Correction to: Acoustic black holes: massless scalar field analytic solutions and analogue Hawking radiation

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In Sect. 2, there is a mistake when we transformed Eq. (16) into Eq. (21), by using the new coordinate given by Eq. (17) and its consequences expressed in Eqs. (18)–(20). Correcting this mistake, the equation corresponding to Eq. (21) should read

$$\begin{aligned} \frac{d^2 R}{dx^2} + \left(\frac{1}{x - x_h} - \frac{1}{2x} \right) \frac{dR}{dx} \\ + \left[\frac{m^2(B^2 + c^2 x_h)}{4c^2 x_h^2} \frac{1}{x} + \frac{-B^2 m^2 - c^2 m^2 x_h + c^2 x_h^2 \omega^2}{4c^2 x_h^2} \frac{1}{x - x_h} \right. \\ \left. + \frac{(cx_h \omega - Bm)^2}{4c^2 x_h} \frac{1}{(x - x_h)^2} \right] R = 0, \end{aligned}$$

where we used another relation for the new radial coordinate instead of Eq. (17), given by

$$x = r^2$$

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in such a way to obtain an equation which is identified with a confluent Heun equation. Now, using this new coordinate, Eq. (18) is written as

$$\Delta = x - \frac{A^2}{c^2}.$$

Thus, the new horizon surface equation of the rotating acoustic black hole given by Eq. (19) should read

$$\Delta = x - x_h = 0,$$

and Eq. (20) turns into

$$x_h = r_h^2 = \frac{A^2}{c^2}$$

which corresponds to the new event horizon of the rotating acoustic black hole. In what follows, let us do homographic substitution given by Eq. (22). Thus, Eq. (23) should read

$$\begin{aligned} & \frac{d^2 R}{dz^2} + \left(\frac{1}{z} + \frac{-1/2}{z-1} \right) \frac{dR}{dz} \\ & + \left\{ \frac{B^2 m^2 + c^2 m^2 x_h - c^2 x_h^2 \omega^2}{4c^2 x_h} \frac{1}{z} + \frac{-m^2(B^2 + c^2 x_h)}{4c^2 x_h} \frac{1}{z-1} \right. \\ & \left. - \left[i \left(\frac{cx_h \omega - Bm}{2c\sqrt{x_h}} \right) \right]^2 \frac{1}{z^2} \right\} R = 0. \end{aligned}$$

The *F-homotopic transformation* of the dependent variable, $R(z) \mapsto U(z)$, is such that

$$R(z) = z^{A_1} U(z),$$

where the coefficient A_1 given by Eq. (25) is now expressed as

$$A_1 = i \left(\frac{cx_h \omega - Bm}{2c\sqrt{x_h}} \right).$$

In this case, Eq. (26) should be changed to

$$\begin{aligned} & \frac{d^2 U}{dz^2} + \left(\frac{2A_1 + 1}{z} + \frac{-1/2}{z-1} \right) \frac{dU}{dz} \\ & + \left[\frac{B^2 m^2 - iBcm\sqrt{h_h} + c^2 m^2 x_h + ic^2 x_h^{3/2} \omega - c^2 x_h^2 \omega^2}{4c^2 x_h} \frac{1}{z} \right. \\ & \left. - \frac{B^2 m^2 - iBcm\sqrt{h_h} + c^2 x_h (m^2 + i\sqrt{x_h} \omega)}{4c^2 x_h} \frac{1}{z-1} \right] U = 0. \end{aligned}$$

This functional form of the general solution of the radial part of the Klein–Gordon equation for a massless scalar field in the spacetime of a rotating acoustic black hole, in the exterior region of the acoustic event horizon, is similar to Eq. (30), and therefore, its solution can be written in terms of the confluent Heun function as

$$R(z) = z^{\frac{\beta}{2}} \{C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; z) + C_2 z^{-\beta} \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; z)\},$$

where C_1 and C_2 are constants, and the parameters α , β , γ , δ , and η , which are given originally by Eqs. (31)–(35), are now given by the following expressions

$$\begin{aligned}\alpha &= 0; \\ \beta &= \frac{i(cx_h\omega - Bm)}{c\sqrt{x_h}}; \\ \gamma &= -\frac{3}{2}; \\ \delta &= -\frac{x_h\omega^2}{4}; \\ \eta &= \frac{1}{4} \left[3 - m^2 \left(1 + \frac{B^2}{c^2 x_h} \right) + x_h\omega^2 \right].\end{aligned}$$

