Geometriae Dedicata (2024) 218:80
https://doi.org/10.1007/s10711-024-00915-1

ORIGINAL PAPER

Fundamental groups and group presentations with bounded relator lengths

Sergio Zamora[1]

Received: 31 January 2021 / Accepted: 12 March 2024

Abstract
We study the geometry of compact geodesic spaces with trivial first Betti number admitting large finite groups of isometries. We show that if a finite group G acts by isometries on a compact geodesic space X whose first Betti number vanishes, then $\mathrm{diam}(X)/\mathrm{diam}(X/G) \leq 4\sqrt{|G|}$. For a group G and a finite symmetric generating set S, $P_k(\Gamma(G,S))$ denotes the 2-dimensional CW-complex whose 1-skeleton is the Cayley graph Γ of G with respect to S and whose 2-cells are m-gons for $0 \leq m \leq k$, defined by the simple graph loops of length m in Γ, up to cyclic permutations. Let G be a finite abelian group with $|G| \geq 3$ and S a symmetric set of generators for which $P_k(\Gamma(G,S))$ has trivial first Betti number. We show that the first nontrivial eigenvalue $-\lambda_1$ of the Laplacian on the Cayley graph satisfies $\lambda_1 \geq 2 - 2\cos(2\pi/k)$. We also give an explicit upper bound on the diameter of the Cayley graph of G with respect to S of the form $O(k^2|S|\log|G|)$. Related explicit bounds for the Cheeger constant and Kazhdan constant of the pair (G,S) are also obtained.

Keywords Geometric group theory · Cayley graph · Metric geometry · Random walks · Spectral gap

1 Introduction

1.1 Diameter

Compact geodesic spaces equipped with large discrete groups of isometries have been objects of great interest for a long time and several problems can be formulated in this setting [5, 9, 21, 22, 36]. One natural source of such spaces are finite-sheeted Galois covers of compact Riemannian manifolds. In 2009, Petrunin asked if one can control in an interesting way the diameter of a compact universal cover [32].

Problem 1 *(Petrunin)* Let M be a compact Riemannian manifold and assume it admits a compact universal cover $\tilde{M}$. What is the smallest upper bound of $\mathrm{diam}(\tilde{M})/\mathrm{diam}(M)$ in terms of $|\pi_1(M)|$?

✉ Sergio Zamora
zamora@mpim-bonn.mpg.de

1 Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

Published online: 10 May 2024

It is not hard to show that $\text{diam}(\tilde{M})/\text{diam}(M) \leq |\pi_1(M)|$ [31], but getting a better bound is non-trivial matter. The goal of this paper is to study this question and the global shape of compact universal covers in general. One of our main results is the following.

Theorem 2 *Let X be a compact geodesic space and $G \leq Iso(X)$ a finite group of isometries. If the first Betti number $b_1(X)$ vanishes, then*

$$\frac{diam(X)}{diam(X/G)} \leq 4\sqrt{|G|}.$$

Asymptotically as $|G| \to \infty$, there is a stronger yet non-effective bound [5].

Theorem 3 (Benjamini–Finucane–Tessera) *Let X_n be a sequence of compact geodesic spaces and $G_n \leq Iso(X_n)$ a sequence of finite groups with $|G_n| \to \infty$ as $n \to \infty$. If the first Betti numbers $b_1(X_n)$ vanish, then for each $\varepsilon > 0$ one has*

$$\frac{diam(X_n)}{diam(X_n/G_n)} = O\left(|G_n|^{\varepsilon}\right).$$

Problem 1 is better handled when reformulated in terms of Cayley graphs. For a group G and a finite symmetric generating set S we denote by $\Gamma(G, S)$ the Cayley graph of G with respect to S.

For a graph Γ and an integer $k \in \mathbb{N}$, as in [18] we denote by $P_k(\Gamma)$ the 2-dimensional CW-complex whose 1-skeleton is Γ and whose 2-cells are m-gons for $0 \leq m \leq k$, defined by the simple graph loops of length m in Γ, up to cyclic permutations.

Proposition 4 (Švarc–Milnor Lemma) *Let X be a proper geodesic space, $p \in X$, $G \leq Iso(X)$ a discrete group, $\delta \geq 0$, and $r \geq 2 \cdot diam(X/G) + \delta$. Then $S := \{g \in G \mid d(gp, p) \leq r\}$ generates G. Moreover, if we equip G with the metric induced from $\Gamma := \Gamma(G, S)$, for all $g, h \in G$ one has*

$$\delta \cdot [d_\Gamma(g, h) - 1] \leq d_X(gp, hp) \leq r \cdot d_\Gamma(g, h).$$

Proposition 5 *Let X and Γ be as in Proposition 4. Then $\pi_1(P_3(\Gamma))$ is a quotient of $\pi_1(X)$.*

A proof of the Švarc–Milnor Lemma can be found in [17], and Proposition 5 will be proven in Sect. 3.3. Using these well known results, Theorem 2 becomes a corollary of its Cayley graph counterpart.

Theorem 6 *Let $k \geq 3$, G be a finite group, and $S \subset G$ a finite symmetric set of generators for which $P_k(\Gamma(G, S))$ has trivial first Betti number. Then*

$$\text{diam}(\Gamma(G, S)) \leq \left(\sqrt{4|G| + 1} - 2\right) \left\lfloor \frac{k+2}{3} \right\rfloor. \tag{1}$$

Remark 7 It is well known that for $k \geq 3$, a group G and a finite symmetric set of generators S, the complex $P_k(\Gamma(G, S))$ is simply connected if and only if G admits a presentation $\langle S \mid R \rangle$ with R consisting of words of length $\leq k$ [18, Section 2][1] Moreover, if one considers the abstract group $\tilde{G} = \langle S \mid R_k \rangle$, where R_k consists of the words of length $\leq k$ representing the identity in G, then $P_k(\Gamma(\tilde{G}, S))$ is the universal cover of $P_k(\Gamma(G, S))$ and the fundamental group of $P_k(\Gamma(G, S))$ is precisely the kernel of the natural map $\tilde{G} \to G$.

By Remark 7, Theorem 6 has the following implication.

Corollary 8 *Let $k \geq 3$, G be a finite group, and $S \subset G$ a finite symmetric set of generators for which G admits a presentation $\langle S \mid R \rangle$ with R consisting of words of length $\leq k$. Then (1) holds.*

[1] see also the primer by Yann Ollivier http://www.yann-ollivier.org/maths/primer.php.

1.2 Kazhdan constant, Cheeger constant, and spectral gap

The Švarc–Milnor Lemma implies that the medium-scale geometric features of X and Γ are closely related to each other. We now focus on such properties. Recall that for a finite group G and a finite symmetric set of generators S, the Kazhdan constant $K(G, S)$, Cheeger constant h, and spectral gap λ_1 are related by the following inequalities

$$\frac{h^2}{|S|^2} \leq \frac{2\lambda_1}{|S|} \leq K(G, S)^2 \leq 2\lambda_1 \leq 4h. \tag{2}$$

We refer the reader to Sect. 3.5 for the definition of such quantities and further comments on (2). For now we just mention that the three non-negative quantities $K(G, S)$, h, and λ_1 measure the connectivity of $\Gamma(G, S)$ in different ways. The other main result of this paper concerns finite abelian groups.

Theorem 9 *Let $k \geq 3$, G a finite abelian group with $|G| \geq 3$, and $S \subset G$ a symmetric set of generators for which $P_k(\Gamma(G, S))$ has trivial first Betti number. Then the Kazhdan constant satisfies*

$$K(G, S) \geq 2 \cdot \sin(\pi/k). \tag{3}$$

Consequently, the Cheeger constant, spectral gap, and diameter satisfy

$$2h \geq \lambda_1 \geq 2 - 2\cos(2\pi/k), \tag{4}$$

$$diam(\Gamma(G, S)) \leq \frac{|S| + 1 - \cos(2\pi/k)}{2(1 - \cos(2\pi/k))} \log|G| + 1. \tag{5}$$

A consequence of Theorem 9 is an upper bound on the mixing time of the walk in the corresponding Cayley graph (see Remark 33). We refer the reader to Sect. 3.6 for the definitions of random walk and mixing time. For now we just mention that $\tau_\Gamma(c)$ is an estimate of how long does one have to wait for heat to propagate evenly (how evenly? quantified by c) along the network Γ.

Corollary 10 *Let $k \geq 4$, G, and S be as in Theorem 9. If $\tau_\Gamma : [0, 2] \to \mathbb{N}$ denotes the mixing time of the Cayley graph $\Gamma(G, S)$, then*

$$\tau_\Gamma(c) \leq \frac{k^2|S|}{32}[\log|G| - 2\log(c)] + 1.$$

Theorem 9 also yields an effective bound on the diameter of the universal cover of a closed Riemannian manifold with finite abelian fundamental group.

Corollary 11 *Let M be a closed n-dimensional Riemannian manifold with* $\mathrm{diam}(M) = D$, *Ricci curvature $\geq \kappa(n-1)$ for some $\kappa \in \mathbb{R}$, and having a point whose injectivity radius is $\geq 2r_0 > 0$. If its fundamental group $\pi_1(M)$ is finite and abelian, then the universal cover $\tilde{M}$ satisfies*

$$\frac{\mathrm{diam}(\tilde{M})}{\mathrm{diam}(M)} \leq 4 + \left\lfloor \left[\frac{2v_n^\kappa(2D + r_0)}{3v_n^\kappa(r_0)} + \frac{1}{3} \right] \log|\pi_1(M)| \right\rfloor,$$

where $v_n^\kappa(r)$ denotes the volume of a ball of radius r in the n-dimensional simply connected space of constant sectional curvature κ.

Considering the situation when $\mathrm{diam}(\Gamma) \to \infty$, there are bounds similar to the ones in Theorem 9 for groups that are not necessarily abelian [9].

