Editorial note to: André Lichnerowicz, Propagators, commutators and anti-commutators in general relativity

Giampiero Esposito¹

Published online: 16 October 2018

© Springer Science+Business Media, LLC, part of Springer Nature 2018

Keywords Mathematical operators in GR · Global differential geometry and analysis · Les Houches School Relativity, Groups and Topology · Lichnerowicz biography · Golden Oldie

1 Introduction

This Golden Oldie is the translation of the lectures given by A. Lichnerowicz at the famous Les Houches School *Relativity, Groups and Topology* [1]. It may be helpful, for the general reader, to outline first the mathematical and physical background, which can be described as follows, starting for completeness from the nineteenth century.

(i) *Mathematical background.* The Laplacian operator occurred naturally in the theory of differential parameters introduced by Beltrami [2] and Darboux [3] when they developed the mathematical theory of surfaces. Shortly after Darboux, when Ricci-Curbastro stressed the need for a coordinate-independent formulation of differential calculus [4], and hence created tensor calculus [5], the Laplacian on scalar functions was written neatly in the language of Riemannian geometry and tensors [6]. On the other hand, the theory of hyperbolic equations, whose investigation began when neither special nor general relativity yet existed, received new input from relativistic physics, the theory of distributions, and the tools of modern analysis and algebra. Some milestones were, in chronological order, the Hadamard lectures on Cauchy's problem for linear partial differential equations [7], the Leray analysis [8] of fundamental solutions for hyperbolic equations and of global hyperbolicity (he studied first

The republication of the original paper can be found in this issue following the editorial note and online via <http://doi.org/10.1007/s10714-018-2433-x>.

✉ Giampiero Esposito
giampiero.esposito@na.infn.it

¹ INFN, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio 6, 80126 Napoli, Italy

vector spaces and then manifolds), the discovery of Sobolev spaces [9] and the Fourès-Bruhat proof that there exists a unique solution of the vacuum Einstein equations with non-analytic Cauchy data [10]. After another quarter of a century, Friedlander [11] applied the theory of distributions on manifolds to a systematic analysis of linear wave equations for scalar, vector and tensor fields, and towards the end of the twentieth century Christodoulou and Klainerman [12] exploited the optical formulation of general relativity and obtained their remarkable proof of the global non-linear stability of Minkowski space-time with the associated timelike geodesic completeness of a new class of asymptotically flat space-times.

(ii) *Physical background.* Nowadays all introductory courses in quantum field theory define the concepts studied in the Lichnerowicz lectures, i.e. the quantum theory of Klein–Gordon and Dirac fields with the associated propagators, and the physical applications of Clifford’s algebra, but the associated space-time is taken to be Minkowski, so that the additional ingredients resulting from Riemann’s curvature, and the possible obstructions to defining spinor structures, are hardly learned. Only general relativists, string theorists and mathematical physicists appreciate the importance of this broader framework.

The Lichnerowicz lectures, which were further expanded by him in another article in French, our Ref. [13], are still very useful in filling this gap in the formation of prospective theoretical physicists, and provide also a very nice blend of the concepts of global differential geometry and analysis (e.g. exterior differentiation and codifferentiation, fibre-bundle techniques, group theory) and of basic Riemannian geometry (covariant derivatives, Riemann and Ricci curvature), another interdisciplinary feature that is sadly missing in modern textbooks, which choose either the former or the latter path instead.

2 Selected features

Hereafter we bring to the attention of the reader a personal choice of unique features of the article, jointly with some remarks that are in order in our opinion.

Equation (1.2) defines covariant p -tensor distributions, in a section devoted to tensorial distributions. It would have taken another 11 years to find a systematic description of distributions on manifolds, i.e. the book by Friedlander [11] mentioned in our Introduction. A simple integration by parts leads to the codifferentiation operator in tensor language, Eq. (1.5). This operator finds a familiar application in writing half of the vacuum Maxwell equations for the field strength: $\delta F = 0$. Another relevant (but less familiar) application consists of writing the electromagnetic potential as the codifferential of an antisymmetric tensor field: $A = \delta P$, see our Ref. [14].

