Geometriae Dedicata (2024) 218:56
https://doi.org/10.1007/s10711-024-00905-3

ORIGINAL PAPER

Left-invariant distributions diffeomorphic to flat distributions

Sebastiano Nicolussi Golo[1] · **Alessandro Ottazzi**[2]

Received: 14 July 2022 / Accepted: 14 February 2024 / Published online: 13 March 2024

Abstract

For a stratified group G, we construct a class of Lie groups endowed with a left-invariant distribution locally diffeomorphic to the flat distribution of G. Vice versa, we show that all Lie groups with a left-invariant distribution that is locally diffeomorphic to the flat distribution of G belong to the class we constructed, if the Lie algebra of G has finite Tanaka prolongation.

Keywords Flat distributions · Tanaka prolongation · Stratified Lie groups · Contact structures · Quasi-conformal maps

Mathematics Subject Classification (2010) 30L10 · 22E25 · 53C30

Contents

Sebastiano Nicolussi Golo has been partially supported by the European Unions Seventh Framework Programme, Marie Curie Actions-Initial Training Network, under Grant Agreement No. 607643, “Metric Analysis For Emergent Technologies (MAnET)”, and by the EPSRC Grant "Sub-Elliptic Harmonic Analysis" (EP/P002447/1), and by University of Padova STARS Project "Sub-Riemannian Geometry and Geometric Measure Theory Issues: Old and New". Alessandro Ottazzi has been partially supported by the ARC Discovery grant DP170103025. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

✉ Alessandro Ottazzi
a.ottazzi@unsw.edu.au

Sebastiano Nicolussi Golo
sebastiano.s.nicolussi-golo@jyu.fi

1 Department of Mathematics and Statistics, University of Jyväskylä, 40014 Jyväskylä, Finland

2 School of Mathematics and Statistics, University of New South Wales (UNSW), Sydney 2052, Australia

1 Introduction

In this article, we consider the following question: given a stratified group G, we wish to characterise those polarised Lie groups that are equivalent to G. Here a *polarisation* on a Lie group is the choice of a left-invariant and bracket-generating subbundle of the tangent bundle (cfr. [11]), and two polarized Lie groups are *equivalent* if there is a locally defined distribution-preserving diffeomorphism between them. Stratified groups carry a canonical polarisation. Given a stratified group, we will construct a class of polarised Lie groups that are equivalent to G, which we will call *modifications* of G. The key tool for our construction will be Tanaka prolongation theory.

Before diving in the technical details of our main results, we first provide some framework. The problem under study is relevant in different areas, such as Tanaka prolongation theory, CR geometry, sub-Riemannian geometry, and control theory. In the setting of Tanaka's theory, it is known that the infinitesimal automorphisms of the polarisation associated to a stratified Lie algebra are encoded by its full Tanaka prolongation (see, e.g., [13, 15, 18]). If the Lie algebra is not stratified, however, all we can conclude is that every infinitesimal automorphism induces an infinitesimal automorphism on its stratified symbol. In this paper, we construct classes of polarised Lie algebras that are not stratified, but that have the same space of infinitesimal automorphisms as their stratified symbol.

Our study has potential applications to geometric control theory. Given a nonholonomic control system, the motion planning problem consists in finding a curve tangent to the polarisation that connects two given points in the ambient space. Nilpotent Lie groups are the widest class of nonholonomic systems for which an exact solution to the motion planning problem is known, see [6]. Distribution-preserving diffeomorphisms are equivalences of motion planning problems. Thus, our method detects classes of non-nilpotent nonholonomic systems that are equivalent to nilpotent ones.

Furthermore, our findings have consequences in metric geometry. On a polarised Lie group, one may define a left-invariant sub-Riemannian distance. In metric geometry, it is natural to study the equivalence of metric spaces up to isometries, bi-Lipschitz mappings, conformal and quasiconformal mappings. For example, if two stratified groups are (locally) quasiconformal, then their Lie algebras are isomorphic [14]. If two nilpotent Lie groups are isometric, then they are isomorphic [7, 9]. It is an open question to determine whether two nilpotent Lie groups that are globally bi-Lipschitz to one another are indeed isomorphic. In [4], the authors study the Lie groups that can be made isometric to a given nilpotent Lie group, endowed with a left-invariant distance. (See also [5] for the Riemannian case.) In this sense, our work follows [4], because distribution-preserving diffeomorphisms are locally bi-Lipschitz.

In sub-Riemannian geometry, one of the major open problems is to determine whether the conclusions of Sard Theorem hold for the endpoint map, which is a canonical map from an infinite dimensional path space to the underlying finite dimensional manifold. The set of critical values for the endpoint map is also known as abnormal set, being the set of endpoints of abnormal extremals leaving the base point. In the context of Lie groups, perhaps the most general positive results have been proved in [11]. Here the authors prove that the abnormal set has measure zero in the case of 2-step stratified groups and several other examples. This property for the abnormal set is preserved by distribution-preserving diffeomorphisms between sub-Riemannian manifolds. It then comes out from our results that every modification of a stratified group satisfies the Sard Theorem, if the stratified group does.

