On cosmological expansion and local physics

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Abstract

We find an exact convergence in the local dynamics described by two supposedly antagonistic approaches applied at the local, solar system scale: one starting from an expanding universe perspective such as FLRW, the other based on a local model ignoring any notion of expansion, such as static Schwarzschild dS. Both models are in complete agreement when the local effects of the expansion are circumscribed to the presence of the cosmological constant. We elaborate on the relevant role of static backgrounds like the Schwarzschild-dS metric in standard form as the most proper coordinatizations to describe physics at the local scale. We also elaborate on the popular expanding 3-space picture—to be distinguished from that of the expanding universe—and point out the confusion of scales which is typically associated with it. Finally, making use of an old and too often forgotten relativistic kinematical invariant, we address some remaining misunderstandings on space expansion, cosmological and gravitational redshifts. As a byproduct we propose a *unique and unambiguous* prescription to match the local and cosmological expression of a specific observable.

Keywords Expansion of the universe · Expansion of space · de Sitter metric · Friedman Laemaitre Robertson Walker cosmological models · Doppler effect · Gravitational redshift · Cosmological redshift

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1 Motivation

Different and opposing views coexist at this moment as to whether the expansion of the universe affects the local dynamics at the scale of the solar system, and the amount of this effect. It is true that the consequence, if any, is too small to be detectable, but the question of principle remains as to what impact has the expansion of the universe on local systems. Here we focus on two—in principle—opposite views that compete in this arena:

- (i) On one side there is the very popular “expanding space” idea which claims that it is the very fabric of 3-space that is growing as time passes by, thus giving rise to the observational effects like the recession of galaxies, see for instance pag.307 in [1]. Imported to the local¹ ground, albeit perhaps with different nuances, there is the view that such an effect may exist. To put it simply, its effect on the local dynamics² of a particle would boil down to an additional repulsive acceleration term, a functional of the scale factor $a(t)$ present in the FLRW background metric, and of the particle’s position.
- (ii) On the other side [2], other analysis ignore every fact about the expanding universe, reducing its local effect to the presence of a non-vanishing cosmological constant. In doing so one applies locally the Schwarzschild de Sitter metric in its static form, leaving no room whatsoever for the very idea of any possible effect of the expansion itself.

¹ By “local” we mean basically the solar system scale, see Stage 1 in Fig. 1. We make also the symplifying assumption of spherical symmetry and a Λ -vacuum as the only ingredient of the energy-momentum tensor.

² We do not claim that these authors endorse the expanding space idea (see Sect. 5), their work just being that of examining the consequences of such an assumption at the local scale.

Fig. 1 Different setups to describe phenomena at different scales

It clearly seems that both pictures can't hold simultaneously. We will try in this paper to elaborate in favor of what we think is the correct standpoint. It is based on an obvious fact, to wit, that when examining the adequacy of a metric in order to describe a certain physical situation, one must ensure its consistency with the right hand side (rhs) of Einstein's equations, that is, the matter energy-momentum tensor.³

Consider for instance the usual layman question: *if space is expanding, does this means that my home is expanding?*, followed with the intriguing: *but, if my measuring stick is expanding too, how can I measure such an effect in the first place?*. If we take a look at the Einstein equations at our local scale, we will find an answer to the former question, which in turn makes the latter void of content. What can one infer from the Einstein equations at our local scale? First and foremost: that, except for the possible presence of a cosmological constant, there is no trace whatsoever of the homogeneous Hubble flow which sources the FLRW metric,⁴ see Fig. 1.

The reason is more than obvious: the Hubble flow is the averaged picture of the distribution of matter that only works at much, much larger scales, than the local one considered here. And therefore, *the FLRW metric is just a broad-brush, coarse-grained, averaged picture of the real metric of spacetime, only apt to describe phenomena at the cosmological scale*. Simply we can not continue to use this concept of an homogeneous Hubble flow at the local scale and simply add to it the local inhomogeneities, which already become non-linear in Stage 2 of Fig. 1. Furthermore there is not such flow at the local scale, Stage 1, except for its possible cosmological constant component. Take for instance the solar system, and adopt as approximately valid the simplifying assumption of spherical symmetry of the matter external to it, then we find ourselves in the framework of the Einstein–Straus approach [3] in which clearly the Hubble flow has no effect—except for the cosmological constant—on the local system.

This paper presents a critical assessment of some of these misunderstandings in cosmology (for a comprehensible introduction see [4]), while we investigate under

³ By “matter” we include ordinary matter, radiation, dark matter, dark energy—understood as the cosmological constant—or in general whatever source that we put in the right hand side of Einstein equations.

⁴ We will adopt in the following the obvious simplification to consider the effect of the CBR or the neutrino background on the FLRW scale factor negligible, as if the intergalactic vacuum were only permeated by the cosmological constant, thus the sources reduce to that of matter and cosmological constant.

which circumstances local and cosmological physics match. In terms of Fig.1 this means that we are concerned with Stage 1 and Stage 4 and do not intend to enter into the subtleties of the intermediate stages 2 and 3, for which we do not make any analytical proposal.

We start by reviewing the de Sitter spacetime and its cosmological incarnations, Sect. 2. In particular we introduce some basic physical requirements to produce in a natural way the changes of coordinates from the static dS metric to its cosmological versions, so that the comoving observers are in geodesic motion. In Sect. 3 we discuss under which circumstances the dynamics for geodesics in a cosmological de Sitter metric is physically equivalent to that induced by the dS component in a static Schwarzschild-dS. This brings the opportunity to discuss, Sect. 4, the role of non-static metrics and the distinguished Schwarzschild-dS coordinatization to describe the local scale. On the other side, at the cosmological scale, we show that the only possible source of the energy-momentum tensor to bring a FLRW metric to a static form is that of a cosmological constant. We further discuss in Sect. 5 the ideas around the notion of expanding 3-space and notice a typical confusion of scales in some of its popular pictures. We end, Sect. 6, by elaborating on the interpretation of gravitational/cosmological redshifts as Doppler effect generalized to General Relativity.

In Appendix A we have gathered the construction of an invariant for the Doppler effect and its generalization to General Relativity. Applications of this generalization to the massive case are discussed in Appendix B.