Theorem 12 (Breuillard–Tointon) *Let G_n be a sequence of finite groups, $S_n \subset G_n$ a sequence of finite symmetric sets of generators, and $\Gamma_n := \Gamma(G_n, S_n)$ the corresponding Cayley graphs. Assume there is a sequence $k_n = o(diam(\Gamma_n))$ such that the first Betti numbers $b_1(P_{k_n}(\Gamma_n))$ vanish. Then for each $\varepsilon > 0$, the quantities*

$$K(G_n, S_n),\ \lambda_1(G_n, S_n),\ h(G_n, S_n),$$

cannot go to zero faster than $(|S_n|/|G_n|)^{\varepsilon}$ as $n \to \infty$.

1.3 Outline

In Sect. 2 we present some computations and examples, and discuss related open problems and potential lines of research.

In Sect. 3 we introduce our notation and the standard theory we will need.

In Sect. 4 we give the proofs of Theorems 2 and 6. Theorem 2 follows from Theorem 6 which in turn depends on an elementary combinatorial argument. We also present a proof of Theorem 3 since it is currently stated in the literature only in the setting of vertex-transitive graphs [5, Theorem 1].

In Sect. 5 we give the proofs of Theorem 9 and Corollaries 10 and 11. An elementary geometric observation yields estimate (3), from which all other results follow.

2 Examples and further problems

2.1 Diameter

Theorem 3 implies that the explicit bound in Theorem 2 is far from being sharp as $|G| \to \infty$. By a fundamental domain argument, even without the first Betti number assumption, one always has

$$\frac{\mathrm{diam}(X)}{\mathrm{diam}(X/G)} \leq 2 \cdot |G|,$$

so Theorem 2 says nothing new for $|G| \leq 4$. However, for a larger number, say, 120, Theorem 2 gives a meaningful bound (again, likely far from sharp). The following example was pointed out by Kuperberg [32].

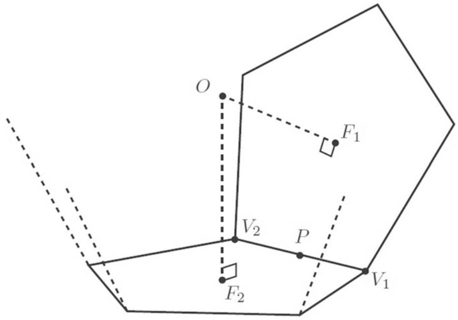
Example 13 Let $\tilde{X} = \mathbb{S}^3$ equipped with its usual metric, and consider $X = (\tilde{X}/\sim)$ the Poincaré sphere [35, Example 1.4.4 and Problem 4.4.17]. Then $\tilde{X}$ is the 120-sheeted universal cover of X, and

$$\frac{\mathrm{diam}(\tilde{X})}{\mathrm{diam}(X)} = \frac{\pi}{\arccos\left(\varphi^2/\sqrt{8}\right)} \approx 8.09,$$

where φ is the golden ratio. On the other hand, the bound provided by Theorem 2 is $4\sqrt{120} \approx 43.81$.

Proof sketch A Voronoi domain of the quotient $\tilde{X} \to X$ is a regular dodecahedron $K \subset \mathbb{S}^3$ with dihedral angles equal to $\pi/3$ [30, Section 3.2.4]. Let $O \in K$ be the center of the dodecahedron, F_1, F_2 be the centers of two adjacent faces of K, and V_1, V_2 be the vertices

Fig. 1 We can use the knowledge of the angles in the triangle OF_1P to deduce the length of the segment OP. We then proceed to compute the length of the segment OV_1 using the length of the segment OP and the known angles of the triangle OPV_1



shared by such faces. Also let P be the midpoint between V_1 and V_2. The diameter of X is attained by $d_{\mathbb{S}^3}(O, V_1)$ (see Fig. 1).

By the symmetry of the dodecahedron,

$$\measuredangle OF_1P = \measuredangle OF_2P = \measuredangle OPV_1 = \measuredangle OPV_2 = \frac{\pi}{2}.$$

Using elementary geometry one can also compute the angles

$$\measuredangle F_1OF_2 = \arccos\frac{1}{3}, \ \measuredangle V_1OV_2 = \arccos\frac{\sqrt{5}}{3}.$$

By the spherical laws of sines and cosines, this is enough information to recover the length $d_{\mathbb{S}^3}(O, V_1) = \arccos(\varphi^2/\sqrt{8})$.

It would also be interesting to investigate how sharp is Theorem 3. The known example in which $\mathrm{diam}(X_n)/\mathrm{diam}(X_n/G_n)$ grows the fastest with respect to $|G_n|$ is the following (again pointed out by Kuperberg [32]), which naturally leads to Conjecture 15 below.

Example 14 Let G_n be the symmetric group (the set of bijections of the set $\{1, \dots, n\}$), and S_n the set of transpositions of consecutive elements of $\{1, \dots, n\}$ (we consider n and 1 not to be consecutive). Setting $\Gamma_n := \Gamma(G_n, S_n)$, we have:

(1) $P_6(\Gamma_n)$ is simply connected for all n.
(2) $\mathrm{diam}(\Gamma_n) = o\left((\log|G_n|)^2\right)$.
(3) $(\log|G_n|)^{2-\varepsilon} = o\left(\mathrm{diam}(\Gamma_n)\right)$ for every $\varepsilon > 0$.

Proof sketch (1): The group G_n can be presented as $\langle S_n | R_n\rangle$, with $S_n = \{\sigma_1, \dots, \sigma_{n-1}\}$, and R_n consisting of the words σ_i^2 for all i, $(\sigma_i\sigma_{i+1})^3$ for all $i = 1, \dots, n-2$, and $(\sigma_i\sigma_j)^2$ with $|i-j| \geq 2$. Since each word in R_n has length ≤ 6, the complex $P_6(\Gamma_n)$ is simply connected.

(2) and (3): Every permutation in G_n can be written as a composition of at most $n(n-1)/2$ elements in S_n, where the maximum is achieved by the permutation

$$i \to (n+1-i), \ i \in \{1, \dots, n\}$$

that "reverses" the order. This means that

$$\mathrm{diam}(\Gamma_n) = n(n-1)/2,$$

while

$$\log|G_n| = \log(n!),$$

which is of the order of $n\log n$.

Conjecture 15 *(Petrunin)* There is $C > 0$ such that if G is a finite group and S a set of generators for which $P_3(\Gamma(G, S))$ is simply connected, then

$$\operatorname{diam}(\Gamma(G, S)) = O\left((\log |G|)^C\right).$$

This question draws resemblance to another well known problem [2].

Conjecture 16 *(Babai)* There is $C > 0$ such that if G is a finite non-abelian simple group and $S \subset G$ is any set of generators, then

$$\operatorname{diam}(\Gamma(G, S)) = O\left((\log |G|)^C\right).$$

Note however, that Babai's Conjecture concerns any set of generators, while Petrunin's Conjecture is about geometrically chosen sets of generators. It would be interesting to investigate how intertwined these two problems are. For instance, does the hypothesis in Conjecture 15 of $P_3(\Gamma(G, S))$ being simply connected imply that the graph $\Gamma(G, S)$ looks like the Cayley graph of a non-abelian simple group? We refer the reader to [20, 24] for recent updates on the state of Conjecture 16.

We would like to also point out that Conjecture 15 is still very interesting when the group G is abelian, in which case C could even be 1. For abelian groups, the known example in which $\operatorname{diam}(\Gamma(G, S))$ grows the fastest with respect to $|G|$ is the following (pointed out by Petrunin [32]).

Example 17 Let $G_n = \mathbb{Z}/(2^n)\mathbb{Z}$, and

$$S_n = \{\pm 1, \pm 2, \pm 2^2, \ldots, \pm 2^{n-1}\}.$$

Setting $\Gamma_n := \Gamma(G_n, S_n)$, we have:

(1) $P_3(\Gamma_n)$ is simply connected for all n.
(2) $\operatorname{diam}(\Gamma_n) = O(\log |G_n|)$.
(3) $\log |G_n| = O(\operatorname{diam}(\Gamma_n))$.

Proof sketch (1): The group G_n can be presented as $\langle S_n | R_n \rangle$, where R_n consists of the expressions $2^j - 2^{j-1} - 2^{j-1} = 0$ for $j \in \{1, \ldots, n-1\}$, and $2^{n-1} + 2^{n-1} = 0$. Since each word in R_n has length ≤ 3, the complex $P_3(\Gamma_n)$ is simply connected.

(2): Any number in $\{1, 2, 3, \ldots, 2^n - 1\}$ can be written (using binary base) as a sum of at most n summands of the form 2^j, $j \in \{1, \ldots, n-1\}$. Hence

$$\operatorname{diam}(\Gamma_n) = O(n) = O(\log |G_n|).$$

(3): Given a sequence of length n of 0's and 1's, one could count the number of "jumps" from one digit to another. E.g., 0001111 has 1 jump, 0011101 has 3 jumps, 1010110 has 5 jumps, etc. By writing an element $x \in G_n$ in binary base, we obtain a sequence of length n of 0's and 1's. One can check that the effect of adding or substracting a power of 2 to x increases the number of such jumps by at most 2.

Expressing $x = 1 + 2^2 + \ldots + 2^{2\lfloor \frac{n-1}{2} \rfloor}$ in binary we find $n - 1$ jumps. Hence at least $\lfloor \frac{n}{2} \rfloor$ elements of S_n are required to write down x. This implies,

$$\log |G_n| = O\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = O(\operatorname{diam}(\Gamma_n)).$$

We conclude this section by pointing out that examples of spaces X with $b_1(X) = 0$ and finite groups $G \leq \operatorname{Iso}(X)$ with $\operatorname{diam}(X)/\operatorname{diam}(X/G)$ of order $\log |G|$ arise in number theory.

Example 18 *(Calegari–Dunfield, Boston–Ellenberg)* There is a closed hyperbolic 3-manifold M_0 admitting a tower of regular m_n-sheeted covers $M_n \to M_0$ with $\mathrm{b}_1(M_n) = 0$ for all n, and

$$c \cdot \log(m_n) \leq \frac{\mathrm{diam}(M_n)}{\mathrm{diam}(M_0)} \leq C \cdot \log(m_n) \tag{6}$$

for some $C > c > 0$.