As a conclusion, the solution of the radial part of the Klein–Gordon equation is formally the same as Eq. (30). In fact α assumes the same value and δ is given, formally, by the same expression as shown in Eq. (34). As a consequence of the fact that the solution is the same, the analogue Hawking radiation given by Eqs. (37)–(60) is all correct and therefore should be preserved.

In Sect. 3, a similar mistake was done when we transformed Eq. (71) into Eq. (78), using the coordinate transformation given by Eq. (72). Now, let us correct this mistake. In order to do this, let us write the line element which describes a canonical acoustic black hole in a more appropriate form given by

$$ds^2 = -c^2 \Delta d\tau^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where in this case Eq. (65) should read

$$\Delta = 1 - \frac{r_0^4}{r^4}.$$

The covariant Klein–Gordon equation in the spacetime of a canonical acoustic black hole, given by Eq. (69), is now written as

$$\left[-\frac{r^2}{c^2 \Delta} \frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial r} \left(r^2 \Delta \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Psi = 0.$$

Thus, substituting Eq. (70) into Eq. (69), we find that Eq. (71) should be written as

$$\frac{d}{dr} \left(r^2 \Delta \frac{dR}{dr} \right) + \left(\frac{\omega^2 r^2}{c^2 \Delta} - \lambda_{lm} \right) R = 0.$$

In order to obtain an equation which should be identified with one of the types of the Heun equation, let us introduce a new coordinate transformation corresponding to Eq. (72), given as follows:

$$x = r^2.$$

Thus, using this new coordinate, Eq. (73) turns into

$$\Delta = 1 - \frac{r_0^4}{x^2}.$$

Therefore, Eq. (75) is simply written as

$$\Delta = (x - x_+)(x - x_-) = 0,$$

whose solutions are the same given by Eqs. (76)–(77), namely,

$$\begin{aligned} x_+ &= +x_0 = +r_0^2, \\ x_- &= -x_0 = -r_0^2, \end{aligned}$$

and correspond to the new acoustic event horizons of the canonical acoustic black hole. Thus, Eq. (78) should read

$$\begin{aligned} \frac{d^2 R}{dx^2} + \left(\frac{1}{x - x_+} + \frac{1}{x - x_-} + \frac{3/2}{x} \right) \frac{dR}{dx} \\ + \left[\frac{A_1}{x - x_+} + \frac{A_2}{x - x_-} + \frac{A_3}{x} + \frac{B_1^2}{(x - x_+)^2} + \frac{B_2^2}{(x - x_-)^2} + \frac{B_3^2}{x^3} \right] R = 0. \end{aligned}$$

From this point on, it is worth calling attention to the fact that the functional form of the radial part of the Klein–Gordon equation, for a massless scalar field in the spacetime of a canonical acoustic black hole, has changed. Then, we need to define some new coefficients, A_1 , A_2 , A_3 , B_1 , B_2 , and B_3 which are given by

$$\begin{aligned} A_1 &= \frac{-c^2 \lambda (x_+^2 + x_-^2) + 2c^2 \lambda x_+ x_- + (x_- - 3x_+) \omega^2}{4c^2 x_+^2 (x_+ - x_-)^3}, \\ A_2 &= \frac{c^2 \lambda (x_+^2 + x_-^2) - 2c^2 \lambda x_+ x_- - (x_+ - 3x_-) \omega^2}{4c^2 x_-^2 (x_+ - x_-)^3}, \\ A_3 &= \frac{-c^2 \lambda (x_+ + x_-) + \omega^2}{4c^2 x_+^2 x_-^2}, \end{aligned}$$

$$\begin{aligned}
 B_1 &= \frac{\omega}{2c\sqrt{x_+}(x_+ - x_-)}, \\
 B_2 &= \frac{\omega}{2c\sqrt{x_-}(x_+ - x_-)}, \\
 B_3 &= \frac{i}{2}\sqrt{\frac{\lambda}{x_+x_-}}.
 \end{aligned}$$