2 de Sitter spacetime: static and cosmological incarnations

Since de Sitter (dS) spacetime plays a relevant role throughout this paper, we review in the sequel its main features. It can be defined [5] by a 4-dimensional embedding in a flat, 5-dimensional Minkowski spacetime, $\mathcal{M}^{(1,4)}$, with coordinates $Z = (x_0, x_4, x_1, x_2, x_3)$ with Lorentzian metric

$$ds_{\mathcal{M}^{(1,4)}}^2 = -dx_0^2 + dx_4^2 + dx_1^2 + dx_2^2 + dx_3^2. \quad (2.1)$$

Then the dS background is described by the hyperboloid submanifold

$$Z^2 = -x_0^2 + x_4^2 + x_1^2 + x_2^2 + x_3^2 = \frac{1}{H^2}, \quad H^2 := \frac{\Lambda}{3}, \quad (2.2)$$

with the cosmological constant $\Lambda > 0$. One can define coordinates (T, x_1, x_2, x_3) for the dS submanifold by

$$\begin{aligned} x_0 &= \frac{\sqrt{1 - H^2 R^2}}{H} \sinh(HT), \quad x_4 = \frac{\sqrt{1 - H^2 R^2}}{H} \cosh(HT), \\ R &= \sqrt{x_1^2 + x_2^2 + x_3^2} \end{aligned} \quad (2.3)$$

so that (2.2) is satisfied and the metric induced on the quadric by the ambient metric (2.1) becomes

$$\begin{aligned} ds^2 &= -(1 - H^2 R^2) dT^2 + \frac{1}{(1 - H^2 R^2)} dR^2 + R^2 d\Omega^2, \\ d\Omega^2 &= d\theta^2 + \sin^2(\theta) d\varphi^2. \end{aligned} \quad (2.4)$$

Two remarks are in order: (i) these coordinates do not cover the whole hyperboloid (2.2). (ii) A static observer in this background, (R, θ, φ) constant, is constantly accelerating.

Starting from (2.4), we introduce, under the guidance of some physical requirements, the cosmological, spherically symmetric versions of dS. These requirements boil down to two key points: first, to obtain a radial coordinate such that the observers comoving with it are time-like geodesics, and second, to take the proper time of these observers as the new time coordinate. Thus our first task is to find the radial geodesics for the background (2.4). In fact we can study a more general case. Consider spherically symmetric static metrics of the form

$$ds^2 = -f(R) dT^2 + \frac{1}{f(R)} dR^2 + R^2 d\Omega^2, \quad (2.5)$$

which include Schwarzschild, dS, AdS, Schwarzschild-dS or Schwarzschild-AdS metrics. We look for the equations for the radial time-like geodesics $(T(s), R(s), \theta_0, \varphi_0)$ in terms of proper time s . The 4-velocity, $V(s) = (T'(s), R'(s), \theta_0, \varphi_0)$, and the proper time condition $V^2 = -1$ implies $T'(s)^2 = \frac{f(R(s)) + (R'(s))^2}{f(R(s))^2}$. Implementing this into the geodesic equation, $V^\mu \nabla_\mu V^\rho = \frac{d}{ds} V^\rho + V^\mu \Gamma_{\mu\nu}^\rho V^\nu = 0$, one obtains the radial equation

$$\frac{1}{2} f'(R(s)) + R''(s) = 0. \quad (2.6)$$

Thus for (2.4) we have

$$R''(s) - H^2 R(s) = 0 \quad \Rightarrow \quad R'(s)^2 - H^2 R(s)^2 = C, \quad (2.7)$$

being C a constant. Time-like radial geodesics are classified according to the sign of this constant. Assuming conventional initial conditions at $s = 0$, the different solutions to (2.7) are:

$$\begin{aligned} C = 0 : \quad & R(s) = r e^{Hs}, \quad e^{-2H T(s)} = e^{-2Hs} - H^2 r^2. \\ C > 0 : \quad & R(s) = r \sinh(Hs), \quad \tanh(H T(s)) = \sqrt{1 + H^2 r^2} \tanh(Hs). \\ C < 0 : \quad & R(s) = r \cosh(Hs), \quad \tanh(H T(s)) = \frac{\tanh(Hs)}{\sqrt{1 - H^2 r^2}}, \end{aligned} \quad (2.8)$$

with $r > 0$ constant and $T(s)$ given in implicit form. For vanishing C there exist in addition the solution $R(s) = re^{-Hs}$ which represents geodesics moving inwards instead of outwards. Notice that, from the viewpoint of the static observers in the background (2.4), the constant C in (2.7) is, in Newtonian language, proportional to the conserved energy of the corresponding geodesic motion.

Equations (2.8) suggest three different changes of coordinates based on the physical requirements mentioned above. In this way the parameter r will be taken as the new radial coordinate and the proper time s as the new coordinate t .

On the whole we obtain three different changes of coordinates $(T, R) \rightarrow (t, r)$ that can be read from (2.8) by writing t instead of s and $R(t, r)$ and $T(t, r)$ in place of $R(s)$, $T(s)$. The metric (2.4) is written, in the new coordinates, as

$$\begin{aligned} C = 0 : \quad ds^2 &= -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2). \\ C > 0 : \quad ds^2 &= -dt^2 + \sinh(Ht)^2 \left(\frac{1}{1 + H^2 r^2} dr^2 + r^2 d\Omega^2 \right), \quad C = H^2 r^2. \\ C < 0 : \quad ds^2 &= -dt^2 + \cosh(Ht)^2 \left(\frac{1}{1 - H^2 r^2} dr^2 + r^2 d\Omega^2 \right), \quad C = -H^2 r^2, \end{aligned} \quad (2.9)$$

which are the well known cosmological dS metrics with the flat, hyperbolic and spherical slices, respectively. The physical arguments used to produce these changes of coordinates will be a guiding principle in the discussions in the next section.

3 Local effects of the expansion

Let us consider the general FLRW cosmology, with metric (see Stage 4 of Fig. 1),

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{1}{1 - \sigma r^2} dr^2 + r^2 d\Omega^2 \right), \quad (3.1)$$

being σ a constant. Although, as mentioned above, this metric is a valid description of spacetime at the cosmological scale, let's us elaborate on the outcomes obtained by assuming for a while its correctness at the local scale. It is straightforward to quantify the local effect of being in such cosmological background. One should find a suitable description of the time evolving physical distance between a comoving observer at a fixed radial coordinate r and the center $r = 0$. Without entering into fine details, is is clear that a function like⁵ $R(t) = a(t)r$ accomplishes this goal.⁶ In fact one realizes that for $\sigma = 0$, it gives the distance from the location (r, θ, φ) to the origin $r = 0$, obtained with the metric induced from (3.1) on the equal time hypersurface.⁷ Clearly

⁵ As a matter of fact there is also an r dependence in R but we do not explicitate it because it will reappear as an integration constant in the ordinary differential equations (3.3), (3.5) with parameter time.

⁶ It is worth noticing that, after checking with (2.9), the trajectories $R(s)$ in (2.8) can be written in this form.

⁷ For $\sigma \neq 0$ that would not be exactly the distance, but it will remain a good approximation as long as $\sigma r^2 < 1$.

this new time dependent radial variable satisfies [6–8],

$$R''(t) = a''(t) r = \frac{a''(t)}{a(t)} R(t), \quad (3.2)$$

and thus it makes the case that the effect of the expansion on the local dynamics, either at the scale of the planetary orbits or at that of the electronic orbits of an atom, is just a repulsive⁸ radial acceleration, proportional to the distance $R(t)$ and to the time-time component of the Ricci tensor $\frac{a''(t)}{a(t)}$. In fact, the more complete discussion in [9], with the use of Fermi coordinates, leads to the same results. Thus one concludes that, *if such FLRW (3.1) models were valid at the local scale, Stage 1 of Fig. 1, then (3.2) would capture the effect of the expansion.* This would make the general derivation of (3.2) just an interesting and valuable academic exercise, except that it turns to be right in a qualified and restricted sense: a contribution to the rhs of (3.2) remains at the local scale when the Hubble flow includes a cosmological constant component. In such case, the cosmological constant will remain as the only local effect of the Hubble flow, because it is present *at all scales*. In addition to that, in the case that $a(t)$ corresponds to dS or AdS spacetimes, and in view of (2.8) and (2.9), the variable R in (3.2) is nothing but the radial variable in the static coordinatization (2.4).