Now we will present our main results in detail. Recall that a Tanaka prolongation of a stratified Lie algebra $\mathfrak{g}$ through a Lie subalgebra $\mathfrak{g}_0$ of the Lie algebra of derivations of $\mathfrak{g}$ that preserve the stratification is the maximal nondegenerate graded Lie algebra that contains $\mathfrak{g} + \mathfrak{g}_0$, where nondegenerate means that the adjoint action of any element of positive weight on $\mathfrak{g}$ is nontrivial.[1] When $\mathfrak{g}_0$ is chosen as the whole set of strata preserving derivations, we obtain the full Tanaka prolongation. When the prolongation algebra is finite dimensional, we obtain a graded Lie algebra $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{q}$ and we say that $\mathfrak{g}$ and any Lie group G with Lie algebra $\mathfrak{g}$ are *rigid*. The term *of finite type* is also common in the literature to denote Lie algebras with finite Tanaka prolongation. Modifications of $\mathfrak{g}$ are then defined to be subalgebras $\mathfrak{s}$ of $\mathfrak{p}$ of the same dimension of $\mathfrak{g}$ that are transversal to $\mathfrak{q}$. It turns out that there is a linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$ of which the modification is the graph. If $\mathfrak{g}_{-1}$ is the first layer of $\mathfrak{g}$, then the set $\{v + \sigma(v) : v \in \mathfrak{g}_{-1}\}$ defines a polarisation on a Lie group whose Lie algebra is the modification $\mathfrak{s}$.

Our first main result draws the connection between modifications of G and Lie groups that are equivalent to G.

Theorem A *Every modification of a stratified Lie group G is equivalent to G. Viceversa, if G is rigid, then every polarised Lie group that is equivalent to G is one of its modification.*

Theorem A is restated and proven in Theorems 4.2 and 4.4. A key tool in the study of local distribution-preserving diffeomorphisms is the quotient manifold $M = P/Q$, where P and $Q < P$ are the Lie groups with Lie algebras $\mathfrak{p}$ and $\mathfrak{q}$ respectively. The polarisation of G induces a polarisation Δ_M on M and G embeds in M as an open subset, see Proposition 2.9. Moreover, if S is a modification of G, then an open neighborhood of the identity in S can be also embedded into M, see Lemma 4.1. The composition of such embeddings induce a local distribution-preserving diffeomorphism between G and S. If the group G is rigid, i.e., its full Tanaka prolongation is finite dimensional, then all distribution-preserving diffeomorphisms between G and S arise in this way, see Theorem 4.4.

The rigid case is particularly favorable because all distribution-preserving diffeomorphisms of G are induced by affine maps on P, see (3) at page 9. We can express this rigidity in terms of local distribution-preserving diffeomorphisms of M. More precisely, we will prove in Theorem 3.3 the following statement:

Theorem B *Suppose G is rigid and let $M = P/Q$ be the manifold described above, where the Lie algebra of P is the full tanaka prolongation of $\mathfrak{g}$. Let $U \subset M$ be open and connected and $f : U \to f(U) \subset M$ a smooth map with $df(\Delta_M) \subset \Delta_M$. Suppose that there exists $x_0 \in U$ such that $df(x_0)$ is non-singular. Then there exists a unique distribution-preserving diffeomorphism $g : M \to M$ such that $g|_U = f$.*

We will also prove that, in the hypothesis of Theorem B, the connected component of the identity in the group of distribution-preserving diffeomorphisms of M is isomorphic to Q, see Theorem 3.6.

[1] Recall that a *grading* of a Lie algebra is a vector space decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$. A grading is a *stratification* if $\mathfrak{g} = \bigoplus_{i \leq -1} \mathfrak{g}_i$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_j] = \mathfrak{g}_{j-1}$ for all $j < 0$.

It remains open whether the second part of Theorem A holds true without asking that the full Tanaka prolongation is finite. While we cannot prove the theorem in this generality, examples suggest that it may be true. More precisely, in Sect. 5.1, we show that all three dimensional sub-Riemannian structures are modifications of the Heisenberg group with respect to a suitable finite dimensional Tanaka prolongation, even though the full prolongation of the Heisenberg Lie algebra is infinite dimensional, see Theorem 5.1. This justifies the following conjecture.

Conjecture *Suppose that G is a stratified Lie group and that S is a polarised Lie group that is equivalent to G. Then there is a finite Tanaka prolongation of* $\mathrm{Lie}(G)$ *in which* $\mathrm{Lie}(S)$ *is a modification of* $\mathrm{Lie}(G)$.