Equation (3.1) defines, with modern nomenclature, an operator of Laplace type, i.e. a Laplacian plus lower-order terms plus a potential term. This operator has been written down in coordinates in all classical papers and books on partial differential equations. Its (formal) adjoint is given by Eq. (3.2), and can be obtained from the condition (3.3). At this stage we are not yet dealing with the rigorous theory of unbounded operators on Hilbert space, but we are extending to Riemannian manifolds the concepts first encountered in the theory of partial differential operators in two independent variables.

The de Rham Laplacian on tensors is instead defined by Eq. (3.4), and it can be re-expressed in tensor language as is shown in Eq. (3.5).

In Sect. 4, Lichnerowicz considers hyperbolic Riemannian manifolds, i.e. pseudo-Riemannian manifolds with a metric of Lorentzian signature. This is the framework of Lorentzian geometry and general relativity. He defines the *emission* of a space-time point x as the union of its chronological future and its chronological past (that he calls future and past emission, respectively). The characteristic conoid at x , discussed at length by Hadamard [7], is here neatly defined as the frontier of the emission of x . The fundamental concept of global hyperbolicity is defined by following the work of Leray [8], i.e. by requiring compactness of the space of timelike paths (nowadays one includes also null paths [15], and one focuses instead on the existence of Cauchy surfaces, but young researchers should be aware that it was a major achievement to *prove* the existence of Cauchy surfaces [16] in a globally hyperbolic space-time). The definitions of elementary kernels of wave operators in Sect. 5 are also inspired by the work of Leray [8].

Section 12 is devoted to the quantum theory of the free electromagnetic field in curved space-time. In Eq. (12.4) one arrives at the commutator of the potential at different space-time points, while Eq. (12.4a) obtains the corresponding commutator of field strengths. This work prepares the ground for Sect. 14, where the commutator for the quantized gravitational field in linearized theory is written in Eq. (14.7). What a pity that Lichnerowicz does not cite explicitly (but only implicitly in his Abstract) the work by DeWitt [17] on invariant commutators for the quantized gravitational field obtained by means of Peierls brackets. With his profound mathematical insight, he might have been able to shed light on this paper by DeWitt, whose contribution to canonical quantum gravity is more widely known only through his work on Dirac-type quantization of gravity, which led to the Wheeler–DeWitt equation [18].

Section 18, devoted to the fibre bundle of spinorial frames, defines contravariant and covariant 1-spinors in Eqs. (18.1) and (18.2), and offers a deeper perspective on the spinorial operations performed in the physics-oriented literature. Sections 19 and 20 are also very pedagogical, and all formulae therein are especially helpful, e.g. the derivation of the Lichnerowicz–Weitzenböck operator in Eq. (20.5). This is a consequence of the fact that, by applying twice an operator of Dirac type, one obtains an operator of Laplace or wave type, depending on whether the metric is Riemannian or Lorentzian, respectively. The use of gamma matrices cannot be avoided if one wants to consider the theory of spinors in arbitrary dimension. Thus, the preference of general relativists for 2-component spinors in four-dimensional space-time is not the same as saying that one can get rid of gamma matrices in all cases of physical interest. On the physical side, we mention the work in our Ref. [19], where the authors studied the quantum theory of interacting gravitational and spinor fields.

Part IV studies the Pétiau–Duffin–Kemmer field in curved space-time, while Part V offers an original and useful perspective on the concepts of positive- and negative-frequency parts of a scalar field in curved space-time. Lichnerowicz proves that the whole theory of annihilation and creation operators can be obtained from his integral formula (31.2) alone, which involves in turn the propagator pertaining to the Klein–Gordon operator and a real symmetric kernel which is a fundamental solution of the Klein–Gordon equation. With hindsight, this might have prepared the ground for

educating research workers to fully understand how to define the basic concepts of quantum field theory in curved space-time [20–22].

Golden Oldies Series Editor's Note

As the above editorial note emphasizes, this paper provides an excellent account of a range of mathematical topics of relevance to theoretical physics, in particular to quantum field theory in curved spaces. It was chosen as a Golden Oldie because it was not readily available, being in French and in a conference proceedings (the rest of which is in English), and had therefore been underappreciated. (Ref. [13], which is contemporary and covers some of the same ground, has been much more frequently cited, although it is also in French.) The article was of course well known to French speakers working in related areas, and Choquet-Bruhat [29] says “This work is a great classic and contains numerous intermediate results that have been used many times”. We hope the translation will bring this article to a wider audience.

