Editorial note to: Manfred Trümper, A threedimensional formulation of the Bianchi identities for vacuum gravitational fields

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Abstract

The value and use of Trümper's formulation of the Bianchi identities in terms of the decomposition of the Weyl tensor into electric and magnetic parts are considered.

Keywords Golden Oldie · Bianchi identities · Gravitational fields

This paper [30], cited in later papers as a preprint, can be considered an adjunct to the “Hamburg Bible”, the series of papers which have appeared as previous Golden Oldies, translated and accompanied by editorial notes and biographies of the authors. For a survey of those papers and some history of the group that wrote them see the editorial note to [14].

The report of which this article formed part was written while its author was an Assistant to Prof. P. Jordan in the Hamburg group. Naturally therefore, the notation and conventions follow the Hamburg bible articles, for example the definitions of kinematic quantities as in [6] and the splitting of the Weyl tensor into electric and magnetic parts. The key coordinate-free reformulation of the Bianchi identities as divergence equations was given in Paper V in the Mainz series ([15]). However the reformulation of those equations in terms of the electric and magnetic parts of the Weyl tensor was not given in that series.

The value of this paper lies in that formulation of the Bianchi identities in terms of the electric and magnetic parts of the Weyl tensor, E_{ab} and H_{ab} , so that they

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are manifestly strongly analogous to the electromagnetic case (see e.g. [18]). The definitions of those parts themselves had been considered before, e.g. in [23], and they appear in chapter 1, section 5, of [14]. The results (equations (9)–(12) of the paper) are stated for the vacuum case but it is relatively straightforward to add the Ricci tensor terms.

The work soon became known to the group in Cambridge led by Dennis Sciama, through the report and a seminar there by Trümper. This led to the first appearance of equations (9)–(12) in a refereed journal in [9], where they were extended to include a perfect fluid Ricci tensor, though [31] had earlier given them as specialized to the pure electric case.¹ The equations with a general Ricci tensor, written in terms of the energy-momentum of a generic imperfect fluid, are given in the appendix to [8]. They have been given again, in various notations, in a number of later publications.

Section 1 of the paper gives the motivation and the outline of later sections. In section 2 the basic equations are given. The split into spatial hyperplanes and a time direction is defined by a congruence of timelike curves with unit tangent vector u^a (cf. [6]). The definitions (6) and (7) of E_{ab} and H_{ab} use the opposite sign convention from [14] and it is the latter convention which is used in later work e.g. [8]. The following unnumbered equation shows how the Weyl tensor can be reconstructed from E_{ab} and H_{ab} (see also [8,9]). The Bianchi identities are then expressed in their 3+1 form in (9)–(12), followed by consideration of special cases to illuminate the structure of these equations.

It should be noted that if one writes Maxwell's equations in a 3+1 decomposition relative to a vector field u^a , terms involving products of kinematical quantities with the field intensities appear in that context also.² Thus the analogy between the Maxwell equations and Bianchi identities is stronger than suggested by Trümper here. Trümper demonstrates how treating the Bianchi equations as if for a test field on a Minkowski space background with a constant u^a leads to a potential satisfying Laplace's equation, like Newton's gravitational potential.

The Bianchi identities (9)–(12) were used by [19] when analysing which components of the derivatives of E_{ab} and H_{ab} are not determined by the Ricci tensor and its derivatives (which E_{ab} and H_{ab} are not): the importance of those components for gravitational waves was pointed out (see also [26]).

The strong analogy between (9) and (12) and Maxwell's equations has been further exploited to study a variety of weak-gravity phenomena (see e.g. [24]) and to improve understanding of gravitational waves ([25]), though there is no analogue in the gravitational case of the constitutive equations or of magnetic permeability and dielectric constant. The equations (11) and (12), with the Ricci tensor terms added, are like the Maxwell equations commonly written as $\nabla E = j$ and $\nabla H = 0$. A series of papers by Hogan and colleagues ([10–12]) established interesting results relating electromagnetic and gravitational perturbations of Robertson-Walker cosmologies, [11] being particularly interesting as it shows how gravitational waves in cosmology can generate electromagnetic waves from a test magnetic field.