Now, by setting the homographic substitution given by Eq. (79), Eq. (80) is now written as

$$\begin{aligned}
 \frac{d^2 R}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1} + \frac{3/2}{z-a} \right) \frac{dR}{dz} \\
 + \left[\frac{-A_1 x_+/a}{z} + \frac{-A_2 x_+/a}{z-1} + \frac{-A_3 x_+/a}{z-a} \right. \\
 \left. + \frac{B_1^2}{z^2} + \frac{B_2^2}{(z-1)^2} + \frac{B_3^2}{(z-a)^2} \right] R = 0,
 \end{aligned}$$

where the singular point a is given by

$$a = \frac{0 - a_1}{a_2 - a_1} = \frac{-x_+}{x_- - x_+}.$$

Thus, the F -homotopic transformation of the dependent variable, $R(z) \mapsto U(z)$, given by Eq. (81), should read, in this case, as

$$R(z) = z^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}(z-a)^{-\frac{3}{4}}U(z).$$

Therefore, Eq. (84) is rewritten as

$$\begin{aligned}
 \frac{d^2 U}{dz^2} + \left\{ \frac{(1 + 4B_1^2)/4}{z^2} + \frac{(1 + 4B_2^2)/4}{(z-1)^2} + \frac{(3 + 16B_3^2)/16}{(z-a)^2} \right. \\
 + \frac{(A_1 + A_3)x_+ + a[-2 + (A_1 + A_2)x_+]}{a} \frac{1}{(z-1)(z-a)} \\
 \left. + \frac{3 + 2a - 4A_1 x_+}{4} \frac{1}{z(z-1)(z-a)} \right\} R = 0.
 \end{aligned}$$

Now, let us consider the general Heun equation, whose canonical form is [59]

$$\frac{d^2 y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0,$$

where $y(z) = \text{HeunG}(a, q; \alpha, \beta, \gamma, \delta; z)$ is the general Heun function. This is a Fuchsian-type equation with regular singularities at $z = (0, 1, a, \infty)$. The parameters $\alpha, \beta, \gamma, \delta, \epsilon, q, a$ are generally complex, arbitrary (except that $a \neq 0, 1$), and related by

$$\gamma + \delta + \epsilon = \alpha + \beta + 1.$$

Its normal form is given by

$$\begin{aligned} \frac{d^2 U}{dz^2} + \left\{ \frac{(2\gamma - \gamma^2)/4}{z^2} + \frac{(2\delta - \delta^2)/4}{(z-1)^2} + \frac{(2\epsilon - \epsilon^2)/4}{(z-a)^2} \right. \\ \left. + \frac{2\alpha\beta - \delta\epsilon - \gamma(\delta + \epsilon)}{2} \frac{1}{(z-1)(z-a)} \right. \\ \left. + \frac{\gamma(a\delta + \epsilon) - 2q}{2} \frac{1}{z(z-1)(z-a)} \right\} R = 0, \end{aligned}$$

where

$$y(z) = z^{-\frac{\gamma}{2}} (z-1)^{-\frac{\delta}{2}} (z-a)^{-\frac{\epsilon}{2}} U(z).$$