In Sect. 5.2 we explicitly compute some modifications of the free nilpotent Lie group with two generators and step four, F_{24}. It comes out that one may construct examples of non-nilpotent Lie groups that are equivalent to F_{24}. We also find a nilpotent, non-stratified, polarised Lie group that is equivalent to F_{24} via a global distribution-preserving diffeomorphism, see Theorem 5.5. Finally, in Sect. 5.3, we study all the modifications of an ultra-rigid stratified group, that is, a stratified group whose only strata-preserving derivation is the infinitesimal generator of dilations. It turns out that such modifications are all solvable and the only nilpotent one is the stratified group itself, see Theorem 5.9.

The paper is organized as follows. In Sect. 2, we fix the notation and establish the framework in which we will be working. We consider stratified algebras and their Tanaka prolongations, we define the corresponding Lie groups and fix a polarisation on them. In Sect. 3, we study distribution-preserving diffeomorphisms of M as affine maps of P and prove Theorem B. In Sect. 4, we define the modifications of a stratified algebra and those of a stratified group, proving Theorem A. Finally, we apply our modification technic to a number of examples in Sect. 5.

2 Notation and preliminaries

2.1 Polarizations and Tanaka prolongations

Given a connected, smooth manifold M, a *polarisation* of M is the choice of a subbundle Δ_M of the tangent bundle TM that is *bracket generating*, i.e., with the property that the sections of Δ_M bracket generate all the sections of TM. Given two polarised manifolds (M, Δ_M) and (N, Δ_N), a *distribution-preserving diffeomorphism* between M and N is a diffeomorphism $f : M \to N$ such that $f_*(\Delta_M) = \Delta_N$. We denote by $\Gamma(TM)$ the space of vector fields on M. A vector field $V \in \Gamma(TM)$ on a polarised manifold (M, Δ_M) is a *contact vector field* if its flow is made of distribution-preserving diffeomorphisms. For a Lie group S, we shall always consider left-invariant polarisations Δ_S. The pair (S, Δ_S) is called a *polarised group*. The identity element will be denoted by e_S, or simply e if no confusion arises. We denote by G a *stratified group*, that is, a connected and simply connected Lie group whose Lie algebra decomposes as $\mathfrak{g} = \bigoplus_{i=-s}^{-1} \mathfrak{g}_i$, with $[\mathfrak{g}_{-1}, \mathfrak{g}_j] = \mathfrak{g}_{j-1}$ for every $-s+1 \leq j \leq -1$. On a stratified group we will always consider the left-invariant polarisation Δ_G for which $(\Delta_G)_{e_G} = \mathfrak{g}_{-1}$. In a stratified group G we consider the strata preserving derivations

$$\begin{aligned}\mathrm{Der}(\mathfrak{g}) := \{u \in \mathrm{End}(\mathfrak{g}) \, : \, & u(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1}, \\ & \text{and } u[X, Y] = [u(X), Y] + [X, u(Y)] \, \forall X, Y \in \mathfrak{g}\}.\end{aligned}$$

Given a subalgebra $\mathfrak{g}_0$ of $\mathrm{Der}(\mathfrak{g})$, we define the *Tanaka prolongation* of $\mathfrak{g}$ through $\mathfrak{g}_0$ as the (possibly infinite) maximal nondegenerate graded Lie algebra $\mathrm{Prol}(\mathfrak{g}, \mathfrak{g}_0) = \bigoplus_{k\geq -s} \mathfrak{g}_k$ which contains $\mathfrak{g} \oplus \mathfrak{g}_0$. When $\mathfrak{g}_0 = \mathrm{Der}(\mathfrak{g})$, we call $\mathrm{Prol}(\mathfrak{g}, \mathfrak{g}_0)$ the *full* Tanaka prolongation of $\mathfrak{g}$. It is not difficult to see that the latter contains all prolongations. We say that $\mathfrak{g}$, or G, is *rigid* if the full Tanaka prolongation has finite dimension. When it is clear from the context and the prolongation under consideration is finite dimensional, we shall denote $\mathrm{Prol}(\mathfrak{g}, \mathfrak{g}_0)$ by $\mathfrak{p}$, the nonnegative part $\bigoplus_{k\geq 0} \mathfrak{g}_k$ by $\mathfrak{q}$, and the positive part $\bigoplus_{k>0} \mathfrak{g}_k$ by $\mathfrak{p}_+$. See [13, 15, 18] for further details on Tanaka prolongation.

2.2 The groups *P* and *Q* and their quotient *M*

In the following, we establish a number of properties of the Lie groups that correspond to the Lie algebras introduced above. Let $\bar{P}$ be the connected and simply connected Lie group whose Lie algebra is a finite dimensional Tanaka prolongation $\mathfrak{p}$ of a stratified Lie algebra $\mathfrak{g}$. Let $\bar{Q}$ be the connected subgroup of $\bar{P}$ whose Lie algebra is $\mathfrak{q}$.