¹ In Hawking's paper, E_{ab} is sometimes printed as ϵ_{ab} .

² We are not sure where the equations in that form first appeared. They are given in [20] and in the source-free case, using a differential form notation, in [13]. A general form similar to (9)–(12) appears in [21].

Sections 3 and 4 consider respectively the cases of pure electric and pure magnetic Weyl tensors and shows they obey analogous theorems. Parts A and B in those theorems concern the geometry of the principal null directions (“eigen null directions”) of the Weyl tensor and the eigenvectors of E_{ab} (resp. H_{ab}), Parts C give specific formulae for those in terms of the eigenvalues of E_{ab} (resp. H_{ab}), and Parts D consider when degeneracy occurs. In each case one of the “ ∇E ” and “ ∇H ” equations yields a constraint on the kinematic quantities.

Each of the pure electric and magnetic cases has received further study in a number of contexts. Results similar to Trümper’s theorems 3 and 4 often appear, though not all the papers make use of his form of the Bianchi identities. In both the magnetic and electric cases there are rather strong restrictions if one assumes that the energy-momentum content is a perfect fluid. Trümper [31] showed that such a fluid is either shearing or rotating, and that such spacetimes are of Petrov type I or D. We now cite a sample of the further results. (A more extensive list of references is contained in the bibliography of [3].)

Lesame [17] discussed the pure electric solutions containing an irrotational perfect fluid: those containing irrotational dust, potentially interesting as ‘silent universes’ [2] in which no sound or gravitational waves can be present, form a rather limited subset [22,27]. If a cosmological constant is added, more solutions can be found [32].

For the pure magnetic case, Trümper in section 4 says “we do not know whether such solutions exist”. Lozanovski and Aarons [16] showed there were no such solutions for irrotational dust. A rather unphysical solution was found in [1]. More recently, somewhat more physical solutions have been found [3].

The “shear-free” conjecture, that a shear-free perfect fluid with a barotropic equation of state such that $\mu + p \neq 0$ must be either irrotational or non-expanding, a conjecture triggered by the results of [7] and [29], has been studied in both the electric and magnetic cases [4,5].

Because the forms of the equations first given here have become pervasive in the many studies using a 3+1 formalism, it is hard to characterize their precise impact just by citing specific uses in later papers. That pervasiveness is itself their essential impact.

Autobiography of the author

An autobiography of the author appeared as part of the editorial note to the republication of [15]. The following is an updated version of it.

I was born on 05 July 1934 in Wernigerode, Germany, as the second child of a vocational school teacher. I graduated from a high school in Bernburg, Germany, in June 1952. In fall 1952 I enrolled as a student in physics at the Martin Luther University of Halle. As a child of bourgeois parents I did not qualify for a higher education, but it happened that in this year Walter Ulbricht, Secretary General of the “United Socialist Party = SED”, had declared a drive to economically overtake capitalist West Germany within the next 10 years. This meant, among other measures, to increase by about a factor 4 the number of physics students. As there were not enough applications

from high school graduates with proletarian background, I was admitted for studies in physics, and so was, a few months later, my older brother Joachim.

Stalin died in early March 1953. In the morning of the next day we students were in the overcrowded physics auditorium, waiting for the main lecture on experimental physics to begin, to be held by Professor Messerschmidt. But a comrade from the communist party leadership appeared to read a message of condolences to be sent to the Central Committee of the soviet communist party. It was to be accepted by acclamation, but all that could be heard was a faint, bored knocking on the tables. Then, when Professor Messerschmidt entered the lecture hall, he was greeted by thunderous applause which did not want to subside. First he appeared startled, then he smiled faintly, as he understood what had triggered the enthusiasm of his students.

In June of 1953, after the uprising of workers against the communist regime, there began a phase with lowered restrictions on travel to West Germany. However, a year later, in spring and summer of 1954, it became clear that the regime was tightening its grip of power and returning to repressive measures. Therefore, in the fall of 1954 I left the workers' paradise and I enrolled at Hamburg University.