Proof sketch: In [15, Theorem 1.6], a sequence of regular finite-sheeted covers $M_n \to M_0$ is constructed, with M_0 a quotient of the hyperbolic space $\mathbb{H}^3$ by an arithmetic lattice $\Gamma \leq PGL_2(\mathbb{C})$, which possesses the Selberg property (see [27, Section 2]).

Since these covers correspond to congruence subgroups of Γ, by the Selberg property their (analytic) spectral gaps satisfy $\lambda_1(M_n) \geq \varepsilon$ for some $\varepsilon > 0$. Then by the work of Brooks [12, Theorem 1], the estimates (6) follow.

In [7] it was then proven that $\mathrm{b}_1(M_n) = 0$ for all n (this fact was proven initially in [15] assuming the Generalized Riemann Hypothesis and Langlands-type conjectures).

2.2 Spectral gap

The bounds in Theorem 9 and Corollary 10 are rather general, so we don't expect them to be fully sharp.

Example 19 Let $G_n = (\mathbb{Z}/2\mathbb{Z})^n$, $S_n = \{e_i \in G_n | i \in \{1, \ldots, n\}\}$, where e_i is the i-th basis vector, and $\Gamma_n = \Gamma(G_n, S_n)$. Then $P_4(\Gamma_n)$ is simply connected for each n so by Corollary 10

$$\tau_{\Gamma_n}(c) \leq \frac{n}{2}[n \log 2 - 2\log(c)] + 1.$$

For fixed $c > 0$, this bound grows quadratically in n. However, a careful computation [19] shows that

$$\tau_{\Gamma_n}(c) = O(n \log n).$$

Example 20 Let $G_n = \mathbb{Z}/n\mathbb{Z}$, $S_n = \{-1, 1\}$, and $\Gamma_n = \Gamma(G_n, S_n)$. For fixed $\varepsilon > 0$, consider the sequence

$$t_n := \left\lfloor n^{2-\varepsilon} \right\rfloor.$$

By the Central Limit Theorem [1, Section 7.3], the random walk in Γ_n after t_n steps will be concentrated in the interval $[-n/\sqrt{t_n}, n/\sqrt{t_n}]$. That is,

$$W^{t_n}\left(\left[-\frac{n}{\sqrt{t_n}}, \frac{n}{\sqrt{t_n}}\right]\right) \to 1 \text{ as } n \to \infty.$$

This implies that for fixed $c < 2$, and large enough n,

$$\tau_{\Gamma_n}(c) \geq t_n \geq n^{2-\varepsilon} - 1.$$

Notice $P_n(\Gamma_n)$ is simply connected for each n, so the bound given by Corollary 10 is the following, not far from being sharp:

$$\tau_{\Gamma_n}(c) \leq \frac{n^2}{16}[\log n - 2\log(c)] + 1.$$

It would be desirable to find explicit bounds similar to the ones of Theorem 9 in the non-abelian setting. However, at the moment the topological condition of $P_k(\Gamma(G, S))$ having trivial first Betti number (or even being simply connected) seems very hard to use when studying isometric actions $G \to \mathrm{Iso}(\mathbb{S}^n)$ with $n \geq 2$. In this direction, there is a universal control on the diameters of quotients of spheres by group actions [22].

Theorem 21 (Gorodski–Lange–Lytchak–Mendes) *There is $\delta > 0$ such that for any $n \geq 2$ and any compact group $G \leq Iso(\mathbb{S}^n)$ not acting transitively, one has*

$$diam(\mathbb{S}^n/G) \geq \delta.$$

With techniques similar to the ones in the work of Mantuano [29], it seems possible to recover, using Theorem 9, effective estimates on spectral gaps and medium-scale isoperimetric inequalities for compact Riemannian manifolds with trivial first Betti number and actions by finite abelian groups with small quotient. Successful results in similar programs have been obtained by Brooks [10, 11], Buser [14], Burger [13], Magee [28], and several others, mostly for surfaces.

2.3 (Lack of) Gromov–Hausdorff precompactness

An interesting problem in the theory of finite groups was to understand the possible limits of finite homogeneous spaces. For instance; can one find a sequence of compact geodesic spaces X_n and finite groups $G_n \leq \mathrm{Iso}(X_n)$ with $\mathrm{diam}(X_n/G_n) \to 0$ such that X_n converges to $\mathbb{S}^2$ in the Gromov–Hausdorff sense? This question was answered negatively by Turing [36], and building upon his work, Gelander [21] proved the following.

Theorem 22 (Gelander) *Let X_n be a sequence of compact geodesic spaces and $G_n \leq Iso(X_n)$ a sequence of finite groups with $diam(X_n/G_n) \to 0$. If X_n converges in the Gromov–Hausdorff sense to a compact space X, then X is a (possibly infinite-dimensional) torus.*

A consequence of Theorem 22 is that a sequence of normalized universal covers cannot have a "limit shape".

Corollary 23 *Let X_n be a sequence of compact geodesic spaces and $G_n \leq Iso(X_n)$ a sequence of finite groups with $diam(X_n/G_n)/diam(X_n) \to 0$. If $b_1(X_n) = 0$ for all n, then the sequence $X_n/diam(X_n)$ diverges in the Gromov–Hausdorff sense.*

Proof Assuming the contrary, $X_n/\mathrm{diam}(X_n)$ converges to a space X of diameter 1. By Theorem 22, X is a torus so it admits a regular covering with Galois group $\mathbb{Z}$. Then by the work of Sormani–Wei [34, Theorem 3.4], there are surjective morphisms $\pi_1(X_n) \to \mathbb{Z}$ for n large enough contradicting the assumption $\mathrm{b}_1(X_n) = 0$. □

Remark 24 Theorems 3 and 12 are proven in a similar fashion. In [5, 9], building upon the structure of approximate groups by Breuillard–Green–Tao [8], it is proven that if one had contradicting subsequences, then the normalized spaces would converge to a finite-dimensional torus, contradicting the lower-semi-continuity of the first Betti number [34].

It would be interesting to further understand what causes the behavior of the sequences $X_n/\mathrm{diam}(X_n)$ in Corollary 23. Recall that some known families of Gromov–Hausdorff divergent sequences such as $X_n = \mathbb{S}^n$ or $X_n = (\mathbb{Z}/2\mathbb{Z})^n$ present a concentration of measure property [25].

Definition 25 We say that a sequence (X_n, d_n, μ_n) of metric probability spaces of diameter 1 is a *Levy family* if for any sequence of 1-Lipschitz maps $f_n : X_n \to \mathbb{R}$, the sequence of variances $\mathrm{Var}((f_n)_*\mu_n)$ goes to 0 as $n \to \infty$.

Conjecture 26 *(Petrunin)* Let (X_n, d_n, μ_n) be a sequence of compact simply connected geodesic probability spaces and $G_n \leq \mathrm{Iso}(X_n)$ a sequence of finite groups of measure preserving isometries with $\mathrm{diam}(X_n/G_n) \to 0$. Then X_n is a Levy family.

3 Preliminaries

3.1 Notation

For a finite-dimensional $\mathbb{C}$-Hilbert space V, we denote by $\mathrm{End}(V)$ the space of linear maps $V \to V$ and by $U(V) \subset \mathrm{End}(V)$ the set of unitary automorphisms. If $V = \mathbb{C}^n$, then we denote $U(V)$ also by $U(n)$. For $A \in \mathrm{End}(V)$, we denote its spectrum by $\sigma(A) \subset \mathbb{C}$ and its adjoint by A^*. The trace operator is denoted by $\mathrm{Tr} : \mathrm{End}(V) \to \mathbb{C}$. When V is 1-dimensional, we will identify $\mathrm{End}(V)$ with $\mathbb{C}$ via $\mathrm{Tr} : \mathrm{End}(V) \to \mathbb{C}$.

For a path connected topological space X, we denote its first Betti number by $\mathrm{b}_1(X)$. Recall that it equals the supremum of the m for which there is a surjective morphism $\pi_1(X) \to \mathbb{Z}^m$.

For a metric space X, $p \in X$, and $r > 0$, we denote by $B(p,r)$ the open ball of radius r around p. For two metric spaces X and Y, we denote their Gromov–Hausdorff distance by $d_{GH}(X, Y)$.

3.2 Graphs and CW-complexes

For the purposes of this paper, a graph always means a locally finite undirected graph without loops or multiple edges. For vertices x, y in a graph, we write $x \sim y$ if there is an edge connecting x to y. For an edge $[x, y]$, we denote by (x, y) its interior.

For a sequence of vertices $v_0, v_1, \ldots, v_m$ in a graph Γ such that $v_{i-1} \sim v_i$ for each $i \in \{1, \ldots, m\}$, we denote by $[v_0, \ldots, v_m]$ the curve $\gamma : [0, m] \to \Gamma$ with $\gamma(i) = v_i$ for every $i \in \{0, \ldots, m\}$, so that $\gamma|_{[i-1,i]}$ travels along the edge $v_{i-1}v_i$. A curve (loop) of this form is called a *graph curve (loop)* of length m.

For vertices x, y in a connected graph Γ, the *graph distance* $d_\Gamma(x, y)$ between x and y is the minimum m for which there is a graph curve of length m connecting them.

For a graph Γ and an integer $k \in \mathbb{N}$, we denote by $P_k(\Gamma)$ the 2-dimensional CW-complex whose 1-skeleton is Γ and whose 2-cells are m-gons for $0 \leq m \leq k$, defined by the simple graph loops of length m in Γ, up to cyclic permutations.

Remark 27 It is not hard to equip $P_k(\Gamma)$ with a geodesic metric that restricted to Γ coincides with its original metric, and such that $d_{GH}(P_k(\Gamma), \Gamma) \leq k$. For instance; for $k = 3$ one can make each 2-cell a Reuleaux triangle.

Let G be a group and $S \subset G$ a symmetric generating subset. The *Cayley graph* $\Gamma(G, S)$ of G with respect to S is defined to be the one with G as its vertex set and such that two distinct elements $g, h \in G$ are adjacent if and only if $g = hs$ for some $s \in S$.

We now state a trivial observation. We include its proof since this same counting argument will be used later (see Claim 2 in the proof of Theorem 6).

