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ORIGINAL PAPER

Trisections obtained by trivially regluing surface-knots

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Abstract
Let S be a P^2-knot which is the connected sum of a 2-knot with normal Euler number 0 and an unknotted P^2-knot with normal Euler number ± 2 in a closed 4-manifold X with trisection T_X. Then, we show that the trisection of X obtained by the trivial gluing of relative trisections of $\overline{\nu(S)}$ and $X - \nu(S)$ is diffeomorphic to a stabilization of T_X. It should be noted that this result is not obvious since boundary-stabilizations introduced by Kim and Miller are used to construct a relative trisection of $X - \nu(S)$. As a corollary, if $X = S^4$ and T_X was the genus 0 trisection of S^4, the resulting trisection is diffeomorphic to a stabilization of the genus 0 trisection of S^4. This result is related to the conjecture that is a 4-dimensional analogue of Waldhausen's theorem on Heegaard splittings.

Keywords 4-manifold · Trisection · Surface-knot · Bridge trisection · Boundary-stabilization

1 Introduction

In 2012, Gay and Kirby [4] introduced the notion of a trisection of a 4-manifold, which is an analogue of a Heegaard splitting of a 3-manifold. A trisection of a 4-manifold with boundary is called a relative trisection. Meier and Zupan [10] introduced the notion of a bridge trisection of a surface-knot, which is an analogue of a bridge decomposition of a classical knot. A surface-knot can be put in a nice position in a 4-manifold, called a bridge position, such that the surface-knot is trisected according to a trisection of the 4-manifold.

Let $T = (X_1, X_2, X_3)$ be a trisection of a 4-manifold X, namely, $X = X_1 \cup X_2 \cup X_3$ and each X_i is a 4-dimensional 1-handlebody. For a 2-knot K in X which is in 1-bridge position, the decomposition of $X - \nu(K)$ into the union of three $X_i - \nu(K)$'s is a relative trisection of $X - \nu(K)$, where $\nu(K)$ is an open tubular neighborhood of K. On the other hand, for a surface-knot S in X which is not a 2-knot, the decomposition of $X - \nu(S)$ is never a relative trisection of $X - \nu(S)$. Kim and Miller [7] introduced a new technique, called a boundary-stabilization, to change the above decomposition of $X - \nu(S)$ into a relative trisection.

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We can construct a new trisection of $X = \overline{\nu(S)} \cup_{id} X - \nu(S)$ by gluing a relative trisection of $\overline{\nu(S)}$ and that of $X - \nu(S)$ constructed above using a gluing technique given by Castro [1]. In this section, the new trisection is called a trisection obtained by trivially gluing $\nu(S)$ and $X - \nu(S)$. This trisection and T are stably diffeomorphic (resp. stably isotopic), namely, they are diffeomorphic (resp. isotopic) after finitely many stabilizations. However, it is not obvious whether this trisection is diffeomorphic, especially isotopic, to a stabilization of T since when we construct a relative trisection of $X - \nu(S)$ from the union of three $X_i - \nu(S)$'s, we use boundary-stabilizations as mentioned above. Thus, we can think about the following question.

Question (Question 5.1) Let S be a surface-knot in a closed 4-manifold X with trisection T. Is a trisection obtained by trivially gluing $\nu(S)$ and $X - \nu(S)$ diffeomorphic, especially isotopic, to a stabilization of T? In particular, if $X = S^4$, does this hold?

The Price twist is a surgery along a P^2-knot P in a 4-manifold X, which yields at most three different 4-manifolds, namely, X, $\Sigma_P(X)$ and a non-simply connected 4-manifold $\tau_P(X)$. The closed 4-manifold $\Sigma_P(S^4)$ is a homotopy 4-sphere. In this paper, we call the twist having X the trivial Price twist. Kim and Miller [7] constructed trisections obtained by the Price twist by attaching a relative trisection of $\overline{\nu(P)}$ obtained from its Kirby diagram to a relative trisection of $X - \nu(P)$ constructed by a boundary-stabilization.

In this paper, we show the following theorem for Question 5.1. Note that a trisection obtained by the trivial Price twist along S corresponds to that obtained by trivially gluing a relative trisection of $\overline{\nu(S)}$ and that of $X - \nu(S)$.

Theorem *(Theorem 5.2) Let X be a closed 4-manifold and S the connected sum of a 2-knot K with normal Euler number 0 and an unknotted P^2-knot with normal Euler number ± 2 in X. Also let $T_{(X,S)}$ be a bridge trisection of (X, S) and T_X the underlying trisection. Suppose that S is in bridge position with respect to T_X. Also let $T_X^{'}$ be the underlying trisection of the bridge trisection obtained by meridionally stabilizing $T_{(X,S)}$ so that S is in 2-bridge position with respect to $T_X^{'}$. Then, the trisection T_S obtained by the trivial Price twist along S is diffeomorphic to a stabilization of $T_X^{'}$. In particular, the trisection T_S is diffeomorphic to a stabilization of T_X.*

In the proof of Theorem 5.2, we will perform handle slides and destabilizations many times (see also [15]).

A P^2-knot S in S^4 is said to be of Kinoshita type if S is the connected sum of a 2-knot and an unknotted P^2-knot. It is conjectured that every P^2-knot in S^4 is of Kinoshita type (see Remark 3.2).

Corollary *(Corollary 5.3) For each P^2-knot S in S^4 that is of Kinoshita type, the trisection obtained by the trivial Price twist along S is diffeomorphic to a stabilization of the genus 0 trisection of S^4.*

This implies that if any two diffeomorphic trisections of S^4 are isotopic, the resulting trisection gives a positive evidence to the conjecture that is a 4-dimensional analogue of Waldhausen's theorem on Heegaard splittings.

Conjucture ([11]) Every trisection of S^4 is isotopic to either the genus 0 trisection or its stabilization.

Organization In Sect. 2, we review trisections, relative trisections and bridge trisections. In Sect. 3, we recall a surgery along a P^2-knot in a 4-manifold, called the Price twist and provide a topic related to a trisection obtained by the Price twist. In Sect. 4, we review the definition of a boundary-stabilization and the way of constructing a relative trisection of the complement of a surface-knot. Finally, in Sect. 5, we raise a question on a stabilization of a trisection obtained by the trivial regluing of a surface-knot and prove our main theorem and its corollary related to the conjecture that is a 4-dimensional analogue of Waldhausen's theorem on Heegaard splittings.

2 Preliminaries

In this paper, we assume that 4-manifolds are compact, connected, oriented, and smooth unless otherwise stated and a surface-knot in a 4-manifold is a closed surface smoothly embedded in the 4-manifold.

2.1 Trisections of 4-manifolds

In this subsection, we review a definition and properties of trisections of closed 4-manifolds introduced in [4]. Let g, k_1, k_2 and k_3 be integers satisfying $0 \leq k_1, k_2, k_3 \leq g$.

Definition 2.1 Let X be a closed 4-manifold. A $(g; k_1, k_2, k_3)$-*trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ into three submanifolds X_1, X_2, X_3 of X satisfying the following conditions:

- For each $i = 1, 2, 3$, there exists a diffeomorphism $\phi_i : X_i \to Z_{k_i}$, where $Z_{k_i} = \natural_{k_i} S^1 \times D^3$.
- For each $i = 1, 2, 3$, $\phi_i(X_i \cap X_{i-1}) = Y^-_{k_i,g}$ and $\phi_i(X_i \cap X_{i+1}) = Y^+_{k_i,g}$, where $Y^\pm_{k_i,g}$ is the genus g Heegaard splitting $\partial Z_{k_i} = Y^-_{k_i,g} \cup Y^+_{k_i,g}$ of ∂Z_{k_i} obtained by stabilizing the standard genus k_i Heegaard splitting of ∂Z_{k_i} $g - k_i$ times.

Note that when X admits a trisection $X = X_1 \cup X_2 \cup X_3$, we call the 3-tuple $T = (X_1, X_2, X_3)$ also a trisection of X. If $k_1 = k_2 = k_3 = k$, the trisection is called a *balanced* trisection, or a (g, k)-trisection; if not, it is called an *unbalanced* trisection. For a (g, k)-trisection, since $\chi(X) = 2 + g - 3k$, we simply call the trisection a *genus* g trisection. For example, the 4-sphere S^4 admits the $(0, 0)$-trisection, namely genus 0 trisection.