The cosmological dS spacetimes have been studied in Sect. 2 and the procedure to deal with AdS spacetime is analogous. In all these cases the same relation holds, $\frac{a''(t)}{a(t)} = \frac{\Lambda}{3}$, with Λ positive for dS and negative for AdS. Thus, either for dS or AdS, (3.2) becomes

$$R''(t) = \frac{\Lambda}{3} R(t). \quad (3.3)$$

Up to here we have analyzed the effects of the cosmological constant in the framework of cosmological FLRW models, Stage 4 in Fig. 1. Now let us turn our attention to the approach [2] where the presence of a cosmological constant at the solar system scale is examined by using a static version of the Schwarzschild-dS metric [10], in which (2.5) is materialized with

$$f(R) = 1 - \frac{2M}{R} - \frac{\Lambda}{3} R^2. \quad (3.4)$$

The radial geodesic equation is obtained from (2.6),

$$R''(s) = -\frac{M}{R(s)^2} + \frac{\Lambda}{3} R(s), \quad (3.5)$$

which contains an attractive Newtonian leading term in addition to a repulsive acceleration coming from the presence of Λ , which we isolate,

$$R''(s) = \frac{\Lambda}{3} R(s). \quad (3.6)$$

⁸ As long as $a''(t) > 0$.

It is centrifugal for dS and centripetal for AdS.⁹

Results in (3.3) and (3.6) match, but this could be just a formal coincidence because of the notation. So we must look closely at the physical meaning attached to (3.3) and (3.6). This has been already pursued in Sect. 2, where it was shown that the time coordinate t in the cosmological dS settings (2.9) is just the proper time s of the radial geodesics of (2.4). We conclude therefore that (3.3) and (3.6) are equations with identical content.

The first lesson we extract from this analysis is that Eq. (3.2) is indeed correct at the local scale if one restricts its application to the only acceptable case, that is, when the Hubble flow is represented—at this scale—by the cosmological constant.

The second lesson is that, at this local scale, cosmological effects are completely captured with a static metric, of which (3.4) is a good example, and there is no need to implement a time dependent metric.

We elaborate slightly more in this second point in the following.

4 Static versus non-static metrics

4.1 A tale at the local scale

Let us take the two versions of dS spacetime, (a) the static (2.4) and (b) the expanding (2.9). While static observers in the former do not observe time evolution in their spacetime, static observers in the latter do notice an expanding universe. The way out to this apparent paradox is that static observers in (a) and (b) do not share the same physical properties. As a matter of fact the static observers in (a) are *constantly accelerating*, whereas the static observers in (b) are *geodesics*. What is the most convenient coordinatization in order to facilitate the quantitative description of physical measurements? To find the answer it may be helpful if we consider Schwarzschild spacetime and examine some coordinate systems available to describe it.

Before getting into details, we may establish two consecutive stages as regards the connection between the mathematical coordinates, quite arbitrary because of diffeomorphism invariance, and the measuring devices.

- (I) In the first stage, observables in General Relativity¹⁰ (GR) are defined through a gauge fixing *i.e.* after a common choice, made by all the observers, of a coordinatization [11–13].
- (II) The second stage is the trickiest one: to connect the coordinates with the physical measuring devices. That is, given the observational features that the particular users are focusing on, to optimize its mathematization through the adoption of coordinate descriptions suited to their measuring devices.

Now consider the solar system scale. As regards considerations of static versus non-static descriptions, we must point out that the success of the standard coordi-

⁹ Notice that the sign of this acceleration is only related to the sign of the cosmological constant and not as to whether the universe is expanding or not. For instance if we take the $C = 0$ metric in (2.9) with $H \rightarrow -H$, it describes a contracting universe and nevertheless the effect of the positive cosmological constant is centrifugal.

¹⁰ Diffeomorphism invariant objects.

natization for the Schwarzschild metric makes such choice the preferable option. As a consequence of Birkoff's theorem the corrections to the solar system Newtonian dynamics are perturbative terms. In the case of the bounded geodesics of the Schwarzschild metric these obtain maximal simplicity—just a single term—within the standard Schwarzschild coordinatization [14], whereas for instance using isotropic coordinates—also static—the corrections are distributed among several terms. As said, static observers in the standard form for Schwarzschild metric are constantly accelerating. In fact, and probably with some degree of retrospection, it is intuitively clear that coordinates for which a static observer is constantly accelerating have a good physical content to describe, precisely, geodesic motion. The reason is inspired in Newtonian physics, in which ideal static observers, placed around the sun, have an internal experience of constant acceleration (to keep them at rest).

By passing let us mention that the authors of [15] have provided with an intrinsic definition of distance, disconnected from Newtonian physics, showing the physical preference of the standard form of the Schwarzschild metric in order to describe the geodesics associated to planetary orbits in general relativity.

If in addition to the Schwarzschild picture we include the presence of a cosmological constant, we may consider Schwarzschild-dS in the forms (a) static, (3.4), and (b) time dependent.¹¹ According to the observations derived from the previous analysis in Sect. 3, in which we saw that the additional acceleration induced by the cosmological constant is already accounted for in the static version of dS, it seems quite clear that the static description (a) is the candidate to be the most convenient one in order to describe physical phenomena at the local scale.

We have considered hitherto the solar system scale up to the galactic scale. In addressing the cosmological scale, we find that there is a drastic limitation for the FLRW metrics that admit static versions. This is the subject of the next section.

4.2 A tale at the cosmological scale

Let us consider the general FLRW cosmology (3.1). We will answer the question as to whether and when such background admits a static metric.¹² With this aim, since we know that observers sitting at a fixed value of the space coordinates in a static metric are constantly accelerating, we should look for the radial trajectories of the constantly accelerated observers. One obtains a first change of variables $r(t, R)$ such that, after a second change $t(T, R)$ we end up with an static metric, being R its new radial coordinate. Examining the metric coefficient for the two-sphere it is clear that $a(t) r(t, R)$ must be a function of R only, the simplest choice being R itself. Thus one can identify $r(t, R) = \frac{R}{a(t)}$. The requirement of being static translates to that of a constantly accelerated observer for constant R . As a consequence one must consider

¹¹ This cosmological Schwarzschild-dS could be constructed in principle by the procedure used in Sect. 2 [16].

¹² The answer was already given in [17] but we take a completely different approach, which we believe is more advantageous and direct.

the trajectory $X = \{t, \frac{R}{a(t)}, \theta_0, \varphi_0\}$. Its 4-velocity with respect to proper time s reads

$$V = \left\{ \frac{1}{\sqrt{\frac{R^2 a'(t)^2}{\sigma R^2 - a(t)^2} + 1}}, -\frac{R a'(t)}{a(t)^2 \sqrt{\frac{R^2 a'(t)^2}{\sigma R^2 - a(t)^2} + 1}}, 0, 0 \right\}, \quad V^2 = -1, \quad (4.1)$$

and the 4-acceleration is computed as $A^\mu = V^\rho \nabla_\rho V^\mu = \frac{d}{ds} V^\mu + V^\rho \Gamma_{\rho\nu}^\mu V^\nu$.