Lemma 28 *Let G be a finite group with $|G| \geq 3$, $S \subset G$ a symmetric set of generators, $\Gamma := \Gamma(G, S)$ the Cayley graph, and $t \in S \backslash \{e\}$. Then $\Gamma \backslash (e, t)$ is connected.*

Proof If t has order $m \geq 3$, then the path $[t, t^2, \ldots, t^m]$ connects the endpoints of the removed edge, so we can assume $t = t^{-1}$. Let C_1 and C_2 denote the connected components of $\Gamma \backslash (e, t)$ containing e and t, respectively. Since multiplication by t exchanges e and t, it sends C_1 to C_2 and vice-versa, so $|C_1| = |C_2|$.

Since $|G| \geq 3$, $S \backslash \{e\}$ contains an element $s \neq t$. Multiplication by s sends (e, t) to (s, st), so sC_1 is the connected component of $\Gamma \backslash (s, st)$ containing s. Since $s \neq t = t^{-1}$, the three segments (s, e), (s, st), and (e, t) are distinct. Hence,

- The path $[s, e, t]$ lies entirely in $sC_1 \subset \Gamma \backslash (s, st)$.
- The path $[e, s, st]$ lies entirely in $C_1 \subset \Gamma \backslash (e, t)$.

If $C_1 \neq C_2$, then $C_1 \cap C_2 = \emptyset$ and the above implies that

- The connected set $[s, e, t] \cup C_2$ lies entirely in $sC_1 \subset \Gamma \backslash (s, st)$.
- $\{s, e\} \cap C_2 = \emptyset$.

Therefore

$$|C_1| = |sC_1| \geq |[s, e, t] \cup C_2| = 2 + |C_2| = 2 + |C_1|.$$

This contradiction finishes the proof of Lemma 28. □

3.3 Constructing covering spaces

In this section we prove Proposition 5. In order to do so, we present a general construction (cf. [23, Section 5D]).

Proposition 29 *Let X be a proper geodesic space, $p \in X$, $G \leq Iso(X)$ a discrete group of isometries, and $r \geq 2 \cdot diam(X/G)$. Then set $S := \{g \in G | d(gp, p) \leq r\}$, and let $\tilde{G}$ be the abstract group generated by S, with relations*

$$s = s_1 s_2 \text{ in } \tilde{G}, \text{ whenever } s, s_1, s_2 \in S \text{ and } s = s_1 s_2 \text{ in } G.$$

Denote the canonical embedding $S \hookrightarrow \tilde{G}$ as $(s \to s^\sharp)$, and by $\Phi : \tilde{G} \to G$ the unique morphism with $\Phi(s^\sharp) = s$ for all $s \in S$. Then there is a regular covering $\tilde{X} \to X$ with Galois group $Ker(\Phi)$.

Proof In order to construct the space $\tilde{X}$, notice that by discreteness of G, there is $\eta > 0$ with $S = \{g \in G | d(gp, p) < r + 2\eta\}$. Set $B := B(p, r/2 + \eta)$. Then $S = \{g \in G | B \cap gB \neq \emptyset\}$. Equip $\tilde{G}$ with the discrete topology, and consider the topological space

$$\tilde{X} := \left(\tilde{G} \times B\right) / \sim,$$

where $\sim$ is the minimal equivalence relation such that

$$(gs^\sharp, x) \sim (g, sx) \text{ whenever } s \in S,\ x, sx \in B. \tag{7}$$

We then obtain a continuous map $\Psi : \tilde{X} \to X$ given by

$$\Psi(g, x) := \Phi(g)(x).$$

Fix $g_0 \in \tilde{G}$ and set $U := \Phi(g_0)(B)$. The proof of [37, Theorem 2.32] carries over (with $V = B$ and $\Gamma = G$) to show that U is evenly covered. As g_0 ranges over $\tilde{G}$, the sets $\Phi(g_0)(B)$ cover X, so Ψ is a covering map. The proof of [37, Theorem 2.32] again carries over to show that Ψ is regular with Galois group $\text{Ker}(\Phi)$.

We now prove Proposition 5. Let X, G, S, Γ be as in the statement of the proposition. Let $\tilde{G}$ be the group with presentation $\langle S \mid R\rangle$, where R consists of the words of length ≤ 3 that represent the trivial element of G. Then $P_3(\Gamma(\tilde{G}, S))$ is the universal cover of $P_3(\Gamma)$, and $\pi_1(P_3(\Gamma))$ is isomorphic to the kernel of the natural map $\tilde{G} \to G$ (see Remark 7). By Proposition 29, there is a regular covering map $\tilde{X} \to X$ with Galois group $\pi_1(P_3(\Gamma))$, so there is a surjective map $\pi_1(X) \to \pi_1(P_3(\Gamma))$.

3.4 Representation theory of finite groups

In this section we recall the results from representation theory we will need. We refer the reader to [33, Chapters 1-2] for proofs and further discussion. Throughout this section, let G be a finite group.

For our purposes, a *(unitary) representation* is a morphism $\rho : G \to U(V)$ for some finite-dimensional $\mathbb{C}$-Hilbert space V. The dimension of V is called the *dimension* of the representation and will be denoted by d_ρ. We say that such representation is *irreducible* if whenever there is a subspace $W \leq V$ invariant under the G-action, either $W = \{0\}$ or $W = V$. The representation ρ is said to be *trivial* if $\rho(g) = \mathrm{Id}_V$ for all $g \in G$.

Given two representations $\rho_1 : G \to U(V_1)$, $\rho_2 : G \to U(V_2)$, we say a linear map $\lambda : V_1 \to V_2$ is *equivariant* if

$$\lambda\rho_1(g) = \rho_2(g)\lambda \text{ for all } g \in G.$$

We say that ρ_1 and ρ_2 are *isomorphic* if there is an equivariant linear isomorphism $\lambda : V_1 \to V_2$. It turns out there are only finitely many isomorphism classes of irreducible representations and they satisfy

$$|G| = \sum_\rho d_\rho^2, \tag{8}$$

where the sum is taken among the isomorphism classes of irreducible unitary representations of G.

Let $\mathbb{C}[G]$ be the space of functions $G \to \mathbb{C}$. For $f \in \mathbb{C}[G]$, we denote by $f^* \in \mathbb{C}[G]$ the function given by $f^*(g) := \overline{f(g^{-1})}$. For $f, h \in \mathbb{C}[G]$, their product is defined as

$$(f * h)(g) := \sum_{u \in G} f(gu)h(u^{-1}).$$

The Hermitian product

$$\langle f, h\rangle := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}$$

makes $\mathbb{C}[G]$ a Hilbert space called the *convolution algebra* of G. It admits a natural action $\rho_0 : G \to U(\mathbb{C}[G])$ given by

$$\rho_0(g)(f)(h) := f(g^{-1}h) \text{ for } f \in \mathbb{C}[G],\ g, h \in G.$$

For $f \in \mathbb{C}[G]$ and a representation $\rho : G \to U(V)$, the *Fourier transform* $\hat{f}(\rho) \in \mathrm{End}(V)$ is defined as

$$\hat{f}(\rho) := \sum_{g \in G} f(g)\rho(g).$$

The Fourier transform is compatible with products and adjoints. That is,

$$\widehat{f * h}(\rho) = \hat{f}(\rho)\hat{h}(\rho) \qquad \widehat{f^*}(\rho) = \left[\hat{f}(\rho)\right]^* \tag{9}$$

for any $f, h \in \mathbb{C}[G]$ and any representation ρ.

It can be shown that for any irreducible representation $\rho : G \to U(V)$, there is a (non-unique!) equivariant isometric embedding $\iota_V : V \to \mathbb{C}[G]$. Moreover, such embeddings among all irreducible representations span $\mathbb{C}[G]$. This leads to the following important result.

Theorem 30 (Plancherel formula) *For $f, h \in \mathbb{C}[G]$ one has*

$$\langle f, h^* \rangle = \frac{1}{|G|^2} \sum_{\rho} d_\rho \, Tr(\hat{f}(\rho)\hat{h}(\rho)),$$

where the sum is taken among the isomorphism classes of irreducible unitary representations of G.

The following result follows from the fact that any unitary representation is totally reducible.

Lemma 31 *For any unitary representation $\rho : G \to U(V)$, there is $x \in V$ with $|x| = 1$ such that the span of the G-orbit of x is irreducible.*

The ensuing result follows from the fact that commuting unitary automorphisms are simultaneously diagonalizable.

Proposition 32 *If G is abelian, then any irreducible unitary representation is 1-dimensional.*

3.5 Kazhdan constant, Cheeger constant, and spectral gap

We refer the reader to [4] for a detailed introduction to the theory of Kazhdan's property (T) and related topics. Throughout this section, let G be a finite group, $S \subset G$ a symmetric generating set, and $\Gamma := \Gamma(G, S)$ the corresponding Cayley graph.

The *Kazhdan constant* of G with respect to S is defined as

$$K(G, S) := \inf_{\rho} \inf_{|x|=1} \sup_{s \in S} d(\rho(s)x, x),$$

where the first infimum is taken among non-trivial irreducible unitary representations $\rho : G \to U(V)$ and the second infimum is taken among unit vectors $x \in V$.

The *Laplacian* is the map $\Delta : \mathbb{C}[G] \to \mathbb{C}[G]$ defined by

$$\Delta(f)(g) := \sum_{s \in S} [f(gs) - f(g)].$$

This map is equivariant, self adjoint, and its spectrum is a finite set of non-positive real numbers

$$\sigma(\Delta) := \{0 = \lambda_0 > -\lambda_1 \geq \ldots \geq -\lambda_{|G|-1}\}.$$

We denote the Δ-eigenspace of an eigenvalue $-\lambda$ by E_λ. The quantity λ_1 is called the *spectral gap* of G with respect to S.