André Lichnerowicz: a brief biography

By M. A. H. MacCallum

André Lichnerowicz (known as Lichné to students and friends) was born on 21st January, 1915 at Bourbon l'Archambault, France. His Polish name came from his paternal grandfather who had fled from Poland in 1860. His father was a professor of literature, though interested in mathematics, and his mother, a mathematics teacher, was one of the first women to be awarded the *agrégation* in mathematics. When young he had rather poor health, including a bout of scarlet fever which left him deaf in one ear.

Image credit: Jean-Pierre Martin, Collège de France

He studied at the École Normale Supérieure in 1933–1936, where his teachers included Élie Cartan, who influenced him considerably, and then until 1939 for his doctorate from the Faculty of Science in Paris, under the supervision of Georges Dar-

mois. His thesis “Problèmes globaux en mécanique relativiste” concerned the initial value problem and the junction conditions named after Darmais. Yvonne Choquet-Bruhat (b. 1923), a former president of the International Society on General Relativity and Gravitation under whose aegis this journal appears, recalls [29] that while she was a high-school student her father spoke highly of Lichnerowicz’s mathematical gifts, profound sense of physics, and teaching ability. (Her father was an influential physicist known especially for his advanced textbooks. He was the deputy director of the École Normale Supérieure at the time of Lichnerowicz’s studies there, and later prompted the writing of the first of Lichnerowicz’s textbooks.)

From 1939–1941 Lichnerowicz was professor for special mathematics classes in Paris and then became one of the first researchers for the newly-formed research council CNRS. In 1941 he was appointed *maître de conférences* (an assistant lecturer) in mechanics at the University of Strasbourg, which had been evacuated at the outbreak of war to Clermont-Ferrand. This was a dangerous period, made more so because for historical reasons the Germans particularly regarded that university as an enemy, and in 1943 during a raid on the university Lichnerowicz was arrested: he escaped and returned with the university to Strasbourg in 1945, rising to Professor of rational mechanics and continuing to work there until 1949. From 1949–1952 he was Professor of mathematical methods in physics at the Faculty of Science in Paris, and from 1952 until his retirement in 1986 he was Professor of mathematical physics at the prestigious Collège de France. He died on 11 December 1998 at the age of 83.

The many tributes on his retirement and, later, after his death emphasize not only that he was an outstanding and creative scholar in mathematics and physics, where he showed depth of vision, but also an exceptional teacher, who “founded an entire new branch of French mathematical physics and differential geometry” [32], and a man of wide cultural knowledge and interests, including work on the philosophy of science.

To summarize his vast output (360 papers and books) properly would take many pages: even his selection from them [23] is over 600 pages. In these works, Lichnerowicz’s co-authors have included Borel, Flato, Fronsdal, and Sternheimer, as well as former students. Bourguignon [31] comments that in several [now important] topics Lichnerowicz made substantial contributions “precisely when these topics were not considered as centre-stage areas”. Here is a very short summary.

During World War II, Lichnerowicz proved a result earlier shown in particular cases by Einstein and Pauli, that there is no solution to the vacuum Einstein equations on a manifold of topology $S \times \mathbb{R}$ that is nontrivial and stationary, where S is spacelike, asymptotically Euclidean and complete or compact. (It is now known there are such solitons in gauge theories where the circle bundle is non trivial, see [29].) In 1944 he rewrote the Hamiltonian constraint of the initial value problem as the Lichnerowicz equation, which is still of importance. He also worked on relativistic kinetic theory, hydrodynamics and relativistic magnetohydrodynamics (with substantial results on shock waves), electromagnetic and gravitational radiation (see [37]), and got under way the theory of spinors on space-times. This work has “been essential for the success of theories of supergravity” [29], in the work of Seiberg and Witten (see [36]).

He made substantial contributions more widely in differential geometry. Berger [28] cites in particular his work on Kähler geometries, where he proved the “now classical” [33] equivalence of various definitions, on Bochner’s ideas giving the “Weitzenböck

formulae” linking curvature and topology, where Lichnerowicz found the first relation between the spectrum of an operator and curvature and found a key result for spinors, on holonomy groups, a notion from Élie Cartan now of great importance in many areas of mathematics and physics, and on homogeneous spaces.