Leaving the DDR was not an easy task. The university administration kept under lock all my academic records, including the high school diploma. I gambled on the hypothesis that I could not be accused of subversion and secret preparation of "Republikflucht" if I told everybody about my plan to move to Hamburg. I said that in Hamburg I wanted to study General Relativity which was not represented in Halle. My fellow students thought I was nuts. Everybody knew that it was virtually impossible to change the place of studies even within the DDR. And I was talking about going to the "Klassenfeind" in the West! So I went to Professor Messerschmidt, who I knew was quite unhappy about the excessive number of physics students which had been forced upon him and which greatly exceeded the capacity of his laboratory. He wrote a strong letter of recommendation in support of my application for an exmatriculation. The university administration was confused when I submitted my application because such a case had not occurred before. People were fleeing by the hundreds a day, but not in this way. Within a day I got my exmatriculation and my academic records and then, in a final gesture of triumph, I went to the office of the youth organization FDJ and resigned from my membership.

My teachers in Hamburg were, among others, L. Collatz, H. Hasse, H. Braun in mathematics, and W. Lenz, P. Jordan, H. Raether, H. Lehmann in physics. I got my doctorate in Physics in 1962 with Pascual Jordan as my thesis advisor. The subject of my thesis dealt with the restrictions imposed on the flow of test matter in a relativistic gravitational field.

I subsequently held the following positions.

Syracuse University, Postdoc with Peter Bergmann

Yeshiva University, Postdoc with Peter Bergmann

Hamburg University, Scientific Assistant to Pascual Jordan

North Texas State University, Visiting Profesor of Physics

Texas A&M, Associate Professor of Physics

Université d'Oran, Algeria, Professor of Physics

MPI for Astrophysics, Munich, Scientific Collaborator

Université de Rennes/France, Visiting Professor

ISP Bukavu/Kivu, Zaire, Professor

ENSSAT Lannion/France, Visiting Professor

From 1992 I worked as a representative for TÜV Süd (a major certification body for product safety and quality management) in Japan and in China. I retired in 1999 and moved in 2000 to the city of Uzès in southern France, where I now live.

Among my scientific achievements are

- About shear-free and irrotational flows in an Einstein vacuum field of type I (1962). “On a Characterization of Non-Degenerate Static Vacuum Fields by Means of Test Particle Motion”, Conférence Internationale sur les Théories Relativistes de la Gravitation, Warszawa 1962, Gauthiers-Villars, (1964)
- Reduction of the full Bianchi identities in the presence of a timelike vector field (1964).

This work is the subject of the present publication. It was distributed as a preprint to various GR groups in Europe and the US. It was presented in seminar talks

- on 15 Dec. 1964 in Cambridge at the invitation by Dennis Sciama and George Ellis
- on 6 Feb. 1965 at the Institute Henri Poincaré in Paris, at the Invitation by Mme. M. A. Tonnelat and A. Lichnerowicz.
- on 8 Feb. 1965 at the Institut de Mathématique, Université Libre de Bruxelles at the invitation by R. Debever and M. Cahen.
- I showed how to furnish the Lagrangian configuration space-time of a holonomic system with a linear connection so that the trajectory of the system is a geodesic line. “Lagrangian Mechanics and the Geometry of Configuration Spacetime”, *Ann. of Physics* 149, 203 (1983).
- I determined the complete tree for the “33 hole central vacancy peg solitaire problem”, unpublished, but described in the German Journal *GEO*, Nr.2/Feb.1993, page 148-149.
- Research on the Collatz Conjecture, “The Collatz Problem in the Light of an Infinite Free Semigroup”, *Chin. J. Math.* 2014, 756917 (2014)

Addendum by the editor.

In addition to the works mentioned above, Dr. Trümper published a number of other papers in physics. Among them, the following are cited in [28]:

“Orthogonal decomposition of axi-symmetric stationary spacetimes”, W. Kundt and M. Trümper, *Z. Phys.* 192, 419 (1966)

“Einsteinische Feldgleichungen für das axialsymmetrische, stationäre Gravitationsfeld im Innern einer starr rotierenden idealen Flüssigkeit”. M. Trümper, *Z. Naturforsch.* 22a, 1347 (1967).

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Declarations

Conflicts of interest The second author is an Editorial Board member of the journal *General Relativity and Gravitation*.

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