The set $\{\delta_\lambda : \lambda > 0\}$ of mappings on $\mathfrak{p}$ defined by $\delta_\lambda(X) = \lambda^i X$ for $X \in \mathfrak{g}_i$ is a one parameter family of automorphisms of $\mathfrak{p}$. By abuse of notation, we write δ_λ for the corresponding automorphisms of the group $\bar{P}$. Such maps exist because $\bar{P}$ is simply connected.

Lemma 2.1 *Denote by* $\exp_P : \mathfrak{p} \to \bar{P}$ *the exponential map of* $\bar{P}$. *Then* $\exp_P$ *is injective on* $\mathfrak{g}$ *and on* $\bigoplus_{k\geq 1} \mathfrak{g}_k$.

Proof Let $v, w \in \mathfrak{g}$ such that $\exp_P(v) = \exp_P(w)$. Since $v, w \in \mathfrak{g}$, then $\lim_{\lambda\to\infty} \delta_\lambda v = \lim_{\lambda\to\infty} \delta_\lambda w = 0$. Let $\lambda \geq 1$ be such that both $\delta_\lambda(v)$ and $\delta_\lambda(w)$ belong to a neighborhood U of 0 in $\mathfrak{p}$ on which the exponential map $\exp_P$ is injective. Then $\exp_P(\delta_\lambda v) = \delta_\lambda(\exp_P(v)) = \delta_\lambda(\exp_P(w)) = \exp_P(\delta_\lambda w)$. By the injectivity of $\exp_P$ on U, we have $\delta_\lambda v = \delta_\lambda w$. Since δ_λ is a linear isomorphism, we conclude that $v = w$. A similar argument proves that $\exp_P$ is injective on $\bigoplus_{k\geq 1} \mathfrak{g}_k$. □

By Lemma 2.1, the canonical immersion $G \hookrightarrow \bar{P}$ induced by $\mathfrak{g} \hookrightarrow \mathfrak{p}$ is injective. We are going to show that G is closed in $\bar{P}$. We prove two lemmas first.

Lemma 2.2 *The intersection of G with* $\bar{Q}$ *is trivial.*

Proof Since $\delta_\lambda(\mathfrak{g}) = \mathfrak{g}$ and $\delta_\lambda(\mathfrak{q}) = \mathfrak{q}$, then $\delta_\lambda(G) = G$ and $\delta_\lambda(\bar{Q}) = \bar{Q}$, for all $\lambda > 0$. Since $\mathfrak{g}$ is nilpotent, $G = \exp_P(\mathfrak{g})$.

Let $x \in G \cap \bar{Q}$; then $x = \exp_P(v)$ for some $v \in \mathfrak{g}$ and $\lim_{\lambda\to\infty} \delta_\lambda(x) = \exp_P(\lim_{\lambda\to\infty} \delta_\lambda v) = e_P$. It follows that the curve $\gamma : (0, 1] \to \bar{P}$, $\gamma(t) = \delta_{t^{-1}} x$, extends to a continuous path $[0, 1] \to \bar{P}$ connecting $\gamma(0) = e_P$ to $\gamma(1) = x$ and laying in G. Since $\delta_\lambda(x) \in \bar{Q}$ for all $\lambda > 0$, then γ lies in $\bar{Q}$ as well.

Since $\mathfrak{g} \oplus \mathfrak{q} = \mathfrak{p}$, there are open neighborhoods $U \subset \mathfrak{g}$ and $V \subset \mathfrak{q}$ of 0 such that $\Omega = \exp_P(U) \exp_P(V)$ is an open neighborhood of e_P in $\bar{P}$ and the following holds: The connected component of $\Omega \cap G$ containing e_P is $\exp_P(U)$, the connected component of $\Omega \cap \bar{Q}$ containing e_P is $\exp_P(V)$, and $\exp_P(U) \cap \exp_P(V) = \{e_P\}$.

Since γ joins x to e_P continuously, then $\gamma([0, 1]) \cap \Omega$ lies in both the connected components of $\Omega \cap G$ and $\Omega \cap \bar{Q}$ containing e_P, i.e., $\gamma([0, 1]) \cap \Omega \subset \exp_P(U) \cap \exp_P(V) = \{e_P\}$. This implies that $x = e_P$. □

Lemma 2.3 (Lemma on Lie groups) *Let G be a Lie subgroup of a Lie group P and let* $\iota : G \hookrightarrow P$ *the inclusion. The image* $\iota(G)$ *is not closed in P if and only if there is a sequence*

$\{g_n\}_{n\in\mathbb{N}} \subset G$ *such that* $\lim_{n\to\infty} g_n = \infty$ *(i.e.,* g_n *eventually escapes every compact set of* G*) and* $\lim_{n\to\infty} \iota(g_n) = e_P$.