Next we should compute the Jerk, defined as the Fermi-Walker covariant derivative of the 4-acceleration [18,19] and implement the equation $\text{Jerk} = 0$, which describes the constantly accelerated trajectory. Due to the symmetry of our setting, one can check that the vanishing of the Jerk is equivalent to the constancy of the norm of the acceleration. One obtains

$$|A|^2 = \frac{1}{a(t)^2} \frac{E_1^2}{E_2^3}, \quad \text{with} \quad \begin{cases} E_1 = R a(t) a''(t) (\sigma R^2 - a(t)^2) + R^3 a'(t)^2 (a'(t)^2 + \sigma), \\ E_2 = a(t)^2 - R^2 (a'(t)^2 + \sigma). \end{cases} \quad (4.2)$$

Requiring $|A| = \text{constant}$ will in general give solutions for $a(t)$ containing R dependences. This translates contrariwise to our starting point, (3.1), into r dependences in $a(t)$.¹³ The only way in order for the equation $|A| = \text{constant}$ not to display an R dependence for its solution $a(t)$, is that the coefficients of different powers of R in the numerator and denominator of (4.2) must sustain a relation of proportionality. With these considerations we set up the following relation

$$\frac{\text{Coefficient}(E_1, R^3)}{\text{Coefficient}(E_1, R)} = \frac{\text{Coefficient}(E_2, R^2)}{\text{Coefficient}(E_2, R^0)}, \quad (4.3)$$

which yields the condition

$$a(t) a''(t) - a'(t)^2 - \sigma = 0. \quad (4.4)$$

We recognize in the above equation the condition for (3.1) to be a maximally symmetric spacetime [20].¹⁴ In addition (4.4) is the EOM for the Lagrangian $\mathcal{L}_\sigma = \frac{a'(t)^2 - \sigma}{a(t)^2}$,

which associated Hamiltonian, $\mathcal{H}_\sigma = \frac{a'(t)^2 + \sigma}{a(t)^2}$, is a constant of motion that is nothing but $\frac{\Lambda}{3}$, being Λ the cosmological constant. Using this fact (4.4) boils down to

$$a''(t) - \frac{\Lambda}{3} a(t) = 0. \quad (4.5)$$

¹³ Notice however that $a(t)$ may indeed depend on σ , which is a parameter already present in the metric.

¹⁴ We can check that the solutions to (4.4), see below, satisfy the requirement $|A| = \text{constant}$.

Notice that this equation is derived from the Lagrangian $\mathcal{L}_\Lambda = a'(t)^2 + \frac{\Lambda}{3} a(t)^2$, in which case σ appears as the new Hamiltonian constant of motion. Thus equations (4.4) and (4.5) are equivalent, and so they are their parent Lagrangians $\mathcal{L}_\sigma$ and $\mathcal{L}_\Lambda$. The solution of (4.5)¹⁵ contains three different cases, depending on the sign of Λ :

- (I) The case $\Lambda > 0$ is dS spacetime which has already been dealt with in Sect. 2. See in particular (2.9).
- (II) The vanishing Λ case is Minkowski spacetime in Milne coordinatization [21].¹⁶

$$ds^2 = -dt^2 + H^2 t^2 \left(\frac{1}{1 + H^2 r^2} dr^2 + r^2 d\Omega^2 \right), \quad (4.6)$$

with H an arbitrary real constant.

- (III) The case $\Lambda < 0$ is AdS spacetime. It needs $\sigma < 0$ and its equal time slices are 3-hyperboloids

$$ds^2 = -dt^2 + (\sin(Ht))^2 \left(\frac{1}{1 + H^2 r^2} dr^2 + r^2 d\Omega^2 \right), \quad H^2 := -\frac{\Lambda}{3} = -\sigma. \quad (4.7)$$

All these solutions share the same description with a static metric,

$$ds^2 = - \left(1 - \frac{\Lambda}{3} R^2 \right) dT^2 + \frac{1}{1 - \frac{\Lambda}{3} R^2} dR^2 + R^2 (d\theta^2 + (\sin \theta)^2 d\varphi^2). \quad (4.8)$$

To summarize, the only source of matter-energy compatible with bringing a given FLRW cosmology to a static metric description is the cosmological constant.

5 Expanding universe and/or expanding 3-space

5.1 Is the 3-space expanding?

For the sake of clarity it is important to neatly distinguish between the notion of expanding universe and that of expanding 3-space. For the first we simply appeal to the observed recession of the galaxies. As for the second, let us give a tentative definition of what is meant by expanding 3-space: *it is the idea, or belief, that there is a physical process of some sort—acting perhaps at an ultra-microscopic scale—that is producing the homogeneous growth of the equal-time hypersurfaces associated in principle to the background (3.1) (Stage 4 of Fig. 1), but now applied down to a scale in which the matter dominated universe consists basically of galaxies, DM, and a possible Λ -vacuum, (Stage 2 of Fig. 1), and in which this growth is still thought as*

¹⁵ We skip the trivial solution $a(t) = \text{constant}$, $\sigma = \Lambda = 0$, which is Minkowski spacetime in standard coordinatization.

¹⁶ The change of variables from Minkowski, $ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2$, to Milne coordinates is given by $T(t, r) = \sqrt{1 + H^2 r^2} t$, $R(t, r) = H r t$.

being dictated by the FLRW scale factor $a(t)$. Then, as a consequence of this idea, galaxies keep separating from each other *because* the intergalactic vacuum 3-space keeps growing and growing.

To address this expanding 3-space picture, let us take a family of non-interacting test particles following radial time-like geodesics inside a Minkowski spacetime, with different velocities but all starting from vanishing radial coordinate at $t = 0$. Suppose we adopt Milne's form (4.6) for the metric so that our chosen test particles are comoving observers, with scale factor $a(t) = Ht$, describing the time-increasing separation between them. Should one conclude from this picture that in this concrete example the 3-space is expanding? [22,23]. Obviously such a notion is a pure artifact of the chosen coordinatization. Now let us translate this observation to the motion of galaxies,¹⁷ Stage 2 of Fig. 1, where the popular 3-space expansion idea applies. There is a strong observational evidence that the galaxies are receding from each other. Provided the intergalactic void is not so different from the strict void described by (4.6),¹⁸ should we believe that the 3-space in our universe expands whereas that in (4.6) does not? We find neither compelling reason nor need to believe in the expansion of 3-space as defined above. This expanding 3-space picture is not sensible and receives the final blow when realizing that if taken seriously, then one is bound to accept the absurd consequence that this growing process holds at the local scale (Stage 1 in Fig. 1), for which there is *no basis at all* when one looks at the rhs of Einstein's equations. Simply put: one can not apply the scale factor $a(t)$ at a vacuum region—we are at the local scale—because, when applied to this region, there is no trace of a homogeneous Hubble flow in the rhs of Einstein equations, except, as said before, for its cosmological constant component.