Marle, in discussing Lichnerowicz’s work on symplectic geometry [30], remarked on his introduction of Poisson manifolds (the subject of the conference at which [33] was first presented), his work on Pfaffian and Jacobi manifolds and on further developments concerning these geometries and “Poisson–Lichnerowicz cohomology”, and the introduction of the idea of deformations of the algebra of functions on a manifold, which provided a new method of quantization [32]. These ideas also “announced the advent of non-commutative geometry” (as later developed by Connes and others). The Lichnerowicz prize in Poisson geometry was created in his honour in 2008.

As well as his research output, he undertook substantial tasks within the mathematics community. He was President of the French Mathematical Society in 1959–1960 and from 1966–1973 headed a Ministerial Commission on the Teaching of Mathematics with great influence in French education (see the full version of [27]). Part of his contribution was a change in the way of supervising theses, and he had many remarkable graduate students, the authors of [28–30] among them, as well as Aubin, Avez, Bel, and others noted by Kerner [32]. He founded the Journées Relativistes (see e.g. [32]), and the Journal of Geometry and Physics. He played a major role, with André Mercier and Marie-Antoinette Tonnelat, in organizing the 1955 conference now recognized as the first in the General Relativity and Gravitation series [38], and with Wheeler, Fock and others he founded in 1957 the International Committee on General Relativity and Gravitation, with Lichnerowicz and Tonnelat as Co-Presidents, which developed into the present-day Society in 1971.

During his career he was awarded many honours, including membership of the French Academy of Sciences at a relatively young age in 1963, and of academies and societies in Milan, Liège, Rome, Madrid, Brussels and the Vatican. He was awarded the Fubini Prize in 1955 and made a commander of the French Order of Merit in 1966. He had married Susa, who had been born in Peru and taught Spanish, and they had a son Jérôme.

Much more comprehensive discussions of Lichnerowicz’s life and work can be found in references [24–36] (with some substantial overlaps). I am grateful to Prof. Yvette Kosmann-Schwarzbach for directing me to these sources, from which the above brief biography has been extracted. I also consulted Lichnerowicz’s entry in the Mac-Tutor archive (www-history.mcs.st-and.ac.uk), which gives additional references and quotations, including some from Lichnerowicz himself.

References

1. DeWitt, C.M., De Witt, B.S. (eds.): *Relativity. Groups and Topology*. Blackie and Son, Glasgow (1964)
2. Beltrami, E.: *Mem. Acc. Sci. Ist. Bologna Serie II*(8), 551 (1868)
3. Darboux, G.: *Leçons sur la Théorie Générale des Surfaces et les Applications Géométriques du Calcul Infinitésimal. Troisième Partie. Lignes Géodésiques et Courbure Géodésique. Paramètres Différentiels, Déformation des Surfaces*. Gauthier-Villars, Paris (1894)
4. Ricci-Curbastro, G.: *Atti Ist. Veneto Serie VII* 4, 1336 (1893)