Proof Recall that G is closed in P if and only if ι is an embedding. So, if such a sequence exists then $\iota(G)$ is not closed in P. We need to prove the converse implication.

Let ρ be any left-invariant Riemannian distance on G. Then ρ is complete and in particular closed balls are compact. Let $\{g_n\}_{n\in\mathbb{N}} \subset G$ be a sequence such that $\lim_{n\to\infty} \iota(g_n) = p \in P$. If there is $R > 0$ such that $\rho(e_G, g_n) \le R$ for all n, then there is a subsequence g_{n_k} converging to some $g_\infty \in G$. Since the immersion $\iota : G \hookrightarrow P$ is continuous, we obtain $\iota(g_\infty) = p$, hence $p \in \iota(G)$.

So, if $\iota(G)$ is not closed, then there is a sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that $\lim_{n\to\infty} \iota(g_n) = p \in P$ but $g_n \to \infty$ in G. Let $\{g_{n_k}\}_k$ be a subsequence such that $\rho(g_{n_k}, g_{n_{k+1}}) > k$ for $k \in \mathbb{N}$ and define $h_k = g_{n_k}^{-1} g_{n_{k+1}}$. Then $h_k \to \infty$ in G, because $\rho(e_G, h_k) = \rho(e_G, g_{n_k}^{-1} g_{n_{k+1}}) = \rho(g_{n_k}, g_{n_{k+1}}) > k$ for all k. However, $\iota(h_k) = \iota(g_{n_k}^{-1})\iota(g_{n_{k+1}}) \to p^{-1}p$ in P as $k \to \infty$. $\square$

Lemma 2.4 *The immersed group* G *is closed in* $\bar{P}$.

Proof We prove that, if $\{v_n\}_{n\in\mathbb{N}} \subset \mathfrak{g}$ is a sequence so that $\lim_{n\to\infty} \exp_P(v_n) = e_P$, then $\lim_{n\to\infty} v_n = 0$. By Lemma 2.3 and $\exp_P(\mathfrak{g}) = G$, this claim implies that G is closed in P.

Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathfrak{g}$ be a sequence with $\lim_{n\to\infty} \exp_P(v_n) = e_P$. Let $U \subset \mathfrak{g}$ and $W \subset \mathfrak{q}$ be open neighborhoods of 0 such that the map $U \times W \to P$, $(u, w) \mapsto \exp_P(u)\exp_P(w)$ is a diffeomorphism onto its image. Then, for n large enough, there are $u_n \in U$ and $w_n \in W$ so that $\exp_P(u_n)\exp_P(w_n) = \exp_P(v_n)$. Therefore, $\exp_P(u_n)^{-1}\exp_P(v_n) = \exp_P(w_n) \in \bar{Q} \cap G$. By Lemma 2.2, we have $\exp_P(u_n) = \exp_P(v_n)$. By Lemma 2.1, we have $u_n = v_n$. Since $\exp_P(u_n) \to e_P$, then $v_n = u_n \to 0$. $\square$

Corollary 2.5 *The immersed group* $\bar{Q}$ *is closed in* $\bar{P}$.

Proof This is a consequence of Lemma 2.4 and part (iii) of Lemma 2.15 in [4] $\square$

Since $\bar{Q}$ is closed, we may consider the homogeneous manifold $M := \bar{P}/\bar{Q}$ with quotient projection $\pi : \bar{P} \to M$. The action of $\bar{P}$ may have a non-trivial kernel

$$K := \{p \in \bar{P} : p.x = x\ \forall x \in M\} = \bigcap_{p\in\bar{P}} p\bar{Q}p^{-1}.$$

Lemma 2.6 *The kernel* K *of the action of* $\bar{P}$ *on* M *is discrete and contained in* $\bar{Q}$*. Moreover, if* $p \in K$*, then* $\delta_\lambda p = p$ *for all* $\lambda > 0$.

Proof Clearly K is a normal and closed subgroup of $\bar{P}$ and it is contained in $\bar{Q}$. Let $v \in \mathrm{Lie}(K)$, the Lie algebra of K. Then for some positive integer ℓ, we may write $v = v_0 + \cdots + v_\ell$, with $v_i \in \mathfrak{g}_i$ for every $i = 0, \dots, \ell$. Since $\mathrm{Lie}(K)$ is an ideal in $\mathfrak{p}$ contained in $\mathfrak{q}$, it follows in particular that for all $i = 0, \dots, \ell$,

$$[[\dots[[v_i, y_1], y_2], \dots], y_{\ell+1}] \in \mathfrak{g}_{i-\ell-1} \cap \mathfrak{q} = \{0\},$$

for every $y_1, \dots, y_{\ell+1} \in \mathfrak{g}_{-1}$. By definition of Tanaka prolongation, this implies that $v = 0$. Therefore, the Lie algebra of K is trivial and so K is discrete.