For a trisection (X_1, X_2, X_3), let $H_\alpha = X_3 \cap X_1$, $H_\beta = X_1 \cap X_2$ and $H_\gamma = X_2 \cap X_3$. Then, the trisection is uniquely determined from $H_\alpha \cup H_\beta \cup H_\gamma$ [9]. The union $H_\alpha \cup H_\beta \cup H_\gamma$ is called the *spine*.

Given a trisection, we can define its diagram, called a trisection diagram. Note that from the definition, we see that the triple intersection $X_1 \cap X_2 \cap X_3$ is an oriented closed surface Σ_g of genus g.

Definition 2.2 Let Σ be a compact, connected, oriented surface, and δ, ϵ collections of disjoint simple closed curves on Σ. The 3-tuples $(\Sigma, \delta, \epsilon)$ and $(\Sigma, \delta', \epsilon')$ are said to be *diffeomorphism and handleslide equivalent* if there exists a self diffeomorphism h of Σ such that $h(\delta)$ and $h(\epsilon)$ are related to δ' and ϵ' by a sequence of handleslides, respectively.

Definition 2.3 A $(g; k_1, k_2, k_3)$-*trisection diagram* is a 4-tuple $(\Sigma_g, \alpha, \beta, \gamma)$ satysfying the following conditions:

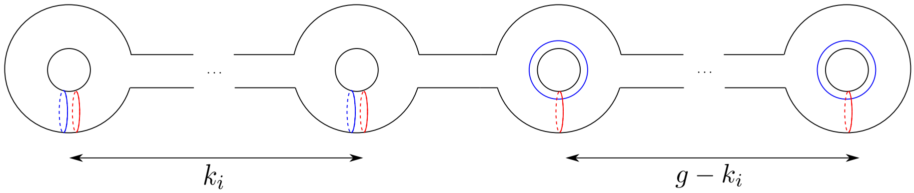
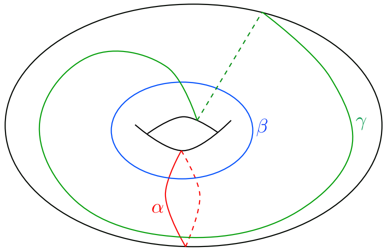


Fig. 1 The standard genus g Heegaard diagram of $\#_{k_i} S^1 \times S^2$

Fig. 2 A (1, 0)-trisection diagram of $\mathbb{C}P^2$



- $(\Sigma_g, \alpha, \beta)$ is diffeomorphism and handleslide equivalent to the standard genus g Heegaard diagram of $\#_{k_1} S^1 \times S^2$.
- $(\Sigma_g, \beta, \gamma)$ is diffeomorphism and handleslide equivalent to the standard genus g Heegaard diagram of $\#_{k_2} S^1 \times S^2$.
- $(\Sigma_g, \gamma, \alpha)$ is diffeomorphism and handleslide equivalent to the standard genus g Heegaard diagram of $\#_{k_3} S^1 \times S^2$.

Figure 1 describes the standard genus g Heegaard diagram of $\#_{k_i} S^1 \times S^2$.

Note that given a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$, α, β and γ are respectively indicated by red, blue and green curves as in Fig. 2.

Example 2.4 Fig. 2 is a (1, 0)-trisection diagram of $\mathbb{C}P^2$ (see also Fig. 6).

Definition 2.5 ([6]) Let X be a closed 4-manifold, and $T = (X_1, X_2, X_3)$ and $T' = (X_1', X_2', X_3')$ trisections of X. We say that T and T' are *diffeomorphic* if there exists a diffeomorphism $h\colon X \to X$ such that $h(X_i) = X_i'$ for each $i = 1, 2, 3$. We say that T and T' are *isotopic* if there exists an isotopy $\{h_t\}_{t\in[0,1]}$ of X such that $h_0 = id$ and $h_1(X_i) = X_i'$ for each $i = 1, 2, 3$.

Note that T and T' are diffeomorphic if and only if trisection diagrams of T and T' are related by handle slides on the same color curves and diffeomorphisms of a surface.

As with the stabilization for a Heegaard splitting, we can define a stabilization for a trisection.

Definition 2.6 Let (X_1, X_2, X_3) be a trisection and C a boundary-parallel arc properly embedded in $X_i \cap X_j$. We define X_i', X_j', and X_k' as follows, where $\{i, j, k\} = \{1, 2, 3\}$.

- $X_i' = X_i - \nu(C)$,
- $X_j' = X_j - \nu(C)$,
- $X_k' = X_k \cup \overline{\nu(C)}$.

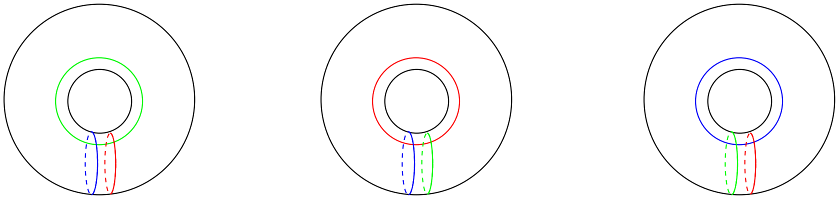


Fig. 3 The unbalanced trisection diagrams of S^4

The replacement of (X_1, X_2, X_3) by (X_1', X_2', X_3') is said to be the *k-stabilization*.

Note that the stabilization does not depend on the choice of an arc since any two boundary-parallel arcs in a 3-dimensional 1-handlebody are isotopic.

We can define a stabilization for a trisection using its trisection diagram.

Definition 2.7 Let $(\Sigma, \alpha, \beta, \gamma)$ be a trisection diagram. The diagram obtainted by connect-summing $(\Sigma, \alpha, \beta, \gamma)$ with one of three diagrams depicted in Fig. 3 is called the *stabilization* of $(\Sigma, \alpha, \beta, \gamma)$.

The diagrams in Fig. 3 are $(1; 1, 0, 0)$, $(1; 0, 1, 0)$, $(1; 0, 0, 1)$-trisection diagrams of S^4 from left to right. Note that for a $(g; k_1, k_2, k_3)$-trisection diagram $(\Sigma, \alpha, \beta, \gamma)$, the diagram obtained by connect-summing $(\Sigma, \alpha, \beta, \gamma)$ with the leftmost (resp. middle, resp. rightmost) diagram in Fig. 3 is a $(g+1; k_1+1, k_2, k_3)$ (resp. $(g+1; k_1, k_2+1, k_3)$, resp. $(g+1; k_1, k_2, k_3+1)$)-trisection diagram. Given a trisection diagram $(\Sigma, \alpha, \beta, \gamma)$, we can define a closed 4-manifold $X(\Sigma, \alpha, \beta, \gamma)$ as follows: We attach 2-handles to $\Sigma \times D^2$ along $\alpha \times \{1\}$, $\beta \times \{e^{\frac{2\pi i}{3}}\}$, and $\gamma \times \{e^{\frac{4\pi i}{3}}\}$, where the framing of each 2-handle is the surface framing. Then, we attach 3, 4-handles. Note that the way of attaching 3, 4-handles is unique up to diffeomorphism [9].

Gay and Kirby [4] showed that every closed 4-manifold X admits a trisection with nice handle decomposition. Moreover, they showed that any two trisections of a fixed closed 4-manifold are stably isotopic. Namely, they are isotopic after finitely many stabilizations. Note that they proved it in the balanced case. In general, an i-stabilized trisection is not isotopic to a j-stabilized trisection when $i \neq j$ [11].

For more details on trisections of closed 4-manifolds, see [4].

2.2 Relative trisections

In this subsection, we review trisections of 4-manifolds with boundary, called relative trisections. Before the definition, we introduce some notations.