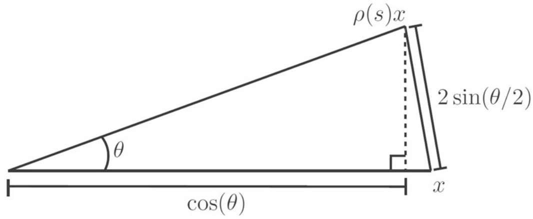


Fig. 2 Basic trigonometry shows that if the angle between the unit vectors x and $\rho(s)x$ is θ, then the distance between the endpoints is $2\sin(\theta/2)$

The *Cheeger (isoperimetric) constant* of G with respect to S is defined by

$$h(G, S) := \inf\left\{\frac{|\partial A|}{|A|} | A \subset G, |A| \frac{1}{2} |v|\right\},$$

where ∂A denotes the set of edges in Γ connecting a vertex in A with a vertex in $G \backslash A$.

Remark 33 The Kazhdan constant $K(G, S)$ quantifies how easy it is to tell apart isometric actions on spheres with fixed points from isometric actions on spheres without fixed points, the spectral gap λ_1 quantifies how fast heat flows through Γ, and h quantifies how bad the bottlenecks of Γ are. Each quantity measures in some way how robust is the network Γ.

Recall that these quantities satisfy the well known relations

$$\frac{h^2}{|S|^2} \leq \frac{2\lambda_1}{|S|} \leq K(G, S)^2 \leq 2\lambda_1 \leq 4h \tag{10}$$

(see [9, Section 1.2] for a similar expression involving $\mathrm{diam}(\Gamma)$). A proof of the first and last inequalities, known as the discrete Cheeger–Buser inequalities, along with historical background can be found in [26, Section 4.2]. To verify the second inequality, take an arbitrary irreducible unitary representation $\rho : G \to U(V)$. Since there exists an equivariant embedding $\iota_V : V \to \mathbb{C}[G]$, we can assume $V \leq \mathbb{C}[V]$. Moreover, since Δ is equivariant, we can further assume $V \leq E_\lambda$ for some $\lambda \neq 0$. Then assume a unit vector $x \in V$ satisfies

$$\sup_{s \in S} d(\rho_0(s)x, x) \leq 2 \cdot \sin(\theta/2)$$

for some $\theta \in [0, \pi]$. Then the angle between x and $\rho_0(s)x$ is at most θ for each $s \in S$ (see Fig. 2). This implies

$$\lambda = \langle -\Delta x, x\rangle = \sum_{s \in S} \langle x - \rho_0(s)x, x\rangle \leq |S|[1 - \cos(\theta)].$$

From the identity $1 - \cos(\theta) = 2 \cdot \sin^2(\theta/2)$, we deduce $[2 \cdot \sin(\theta/2)]^2 \geq 2\lambda/|S|$. Since x was arbitrary and $\lambda \geq \lambda_1$, the second inequality of (10) follows.

To verify the third inequality of (10), recall that by Lemma 31, there is a unit vector $x \in E_{\lambda_1}$ for which its G-orbit spans an irreducible representation. By definition there is $t \in S$ with $d(\rho_0(t)x, x) \geq K(G, S)$. Set $K(G, S) = 2 \cdot \sin(\theta/2)$ with $\theta \in [0, \pi]$. Then since $|(\langle \rho(s)x, x\rangle| \leq 1$ for all $s \in S \backslash \{t\}$, one has

$$\lambda_1 = \langle -\Delta x, x\rangle = \sum_{s \in S} \langle x - \rho_0(s)x, x\rangle \geq 1 - \cos(\theta) = \frac{K(G, S)^2}{2}.$$

3.6 Random walks on Cayley graphs

We refer to [19, Chapter 3] for an introduction on the theory of random walks in finite groups. Throughout this section, let G be a finite group, $S \subset G$ a symmetric generating set, and $\Gamma := \Gamma(G, S)$ the corresponding Cayley graph.

For $\alpha \geq |S|$, the *random walk* on Γ is the G-valued Markov process $\{W_\alpha^t\}_{t\in\mathbb{N}}$ such that $W_\alpha^0 \equiv e$, and at each time, if the walker is at $g \in G$, then it stays at g with probability $1 - |S|/\alpha$ and jumps to a neighbor uniformly at random with probability $|S|/\alpha$. This gives rise to the law

$$\mathbb{P}\left[W_\alpha^{t+1} = g\right] = \frac{1}{\alpha}\left[(\alpha - |S|)\mathbb{P}\left[W_\alpha^t = g\right] + \sum_{s\in S}\mathbb{P}\left[W_\alpha^t = gs\right]\right].$$

We will denote W_α^1 simply by W_α. We can identify $\mathbb{C}[G]$ with the set of complex valued measures on G via the correspondence

$$\mu \in \mathbb{C}[G] \longleftrightarrow \mu(A) := \sum_{g\in A}\mu(g).$$

Note that after this identification,

$$W_\alpha = \delta_e + \frac{1}{\alpha}\Delta(\delta_e), \tag{11}$$

where δ_e denotes the Dirac mass at e. It is also straightforward to verify that

$$W_\alpha^s * W_\alpha^t = W_\alpha^{s+t} \qquad \left(W_\alpha^t\right)^* = W_\alpha^t \tag{12}$$

for all $s, t \in \mathbb{N}$. If $\alpha > |S|$, the distribution W_α^t converges to the uniform distribution U on G as $t \to \infty$. One can quantify how fast this convergence occurs with the quantity

$$\varepsilon_\alpha(t) := \sum_{g\in G}|W_\alpha^t(g) - \frac{1}{|G|}|.$$

The *mixing time* $\tau_\Gamma^\alpha : [0, 2] \to \mathbb{N}$ of the process $\{W_\alpha^t\}_{t\in\mathbb{N}}$ is defined as

$$\tau_\Gamma^\alpha(c) := \inf\{t \in \mathbb{N} \mid \varepsilon_\alpha(t) \leq c\}.$$

A direct computation shows that for all $t \in \mathbb{N}$,

$$\langle W_\alpha^t, U\rangle = \langle U, U\rangle = \frac{1}{|G|^2}. \tag{13}$$

If $\mathbf{1} : G \to U(1)$ denotes the trivial representation, then (9) and (12) imply

$$\widehat{W_\alpha^t}(\mathbf{1}) = \hat{W}_\alpha^t(\mathbf{1}) = \mathrm{Id}_{\mathbb{C}} \tag{14}$$

for all $t \in \mathbb{N}$. Then one has

$$\begin{aligned} \langle W_\alpha^t - U, W_\alpha^t - U\rangle &= \langle W_\alpha^t, W_\alpha^t\rangle - \frac{1}{|G|^2} \\ &= \frac{1}{|G|^2}\sum_\rho d_\rho \operatorname{Tr}(\hat{W}_\alpha^{2t}(\rho)) - \frac{1}{|G|^2} \\ &= \frac{1}{|G|^2}\sum_{\rho\neq 1} d_\rho \operatorname{Tr}(\hat{W}_\alpha^{2t}(\rho)), \end{aligned} \tag{15}$$

where the last sum is taken over isomorphism classes of non-trivial irreducible unitary representations; the first equality uses (13), the second one follows from the Plancherel formula and (12), and the third one uses (14). Finally, by the Cauchy-Schwarz inequality, (15) implies

$$|\varepsilon_\alpha(t)|^2 \leq \sum_{\rho\neq 1} d_\rho \operatorname{Tr}(\hat{W}_\alpha^{2t}(\rho)). \tag{16}$$

When $\alpha = 2|S|$, we denote W_α^t by W^t, W_α by W, $\varepsilon_\alpha(t)$ by $\varepsilon(t)$, and τ_Γ^α by τ_Γ.

4 Diameter bounds

In this section we prove Theorems 2, 3, and 6. For a group G and a symmetric generating set $S \subset G$, we set $S_k := S \cup S^2 \cup \ldots \cup S^{\left\lfloor \frac{k+2}{3} \right\rfloor}$. It is straightforward to check that

$$\operatorname{diam}(\Gamma(G, S)) \leq \operatorname{diam}(\Gamma(G, S_k)) \left\lfloor \frac{k+2}{3} \right\rfloor. \tag{17}$$

The ensuing result [3] reduces the proof of Theorem 6 to the case $k = 3$.

Lemma 34 (Behr) *Let G a group, and $S \subset G$ a finite symmetric set of generators. Then for each $k \geq 3$, there is a surjective map*

$$\pi_1(P_k(\Gamma(G, S))) \to \pi_1(P_3(\Gamma(G, S_k))).$$

Proof First notice that if an edge ω of $\Gamma(G, S_k)$ corresponds to an element of S^m with $m \leq \lfloor \frac{k+2}{3} \rfloor$ then there is an endpoint-preserving homotopy in $P_3(\Gamma(G, S_k))$ taking ω to a concatenation of m edges in $\Gamma(G, S)$ (see Fig. 3). Since the fundamental group of a CW-complex is generated by the loops in its 1-skeleton, the above observation implies that the inclusion

$$\Gamma(G, S) \to P_3(\Gamma(G, S_k)) \tag{18}$$

induces a surjective map at the level of fundamental groups. It remains to check that (18) extends to a continuous map $P_k(\Gamma(G, S)) \to P_3(\Gamma(G, S_k))$. This boils down to the fact that any word of length $\leq k$ representing the identity in G using the elements of S as letters, can be written as a concatenation of words of length ≤ 3 representing the identity in G using the elements of S_k as letters (see [16, Lemma 7.A.8] for further details). □

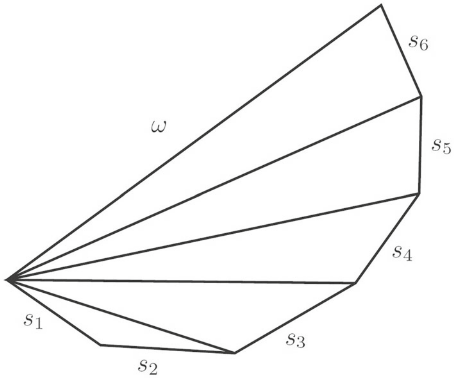
The following elementary observation will be required at the end of the proof of Theorem 6.