Since $K = \bigcap_{x\in\bar{P}} x\bar{Q}x^{-1}$, it is clear that $\delta_\lambda(K) \subset K$ for all $\lambda > 0$. However, since $\lambda \mapsto \delta_\lambda p$ is a continuous curve passing through p, we must have $\delta_\lambda p = p$ when $p \in K$. $\square$

From Lemma 2.6 it follows that $P := \bar{P}/K$ and $Q := \bar{Q}/K$ are Lie groups, that Q is closed in P and $M = P/Q$. Moreover, the maps δ_λ are automorphisms of P as well, for all $\lambda > 0$. Since $G \cap K = \{e\}$, the group G is embedded in P with $G \cap Q = \{e\}$.

Remark 2.7 If we are given G and Q inside P, for instance as matrix groups, we may want to visualise the action of P on M as a local action of P on G. In other words, if $p \in P$, then there may be open subsets $U_p, V_p \subset G$ and a distribution-preserving diffeomorphism $f_p : U_p \to V_p$ that corresponds to the action of p on M, i.e., $f_p(g_1)$ is the only $g_2 \in G$, if it exists, such that $\{g_1 Q\} \cap G = \{g_2\}$. In general, such construction is not possible for all $p \in P$, but if p is near enough to e_P, then U_p, V_p and f_p do exist. The fact that such f_p is a distribution-preserving diffeomorphism will be proved in Proposition 2.8.

2.3 Polarizations on *G*, *P* and *M*

We denote by $\pi : P \to M$ the quotient map, with $M = P/Q$. If $p \in P$ and $m \in M$, we use the notation $p.m$ or $p(m)$ for the action of p on m. In such contexts, we will identify elements $p \in P$ with smooth diffeomorphisms $p : M \to M$.

Recall that on G we have the polarisation Δ_G with $(\Delta_G)_e = \mathfrak{g}_{-1}$. We define on P the polarisation Δ_P such that $(\Delta_P)_{e_P} = \mathfrak{g}_{-1} \oplus \mathfrak{q}$. Notice that $\Delta_G = \Delta_P \cap TG$. Define $\Delta_M := d\pi(\Delta_P)$ which is a subset of TM. We shall prove that Δ_M is a P-invariant polarisation on M.

Proposition 2.8 *The set $\Delta_M \subset TM$ is a P-invariant, bracket generating subbundle of M. In particular, (M, Δ_M) is a polarised manifold and the diffeomorphisms $p : M \to M$ for $p \in P$ are distribution-preserving diffeomorphisms.*

Proof Notice that Δ_M is a P-invariant subset of TM. In order to show that Δ_M is a subbundle, we need to prove that, if $p_1, p_2 \in P$ are such that $\pi(p_1) = \pi(p_2)$, then

$$d\pi((\Delta_P)_{p_1}) = d\pi((\Delta_P)_{p_2}). \tag{1}$$

Since $p \circ \pi = \pi \circ L_p$ for all $p \in P$, then (1) is equivalent to $d(p_2^{-1} \circ \pi \circ L_{p_1})[(\Delta_P)_e] = d\pi[(\Delta_P)_e]$. Let $p = p_1$ and choose $q \in Q$ so that $p_2 = p_1 q$. Then $p_2^{-1} \circ \pi \circ L_{p_1} = \pi \circ L_{q^{-1}}$ and thus (1) is also equivalent to

$$\mathrm{Ad}_q[(\Delta_P)_e] \mod \mathfrak{q} = (\Delta_P)_e \mod \mathfrak{q}. \tag{2}$$

Since Ad is a homomorphism and every $q \in Q$ is the finite product of exponential elements, it's enough that we show (2) for $q = \exp y$, $y \in \mathfrak{q}$. Denote by y_0 the projection of y on $\mathfrak{g}_0$. Let $w \in \mathfrak{g}_{-1} \oplus \mathfrak{q}$ and denote by w_{-1} its projection on $\mathfrak{g}_{-1}$. Then

$$\begin{aligned} \mathrm{Ad}_q w \mod \mathfrak{q} &= e^{\mathrm{ad}(y)} w \mod \mathfrak{q} \\ &= e^{\mathrm{ad}(y_0)} w_{-1} \mod \mathfrak{q}. \end{aligned}$$

Since $e^{\mathrm{ad}(y_0)} : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is a bijection, we conclude that $\mathrm{Ad}_q[(\Delta_P)_e] \mod \mathfrak{q} = \mathfrak{g}_{-1} \mod \mathfrak{q}$. This proves (2) and therefore (1).