Let g, k, p and b be non-negative integers with $b \geq 1$ and $g+p+b-1 \geq k \geq 2p+b-1$. Also let Σ_p^b be a compact, connected, oriented genus p surface with b boundary components and $l = 2p + b - 1$. We define D, $\partial^- D$, $\partial^0 D$, and $\partial^+ D$ as follows:

$$D = \{(r, \theta) \mid r \in [0, 1],\ \theta \in [-\pi/3, \pi/3]\},\quad \partial^- D = \{(r, \theta) \mid r \in [0, 1],\ \theta = -\pi/3\},$$
$$\partial^0 D = \{(r, \theta) \mid r = 1,\ \theta \in [-\pi/3, \pi/3]\},\quad \partial^+ D = \{(r, \theta) \mid r \in [0, 1],\ \theta = \pi/3\}.$$

Then, $\partial D = \partial^- D \cup \partial^0 D \cup \partial^+ D$ holds. We write P for Σ_p^b and U for $D \times P$. Then, from the decomposition of ∂D, we have $\partial U = \partial^- U \cup \partial^0 U \cup \partial^+ U$, where

$$\partial^\pm U = \partial^\pm D \times P,\ \partial^0 U = P \times \partial^0 D \cup \partial P \times D.$$

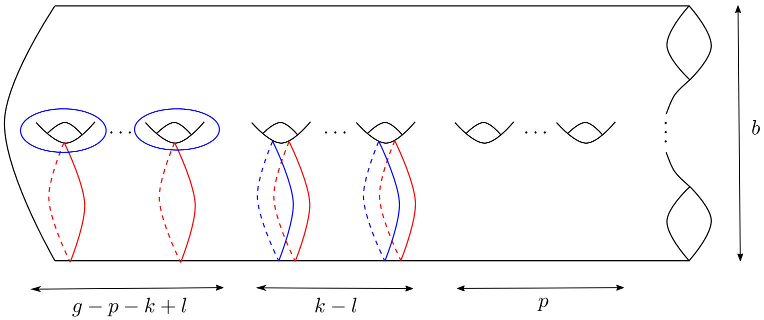


Fig. 4 The standard diagram for a relative trisection diagram. Note that $l = 2p + b - 1$

For an integer $n > 0$, let $V_n = \natural_n S^1 \times D^3$ and $\partial V_n = \partial^- V_n \cup \partial^+ V_n$ be the standard genus n Heegaard splitting of ∂V_n. Moreover, for an integer $s \geq n$, the Heegaard splitting of ∂V_n obtained by stabilizing the standard Heegaard splitting is denoted by $\partial V_n = \partial_s^- V_n \cup \partial_s^+ V_n$. Henceforth, let $n = k-2p-b+1 = k-l$, $s = g-k+p+b-1$ ($V_n = V_{k-2p-b+1} = V_{k-l}$).

Lastly, we define $Z_k = U \natural V_n$, where the boundary sum is taken by identifying the neighborhood of a point in int($\partial^- U \cap \partial^+ U$) with the neighborhood of a point in int($\partial_s^- V_n \cap \partial_s^+ V_n$). Here, we define $Y_k = \partial Z_k = \partial U \# \partial V_n$. Then, from the above decomposition, we have $Y_k = Y^-_{g,k;p,b} \cup Y^0_{g,k;p,b} \cup Y^+_{g,k;p,b}$, where $Y^{\pm}_{g,k;p,b} = \partial^{\pm} U \natural \partial_s^{\pm} V_n$ and $Y^0_{g,k;p,b} = \partial^0 U = P \times \partial^0 D \cup \partial P \times D$.

Using these notations, we can define a relative trisection as follows.

Definition 2.8 Let X be a 4-manifold with connected boundary. The decomposition $X = X_1 \cup X_2 \cup X_3$ of X satisfying the following conditions is called a $(g, k; p, b)$-*relative trisection*:

- For each $i = 1, 2, 3$, there exists a diffeomorphism $\phi_i : X_i \to Z_k$.
- For each $i = 1, 2, 3$, $\phi_i(X_i \cap X_{i-1}) = Y^-_{g,k;p,b}$, $\phi_i(X_i \cap X_{i+1}) = Y^+_{g,k;p,b}$ and $\phi_i(X_i \cap \partial X) = Y^0_{g,k;p,b}$, where $X_4 = X_1$ and $X_0 = X_3$.

Note that this definition is that of a *balanced* relative trisection. As with the definition 2.1, we can define an *unbalanced* relative trisection. Moreover in Definition 2.8, $X_i \cap X_j \cap \partial X \cong \Sigma_p^b$ must be connected since ∂X is assumed to be connected. This fact is used in Sect. 4 to consider a relative trisection of the complement of a surface-knot.

Given a relative trisection, we can define a relative trisection diagram.

Definition 2.9 A $(g, k; p, b)$-*relative trisection diagram* is a 4-tuple $(\Sigma_g^b, \alpha, \beta, \gamma)$ satysfying the following conditions:

- α, β and γ are respectively $(g-p)$-tuples of curves on Σ_g^b.
- Each of the 3-tuples $(\Sigma_g^b, \alpha, \beta)$, $(\Sigma_g^b, \beta, \gamma)$, $(\Sigma_g^b, \gamma, \alpha)$ is diffeomorphism and handleslide equivalent to the diagram described in Fig. 4.

Lemma 2.10 *(Lemma 11 in [2]) A $(g, k; p, b)$-relative trisection of a 4-manifold X with non-empty boundary induces an open book decomposition on ∂X with page Σ_p^b (hence binding $\partial \Sigma_p^b$).*

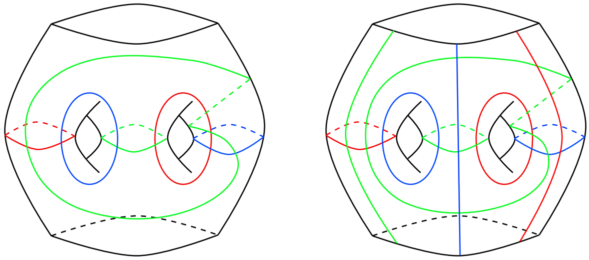


Fig. 5 (Left) A (2, 1; 0, 2)-relative trisection diagram of the D^2 bundle over S^2 with Euler number -1. (Right) Its arced relative trisection diagram

If we want to glue several relative trisection diagrams, we must describe a diagram with arcs, called an *arced relative trisection diagram*. There exists an algorithm for drawing such arcs.

Lemma 2.11 *(Theorem 6 in* [1]*) For $i = 1, 2$, let X_i be a 4-manifold with nomempty and connected boundary, and T_i a relative trisection of X_i. Also let $\mathcal{O}X_i$ be the open book decomposition on ∂X_i induced by T_i. If $f : \partial X_1 \to \partial X_2$ is an orientation reversing diffeomorphism which takes $\mathcal{O}X_1$ to $\mathcal{O}X_2$, then we obtain a trisection of $X = X_1 \cup_f X_2$ by gluing T_1 and T_2.*

Note that if there exists a diffeomorphism f as above, the page of $\mathcal{O}X_1$ is diffeomorphic to the page of $\mathcal{O}X_2$ via f. Thus, if T_i is the $(g_i, k_i; p_i, b_i)$-relative trisection, then $p_1 = p_2$ and $b_1 = b_2$.

Let $(\Sigma(i), \alpha(i), \beta(i), \gamma(i), a(i), b(i), c(i))$ be an arced relative trisection diagram of X_i. If there exists f in Lem 2.11, we can obtain three kinds of new simple closed curves in $\Sigma(1) \cup_f \Sigma(2)$, i.e. $a(1) \cup a(2)$, $b(1) \cup b(2)$ and $c(1) \cup c(2)$ via f. Thus, we have the following proposition, where $\Sigma = \Sigma(1) \cup_f \Sigma(2)$ and $\tilde{\alpha}$ (resp. $\tilde{\beta}$, resp. $\tilde{\gamma}$) $= (a(1)_j \cup_\partial a(2)_j)_j$ (resp. $(b(1)_j \cup_\partial b(2)_j)_j$, resp. $(c(1)_j \cup_\partial c(2)_j)_j$.

