - u_i \nabla_j \dot{u}_k) + \lambda (\dot{u}_j \nabla_i u_k - \dot{u}_i \nabla_j u_k) \end{aligned} \tag{59}$$

(6) Contraction with u^i and elimination of \ddot{u}_k :

$$\begin{aligned} 0 &= \dot{u}_j (\dot{\lambda} u_k + \frac{\dot{u}^p \nabla_p \lambda}{\eta} \dot{u}_k) + 2\lambda \dot{u}_j \dot{u}_k + \lambda (u_j \ddot{u}_k + \nabla_j \dot{u}_k) \\ &= \dot{u}_j (\dot{\lambda} u_k + \frac{\dot{u}^p \nabla_p \lambda}{\eta} \dot{u}_k) + 2\lambda \dot{u}_j \dot{u}_k + \lambda \eta u_j u_k + \dot{\lambda} u_j \dot{u}_k + \lambda \nabla_j \dot{u}_k \end{aligned}$$

We then obtain:

$$\lambda \nabla_j \dot{u}_k = -\lambda \eta u_j u_k - \dot{\lambda} (\dot{u}_j u_k + u_j \dot{u}_k) - \dot{u}_j \dot{u}_k \left(2\lambda + \frac{\dot{u}^p \nabla_p \lambda}{\eta} \right) \tag{60}$$

and the contraction with \dot{u}^k : $\frac{1}{2}\lambda\nabla_j\eta = -\dot{\lambda}\eta u_j - \dot{u}_j(2\lambda\eta + \dot{u}^p\nabla_p\lambda)$. In particular,

$$2\lambda + \lambda\frac{\dot{u}^j\nabla_j\eta}{2\eta^2} + \frac{\dot{u}^p\nabla_p\lambda}{\eta} = 0 \tag{61}$$

This relation in (58) and in (60) respectively gives Eqs. (25) and (26).

(7) Contraction of (59) with \dot{u}^k is:

$$0 = (\dot{u}_i u_j - u_i \dot{u}_j)(\dot{u}^p\nabla_p\lambda) + \lambda\eta(\nabla_i u_j - \nabla_j u_i) + \frac{1}{2}\lambda(u_j\nabla_i\eta - u_i\nabla_j\eta) - \lambda(\dot{u}_j\ddot{u}_i - \dot{u}_i\ddot{u}_j)$$

Now specify \ddot{u}_k and $\nabla_k\eta$: $0 = (u_i\dot{u}_j - u_j\dot{u}_i) + (\nabla_i u_j - \nabla_j u_i)$. This statement means that the velocity is vorticity-free.

(8) Contraction of (59) with \dot{u}^i :

$$\begin{aligned} 0 &= u_j(\dot{\lambda}\eta u_k + \dot{u}_k\dot{u}^p\nabla_p\lambda) + \lambda\dot{u}^i(\nabla_i u_j - \nabla_j u_i)\dot{u}_k \\ &\quad + \lambda(u_j\dot{u}^i\nabla_i\dot{u}_k + \dot{u}_j\dot{u}^i\nabla_i u_k - \eta\nabla_j u_k) \\ &= u_j(\dot{\lambda}\eta u_k + \dot{u}_k\dot{u}^p\nabla_p\lambda) + \lambda\eta u_j\dot{u}_k \\ &\quad + \lambda(u_j\dot{u}^i\nabla_i\dot{u}_k + \dot{u}_j\dot{u}^i\nabla_i u_k - \eta\nabla_j u_k) \end{aligned}$$

Note that $\dot{u}^i\nabla_i\dot{u}_k = \dot{u}^i\nabla_k\dot{u}_i = \frac{1}{2}\nabla_k\eta = -(\dot{\lambda}/\lambda)\eta u_k - \dot{u}_k[2\eta + (\dot{u}^p\nabla_p\lambda)/\lambda]$.

Next, a result in (3) is: $\dot{u}^i\nabla_i u_k = -\ddot{u}_k + \eta u_k = -(\dot{\lambda}/\lambda)\dot{u}_k$. We then obtain:

$$\begin{aligned} \lambda\eta\nabla_j u_k &= u_j(\dot{u}_k\dot{u}^p\nabla_p\lambda + \lambda\eta\dot{u}_k) - u_j\dot{u}_k(2\eta\lambda + \dot{u}^p\nabla_p\lambda) \\ &\quad - \dot{\lambda}\dot{u}_j\dot{u}_k = -\lambda\eta u_j\dot{u}_k - \dot{\lambda}\dot{u}_j\dot{u}_k \end{aligned}$$

Then: $\nabla_j u_k = -\frac{\dot{\lambda}}{\lambda}\frac{\dot{u}_j\dot{u}_k}{\eta} - u_j\dot{u}_k$.

Now we prove the statement the way back. Let's evaluate with conditions (24)–(26) and closed \dot{u}_i :

$$\begin{aligned} \nabla_i\mathcal{E}_{jk} - \nabla_j\mathcal{E}_{ik} &= (u_j\dot{u}_k + \dot{u}_j u_k)\nabla_i\lambda - (u_i\dot{u}_k + \dot{u}_i u_k)\nabla_j\lambda \\ &\quad + \lambda[\nabla_i(u_j\dot{u}_k + \dot{u}_j u_k) - \nabla_j(u_i\dot{u}_k + \dot{u}_i u_k)]. \end{aligned}$$

In the first line we use (25):

$$\begin{aligned} &- \left[u_i\dot{\lambda} + \lambda\dot{u}_i \left(2 + \frac{\dot{u}^p\nabla_p\eta}{2\eta^2} \right) \right] (u_j\dot{u}_k + \dot{u}_j u_k) + \left[u_j\dot{\lambda} + \lambda\dot{u}_j \left(2 + \frac{\dot{u}^p\nabla_p\eta}{2\eta^2} \right) \right] \\ &\quad (u_i\dot{u}_k + \dot{u}_i u_k) \\ &= -(u_i\dot{u}_j - u_j\dot{u}_i) \left[u_k\dot{\lambda} - \lambda\dot{u}_k \left(2 + \lambda\dot{u}_k\frac{\dot{u}^p\nabla_p\eta}{2\eta^2} \right) \right]. \end{aligned}$$

In the second line, we use closedness, (24) and (26):

$$\begin{aligned} & \lambda[(\nabla_i u_j - \nabla_j u_i)\dot{u}_k + \dot{u}_j \nabla_i u_k - \dot{u}_i \nabla_j u_k + u_j \nabla_i \dot{u}_k - u_i \nabla_j \dot{u}_k] \\ &= \lambda \left[(-u_i \dot{u}_j + u_j \dot{u}_i)\dot{u}_k - \dot{u}_j u_i \dot{u}_k + \dot{u}_i u_j \dot{u}_k \right] + (u_i \dot{u}_j - \dot{u}_i u_j) \left(\dot{\lambda} u_k - \lambda \dot{u}_k \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \right) \\ &= (u_i \dot{u}_j - u_j \dot{u}_i) \left[-2\lambda \dot{u}_k + \dot{\lambda} u_k - \lambda \dot{u}_k \frac{\dot{u}^p \nabla_p \eta}{2\eta^2} \right]. \end{aligned}$$

The addends cancel and $\nabla_i \mathcal{C}_{jk} - \nabla_j \mathcal{C}_{ik} = 0$. \square

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