Finally, we need to show that Δ_M is bracket generating. Recall that, for an analytic subbundle of an analytic manifold, being bracket generating is equivalent to being connected by curves tangent to the subbundle, and that quotients of Lie groups and invariant subbundles are all analytic. Thus, let $m_0 = \pi(p_0)$ and $m_1 = \pi(p_1)$ in M. Then there is a C^1-curve $\gamma : [0, 1] \to P$ such that $\gamma(0) = p_0$, $\gamma(1) = p_1$ and $\gamma'(t) \in \Delta_P$ for all $t \in [0, 1]$. Hence, the curve $\pi \circ \gamma : [0, 1] \to M$ goes from m_0 to m_1 and is clearly tangent to Δ_M. □

Proposition 2.9 *The restriction $\pi|_G : (G, \Delta_G) \to (M, \Delta_M)$ is a distribution-preserving diffeomorphism onto its image, which is an open subset of M.*

Proof First, we show that $\pi|_G$ is injective. Let $a, b \in G$ such that $\pi(a) = \pi(b)$. Then $\pi(e) = \pi(a^{-1}a) = a^{-1}.\pi(a) = a^{-1}\pi(b) = \pi(a^{-1}b)$, i.e., $a^{-1}b \in Q$. Since $a^{-1}b \in G$ and $G \cap Q = \{e\}$, then $a = b$.

Second, we show that $\pi|_G$ is an immersion. Since $\ker(d\pi|_g) = dL_g(\mathfrak{q})$ and $dL_g(\mathfrak{q}) \cap T_gG = DL_g(\mathfrak{q}) \cap DL_g(\mathfrak{g}) = \{0\}$, then $d(\pi|_G)|_g = (d\pi|_g)|_{T_gG}$ is injective, for all $g \in G$.

Third, we claim $d\pi|_G(\Delta_G) = \Delta_M \cap T(\pi(G))$. Since $\Delta_G \subset \Delta_P$ and since $\Delta_M = d\pi(\Delta_P)$ by definition, it follows that $d\pi|_G(\Delta_G) \subset \Delta_M \cap T(\pi(G))$. Moreover, since Δ_M is P-invariant by Proposition 2.8, for all $x \in M$, $\dim(\Delta_M)_x = \dim(\Delta_M)_{\pi(e)} = \dim((\mathfrak{g}_{-1} \oplus \mathfrak{q})/\mathfrak{q}) = \dim(\mathfrak{g}_{-1})$. Therefore, we obtain the claim by comparing the dimensions.

Finally, the fact that $\pi(G)$ is open in M and the fact that $\pi|_G$ is an embedding are both consequences of $(\pi|_G)$ being an immersion and the fact that M and G have the same dimension. □

Remark 2.10 A first consequence of Proposition 2.9 is that any local distribution-preserving diffeomorphism on M is in fact a local distribution-preserving diffeomorphism on G. Indeed, by the action of P on M and via the map $\pi|_G$, any local distribution-preserving diffeomorphism of M defines a local distribution-preserving diffeomorphism of G. Similarly, contact vector fields on M define contact vector fields on G.

In case G is a rigid stratified group and $\mathfrak{p}$ is the full Tanaka prolongation of $\mathfrak{g}$, these relations are stronger, see Sect. 3.

3 Distribution-preserving diffeomorphisms of *M* when *G* is rigid

This section contains Theorem 3.3 for distribution-preserving diffeomorphisms in the rigid case.

Relative to a vector $X \in T_eP$, we denote by $\tilde{X}$ the left-invariant vector field $\tilde{X}(p) = dL_p|_e[X]$, and by $X^\dagger$ the right-invariant vector field $X^\dagger(p) = dR_p|_e[X]$. Similarly, we denote by $\tilde{\mathfrak{p}}$ the Lie algebra of left-invariant vector fields and by $\mathfrak{p}^\dagger$ the Lie algebra of right-invariant vector fields on P. Moreover, as in the previous sections, the manifold M is the quotient P/Q and we denote by o the point $\pi(e) \in M$.

Lemma 3.1 *Let $\ell : \mathfrak{p} \to \mathfrak{p}$ be a Lie algebra automorphism with $\ell(\mathfrak{q}) = \mathfrak{q}$ and $\ell(\mathfrak{g}_{-1} \oplus \mathfrak{q}) = \mathfrak{g}_{-1} \oplus \mathfrak{q}$. Then there is a unique distribution-preserving Lie group automorphism $L : P \to P$ with $L_* = \ell$ and a unique distribution-preserving diffeomorphism $L^\pi : M \to M$ with $L^\pi \circ \pi = \pi \circ L$.*

Proof If $\ell : \mathfrak{p} \to \mathfrak{p}$ is a Lie algebra automorphism with $\ell(\mathfrak{q}) = \mathfrak{q}$, then the induced Lie group automorphism $\bar{L} : \bar{P} \to \bar{P}$ has the property that $\bar{L}(K) = K$, where K is the kernel of the action of $\bar{P}$ on M. It follows that there is a Lie group automorphism $L : P \to P$ such that $L_* = \ell$.