Proposition 2.12 *(Proposition 2.12 in* [3]*) In addition to the assumptions in Lem 2.11, let $(\Sigma(i), \alpha(i), \beta(i), \gamma(i), a(i), b(i), c(i))$ be an arced relative trisection diagram of X_i. Then, the 4-tuple $(\Sigma, \alpha, \beta, \gamma)$ is a trisection diagram of X, where $\alpha = \alpha(1) \cup \alpha(2) \cup \tilde{\alpha}$.*

Proposition 2.13 *(Theorem 5 in* [2]*) Let $(\Sigma, \alpha, \beta, \gamma)$ be a $(g, k; p, b)$ relative trisection diagram and Σ_α the surface obtained by performing the surgery along α. Suppose that this operation comes with an embedding $\phi_\alpha : \Sigma - \alpha \to \Sigma_\alpha$. Consider the following step.*

1. *Choose a collection of arcs a such that a is disjoint from α in Σ and $\phi_\alpha(a)$ cuts Σ_α into a disk. Note that a consists of $2p + b - 1$ arcs.*
2. *Choose b by handle sliding a over α so that b is disjoint from β. If necessary, we slide β_i over β_j. In this case, the β is denoted by β'. If handle slides are not needed, $\beta' = \beta$.*
3. *Choose c by handle sliding b over β' so that c is disjoint from γ. If necessary, we slide γ_i over γ_j. In this case, the γ is denoted by γ'. If handle slides are not needed, $\gamma' = \gamma$.*

Then, $(\Sigma, \alpha, \beta', \gamma', a, b, c)$ is an arced relative trisection diagram.

Example 2.14 Fig. 5 is a (2, 1; 0, 2)-relative trisection diagram of the D^2 bundle over S^2 with Euler number -1 and its arced relative trisection diagram constructed from the algorithm.

For more details on relative trisections, see [1–3].

2.3 Bridge trisections

In this subsection, we review trisections of surface-knots, called bridge trisections.

Definition 2.15 Let V be a 4-dimensional 1-handlebody and $\mathcal{D}$ a collection of disks properly embedded in V. We say that $\mathcal{D}$ is *trivial* if the disks of $\mathcal{D}$ are simultaneously isotoped into ∂V.

Definition 2.16 Let H be a 3-dimensional 1-handlebody and $\tau = \{\tau_i\}$ a collection of arcs properly embedded in H. We say that τ is *trivial* if τ_i is isotoped into ∂H for each i. Or equivalently, there exists a collection $\Delta = \{\Delta_i\}$ of disks in H with $\Delta_i \cap \Delta_j = \emptyset$ such that $\partial \Delta_i = \tau_i \cup \tau_i^{'}$ for some arc $\tau_i^{'} \subset \partial H$. We call τ, Δ and $\tau_i^{'}$ *trivial tangles*, *bridge disks* and a *shadow* of τ_i respectively.

Definition 2.17 ([10]) Let $X = X_1 \cup X_2 \cup X_3$ be a $(g; k_1, k_2, k_3)$-trisection of a closed 4-manifold X, and S a surface-knot in X. A decomposition $(X, S) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_3, \mathcal{D}_3)$ is a $(g; k_1, k_2, k_3; b; c_1, c_2, c_3)$-*bridge trisection* of (X, S) if

- For each $i = 1, 2, 3$, $\mathcal{D}_i$ is a collection of trivial c_i disks in X_i.
- For $i \neq j$, $\mathcal{D}_i \cap \mathcal{D}_j$ form trivial b tangles in $X_i \cap X_j$.

We say that S is in $(b; c_1, c_2, c_3)$-*bridge position* with respect to (X_1, X_2, X_3) if $(X, S) = (X_1, S \cap X_1) \cup (X_2, S \cap X_2) \cup (X_3, S \cap X_3)$ is a $(g; k_1, k_2, k_3; b; c_1, c_2, c_3)$-bridge trisection.

We call the trisection (X_1, X_2, X_3) the *underlying trisection* of the bridge trisection.

Remark 2.18 In Definition 2.17, if $X = S^4$, then the trisection is the $(0, 0)$-trisection [12, Definition 1.2].

As with a balanced trisection, when $k_1 = k_2 = k_3 = k$ and $c_1 = c_2 = c_3 = c$, we say that the decomposition of (X, S) is a $(g, k; b, c)$-*bridge trisection* and S is in (b, c)-*bridge position*. Note that if S is in $(b; c_1, c_2, c_3)$-bridge position, then $\chi(S) = c_1 + c_2 + c_3 - b$. So, when $c_1 = c_2 = c_3$, we often say that S is in b-bridge position.

Meier and Zupan [10] showed that every pair of a 4-manifold X and a surface-knot S in X admits a bridge trisection, using a technical operation called *meridional stabilization*.

Definition 2.19 Let $(X, S) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_3, \mathcal{D}_3)$ be a bridge trisection and C an arc in $\mathcal{D}_i \cap \mathcal{D}_j$ whose endpoints are in distinct components of $\mathcal{D}_k$. We define $(X_\ell^{'}, \mathcal{D}_\ell^{'})$ as follows, where $\{i, j, k\} = \{1, 2, 3\}$.

- $(X_i^{'}, \mathcal{D}_i^{'}) = (X_i - \nu(C), \mathcal{D}_i - \nu(C))$
- $(X_j^{'}, \mathcal{D}_j^{'}) = (X_j - \nu(C), \mathcal{D}_j - \nu(C))$
- $(X_k^{'}, \mathcal{D}_k^{'}) = (X_k \cup \overline{\nu(C)}, \mathcal{D}_k \cup (\overline{\nu(C)} \cap S)$

The replacement of $(X_\ell, \mathcal{D}_\ell)$ by $(X_\ell^{'}, \mathcal{D}_\ell^{'})$ for all ℓ is said to be a k-meridionally stabilization.

Note that when we meridionally stabilize a bridge trisection of (X, S), for the underlying trisection of X, we simply stabilize it. This observation is used in the proof of our main theorem.

Theorem 2.20 *(Theorem 2 in [10]) Let S be a surface-link in a closed 4-manifold X with a (g, k)-trisection T. Then, the pair (X, S) admits a $(g, k; b, n)$-bridge trisection with $b = 3n - \chi(S)$, where n is the number of connected components of S.*

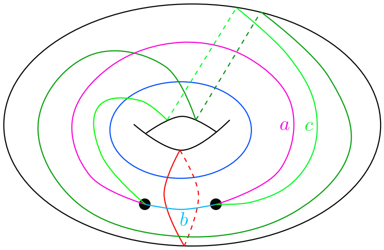


Fig. 6 A doubly pointed trisection diagram of $(\mathbb{C}P^2, \mathbb{C}P^1)$. The red, blue, and green curves describe a (1, 1)-trisection diagram of $\mathbb{C}P^2$ and the arcs a, b, and c describe $\mathbb{C}P^1$. Note that in a doubly pointed trisection diagram, we do not need to draw the arcs since there is a unique way to describe them

Note that in Theorem 2.20, if S is a 2-knot, then S can be in 1-bridge position with respect to a trisection obtained by stabilizing T. Furthermore if S is a P^2-knot, then S can be in 2-bridge position.

A surface-knot in S^4 can be described by a triplane diagram introduced by Meier and Zupan [12]. On the other hand, it is difficult to describe a surface-knot in a general 4-manifold in the same way. Therefore, Meier and Zupan [10] developed another diagram using shadows in Definition 2.16. It is called a shadow diagram.

Definition 2.21 Let $(X, S) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_3, \mathcal{D}_3)$ be a bridge trisection. A 4-tuple $(\Sigma, (\alpha, a), (\beta, b), (\gamma, c))$ is called a *shadow diagram* if the 4-tuple $(\Sigma, \alpha, \beta, \gamma)$ is a trisection diagram of (X_1, X_2, X_3), and a, b and c are shadows of $\mathcal{D}_1 \cap \mathcal{D}_2$, $\mathcal{D}_2 \cap \mathcal{D}_3$ and $\mathcal{D}_3 \cap \mathcal{D}_1$ respectively. In particular, a, b and c are a shadow of $\mathcal{D}_i \cap \mathcal{D}_j$, the shadow diagram is called a *doubly pointed trisection diagram*.