5.2 The rubber balloon picture

Let us add some considerations on the very popular rubber balloon picture, which in our view is at the origin of many misunderstandings on the subject of expanding 3-space. This rubber balloon picture, which is so helpful in giving an immediate—and computationally correct—explanation of the cosmological redshift—through the idea that the wave function of traveling photons stretch as 3-space expands—is deeply flawed *when used in support—or illustration—of the expansion of 3-space*. The reason is the following.

The basis for this popular picture is only found at the highest, cosmological scale, Stage 4 of Fig. 1, in which it is legitimate to consider the distribution of galaxies, Dark Matter, Dark Energy (Λ) at a given—cosmological—time, as a *continuous* and homogeneous distribution. But this is *not* the picture offered with the rubber balloon, which is a vacuum balloon—perhaps with a nonvanishing cosmological constant—punctuated by dots, equally distributed, classically representing the galaxies (Although the presence of DM is obviously necessary too), Stage 2 of Fig. 1. The crucial point

¹⁷ This is the popular picture, of course at least the DM halos of these galaxies should be included in the picture.

¹⁸ Of course the former is crossed by electromagnetic radiation, neutrinos, cosmic rays, DM...and feels the pervading presence of a tiny cosmological constant.

is that this distribution is not homogeneous in the sense of a continuous one as used in the FLRW models, but it is only a discrete distribution.

If within this discrete distribution of point-like galaxies represented by the rubber balloon, we focus on a vacuum region, it is obvious that the rhs of Einstein equations will only contain the energy-momentum tensor of such vacuum, that is, nothing, or at most just the cosmological constant, so in this region the solution of the Einstein equations—with the appropriate boundary conditions—will be a vacuum or Lambda-vacuum solution. But one can not build out of it the scale factor (used to describe the "growing of the 3-space") that is derived when using the FLRW metric, because for that we need in the rhs of Einstein equations a continuous homogeneous distribution. It is a fact that the FLRW metric is an approximation valid at the cosmological scale, Stage 4 of Fig. 1, in which, in order to get a continuous homogeneous distribution, an average density is taken out of galaxies, DM and Λ -vacuum.

It is obvious therefore that the rubber balloon picture, as long as it is used to support the expansion of 3-space, is deeply flawed, misleading to say the least, and plain wrong to use more accurate words, despite its pedagogical role for instance in the quick explanation of the cosmological redshift.¹⁹ One can trace the problem with this picture to the confusion and mixing of two different scales (Stages 4 and 2 in Fig. 1): One, the cosmological scale in which we apply the approximation of a continuous and homogeneous distribution of matter, and another, much lower scale, in which the distribution becomes discrete, essentially galaxies, DM and Λ -vacuum,²⁰ and thus homogeneity is only considered at a discrete level.

6 Spectral shifts as a GR Doppler effect. Some clarifications

6.1 Cosmological and gravitational redshift versus Doppler effect

Take two test observers in Minkowski spacetime, A and B, simultaneously departing from the origin of coordinates at $T = 0$ and traveling radially in different directions. They correspond, see (4.6), to comoving observers located at fixed $\{r_a, \theta_a, \varphi_a\}$ and $\{r_b, \theta_b, \varphi_b\}$ in the Milne coordinatization. One can compute the redshift of a photon emitted by A and detected by B using the Special Relativity (SR) formulas for the Doppler effect, when sender and receiver are in different inertial reference systems. Although this redshift can also be computed with the standard formulas for the cosmological redshift [see Eq. (6.7) below] for which the "expansion of space" idea is very well suited, it is clearly nothing else than a Doppler effect.

The Doppler effect as introduced above in the framework of SR can be generalized to GR as relating the frequencies of the emitted and received photons by arbitrary Sources and Observers. This was done long time ago, first by Schrodinger [24] and Synger [25], independently, and later re-elaborated by Narlikar [26], see also [27]. In this respect our

¹⁹ This, say, fortunate circumstance is explained by the fact that indeed a photon in a cosmological trip experiences an effect from the background which *on average* is accounted for by the FLRW approximation of an homogeneous distribution of matter.

²⁰ Note that one can not apply at this scale the approximation in which the inhomogeneities are taken as fluctuations around the homogeneous Hubble flow.

results are not new at all but we consider that including our derivations, beyond their pedagogical flavour, make the connection with the new results presented in Appendices A and B. Once this generalization of the Doppler effect to curved spacetime is at our disposal, it is legitimate to include the cosmological and gravitational spectral shifts within this broad, GR extended version, of the Doppler effect. In doing so, all considerations of a redshift that is not Doppler but only cosmological, or gravitational, become essentially a matter of taste. In our view, it is not wrong to talk on cosmological redshift, or gravitational redshift, but it is also true that they can be derived from an unique concept of Doppler redshift, now understood as an extension to GR of the SR effect. This extension is reviewed in detail in the Appendix A. Whether this generalized effect must be called Doppler or something else has proved to be a matter of some semantic quarreling. As stated in page 123 of [25]: “Arguments about this are completely futile because they are merely windy warfare conducted without any attempt to analyze the meanings of the terms employed”.

In the next two subsections we examine the cosmological and gravitational redshifts within the framework of this generalization. It is worth noticing in these derivations the relevance of identifying an affine parameter for the photon trajectory. In Appendix B we discuss the massive particle case, and show that we recover in the massless limit the results shown below, thus becoming an alternative derivation of the frequency shifts for the photon, only relying on the notion of proper time for massive particles.

6.1.1 The cosmological redshift as a General Relativity Doppler effect

As is elaborated in depth in Appendix A the spectral shift, z , relates the frequency ν_s of the photon as seen by the Source at the time of emission, to the frequency ν_o of the photon as seen by the Observer at the time of reception.²¹ This relation can be cast in an invariant way as (A.6)

$$1 + z = \frac{\nu_s}{\nu_o} = \frac{E_s}{E_o} = \frac{V_s(0) \cdot U(0)}{V_o(t) \cdot U(t)}, \quad (6.1)$$

with $t = 0$ at the emission event. The Source and Observer's velocities, $V_s(t)$ and $V_o(t)$, computed with respect to their respective proper time, are parametrized here by the coordinate time in the spacetime. While $U(t)$ is the 4-velocity of the photon, still parametrized by the coordinate time, but computed with respect to an affine parameter,²² see Eqs. (6.2), (6.3) and (6.4).

In the sequel we shall apply the previous expression, (6.1), to verify that it describes the cosmological redshift for the FLRW, (3.1). We consider both the Source and the Observer as comoving in the Hubble flow. The Source is located at $r = 0$, in this way the spherical symmetry imposes that all geodesics passing through $r = 0$ must be radial. In addition, the maximal symmetry of the equal time slices allows that any non-radial geodesic become automatically included in the analysis through a change of

²¹ In both cases the frequency is proportional to the kinetic energy.