5. Ricci-Curbastro, G.: *Lezioni sulla Teoria delle Superficie*. Drucker, Padova (1898)
6. Bianchi, L.: *Lezioni di Geometria Differenziale*, vol. 1. Enrico Spoerri, Pisa (1902)
7. Hadamard, J.: *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, New Haven (1923)
8. Leray, J.: *Hyperbolic Differential Equations*. Princeton University Press, Princeton (1952)
9. Sobolev, S.L.: *Am. Math. Soc. Transl. Ser. 2*, 39 (1963)
10. Fourès-Bruhat, Y.: *Acta Math.* **88**, 141 (1952)
11. Friedlander, F.G.: *The Wave Equation on a Curved Space-Time*. Cambridge University Press, Cambridge (1975)
12. Christodoulou, D., Klainerman, S.K.: *The Global Nonlinear Stability of the Minkowski Space*. Princeton University Press, Princeton (1993)
13. Lichnerowicz, A.: *Bull. S.M.F.* **92**, 11 (1964)
14. Cohen, J.M., Kegeles, L.S.: *Phys. Rev. D* **10**, 1070 (1974)
15. Hawking, S.W., Ellis, G.F.R.: *The Large-Scale Structure of Space-Time*. Cambridge University Press, Cambridge (1973)
16. Geroch, R.P.: *J. Math. Phys.* **11**, 437 (1970)
17. DeWitt, B.S.: *Phys. Rev. Lett.* **4**, 317 (1960)
18. DeWitt, B.S.: *Phys. Rev.* **160**, 1113 (1967)
19. DeWitt, B.S., DeWitt-Morette, C.: *Phys. Rev.* **87**, 116 (1952)
20. Birrell, N.D., Davies, P.C.W.: *Quantum Fields in Curved Space*. Cambridge University Press, Cambridge (1982)
21. Fulling, S.A.: *Aspects of Quantum Field Theory in Curved Space-Time*. Cambridge University Press, Cambridge (1989)
22. Parker, L.E., Toms, D.J.: *Quantum Field Theory in Curved Space-Time*. Cambridge University Press, Cambridge (2009)
23. Lichnerowicz, A.: *Choix d'œuvres mathématiques*. Hermann, Paris (1982)
24. Schmidt, M.: *Hommes de Science*, 28 portraits, p. 170. Hermann, Paris (1990) (This contains a portrait and an interview with Lichnerowicz)
25. Choquet-Bruhat, Y., Bernard, D. (eds): *Physique quantique et géométrie. Colloque Géométrie et Physique de 1986 en l'honneur de André Lichnerowicz. Travaux en cours*, vol. 32. Hermann, Paris (1988) (Pages 1–42 of this volume are articles by Choquet-Bruhat, M. Berger and C.-M. Marle discussing Lichnerowicz's work)
26. Revuz, A.: *André Lichnérowicz (1915–1998)*. *Gazette des Mathématiciens* **81**, 94 (1999)
27. Revuz, A.: *Lichnérowicz et la réforme des mathématiques*. *Gazette des Mathématiciens* **82**, 90 (1999). Translated to English and adapted. *Not. Am. Math. Soc.* **46**(11), 1387 (1999)
28. Berger, M.: *Lichnérowicz et la géométrie différentielle*. *Gazette des Mathématiciens* **82**, 93 (1999). Translated to English and adapted. *Not. Am. Math. Soc.* **46**(11), 1388 (1999)
29. Choquet-Bruhat, Y.: *Lichnérowicz et la relativité générale*. *Gazette des Mathématiciens* **82**, 99 (1999). Translated to English and adapted. *Not. Am. Math. Soc.* **46**(11), 1391 (1999)
30. Marle, C.-M.: *L'œuvre d'André Lichnérowicz en géométrie symplectique*. *Gazette des Mathématiciens* **82**, 102 (1999). Translated to English and adapted. *Not. Am. Math. Soc.* **46**(11), 1393 (1999)
31. Bourguignon, J.-P.: *Not. Am. Math. Soc.* **46**(11), 1388 (1999)
32. Kerner, R.: *Tribute to André Lichnerowicz*. *Ann. d. Phys. (Leipzig)* **9**(3–5), 169 (2000)
33. Kosmann-Schwarzbach, Y.: *Hommage à André Lichnerowicz (1915–98)*. *Gazette des Mathématiciens* **118**, 52 (2008). English translation: *Not. Am. Math. Soc.* **56**, 244 (2009). Reprinted in *Newletter of the European Mathematical Society* **71**, 21 (2009)
34. Cahen, M.: *André Lichnerowicz (1915–98)*. *Acad. Roy. Belg. Bull. Cl. Sci.* **10**(6), 13 (1999)
35. Campos, A.: *André Lichnerowicz (1915–98)*. *Lect. Mat.* **21**(2), 143 (2000)
36. Connes, A.: *Hommage at <http://www.college-de-france.fr/site/andre-lichnerowicz>*. A bibliography of Lichnerowicz's works is also available at that URL
37. Lichnerowicz, A.: *Ondes et radiations électromagnétiques et gravitationnelles en relativité générale*. *Ann. Math. Pura Appl.* **50**, 1 (1960)
38. Lalli, R.: *Building the General Relativity and Gravitation Community During the Cold War*. Springer, Heidelberg (2017)