Each 2-knot in a closed 4-manifold admits a doubly pointed trisection diagram since it can be put in 1-bridge position. Note that for a 2-knot K in 1-bridge position with respect to a trisection T of X, the underlying trisection diagram of (X, K) is the diagram of T. For example, Fig. 6 describes a doubly pointed trisection diagram of $(\mathbb{C}P^2, \mathbb{C}P^1)$. We call the two black points of a doubly pointed trisection diagram *base points* in the proof of our main theorem.

For more details on bridge trisections, see [10, 12].

3 The price twist

In this section, we review a surgery along a P^2-knot in a closed 4-manifold, called the Price twist.

Let S be a P^2-knot, that is, a real projective plane smoothly embedded in a closed 4-manifold X, with normal Euler number $e(S) = \pm 2$. Note that when $X = S^4$, from Whitney-Massey's theorem [13, 18], each P^2-knot S satisfies $e(S) = \pm 2$. Then, for a tubular neighborhood $\nu(S)$ of S in X, the boundary $\partial\nu(S)$ is a Seifert-fibered space Q over S^2 with three singular fibers labeled S_0, S_1 and S_{-1}, where these indices are respectively ± 2, ± 2 and ∓ 2 when $e(S) = \pm 2$. Since $\partial(X - \nu(S)) \cong Q$, $\partial(X - \nu(S))$ has the same label with $\partial\nu(S)$. Price [16] showed that there exist three kinds of self-homeomorphism of $\partial\nu(S)$ up to isotopy, that is, $S_{-1} \mapsto S_{-1}$, $S_{-1} \mapsto S_0$ and $S_{-1} \mapsto S_1$. Thus, when we reglue $\nu(S)$ deleted from X according to $\phi\colon \partial\nu(S) \to \partial(X - \nu(S))$, we can obtain the following at most (see below) three 4-manifolds up to diffeomorphism (the notation follows [7]):

- If $\phi(S_{-1}) = S_{-1}$, the resulting manifold is X.
- If $\phi(S_{-1}) = S_0$, the resulting manifold is denoted by $\tau_S(X)$.
- If $\phi(S_{-1}) = S_1$, the resulting manifold is denoted by $\Sigma_S(X)$.

This operation is called the *Price twist* of X along S. Especially, in this paper, we call the first twist, that is, the twist having the original manifold X, the *trivial Price twist*. Note that $\Sigma_S(S^4)$ is a homotopy 4-sphere. Let $\Sigma_K^G(X)$ be the 4-manifold obtained by the Gluck twist along K, where K is a 2-knot in X. Then, from [8], we see that for a P^2-knot $S = K\#P_{\pm}$, $\Sigma_S(X) \cong \Sigma_K^G(X)$ holds, where $P_{\pm}$ is an unknotted P^2-knot with normal Euler number ± 2 in X. So, for a 2-knot K satisfying $\Sigma_K^G(S^4) \cong S^4$ such as a twist spun 2-knot, we have $\Sigma_S(S^4) \cong S^4$. Thus, we can ask whether the conjecture that is a 4-dimensional analogue of Waldhausen's theorem on Heegaard splittings [11, Conjecture 3.11] holds for such $\Sigma_S(S^4)$ ([7, Question 6.2]). Note that [7, Question 6.2] is a specific case of [11, Conjecture 3.11].

Question (Question 6.2 in [7]) Let S be a P^2-knot in S^4 so that $\Sigma_S(S^4) \cong S^4$. Is a trisection of $\Sigma_S(S^4)$ obtained from the algorithm of Section 5 in [7] isotopic to a stabilization of the genus 0 trisection of S^4?

Conjucture (Conjecture 3.11 in [11]) Every trisection of S^4 is isotopic to either the genus 0 trisection or its stabilization.

Remark 3.1 From [15], we immediately see that Figure 19 right of [7], that is, a $(6, 2)$-trisection diagram of $\Sigma_{P_-}(S^4)$, is a stabilization of the $(0, 0)$-trisection diagram of S^4 up to handle slides and diffeomorphisms.

Remark 3.2 For a 2-knot K and an unknotted P^2-knot P in S^4, the P^2-knot S admits the decomposition $K\#P$ is said to be *of Kinoshita type*. It is not known whether every P^2-knot in S^4 is of Kinoshita type. This question is called the Kinoshota question or the Kinoshita conjecture. We may answer the question with $\tau_S(S^4)$ [7]. Note that in [7, Question 6.2], if S is of Kinoshita type, then trisections in the question are diffeomorphic to trisections obtained by the Gluck twist [15]. In particular, if S is the connected sum of the unknotted P^2-knot and a spun or twist spun 2-knot, [7, Question 6.2] reduces to [5, Question 6.4] in the sense of diffeomorphic trisections.

Question (Question 6.4 in [5]) Is the trisection diagram constructed by [14] and [5, Lemma 5.5] for the Gluck twist along a spun or twist spun 2-knot a stabilization of the $(0, 0)$-trisection diagram of S^4?

This question is not answered even in the case of the spun trefoil, which can be regarded as the simplest non trivial spun 2-knot.

By the following theorem, called Waldhausen's theorem, we can see the reason that [11, Conjecture 3.11] is a 4-dimensional analogue of Waldhausen's theorem on Heegaard splittings.

Theorem 3.3 *([19], [17]) The 3-sphere S^3 admits a unique Heegaard splitting up to isotopy for each genus.*

For more details on the Price twist and a trisection obtained by the Price twist, see [7, 16].

4 A boundary-stabilization

In this section, we review a boundary-stabilization for a 4-manifold with boundary introduced in [7].

Definition 4.1 Let $Y = Y_1 \cup Y_2 \cup Y_3$ be a 4-manifold with $\partial Y \neq \emptyset$, where $Y_i \cap Y_j = \partial Y_i \cap \partial Y_j$, and C an arc properly embedded in $Y_i \cap Y_j \cap \partial Y$ whose endpoints are in $Y_1 \cap Y_2 \cap Y_3$. Also let $\nu(C)$ be a fixed open tubular neighborhood of C. Then, we define $\tilde{Y}_i, \tilde{Y}_j, \tilde{Y}_k$ as follows:

- $\tilde{Y}_i = Y_i - \nu(C)$,
- $\tilde{Y}_j = Y_j - \nu(C)$,
- $\tilde{Y}_k = Y_k \cup \overline{\nu(C)}$.

The replacement of (Y_1, Y_2, Y_3) by $(\tilde{Y}_i, \tilde{Y}_j, \tilde{Y}_k)$ is said to be a *boundary-stabilization* along C. In this case, we say that $\tilde{Y}_k$ has been obtained by boundary-stabilizing Y_k along C.

As we have seen in Sect. 1, we need a boundary-stabilization in order to construct a relative trisection of the complement of a surface-knot in a closed 4-manifold. The following explanation is more precise.

Let S be a surface-knot in a closed 4-manifold X with trisection (X_1, X_2, X_3). Suppose that S is in (b, c)-bridge position with respect to (X_1, X_2, X_3). Let $X_i^{'} = X_i - \nu(S)$. Then, $X - \nu(S)$ admits a natural decomposition $X - \nu(S) = X_1^{'} \cup X_2^{'} \cup X_3^{'}$. However, this decomposition of $X - \nu(S)$ can be a relative trisection if and only if S is a 2-knot and S is in 1-bridge position, that is, $b = 1$. This is because if $b > 1$, then the triple intersection $X_i^{'} \cap X_j^{'} \cap \partial(X - \nu(S))$, which is diffeomorphic to the disjoint union $\sqcup_b S^1 \times I$ of b annuli, is disconnected. This contradicts the fact that for a relative trisection (Y_1, Y_2, Y_3), if ∂Y is connected, then $Y_i \cap Y_j \cap \partial Y$ must be connected. So, for all S except 2-knots, $X - \nu(S)$ cannot admit $(X_1^{'}, X_2^{'}, X_3^{'})$ as a relative trisection. Although, we can refine the decomposition by boundary-stabilizing each $X_i^{'}$ so that $X - \nu(S)$ admits a relative trisection for each S. Put briefly, the way is the following:

In this paper, since we focus on a P^2-knot, we first review a boundary-stabilization of the complement of a P^2-knot. In the above situation, suppose also that S is a P^2-knot and $b = 2$ (Theorem 2.20). For each $i = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$, we define C_i to be an arc in $X_j^{'} \cap X_k^{'} \cap \partial(X - \nu(S))$ whose endpoints are in $X_1^{'} \cap X_2^{'} \cap X_3^{'}$ which intersects two distinct connected components of $\partial(X_1^{'} \cap X_2^{'} \cap X_3^{'})$. Take C_1, C_2 and C_3 so that they have different endpoints. Then, if we boundary-stabilize $X_\ell^{'}$ along C_ℓ, we obtain the decomposition $X - \nu(S) = \tilde{X}_1 \cup \tilde{X}_2 \cup \tilde{X}_3$, where $\tilde{X}_\ell$ is the submanifold of $X - \nu(S)$ obtained by boundary-stabilizing $X_\ell^{'}$ along C_1, C_2, and C_3. We see that $\tilde{X}_i \cap \tilde{X}_j \cap \partial(X - \nu(S))$ is connected and if we furthermore check on the structure of an open book decomposition which will be induced, we have the following proposition.