²² Lacking of the concept of proper time for the massless particle, this is the *only* way to ensure that the quotient in (6.1) is indeed an invariant.

coordinates in the equal-time slices. With the aforementioned conditions, the photon trajectory is given by $X(t) = (t, r(t), \theta_0, \phi_0)$, with $r(0)=0$ and velocity with respect to coordinate time $\frac{dX}{dt} = (1, r'(t), 0, 0)$. The velocity with respect to the affine parameter s is,

$$U(t) = \frac{dt}{ds} \frac{dX}{dt} \equiv h(t) (1, r'(t), 0, 0), \quad \frac{dt}{ds} := h(t(s)), \quad (6.2)$$

where h is as yet an unknown function. The equation for $h(t)$ is found by inserting $U(t)$ into the geodesic equation,

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\nu\rho} U^\nu U^\rho = h(t) \frac{dU^\mu}{dt} + \Gamma^\mu_{\nu\rho} U^\nu U^\rho = 0, \quad (6.3)$$

from which we obtain a differential equation for $h(t)$, $h(t) a'(t) + a(t) h'(t) = 0$, with solution²³

$$h(t) = \frac{1}{a(t)}. \quad (6.4)$$

Imposing the light-like condition $\left(\frac{dX}{dt}\right)^2 = 0$, which becomes $\frac{a(t)^2 r'(t)^2}{1 - \sigma r(t)^2} = 1$, one determines the trajectory²⁴

$$r(t) = \frac{1}{\sqrt{\sigma}} \sin \left(\sqrt{\sigma} \int_0^t \frac{1}{a(\tau)} d\tau \right) \quad (6.5)$$

which holds for both positive or negative σ and also in the limit of vanishing σ . Thus we end up with

$$U(t) = \left(\frac{1}{a(t)}, \frac{\cos \left(\sqrt{\sigma} \int_0^t \frac{1}{a(\tau)} d\tau \right)}{a(t)^2}, 0, 0 \right). \quad (6.6)$$

The comoving Source at the time of emission $t = 0$ has proper velocity $V_s(0) = (1, 0, 0, 0)$, and thus $V_s(0) \cdot U(0) = \frac{1}{a(0)}$. On the other hand, the comoving Observer at the time t of reception has proper velocity $V_o(t) = (1, 0, 0, 0)$, hence $V_o(t) \cdot U(t) = \frac{1}{a(t)}$. All in all, from (6.1) we get

$$1 + z = \frac{v_s}{v_o} = \frac{V_s(0) \cdot U(0)}{V_o(t) \cdot U(t)} = \frac{a(t)}{a(0)}, \quad (6.7)$$

²³ The arbitrary constant factor of the solution will be discussed later on.

²⁴ Note that $\int_0^t \frac{1}{a(\tau)} d\tau$ is the conformal time η , with $dt = a(t(\eta))d\eta$.

which is the standard formula for the cosmological redshift.

This result shows that the cosmological redshift is just the manifestation of the Doppler effect, once extended from SR to GR [28]. Nothing more, nothing less. Of course one can use the rubber balloon picture [29] as a metaphor, but to claim that a notion of expanding 3-space, in the sense of our tentative definition given in Sect. 5, is necessary, is a mistake.

6.1.2 The gravitational redshift as a general relativity Doppler effect

Once the Doppler effect has been properly extended to GR, we can conclude that cosmological and gravitational redshifts have a common origin. This result was already derived in [24–26] and in the following we make a detailed treatment for the general Schwarzschild-dS spacetime.

Consider the general metric (2.5) with $f(R)$ given in (3.4). We are interested in the radial emission of a photon from a location (R_0, θ_0, ϕ_0) and its latter detection along the same radial line at (R_1, θ_0, ϕ_0) with $R_1 > R_0$. The Source (Observer) position and velocity with respect to proper time²⁵ are

$$\text{Source: } \begin{cases} X_s = (T, R_0, \theta_0, \phi_0) \\ V_s = (\frac{1}{\sqrt{f(R_0)}}, 0, 0, 0) \end{cases}, \quad \text{Observer: } \begin{cases} X_o = (T, R_1, \theta_0, \phi_0) \\ V_o = (\frac{1}{\sqrt{f(R_1)}}, 0, 0, 0) \end{cases}. \quad (6.8)$$

While the photon trajectory²⁶ is $X_{ph} = (T, R(T), \theta_0, \phi_0)$, with $R(0) = R_0$ and $R(T) = R_1$. Its 4-velocity, $(1, R'(T), 0, 0)$, being a null vector determines the equation $R'(T) = f(R(T))$ and with respect to proper time boils down to $U(T) = h(T)(1, R'(T), 0, 0)$, with $h(T) = \frac{dT}{ds}$ and s an affine parameter. Similarly to the previous case, the geodesic condition fixes $h(T)$ and $U(T)$ becomes

$$U(T) = \left(\frac{1}{f(R(T))}, 1, 0, 0 \right). \quad (6.9)$$

At any $T > 0$, with $R(T) = R_1$, the scalar products are found to be

$$V_o(T) \cdot U(T) = -\frac{1}{\sqrt{f(R_1)}}, \quad V_s \cdot U(0) = -\frac{1}{\sqrt{f(R_0)}}.$$

Adapting (6.7) to the case at hand

$$z = \frac{V_s \cdot U(0)}{V_o(T) \cdot U(T)} - 1 = \sqrt{\frac{f(R_1)}{f(R_0)}} - 1, \quad (6.10)$$

which extends the usual formula for the gravitational redshift for radial photons in the Schwarzschild metric to Schwarzschild-dS or Schwarzschild-AdS backgrounds. This formula also holds for pure dS or AdS backgrounds.

²⁵ Thus normalized to $V^2 = -1$.

²⁶ Not described yet with the necessary affine parameter.

Notice that contrariwise to the previous discussion on the cosmological redshift the Source and Observer are no longer geodesics but constantly accelerating. One can nevertheless consider the Source as belonging to a radial geodesics which happens to be at R_0 when $t = 0$ and such that $R'(0) = 0$, and the same can be done at time t for the Observer [30]. The GR Doppler formula still holds because the quotient of the scalar products in (6.7) is always an invariant regardless of the fact that Source and Observer be geodesics or not. In addition, once the Doppler effect has been extended to GR, the curvature of spacetime contributes to this effect, making it detectable even in cases where Source and Observer are at rest.²⁷

7 Summary of results and concluding remarks

Let us summarize our main results, with additional comments on their relevance.

1. We give a detailed proof of the fact that two different approaches, unconnected so far, to the study of possible local effects of the cosmological expansion, one with a time dependent cosmological metric, the other with a static metric, do coincide when we take into account the contents of the rhs of Einstein equations at every scale, Fig. 1. Thus the only contribution of the Hubble flow that survives at the local, solar system scale is that of the cosmological constant. To match the results of the two approaches one must carefully identify the physical content of the coordinates used in each approach and how are they related. In this way, approaches in the literature that until now seemed to be incompatible can be reconciled when applied to plausible physical scenarios.
2. We review an old, but not yet as popular as it deserves, unified presentation of the GR Doppler effect—that is, the generalization of the Doppler effect from SR to curved spacetime—with a single formula which encompasses all circumstances. It's common theme being that of energy gain or energy loss for a particle, either massive or massless, in geodesic motion from the Source to the Observer. Under this unifying theme, all energy shifts, including the cosmological and gravitational ones, appear as particular cases of this GR Doppler effect. Two novelties of our presentation are the following: (A) by including the massive case, we show how to retrieve the massless one by taking the appropriate limit. In this sense, the essential role of the affine parameter in the computations associated with the massless case can be circumvented. And (B) since, for instance for the cosmological redshift, one works with metrics at the cosmological scale, like FLRW, but also, from the viewpoint of, say, the receiver of the signal, a local SR setting is typically applied, we claim that the matching of an invariant at the local and cosmological scale provides an unambiguous and unique way to relate observables at both scales, which use different metrics. We think that the strict necessity of this step, in connecting both descriptions, has been largely overlooked.
3. We give a new, simple and elegant proof that the only FLRW models admitting a static metric are those of Milne, dS and AdS. The proof is based in identifying the consistency properties we must require to the differential equation posed by

²⁷ This is a coordinate dependent statement.

setting up the problem—that is to look for a possible static version of a given FLRW metric—and then obtain a well known equation which describes the maximally symmetric FLRW metrics.