Thus, the general solution of the radial part of the Klein–Gordon equation for a massless scalar particle in the spacetime of a canonical acoustic black hole, in the exterior region of the acoustic event horizon, has changed and hence Eq. (87) should read

$$\begin{aligned} R(z) = z^{\frac{1}{2}(\gamma-1)} (z-1)^{\frac{1}{2}(\delta-1)} (z-a)^{\frac{1}{2}(\epsilon-2)} \{ C_1 \text{HeunG}(a, q; \alpha, \beta, \gamma, \delta; z) \\ + C_2 z^{1-\gamma} \text{HeunG}(a, q_1; \alpha_1, \beta_1, \gamma_1, \delta; z) \}, \end{aligned}$$

where C_1 and C_2 are constants, and the parameters $\alpha, \beta, \gamma, \delta, \epsilon$, and q , which are given by Eqs. (88)–(92), are now given as follows:

$$\begin{aligned} \alpha &= \frac{1}{4a} \{ \sqrt{25a^2 - 16a^2(A_1x_+ + A_2x_+ + B_1^2 + B_2^2 + B_3^2) - 16ax_+(A_1 + A_3)} \\ &\quad + a[4i(B_1 + B_2) + 4 + \sqrt{1 - 16B_3^2}] \}; \\ \beta &= \frac{1}{2} [4i(B_1 + B_2) + \sqrt{1 - 16B_3^2} + 4] - \alpha; \\ \gamma &= 1 + 2iB_1; \\ \delta &= 1 + 2iB_2; \\ \epsilon &= 1 + \frac{1}{2} \sqrt{1 - 16B_3^2}; \\ q &= \frac{1}{4} \{ 2iB_1[a(2 + 4iB_2) + \sqrt{1 - 16B_3^2} + 2] + 4iaB_2 + 4A_1x_+ \\ &\quad + \sqrt{1 - 16B_3^2} - 1 \}. \end{aligned}$$

The parameters α_1 , β_1 , γ_1 , and q_1 are given by the relations

$$\begin{aligned}\alpha_1 &= \alpha + 1 - \gamma, \\ \beta_1 &= \beta + 1 - \gamma, \\ \gamma_1 &= 2 - \gamma. \\ q_1 &= q + (\alpha\delta + \epsilon)(1 - \gamma).\end{aligned}$$

Thus, the solution of the radial part of the Klein–Gordon equation has changed. It is no more given by a confluent Heun function, but now it is given by a general Heun function. As a consequence, the analogue Hawking radiation is modified. Then, in order to explicit this modification, we need to discuss the expansion of the general Heun function, as follows: If $\gamma \neq \{0, -1, -2, \dots\}$, then from the Fuchs–Frobenius Theory, it follows that $\text{HeunG}(a, q; \alpha, \beta, \gamma, \delta; z)$ exists, is analytic in the disk $|z| < 1$, corresponds to exponent 0 at $z = 0$ and assumes the value 1 there, and has the Maclaurin expansion

$$\text{HeunG}(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{j=0}^{\infty} b_j z^j,$$

where $b_0 = 1$, and

$$\begin{aligned}a\gamma b_1 - qb_0 &= 0, \\ X_j b_{j+1} - (Q_j + q)b_j + P_j b_{j-1} &= 0, \quad j \geq 1,\end{aligned}$$

with

$$\begin{aligned}P_j &= (j - 1 + \alpha)(j - 1 + \beta), \\ Q_j &= j[(j - 1 + \gamma)(1 + a) + a\delta + \epsilon], \\ X_j &= a(j + 1)(j + \gamma).\end{aligned}$$

Thus, when $r \rightarrow r_+ \Rightarrow x \rightarrow x_+ \Rightarrow z \rightarrow 0$, Eq. (93) should have been

$$R(r) \sim C_1 (r - r_+)^{\frac{1}{2}(\gamma-1)} + C_2 (r - r_+)^{-\frac{1}{2}(\gamma-1)}.$$

Then, considering the time factor, near the canonical acoustic black hole event horizon r_0 , Eq. (94) should be written as

$$\Psi = e^{-i\omega t} (r - r_-)^{\pm \frac{1}{2}(\gamma-1)},$$

and thus Eq. (95) should have been done by

$$\frac{1}{2}(\gamma - 1) = \frac{i}{2\kappa_0}\omega,$$

where

$$\kappa_0 = 2cr_0^3.$$

From this point on, the analysis of the analogue Hawking radiation expressed by Eqs. (97)–(111) is all correct.

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