Proposition 4.2 *([7]) The 3-tuple $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is a relative trisection of $X - \nu(S)$.*

For a surface-knot S except P^2-knots, we can construct a relative trisection of the complement of S as with the case of a P^2-knot. The differences are that for each $i = 1, 2, 3$, we take C_i to be a collection of $2 - \chi(S)$ arcs and take each arc in C_i so that the arc is parallel to a different one in $\nu(S) \cap X_j \cap X_k$.

Note that unlike a stabilization of a trisection, a boundary-stabilization depends on the choice of an arc. If S is a P^2-knot, the type of a relative trisection of $X - \nu(S)$ obtained by boundary-stabilizations as above is either $(g, k; 0, 3)$ or $(g^{'}, k^{'}; 1, 1)$. In Sect. 5, since we glue a (2, 2; 0, 3)-relative trisection of $\overline{\nu(S)}$ and a relative trisection of $X - \nu(S)$ from boundary-stabilizations, we need to boundary-stabilize $X - \nu(S) = \bigcup_{i=1}^{3} X_i - \nu(S)$ so that the type of the resulting relative trisection is $(g, k; 0, 3)$ for some g and k.

Kim and Miller developed an algorithm to describe a relative trisection diagram of the complement of a surface-knot using the shadow diagram; see [7, Section 4].

For more details on boundary-stabilizations and a relative trisection of the complement of a surface-knot, see [7].

5 Main theorem

As we have seen in Sect. 1, we can think about the following question.

Question 5.1 Let S be a surface-knot in a closed 4-manifold X with trisection T. Is a trisection obtained by trivially gluing $\nu(S)$ and $X - \nu(S)$ diffeomorphic, especially isotopic, to a stabilization of T? In particular, if $X = S^4$, does this hold?

For the restricting case, we answer Question 5.1 affirmatively in Theorem 5.2, our main theorem.

Theorem 5.2 *Let X be a closed 4-manifold and S the connected sum of a 2-knot K with normal Euler number 0 and an unknotted P^2-knot with normal Euler number ± 2 in X. Also let $T_{(X,S)}$ be a bridge trisection of (X, S) and T_X the underlying trisection. Suppose that S is in bridge position with respect to T_X. Also let $T_X^{'}$ be the underlying trisection of the bridge trisection obtained by meridionally stabilizing $T_{(X,S)}$ so that S is in 2-bridge position with respect to $T_X^{'}$. Then, the trisection T_S obtained by the trivial Price twist along S is diffeomorphic to a stabilization of $T_X^{'}$. In particular, the trisection T_S is diffeomorphic to a stabilization of T_X.*

Proof Let $\mathcal{D}_Y$ be a relative trisection diagram of a 4-manifold Y. Also let P_+ and P_- be unknotted P^2-knots in X with normal Euler number 2 and -2, respectively.

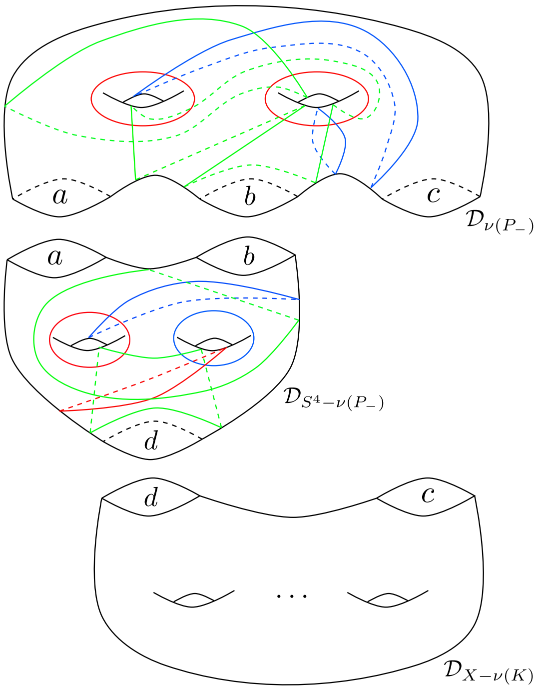
Since the preferred diagram $\mathcal{D}_{\nu(P_+)}$ and $\mathcal{D}_{S^4-\nu(P_+)}$ in [7] are the mirror images of $\mathcal{D}_{\nu(P_-)}$ and $\mathcal{D}_{S^4-\nu(P_-)}$, respectively, it suffices to proof Theorem 5.2 only for $S = K \# P_-$. □

Constructing T_S It follows from [7] that $\mathcal{D}_{X-\nu(S)}$ is the union of $\mathcal{D}_{S^4-\nu(P_-)}$ and $\mathcal{D}_{X-\nu(K)}$. Thus, the gluing $\mathcal{D}_{\nu(P_-)}$ and $\mathcal{D}_{X-\nu(S)}$ together by the trivial Price twist is described as Fig. 7. Note that we construct $\mathcal{D}_{\nu(P_-)}$ in Fig. 7 by deforming the preferred diagram of $\nu(P_-)$ in [7] so that the gluing is described as Fig. 7. In Fig. 7, if we draw arcs of $\mathcal{D}_{\nu(P_-)}$ and $\mathcal{D}_{S^4-\nu(P_-)}$, then we can obtain Fig. 8. The diagram depicted in Fig. 8 corresponds to T_S. It should be noted that we do not draw curves and arcs on the surface of $\mathcal{D}_{X-\nu(K)}$ in Fig. 7, but $\mathcal{D}_{X-\nu(K)}$ has them.

From now on, we deform trisection diagrams specifically. Note that from Fig. 8–18, the undrawn part describes $\mathcal{D}_{X-\nu(K)}$ with arcs and if necessary, let two arcs of $\mathcal{D}_{X-\nu(K)}$ be parallel by performing handle slides. Also note that for a α curve α_i, we call a curve obtained by sliding α_i over another α curve also α_i. The same is true for β and γ curves.

The first destabilization In Fig. 8 (or Fig. 9), we will destabilize α_1, β_1 and γ_1. To do this, we slide γ_2 over γ_3 so that the geometric intersection number of γ_2 and α_1 is 2. Then, we slide γ_2, γ_3 and γ_4 over γ_1 in this order. After that, we slide γ_2 over γ_4. As a result, γ_2 does not intersect α_1. We also slide γ_4 over γ_3 so that γ_4 does not intersect α_1. Finally, we slide γ_3 over γ_1, so that all γ curves except γ_1 do not meet α_1 and β_1. Then, we obtain Fig. 9. In

Fig. 7 The gluing diagram of $\mathcal{D}_{\nu(P_-)}$ and $\mathcal{D}_{X-\nu(S)}$ by the trivial Price twist along $S = K\#P_-$ in X. We glue $\mathcal{D}_{\nu(P_-)}$, $\mathcal{D}_{S^4-\nu(P_-)}$, and $\mathcal{D}_{X-\nu(K)}$ along the boundary components of the corresponding characters a, b, c and d