4. We make some considerations regarding the expanding 3-space idea, which still keeps some popularity among practitioners in cosmology. In particular we insist that this picture is misleading and wrong because it goes against the very tenets of Einstein GR equations, which require the strict matching, at every spacetime point—and obviously at every neighborhood—of its left hand side—the Einstein tensor derived from the metric—with the rhs—provided by the energy-momentum tensor of matter. This obvious requirement from Einstein equations is violated when one assumes that the scale factor $a(t)$ works at any neighborhood within a vacuum region of 3-space. We think that, typically, supporters of this expanding 3-space idea mix two very different scales, Stages 2 and 4 in Fig. 1, and completely ignore—or forget—this need of consistency mentioned above, which is at the very basis of Einstein GR.
5. We have also argued on the convenience, at the local scale, of using a static metric, instead of other coordinatizations that carry a more cosmological flavor. This means for instance at the local solar system scale, that the static Schwarzschild solution, or the static Schwarzschild-deSitter solution, are the preferable metrics to use, as long as the assumption of spherical symmetry can be taken as approximately valid.

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Appendices

A The Doppler effect as energy gain or energy loss

A.1 From special relativity...

The Doppler spectral shift of light in the framework of SR can be understood as a change of the energy of a traveling photon between emission and reception.²⁸ We will skip the usual derivation of this effect and examine what is basically equivalent: the

²⁸ It goes without saying that we refer to energies as measured by the emitter or the receiver, respectively. Synge showed in [31] the proportionality of energy and frequency for a photon independently of quantum mechanical considerations.

energy gain/loss of a particle—either massive or massless—in free motion from the Source to the Observer.

We consider the inertial reference system of the Observer, placed at the origin of spatial coordinates, whereas the Source and the massive particle are moving with respect to it at speeds $\vec{v}$, $\vec{u}$ respectively. Their respective 4-velocities with respect to proper time are

$$V_o = (1, \vec{0}), \quad V_s = \left(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}} \right), \quad U = \left(\frac{1}{\sqrt{1-u^2}}, \frac{\vec{u}}{\sqrt{1-u^2}} \right), \quad (\text{A.1})$$

with $v = |\vec{v}|$, $u = |\vec{u}|$. It is assumed that the particle intersects the Source and Observer trajectories at different points in Minkowski spacetime.

If we set the mass of the particle to $m = 1$ its energy can be expressed in terms of an invariant form from both rest systems, Observer and Source

$$E_o = -V_o \cdot U = \frac{1}{\sqrt{1-u^2}}, \quad E_s = -V_s \cdot U = \frac{1 - \vec{v} \cdot \vec{u}}{\sqrt{(1-v^2)(1-u^2)}}, \quad (\text{A.2})$$

and therefore, the invariant ratio of energies, E_s/E_o , for the massive particle is

$$\frac{E_s}{E_o} (\text{massive}) = \frac{1 - \vec{v} \cdot \vec{u}}{\sqrt{1-v^2}} = \frac{1 - v u \cos \alpha}{\sqrt{1-v^2}}. \quad (\text{A.3})$$

In the massless limit, $u \rightarrow 1$, we get the standard formula for the energy shift of the photon,²⁹

$$\frac{E_s}{E_o} (\text{massless}) = \frac{1 - v \cos \alpha}{\sqrt{1-v^2}}, \quad (\text{A.4})$$

which, for $\alpha = \pi$, gives the usual longitudinal Doppler redshift when the motions of Source and Observer are aligned and in opposite directions.

Summing up, in the SR framework, the Doppler effect or in general, the ratio of the particle's energy seen from the Source rest frame to the particle's energy seen from the Observer rest frame³⁰ is always described by the invariant

$$\frac{E_s}{E_o} = \frac{V_s \cdot U_s}{V_o \cdot U_o}, \quad (\text{A.5})$$

with $U_s = U_o = U$ in this case.

A.2 ...to general relativity

There are many definitions within the SR framework that can be extended to GR. Take for instance the geodesic motion, which is extended to GR by basically replacing the

²⁹ Obviously the same result is obtained by using directly for the massless particle the velocity $U = (1, \vec{\omega})$ with $|\vec{\omega}| = 1$.

³⁰ Be the particle either massive or massless.

ordinary derivative for the covariant one. Or the concept of the constantly accelerated observer, that can be brought to GR by keeping the requirement of constancy [19,32,33] for some curvature scalars that generalize the Frenet–Serret formalism [34]. These cases bear in common that only the point and its neighborhood in a world line trajectory are necessary ingredients. Other concepts, like the Doppler effect, require more refined considerations because points of different trajectories are involved. Luckily enough (A.5) is easily exported to GR. In such case the particle travels through a geodesic with a 4-velocity computed either with respect to proper time for massive particles or with respect to an affine parameter for massless ones. Its velocity U is typically different when evaluated at the Source location, U_s , than when evaluated at the point of reception by the Observer, U_o . Unlike the massive case, the affine parameter for photons is only determined up to an arbitrary constant factor, the consequence being that whereas for the massive case both scalar products, $V_s \cdot U_s$ and $V_o \cdot U_o$, are invariant,³¹ it is only their quotient which is invariant for massless particles

$$\frac{E_s}{E_o} = \frac{V_s \cdot U_s}{V_o \cdot U_o}. \quad (\text{A.6})$$

The above expression captures the Doppler effect and its extension to the massive case as a ratio between some data from the emission event, $V_s \cdot U_s$, and some data from the reception event, $V_o \cdot U_o$.³² Thus (A.6) can be taken as the definition of the Doppler effect—interpreted as an energy shift and also extended to the massive case—in GR.

Let us notice that in adopting (A.6), hence including a computational prescription, for the evaluation of the GR Doppler effect, the notion of the relative velocity between Source and Observer, which is crucial in the SR derivation, has disappeared.

B Application to massive particles

B.1 The cosmological energy shift for massive particles

We continue in the cosmological FLRW setting (3.1), but considering the emission of a massive particle from a comoving Source located at $r = 0$. Its geodesic trajectory and velocity with respect to proper time, s , are

$$X(t) = (t, r(t), \theta_0, \phi_0), \quad U(t) = h(t)(1, r'(t), 0, 0), \quad (\text{B.1})$$

³¹ We mean invariants under general changes of coordinates. In the passive interpretation of diffeomorphisms a scalar computed at a given point becomes an invariant, in the sense that its value is independent of the coordinates used to describe such point [35].