Lemma 35 *Let Γ be a connected graph, $T \subset \Gamma$ a connected subgraph, and $C_1, \ldots, C_\ell \subset \Gamma$ the connected components of $\Gamma \backslash T$. Then for each $j \in \{1, \ldots, \ell\}$, the graph $\Gamma \backslash C_j$ is connected.*

Proof If the result is false, there are $w_0, w_1 \in \Gamma \backslash C_j$ such that any path connecting them passes through C_j. Since Γ is connected, there is a path $[w_0 = v_0, v_1, \ldots, v_k = w_1]$, which by assumption passes through C_j. Let i_1 be the first index such that v_{i_1+1} is in C_j and let i_2 be the last index such that v_{i_2-1} lies in C_j. Then both v_{i_1} and v_{i_2} lie in T, and by connectedness of T there is a path $[v_{i_1} = a_0, a_1, \ldots, a_m = v_{i_2}]$ in T. Then it is easy to check that the path

$$[w_0 = v_0, \ldots, v_{i_1}, a_1, \ldots, a_{m-1}, v_{i_2}, \ldots, v_k = w_1]$$

Fig. 3 If $\omega = s_1 \cdots s_m$, then the decomposition $w = (s_1 \cdots s_{m-1})(s_m)$ yields an endpoint preserving homotopy in $P_3(\Gamma(G, S_k))$ from w to the concatenation of an edge corresponding to an element in S^{m-1} and one in S. Proceeding inductively yields the desired homotopy. In the picture $m = 6$



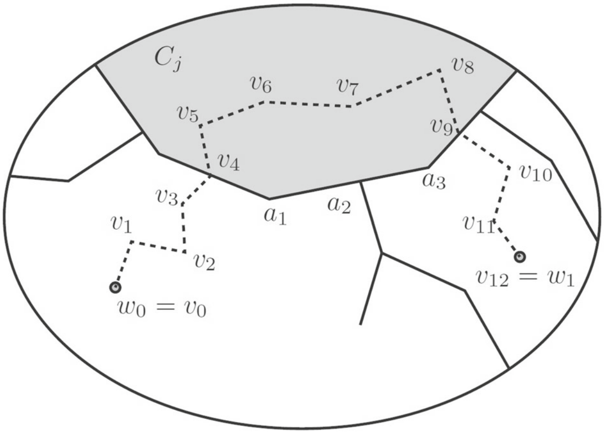


Fig. 4 The portion $[v_{i_1}, v_{i_1+1}, \ldots, v_{i_2-1}, v_{i_2}]$ can be replaced by $[a_0, \ldots, a_m]$. In the picture the shadowed region represents C_j, the solid line represents T, $i_1 = 4$, and $i_2 = 9$

does not intersect C_j, contradicting our assumption (see Fig. 4).

□

Theorem 6 By Lemma 34 we have $\mathrm{b}_1(P_3(\Gamma(G, S_k))) = 0$. Combining this with (17) we can assume $k = 3$ without loss of generality.

Let $e \in G$ be the neutral element and $\Gamma = \Gamma(G, S)$. Take $h \in G$ with $d_\Gamma(h, e) = m = \mathrm{diam}(\Gamma)$ and a minimizing path $[e = g_0, g_1, \ldots, g_m = h]$. For each i, set $\Sigma_i \subset \Gamma$ as the subgraph induced by the set of vertices $\{g \in G \mid d_\Gamma(g, e) = i\}$ and let T_i be the connected component of Σ_i containing g_i.

Claim 1: For each $i_0 \in \{1, \ldots, m-1\}$, the vertices e and h lie in distinct connected components of $\Gamma \backslash T_{i_0}$.

Let $Y_0 = Y_1 \subset \Gamma$ be the subgraph induced by $\bigcup_{j=0}^{i_0-1} \Sigma_j$, $Y_{1/4} \subset \Gamma$ the subgraph induced by T_{i_0}, $Y_{1/2} \subset \Gamma$ the subgraph induced by $\bigcup_{j=i_0+1}^{m} \Sigma_j$, and $Y_{3/4} \subset \Gamma$ the subgraph induced by $\Sigma_{i_0} \backslash T_{i_0}$. Since $d_\Gamma(\cdot, e)$ is 1-Lipschitz, there are no edges between Y_0 and $Y_{1/2}$, and by the definition of T_{i_0} there are no edges between $Y_{1/4}$ and $Y_{3/4}$.

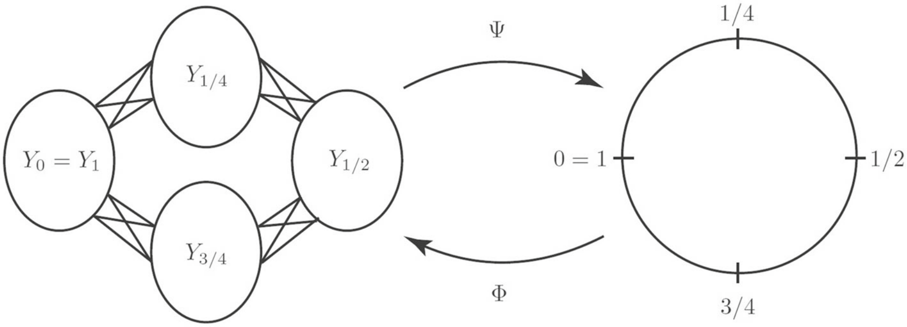


Fig. 5 Each edge of Γ is sent via ψ to either a point or an interval of length $\leq 1/4$, so ψ extends to a map $\Psi : P_3(\Gamma) \to \mathbb{R}/\mathbb{Z}$

Then construct a map $\psi : \Gamma \to \mathbb{R}/\mathbb{Z}$ that restricted to Y_s equals s for $s \in \{0, 1/4, 1/2, 3/4\}$, and takes all edges joining Y_0 with $Y_{1/4}$ to the interval $[0, 1/4]$, doing the same for the intervals $[1/4, 1/2]$, $[1/2, 3/4]$, and $[3/4, 1]$.

By construction, ψ sends each edge of Γ to either a point or an interval of length 1/4 in $\mathbb{R}/\mathbb{Z}$. Also recall that each 2-cell α of $P_3(\Gamma)$ is attached to Γ via a simple loop $\partial\alpha$ of length ≤ 3. Therefore, $\psi(\partial\alpha) \subset \mathbb{R}/\mathbb{Z}$ is a loop of length $\leq 3/4$ hence nullhomotopic for each α, and ψ extends to a map $\Psi : P_3(\Gamma) \to \mathbb{R}/\mathbb{Z}$ (see Fig. 5).

Assume the claim is false and take a minimizing path $[e = g'_0, g'_1, \ldots, g'_{m'} = h]$ in $\Gamma \backslash T_{i_0}$. Consider the map $\Phi : \mathbb{R}/\mathbb{Z} \to P_3(\Gamma)$ that sends the interval $[0, 1/2]$ to the path $[g_0, g_1, \ldots, g_m]$ and $[1/2, 1]$ to the path $[g'_{m'}, \ldots, g'_1, g'_0]$.

Since $\Psi \circ \Phi(0) = 0$, $\Psi \circ \Phi(1/2) = 1/2$, $\Psi \circ \Phi|_{[0,1/2]}$ misses 3/4, and $\Psi \circ \Phi|_{[1/2,1]}$ misses 1/4, the composition $\Psi \circ \Phi$ is homotopic to the identity in $\mathbb{R}/\mathbb{Z}$, meaning that the induced map $\Psi_* : \pi_1(P_3(\Gamma)) \to \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$ is surjective, contradicting the hypothesis $\mathrm{b}_1(P_3(\Gamma)) = 0$. This finishes the proof of the claim.

Claim 2: For each $i_0 \in \{1, \ldots, m-1\}$, either $g_{i_0}^{-1} T_{i_0}$ or $h g_{i_0}^{-1} T_{i_0}$ intersect T_{i_0}.

Let $C_1, \ldots, C_\ell \subset \Gamma$ denote the connected components of $\Gamma \backslash T_{i_0}$, with $e \in C_1$, $h \in C_2$. Observe that $g_{i_0}^{-1} C_1, \ldots, g_{i_0}^{-1} C_\ell$ are the connected components of $\Gamma \backslash g_{i_0}^{-1} T_{i_0}$, and $h g_{i_0}^{-1} C_1, \ldots, h g_{i_0}^{-1} C_\ell$ are the connected components of $\Gamma \backslash h g_{i_0}^{-1} T_{i_0}$.

Assume $T_{i_0} \cap g_{i_0}^{-1} T_{i_0} = \emptyset$. Since $g_{i_0}^{-1} T_{i_0}$ is connected and contains e, it is contained in C_1. By Lemma 35, $\Gamma \backslash C_1 = T_{i_0} \cup C_2 \cup \ldots \cup C_\ell$ is connected, and since it doesn't intersect $g_{i_0}^{-1} T_{i_0}$, it is contained in $g_{i_0}^{-1} C_{j_1}$ for some j_1. Therefore

$$|C_j| < |C_{j_1}| \text{ for all } j \neq 1.$$

This is only possible if $j_1 = 1$, and in particular we have

$$|C_2| < |C_1|. \tag{19}$$

Similarly, if $T_{i_0} \cap h g_{i_0}^{-1} T_{i_0} = \emptyset$, then $h g_{i_0}^{-1} T_{i_0}$ is contained in C_2. By Lemma 35, $\Gamma \backslash C_2 = T_{i_0} \cup C_1 \cup C_3 \cup \ldots \cup C_\ell$ is connected, and since it doesn't intersect $h g_{i_0}^{-1} T_{i_0}$, it is contained in $h g_{i_0}^{-1} C_{j_2}$ for some j_2, meaning that

$$|C_j| < |C_{j_2}| \text{ for all } j \neq 2.$$

This implies that $j_2 = 2$ and

$$|C_1| < |C_2|. \tag{20}$$

Assuming the claim is false, both (19) and (20) would hold; a contradiction.