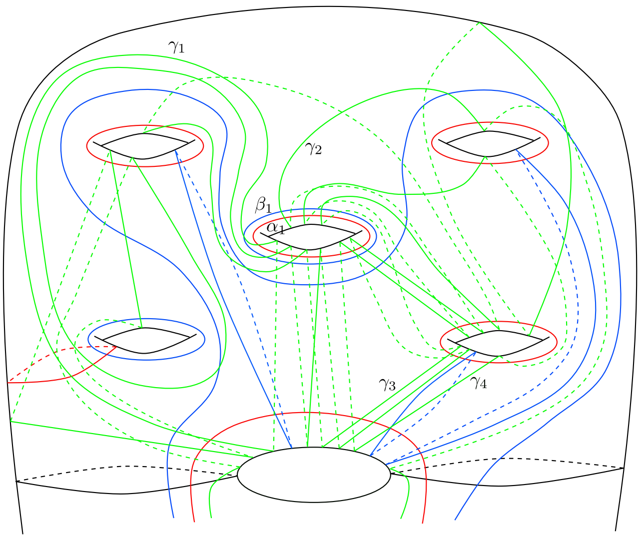


Fig. 8 Starting diagram

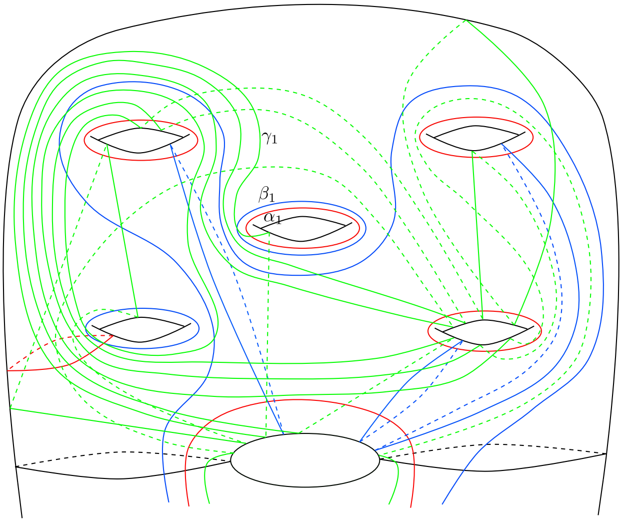


Fig. 9 Before the first destabilization

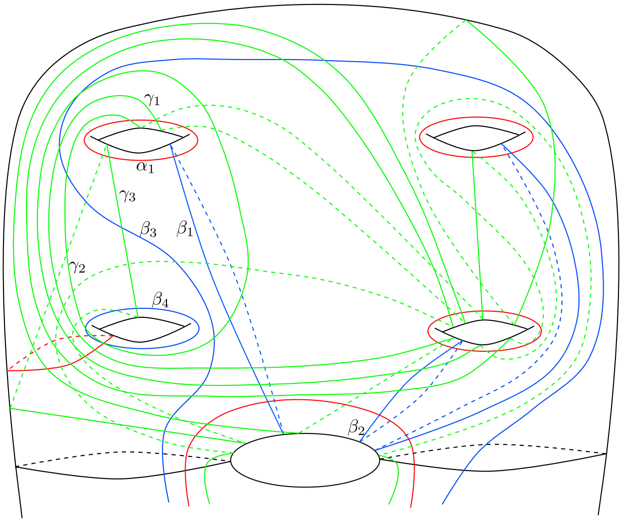


Fig. 10 After the first destabilization

Fig. 9, by destabilizing α_1, β_1 and γ_1, that is, erasing γ_1 and surgering α_1 or β_1 (if we choose α_1, then we erase β_1 and vise versa), we get Fig. 10.

The second destabilization In Fig. 10 (or Fig. 11), we will destabilize α_1, β_1 and γ_1. To do this, we firstly need to make β_1 parallel to γ_1. We slide β_3 over β_4 and β_1 over β_2. We again slide β_1 over β_2 so that β_1 is parallel to γ_1. After that, we slide γ_2 and γ_3 over γ_1 in order to remove the crossings of γ_2, γ_3 and α_1. As a result, we obtain Fig. 11. In Fig. 11, by destabilizing α_1, β_1 and γ_1, that is, erasing β_1 and γ_1 and surgering α_1, we get Fig. 12.

The third destabilization In Fig. 12 (or Fig. 13), we will destabilize α_1, β_1 and γ_1. To do this, we need to make β_1 parallel to γ_1. We slide γ_1 over γ_2 so that γ_1 does not intersect α_2.

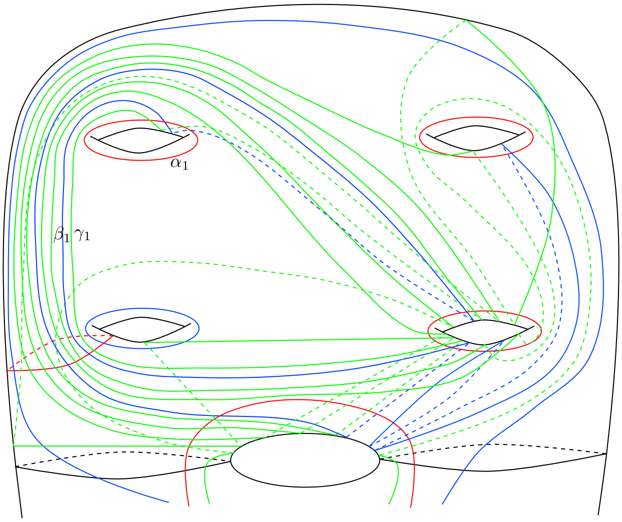


Fig. 11 Before the second destabilization

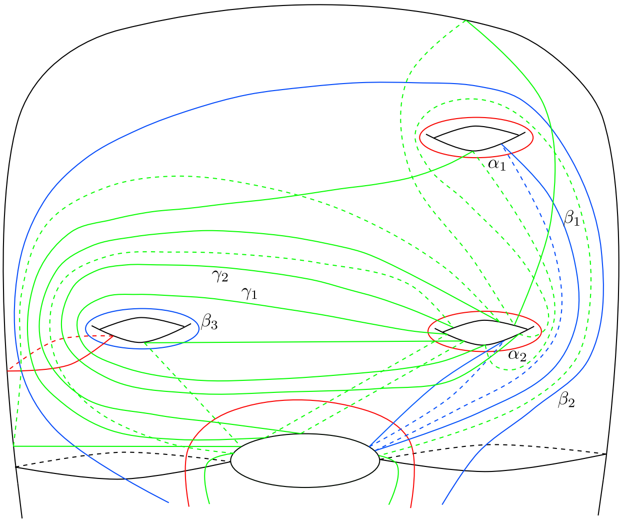


Fig. 12 After the second destabilization

Then, we slide β_1 over β_2 and β_3, so that β_1 is parallel to γ_1. As a result, we obtain Fig. 13. In Fig. 13, by destabilizing α_1, β_1 and γ_1, we get Fig. 14.

The fourth destabilization In Fig. 14 (or Fig. 15), we will destabilize α_1, β_1 and γ_1. To do this, we need to make β_1 parallel to α_1. We slide β_1 over β_2 and α_1 over α_2, so that α_1 is parallel to β_1. As a result, we obtain Fig. 15. In Fig. 15, by destabilizing α_1, β_1 and γ_1, we get Fig. 16.

The fifth destabilization In Fig. 16, we make γ_1 and α_1 be parallel by isotopies. Then, we obtain Fig. 17. In Fig. 17, by destabilizing α_1, β_1 and γ_1, we get Fig. 18.