³² One can go one step further and parallel transport the data from the Source to the Observer's location. Since U_s is transported to U_o , one can see that the whole effect originates from the fact that V_s is not transported to V_o .

where $h(t) := \frac{dt}{ds}$ is obtained by requiring $U(t)^2 = -1$,

$$h(t) = \left(1 - \frac{a(t)^2 r'(t)^2}{1 - \sigma r(t)^2}\right)^{-1/2}. \quad (\text{B.2})$$

To compute $r(t)$, we formulate the geodesic equation (6.3) with the normalized velocity (B.1), obtaining

$$r(t) = \frac{1}{\sqrt{\sigma}} \sin \left(\sqrt{\sigma} \int_0^t \frac{1}{a(\tau) \sqrt{1 + C a(\tau)^2}} d\tau \right), \quad (\text{B.3})$$

with $C > 0$ an integration constant related to the initial condition $r'(0)$. Similarly to (6.5), (B.3) holds also for null or negative σ . In addition, the massless case can be recovered in the limit $C \rightarrow 0$.

With this at hand the expression for the proper 4-velocity³³ becomes

$$U(t) = \frac{1}{m} \left(\sqrt{m^2 + \frac{a(0)^2}{a(t)^2} p^2}, \frac{a(0)}{a(t)^2} p \cos \left(\sqrt{\sigma} \int_0^t \frac{1}{a(\tau) \sqrt{1 + m^2 \frac{a(\tau)^2}{a(0)^2 p^2}}} d\tau \right), 0, 0 \right), \quad (\text{B.4})$$

where, for convenience, we restored the mass m of the particle and defined p through $C = \left(\frac{m}{a(0)p}\right)^2$.

Let's interpret the invariants:

- (i) At the particular time of emission $t = 0$ the energy of the massive particle is given by

$$E_s = V_s \cdot (m U(0)) = \sqrt{m^2 + p^2}. \quad (\text{B.5})$$

Thus p is interpreted as the initial momentum of the particle as measured by the Source.³⁴

- (ii) At the time t of reception we obtain the energy of the particle from the invariant

$$E_o = V_o \cdot (m U(t)) = \sqrt{m^2 + \frac{a(0)^2}{a(t)^2} p^2} \equiv \sqrt{m^2 + p(t)^2}, \text{ with } p(t) := \frac{a(0)}{a(t)} p, \quad (\text{B.6})$$

and $p(t)$ is interpreted as the momentum of the particle a time t , as as measured by the Observer.

³³ Notice that although expressed in terms of the cosmological time it is a proper velocity, $U(t)^2 = -1$.

³⁴ We elaborate on this interpretation in the next subsection.

B.2 The necessary connection between two scales

The result (B.4) has been obtained using the cosmological FLRW metric (3.1), which works at the cosmological scale. Evidently if we just consider a small region in the close neighborhood of the Source there is no trace of the homogeneous Hubble flow that sources the background and therefore (3.1) is not applicable at this scale. Instead, what is applicable at this local scale are the kinematic relations of SR, as it is stated by the equivalence principle. But then the question arises as how can we proceed in order to connect both settings, cosmological and local one. It is not a coordinate transformation because we are talking about different metrics: on one side, the broad-brush FLRW metric, obtained by averaging the density of matter-radiation on very large scales and assuming homogeneity; on the other side, the approximate SR Minkowski metric that holds in every small neighborhood of spacetime. Both pictures are correct, the only caveat being, as said, that they are applicable at completely different scales.

To our understanding, there is a unique way to physically connect the two scales: *one must retain the values of the invariants found above when moving from the cosmological scale description to the local SR one, or vice versa.* Now made explicit, this is the assumption that was already implicit in the previous subsection.

In the local SR frame at the Source we have $V_s = (1, \vec{v})$ and $U(0) = \frac{1}{m}(\sqrt{m^2 + p^2}, \vec{p})$ with $\vec{p} = m \frac{\vec{v}}{\sqrt{1-v^2}}$ so that $V_s \cdot U(0) = \frac{1}{m}\sqrt{m^2 + p^2}$. Analogously we will have at the point of reception, at time t , $V_o \cdot U(t) = \frac{1}{m}\sqrt{m^2 + p(t)^2}$. Thus

$$\frac{E_s}{E_o} = \frac{V_s \cdot U(0)}{V_o \cdot U(t)} = \frac{\sqrt{m^2 + p^2}}{\sqrt{m^2 + p(t)^2}} = \frac{\sqrt{m^2 + p^2}}{\sqrt{m^2 + p^2 \left(\frac{a(0)}{a(t)}\right)^2}}. \quad (\text{B.7})$$

Notice that in the massless limit, $m \rightarrow 0$, we recover (6.7), and the same happens for large momentum, $p \rightarrow \infty$, as well.

Once established the connection between the two scales, we infer that p is indeed the momentum of the particle as seen by the comoving Source at the time of emission, and $p(t)$ is indeed the momentum of the particle as seen by the comoving Observer at the time of reception. Thus, independently of the particle being massive or massless, the following relation always holds

$$p(t) a(t) = p(0) a(0). \quad (\text{B.8})$$

B.3 The gravitational energy shift for massive particles

Similarly to the massless case, sending and receiving a massive particle will also exhibit a shift in its kinetic energy. With the same setup of Sect. 6.1.2, and working directly with proper time s , the trajectory and velocity will be denoted as

$$X(s) = (T(s), R(s), \theta_0, \phi_0), \quad U(s) = (T'(s), R'(s), \theta_0, \phi_0), \quad (\text{B.9})$$

with $U(s)^2 = -1$, which implies $T'(s) = f(R(s))^{-1} \sqrt{f(R(s)) + (R'(s))^2}$. The geodesic equation gives $f'(R(s)) + 2R''(s) = 0$, which can be integrated to $f(R(s)) + (R'(s))^2 = G^2$, with $G > \sqrt{f(R(s))}$ a constant related with the initial conditions. Hence

$$U(s) = (T'(s), R'(s), \theta_0, \phi_0) = \left(\frac{G}{f(R(s))}, \sqrt{G^2 - f(R(s))}, 0, 0 \right). \quad (\text{B.10})$$

If we set $s = 0$ for $T = 0$ and using (6.8) one can compute the invariant

$$V_o(s) \cdot U(s) = -\frac{G}{\sqrt{f(R(s))}},$$

which implies

$$1 + z = \frac{E(R_0)}{E(R_1)} = \frac{V_o(0) \cdot U(0)}{V_o(s) \cdot U(s)} = \sqrt{\frac{f(R_1)}{f(R_0)}}, \quad (\text{B.11})$$

where $E(R_0)$ is the kinetic energy of the emitted particle as seen by the Source and $E(R_1)$ is the kinetic energy of the received particle as seen by the Observer. Unlike the cosmological case above, this result is independent of the mass of the particle and it directly admits the massless limit yielding (6.10) in which case these energy ratios can also be read as quotients of frequencies of the photon.

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