From the second claim, we deduce $\mathrm{diam}(T_i) \geq \min\{i, m-i\}$ for each $i \in \{1, \ldots, m-1\}$, and since the T_i's are disjoint, we conclude that

$$|G| \geq \Sigma_{i=0}^{m} |T_i| \geq \Sigma_{i=0}^{m} \min\{i+1, m-i+1\}.$$

This implies that $m \leq \sqrt{4|G|+1} - 2$, which is the required inequality. □

Theorem 2 Let $p \in X$ and define S as in Proposition 4 with $\delta = 0$. By Proposition 5, the first Betti number of $P_3(\Gamma(G, S))$ vanishes. For $x, y \in X$, take $g_1, g_2 \in G$ with $d_X(g_1 p, x), d_X(g_2 p, y) \leq \mathrm{diam}(X/G)$. Then

$$\begin{aligned} d_X(x, y) &\leq 2 \cdot \mathrm{diam}(X/G) + d_X(g_1 p, g_2 p) \\ &\leq 2 \cdot \mathrm{diam}(X/G)[1 + d_\Gamma(g_1, g_2)] \\ &\leq 2 \cdot \mathrm{diam}(X/G)[\sqrt{4|G|+1} - 1], \end{aligned}$$

where the second inequality follows from Švarc–Milnor Lemma and the third one from Theorem 6. The result follows since $\sqrt{4|G|+1} \leq 2\sqrt{|G|} + 1$. □

Theorem 3 Pick $p_n \in X_n$, set $S_n := \{g \in G_n | d(gp_n, p_n) \leq 2 \cdot \mathrm{diam}(X_n/G_n)\}$, and let $\Gamma_n := \Gamma(G_n, S_n)$. If the result fails, there is $\varepsilon > 0$ such that after taking a subsequence one has

$$|G_n|^\varepsilon = O\left(\frac{\mathrm{diam}(X_n)}{\mathrm{diam}(X_n/G_n)}\right).$$

By the Švarc–Milnor Lemma, this would imply $|G_n|^\varepsilon = O\left(\mathrm{diam}(\Gamma_n)\right)$. Then by [5, Theorem 1], after further taking a subsequence, $\Gamma_n/\mathrm{diam}(\Gamma_n)$ converges to an m-dimensional torus X. By Remark 27, the sequence $P_3(\Gamma_n)/\mathrm{diam}(P_3(\Gamma_n))$ also converges to X and by [34, Theorem 2.1] there are surjective morphisms $\pi_1(P_3(\Gamma_n)) \to \pi_1(X) = \mathbb{Z}^m$ for large enough n, contradicting Proposition 5. □

5 Fourier analysis in abelian groups

In this section we prove Theorem 9 and Corollaries 10 and 11. Let k, G, S be as in the statement of Theorem 9 and let $\Gamma := \Gamma(G, S)$. Assuming the estimate (3) fails to hold, there is an irreducible non-trivial unitary representation $\rho : G \to U(m)$ and $x \in \mathbb{S}^{2m-1}$ with

$$\sup_{s \in S} d(\rho(s)x, x) < 2 \cdot \sin(\pi/k). \tag{21}$$

By Proposition 32 we have $m = 1$, and since the metric on $\mathbb{S}^1$ is bi-invariant, we can assume $x = 1$. Let $\psi : \Gamma \to \mathbb{S}^1$ be the map that restricted to G coincides with ρ, and restricted to an edge $[g, h] \subset \Gamma$ is a minimizing geodesic from $\rho(g)$ to $\rho(h)$.

If $g, h \in G$ are such that $g = hs$ for some $s \in S$, then (21) implies that the angle between $\rho(g)$ and $\rho(h)$ is less than $2\pi/k$. Hence, for any simple loop of length $\leq k$ in Γ, its image under ψ has length less than 2π and is contractible. Therefore, ψ extends to a map $\Psi : P_k(\Gamma) \to \mathbb{S}^1$.

Since ρ is non-trivial, there is $s \in S$ with $\rho(s) \neq 1$. Then the image under Ψ of the loop $[e, s, s^2, \ldots, s^{|G|} = e]$ is a loop in $\mathbb{S}^1$ that winds around at least once; counterclockwise if $\mathrm{Re}(\rho(s)) > 0$ and clockwise if $\mathrm{Re}(\rho(s)) < 0$. This means the map $\Psi_* : \pi_1(P_k(\Gamma)) \to \pi_1(\mathbb{S}^1) = \mathbb{Z}$ is non-trivial, contradicting the assumption $\mathrm{b}_1(P_k(\Gamma)) = 0$. This finishes the proof of (3).

We now proceed to prove (4). Notice that if we simply apply (2) naively to (3) we would get a weaker result. Let $-\lambda \in \sigma(\Delta)\backslash\{0\}$, and $E_\lambda \leq \mathbb{C}[G]$ the corresponding eigenspace. By Lemma 31, there is $x \in E_\lambda$ with $|x| = 1$ such that the span of its orbit is an irreducible representation $\rho : G \to \mathbb{S}^1$. Since $\lambda \neq 0$, it follows that ρ is non-trivial.

Case 1 $\rho(s) \neq -1$ for all $s \in S$.

By (3), there is $t \in S$ with $d(\rho(t), 1) \geq 2 \cdot \sin(\pi/k)$. This implies $\mathrm{Re}(\rho(t)) = \mathrm{Re}(\rho(t^{-1})) \leq \cos(2\pi/k)$. Since $\rho(t) \neq -1$, one has $t \neq t^{-1}$ so

$$\begin{aligned}\lambda = \langle -\Delta x, x\rangle &= \sum_{s\in S}[1-\rho(s)]\\ &= |S| - \sum_{s\in S}\mathrm{Re}(\rho(s))\\ &\geq 2 - 2\cos(2\pi/k),\end{aligned}$$

where in the last line we used that $\mathrm{Re}(\rho(s)) < 1$ for all $s \in S\backslash\{t, t^{-1}\}$.

Case 2 $k = 3$ and $\rho(t) = -1$ for some $t \in S$.

We claim there are $s_1, s_2 \in S\backslash\{e, t\}$ such that $s_1 s_2 = t$. Assume otherwise; then the edge $[e, t] \subset \Gamma$ does not belong to a 2-cell of $P_3(\Gamma)$. Let x be the midpoint of the edge $[e, t]$, $A := [e, t]$, and $B := P_3(\Gamma)\backslash\{x\}$. Then $A \cap B = [e, x) \cup (x, t]$ and by Lemma 28, B is connected, so the portion of the Mayer–Vietoris sequence (with real coefficients)

$$H_1(P_3(\Gamma)) \to H_0(A\cap B) \to H_0(A) \oplus H_0(B) \to H_0(P_3(\Gamma)) \to 0$$

yields the exact sequence (using $\mathrm{b}_1(P_3(\Gamma) = 0)$)

$$0 \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R} \to 0.$$

This is impossible by dimension arguments. Then there are $s_1, s_2 \in S\backslash\{e, t\}$ (not necessarily distinct) with $s_1 s_2 = t$. Since $\rho(s_1)\rho(s_2) = -1$, without loss of generality we can assume $\mathrm{Re}(\rho(s_1)) \leq 0$. Then

$$\begin{aligned}\lambda &= |S| - \sum_{s\in S}\mathrm{Re}(\rho(s)) \geq 2 - \mathrm{Re}(\rho(t) + \rho(s_1))\\ &\geq 3 = 2 - 2\cos(2\pi/3).\end{aligned}$$

Case 3 $k \geq 4$ and $\rho(t) = -1$ for some $t \in S$.

Directly compute

$$\lambda = |S| - \sum_{s\in S}\mathrm{Re}(\rho(s)) \geq 2 \geq 2 - 2\cos(2\pi/k).$$

This, together with (2), finishes the proof of (4).

For notational convenience, we set

$$\xi_k := 1 - \cos(2\pi/k) \text{ and } \alpha_k := |S| + \xi_k.$$

To prove (5), we look at the random walk $W^t_{\alpha_k}$ in Γ.

Lemma 36 *For any non-trivial irreducible unitary representation $\rho : G \to \mathbb{S}^1$,*

$$\hat{W}_{\alpha_k}(\rho) \in \left[-1 + \frac{2\xi_k}{\alpha_k}, 1 - \frac{2\xi_k}{\alpha_k}\right].$$

Proof This is a direct computation using (11). On one hand we have

$$\hat{W}_{\alpha_k}(\rho) = 1 + \frac{1}{\alpha_k}\sum_{s\in S}[\rho(s) - 1] \leq 1 - \frac{2\xi_k}{\alpha_k},$$

where we first used the identity $\hat{\delta}_s(\rho) = \rho(s)$ and then the estimate $\lambda_1 \geq 2\xi_k$. For the other inequality, notice that $\alpha_k - 2|S| = 2\xi_k - \alpha_k$, then

$$\hat{W}_{\alpha_k}(\rho) = 1 + \frac{1}{\alpha_k}\sum_{s\in S}[\rho(s) - 1] \geq \frac{\alpha_k - 2|S|}{\alpha_k} = \frac{2\xi_k}{\alpha_k} - 1,$$

where we used $\mathrm{Re}(\rho(s)) \geq -1$ for all $s \in S$ in the inequality. □

Then by (15), for $t \in \mathbb{N}$ we have

$$\langle W^t_{\alpha_k} - U, W^t_{\alpha_k} - U\rangle = \frac{1}{|G|^2}\sum_{\rho\neq 1} d_\rho \operatorname{Tr}\left(\hat{W}^{2t}_{\alpha_k}\right) < \frac{1}{|G|}\left|1 - \frac{2\xi_k}{\alpha_k}\right|^{2t},$$

where we used (8) for the inequality. Now assume $t \leq \mathrm{diam}(\Gamma)$. Since $W^t_{\alpha_k}$ is supported in the ball of radius t around e, then the left hand side of the equation is at least $\frac{1}{|G|^3}$. Hence by taking logarithm and using the identity $\log(1 + u) \leq u$ we get

$$-2\log|G| < 2t\log\left|1 - \frac{2\xi_k}{\alpha_k}\right| \leq -\frac{4t\xi_k}{\alpha_k}.$$

Rearranging terms,

$$t < \frac{\alpha_k \log|G|}{2\xi_k}.$$

This implies

$$\mathrm{diam}(\Gamma) \leq \frac{\alpha_k}{2\xi_k}\log|G| + 1.$$

This concludes the proof of Theorem 9. Notice that if $k = 3$, then (5) simplifies to

$$\mathrm{diam}(\Gamma) \leq \left[\frac{|S|}{3} + \frac{1}{2}\right]\log|G| + 1. \tag{22}$$

In order to prove Corollary 10, we need to establish an analogue of Lemma 36.