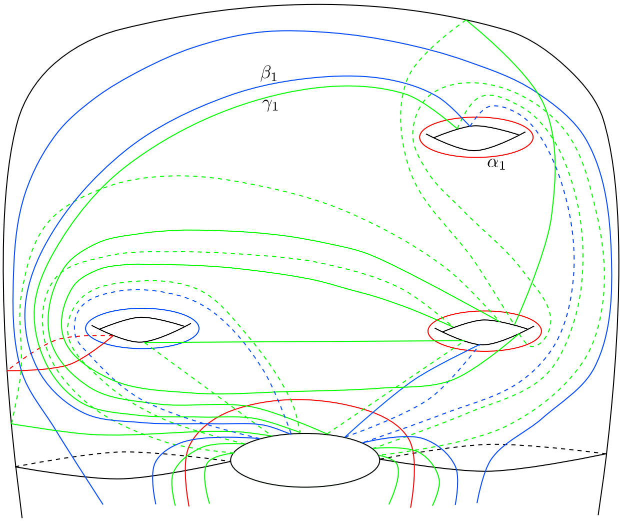


Fig. 13 Before the third destabilization

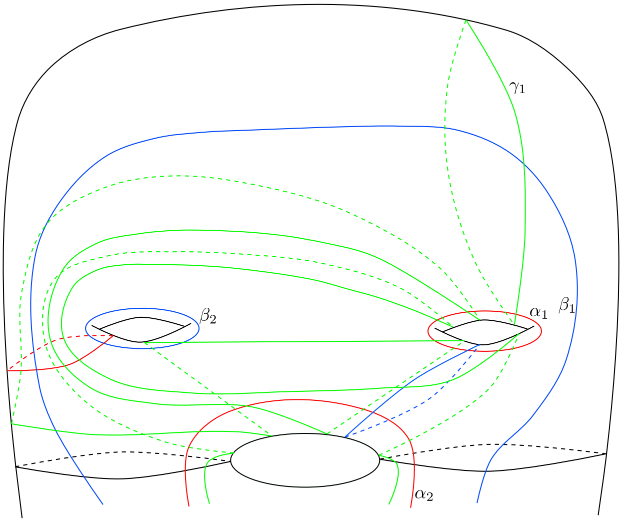


Fig. 14 After the third destabilization

The sixth destabilization In Fig. 18, the trisection of $X - \nu(K)$ is 0-annular since the normal Euler number of K is 0. Thus, the monodromy of the open book decomposition is the identity, that is, α_1 and γ_1 in Fig. 18 can be parallel. By destabilizing α_1, β_1 and γ_1, we have a diagram $\mathcal{D}$.

The diagram $\mathcal{D}$ is obtained by attaching two disks to the two boundary components of the surface of $\mathcal{D}_{X-\nu(K)}$ since we surger along α_1 when we destabilize α_1, β_1 and γ_1 in Fig. 18. In fact, $\mathcal{D}_{X-\nu(K)}$ is the diagram obtained by removing the open neighborhood of base points of the doubly pointed trisection diagram of (X, K). Thus, $\mathcal{D}$ is the diagram obtained by simply deleting the base points. (Note that the surface erased the base points has

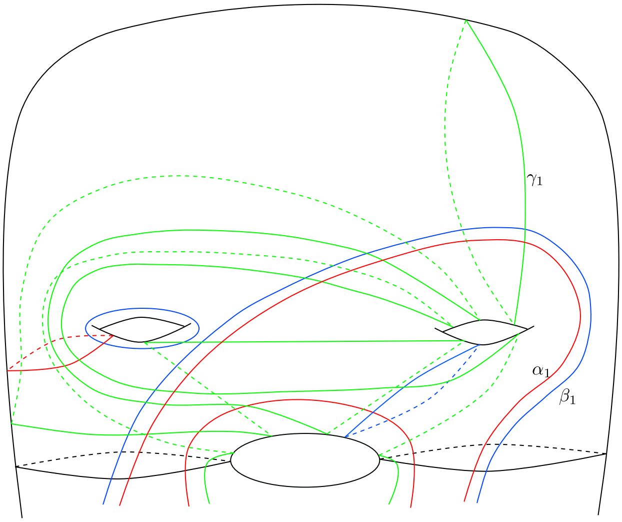


Fig. 15 Before the fourth destabilization

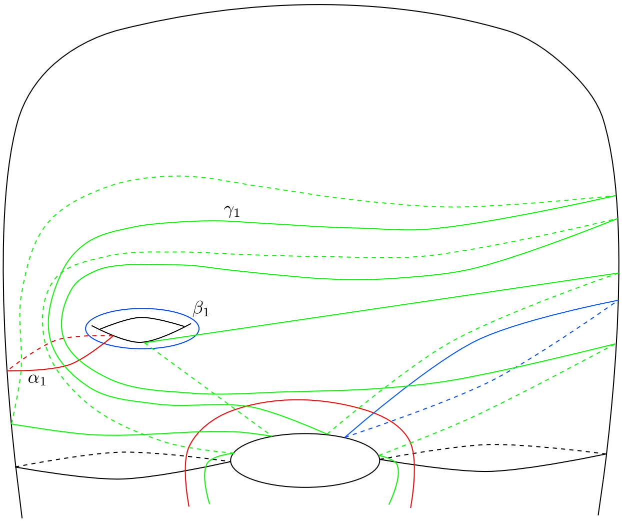


Fig. 16 After the fourth destabilization

no punctures.) In addition, the underlying trisection diagram of the doubly pointed trisection diagram of (X, K) is the diagram of $T_X^{'}$. It can be seen from the way of boundary-stabilizations performed to construct a relative trisection diagram of $X - \nu(S)$ [7]. This means that $\mathcal{D}$ is just the diagram of $T_X^{'}$. Therefore, T_S is diffeomorphic to a stabilization of $T_X^{'}$. Moreover, a meridional stabilization of a bridge trisection corresponds to a stabilization for the underlying trisection. Thus, $T_X^{'}$ is a stabilization of T_X. This completes the proof of Theorem 5.2. □

Corollary 5.3 *For each P^2-knot S in S^4 that is of Kinoshita type, the trisection obtained by the trivial Price twist along S is diffeomorphic to a stabilization of the genus 0 trisection of S^4.*

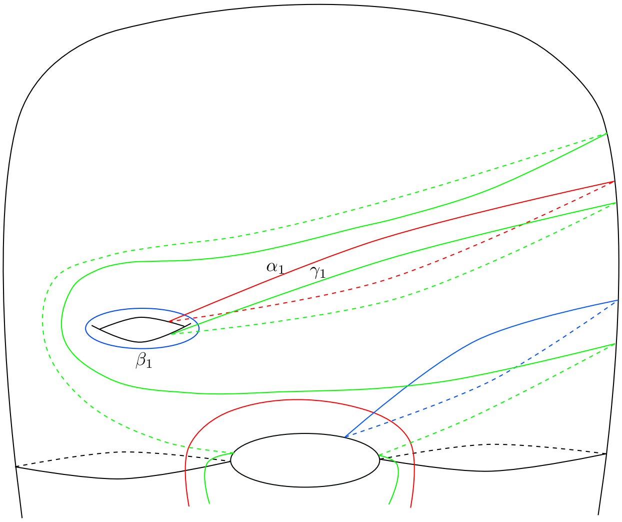


Fig. 17 Before the fifth destabilization

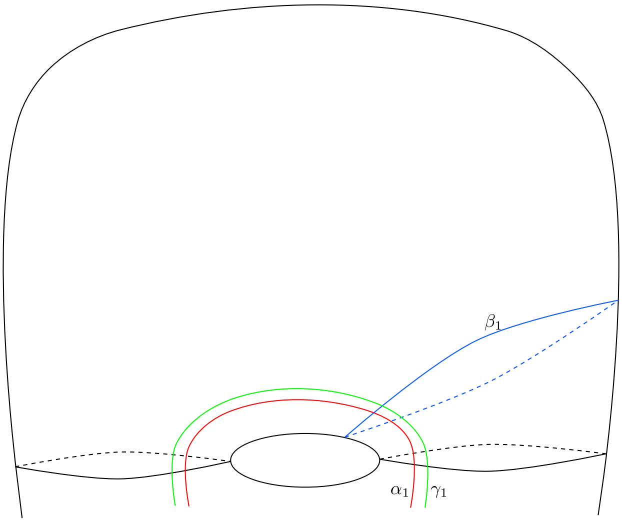


Fig. 18 After the fifth destabilization

Proof In Theorem 5.2, if $X = S^4$, then T_X is the genus 0 trisection of S^4 (see Remark 2.18). □

Lastly, as we have seen in Sect. 1, if any two diffeomorphic trisections of S^4 are isotopic, it follows from corollary 5.3 that the trisection obtained by the trivial Price twist along a P^2-knot which is of Kinoshita type is isotopic to a stabilization of the genus 0 trisection of S^4. Namely, Conjecture 3.11 in [11], i.e. the conjecture that is a 4-dimansional analogue of Waldhausen's theorem on Heegaard splittings, is correct for this trisection.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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