Lemma 37 *Under the hypothesis of Corollary 10, for any non-trivial irreducible unitary representation $\rho : G \to \mathbb{S}^1$ one has*

$$\hat{W}(\rho) \in \left[0, 1 - \frac{\xi_k}{|S|}\right].$$

Proof This is again a direct computation. Using $\lambda_1 \geq 2\xi_k$ we get

$$\hat{W}(\rho) = 1 + \frac{1}{2|S|}\sum_{s\in S}[\rho(s) - 1] \leq 1 - \frac{\xi_k}{|S|}.$$

On the other hand, simply using $\mathrm{Re}(\rho(s)) \geq -1$ for all $s \in S$ we conclude

$$\hat{W}(\rho) = 1 + \frac{1}{2|S|}\sum_{s\in S}[\rho(s) - 1] \geq 0.$$

□

Corollary 10 By (16), using Lemma 37 and (8), we have

$$\varepsilon(t)^2 \leq \sum_{\rho\neq 1} d_\rho \,\mathrm{Tr}(\hat{W}^{2t}) < |G| \left|1 - \frac{\xi_k}{|S|}\right|^{2t}$$

for $t \in \mathbb{N}$. If $\varepsilon(t) \geq c \in [0, 2]$, then taking logarithms as above we get

$$2\log(c) < \log|G| - \frac{2t\xi_k}{|S|}.$$

Rearranging terms we get

$$t < \frac{|S|}{2\xi_k}[\log|G| - 2\log(c)].$$

Since $\xi_k \geq 16/k^2$ the result follows. □

Corollary 11 Take a point $p \in M$ with injectivity radius $\geq 2r_0$, $\tilde{p} \in \tilde{M}$ in its preimage, and set

$$S := \{g \in \pi_1(M)\backslash\{e\} \,|\, d(g\tilde{p}, \tilde{p}) \leq 2D\}.$$

By the injectivity radius condition, for $g, h \in \pi_1(M)$ distinct, the balls $B(g\tilde{p}, r_0)$ and $B(h\tilde{p}, r_0)$ are isometric and disjoint. Since $g \in S \cup \{e\}$ implies $B(g\tilde{p}, r_0) \subset B(\tilde{p}, 2D + r_0)$, we have

$$|S| + 1 \leq \frac{\mathrm{Vol}(B(\tilde{p}, 2D + r_0))}{\mathrm{Vol}(B(\tilde{p}, r_0))}.$$

By the Bishop–Gromov inequality [6, Section 11.10], the right hand side of the equation is less or equal than $v_n^\kappa(2D + r_0)/v_n^\kappa(r_0)$. By Proposition 5, $P_3(\Gamma(\pi_1(M), S))$ is simply connected, so (22) holds;

$$\mathrm{diam}(\Gamma(\pi_1(M), S)) \leq \left[\frac{v_n^\kappa(2D + r_0)}{3v_n^\kappa(r_0)} + \frac{1}{6}\right]\log|\pi_1(M)| + 1. \tag{23}$$

Arguing as in the proof of Theorem 2, we have

$$\frac{\mathrm{diam}(\tilde{M})}{\mathrm{diam}(M)} \leq 2 + 2 \cdot \mathrm{diam}(\Gamma(\pi_1(M), S)). \tag{24}$$

Combining (23) and (24) the result follows. □

Acknowledgements The author would like to express his thanks to Cameron Rudd and Andrew Ng for suggesting Example 18, and Jacob Bradd, Ana C. Chávez-Cáliz, and Anton Petrunin for lengthy discussions and helpful comments about previous versions of this paper. He is also grateful to Sebastian Gouzel for pointing out the ideas in [10, 11, 13, 28] to him, stimulating his interest in Theorem 9 even further, and an anonymous reviewer whose contributions have significantly improved not only the presentation of this paper but the quality of the results. Starting from Summer 2022, the author holds a Postdoctoral Fellowship at the Max Planck Institute for Mathematics at Bonn.

Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The author declares that he has no Conflict of interest.

References

1. Ash, R. B.: Probability and Measure Theory. Second edition. With contributions by Catherine Doléans-Dade Harcourt/Academic Press, Burlington (2000). pp. 516. ISBN:0-12-065202-1
2. Babai, L., Seress, Á.: On the diameter of permutation groups. Eur. J. Combin. **13**(4), 231–243 (1992)
3. Behr, H.: Endliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkörpern. Invent. Math. **7**, 1–32 (1969)
4. Bekka, B., de la Harpe, P., Valette, A.: Kazhdan's property (T). New Math. Monogr., 11. Cambridge University Press, Cambridge, (2008), pp. 472. ISBN:978-0-521-88720-5
5. Benjamini, I., Finucane, H., Tessera, R.: On the scaling limit of finite vertex transitive graphs with large diameter. Combinatorica **37**(3), 333–374 (2017)
6. Bishop, R. L., Crittenden, R. J.: Geometry of Manifolds. Reprint of the 1964 original, pp. 273. AMS Chelsea Publishing, Providence (2001). ISBN:0-8218-2923-8
7. Boston, N., Ellenberg, J.S.: Pro-p groups and towers of rational homology spheres. Geom. Topol. **10**, 331–334 (2006)
8. Breuillard, E., Green, B., Tao, T.: The structure of approximate groups. Publ. Math. Inst. Hautes Études Sci. **116**, 115–221 (2012)
9. Breuillard, E., Tointon, M.C.H.: Nilprogressions and groups with moderate growth. Adv. Math. **289**, 1008–1055 (2016)
10. Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. **56**(4), 581–598 (1981)
11. Brooks, R.: The spectral geometry of a tower of coverings. J. Differ. Geom. **23**(1), 97–107 (1986)
12. Brooks, R.: Some remarks on volume and diameter of Riemannian manifolds. J. Differ. Geom. **27**(1), 81–86 (1988)
13. Burger, M.: Spectre du laplacien, graphes et topologie de Fell. Comment. Math. Helv. **63**(2), 226–252 (1988)
14. Buser, P.: A note on the isoperimetric constant. Ann. Sci. École Norm. Sup. (4) **15**(2), 213–230 (1982)
15. Calegari, F., Dunfield, N.M.: Automorphic forms and rational homology 3-spheres. Geom. Topol. **10**, 295–329 (2006)
16. Cornulier, Y., de la Harpe, P.: Metric geometry of locally compact groups. EMS Tracts Math., 25 European Mathematical Society (EMS), Zürich, pp. 235 (2016). ISBN:978-3-03719-166-8
17. de la Harpe, P.: Topics in geometric group theory. Chicago Lectures in Math. University of Chicago Press, Chicago, pp. 310 (2000). ISBN:0-226-31719-6, ISBN:0-226-31721-8
18. de la Salle, M., Tessera, R.: Characterizing a vertex-transitive graph by a large ball. J. Topol. **12**(3), 705–743 (2019)
19. Diaconis, P.: Group representations in probability and statistics. IMS Lecture Notes Monogr. Ser., 11, pp. 198. Institute of Mathematical Statistics, Hayward (1988). ISBN:0-940600-14-5
20. Eberhard, S., Jezernik, U.: Babai's conjecture for high-rank classical groups with random generators. Invent. Math. **227**(1), 149–210 (2022)
21. Gelander, T.: Limits of finite homogeneous metric spaces. Enseign. Math. (2) **59**(1–2), 195–206 (2013)

22. Gorodski, C., Lange, C., Lytchak, A., Mendes, R.A.: A diameter gap for quotients of the unit sphere. J. Eur. Math. Soc. (JEMS) **25**(9), 3767–3793 (2023)
23. Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces. Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates, pp. 585. Reprint of the 2001 English edition. Mod. Birkhäuser Class. Birkhäuser Boston, Inc., Boston (2007). ISBN:978-0-8176-4582-3, ISBN:0-8176-4582-9
24. Halasi, Z.: Diameter of Cayley graphs of SL(n, p) with generating sets containing a transvection. J. Algebra **569**, 195–219 (2021)
25. Ledoux, M.: The concentration of measure phenomenon, pp. 181. Math. Surveys Monogr., 89. American Mathematical Society, Providence (2001). ISBN:0-8218-2864-9
26. Lubotzky, A.: Discrete groups, expanding graphs and invariant measures, pp. 195. With an appendix by Jonathan D. Rogawski. Progr. Math., 125. Birkhäuser Verlag, Basel (1994). ISBN:3-7643-5075-X
27. Lubotzky, A.: Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. Ann. Math. (2) **144**(2), 441–452 (1996)
28. Magee, M.: On Selberg's eigenvalue conjecture for moduli spaces of abelian differentials. Compos. Math. **155**(12), 2354–2398 (2019)
29. Mantuano, T.: Discretization of compact Riemannian manifolds applied to the spectrum of Laplacian. Ann. Global Anal. Geom. **27**(1), 33–46 (2005)
30. Martelli, B.: An introduction to geometric topology. arXiv preprint arXiv:1610.02592 (2016)
31. Petrunin, A.: Diameter of m-fold cover. https://mathoverflow.net/questions/7732/diameter-of-m-fold-cover
32. Petrunin, A.: Diameter of universal cover. https://mathoverflow.net/questions/8534/ diameter-of-universal-cover
33. Serre, J.-P.: Linear representations of finite groups, pp. 170. Translated from the second French edition by Leonard L. Scott. Grad. Texts in Math., Vol. 42. Springer-Verlag, New York-Heidelberg (1977). ISBN:0-387-90190-6
34. Sormani, C., Wei, G.: Hausdorff convergence and universal covers. Trans. Am. Math. Soc. **353**(9), 3585–3602 (2001)
35. Thurston, W. P.: Three-dimensional geometry and topology, pp. 311. Vol. 1. Edited by Silvio Levy Princeton Math. Ser., 35. Princeton University Press, Princeton (1997). ISBN:0-691-08304-5
36. Turing, A.M.: Finite approximations to Lie groups. Ann. Math. (2) **39**(1), 105–111 (1938)
37. Zamora, S.: *Limits of almost homogeneous spaces and their fundamental groups*. to appear in Groups, Geometry and Dynamics