If $L : P \to P$ is a Lie group automorphism with $L(Q) = Q$, then it is well known that there is a unique diffeomorphism $L^\pi : M \to M$ such that $L^\pi \circ \pi = \pi \circ L$.

Now, suppose that $\ell(\mathfrak{g}_{-1} \oplus \mathfrak{q}) = \mathfrak{g}_{-1} \oplus \mathfrak{q}$, i.e., $L_*(\Delta_P)_e = (\Delta_P)_e$. Since Δ_P is left-invariant, then L is a distribution-preserving diffeomorphism of (P, Δ_P). Finally, we prove that L^π is a distribution-preserving diffeomorphism. Let $X \in \Delta_P$ and $x \in P$. Then

$$dL^\pi|_{\pi(x)}[d\pi|_x[\tilde{X}_x]] = d(L^\pi \circ \pi)|_x[\tilde{X}_x] = d(\pi \circ L)|_x[\tilde{X}_x] \in \Delta_M|_{L^\pi(x)}.$$

□

Let $\mathtt{Aut}(\mathfrak{p},\mathfrak{g})$ be the group of Lie algebra automorphisms of $\mathfrak{p}$ that induce distribution-preserving diffeomorphism on M. By Lemma 3.1, we have

$$\mathtt{Aut}(\mathfrak{p},\mathfrak{g}) = \{\phi \in \mathtt{Aut}(\mathfrak{p}) : \phi(\mathfrak{q}) = \mathfrak{q},\ \phi(\mathfrak{g}_{-1} \oplus \mathfrak{q}) = \mathfrak{g}_{-1} \oplus \mathfrak{q}\}.$$

This group plays a crucial role in the classification of modifications of a stratified Lie algebra, as we shall show in Theorem 4.6.

For the following claim, see [15, Sect. 6] and [18].

Theorem 3.2 (Tanaka) *If $\mathfrak{g}$ is rigid and $\mathfrak{p}$ is the full Tanaka prolongation, then $\pi_*\mathfrak{p}^\dagger \subset \Gamma(TM)$ is the set of all germs of contact vector fields on M. More precisely, on the one hand $\pi_*\mathfrak{p}^\dagger$ are contact vector fields of (M, Δ_M); On the other hand, if $U \subset M$ is open and connected, and $V \in \Gamma(TU)$ is a contact vector field, then there is a unique $X \in \mathfrak{p}$ such that $V = \pi_* X^\dagger|_U$.*

Denote by $\mathtt{Cont}(U)$ the space of contact vector fields on an open set $U \subset M$. Notice that $\mathtt{Cont}(U)$ is a Lie algebra. Theorem 3.2 can be restated as: *if $U \subset M$ is open and connected, then $\pi_*|U : X^\dagger \mapsto \pi_* X^\dagger|_U$ is a Lie algebra isomorphism between $\mathfrak{p}^\dagger$ and* $\mathtt{Cont}(U)$. With Theorem 3.2, we can prove the following result:

Theorem 3.3 *Suppose $\mathfrak{g}$ is rigid and let $\mathfrak{p}$ be its full Tanaka prolongation. Let $U \subset M$ be open and path-connected and $f : U \to f(U) \subset M$ be a smooth map with $df(\Delta_M) \subset \Delta_M$ and suppose that there exists $x_0 \in U$ such that $df(x_0)$ is non-singular. Then there exists a unique a Lie group automorphism $L^f : P \to P$ such that $(L^f)_* \in \mathtt{Aut}(\mathfrak{p},\mathfrak{g})$ and $p^f \in P$ such that, for every $p \in P$ with $\pi(p) \in U$, we have*

$$f(\pi(p)) = \pi(p^f L^f(p)). \tag{3}$$

In particular, there exists a unique distribution-preserving diffeomorphism $g : M \to M$ such that $g|_U = f$.

The proof of Theorem 3.3 requires the following preliminary lemma.

Lemma 3.4 *Suppose $\mathfrak{g}$ is rigid and let $\mathfrak{p}$ be its full Tanaka prolongation. Let $U \subset M$ be open and path-connected and $f : U \to f(U) \subset M$ be a smooth diffeomorphism with $df(\Delta_M) \subset \Delta_M$. Then there exist a unique $p_f \in P$ and a Lie group automorphism $L^f : P \to P$ with $(L^f)_* \in \mathtt{Aut}(\mathfrak{p},\mathfrak{g})$ such that, for every $x \in P$ with $\pi(x) \in U$, we have*

$$f(\pi(x)) = \pi(p_f L^f(x)). \tag{4}$$

In particular, there exists a unique distribution-preserving diffeomorphism $g : M \to M$ such that $g|_U = f$.

Proof By Theorem 3.2, the map $\ell^f = \pi_*|_{f(U)}^{-1} \circ f_* \circ \pi_*|_U : \mathfrak{p}^\dagger \to \mathfrak{p}^\dagger$ is a composition of Lie algebra isomorphisms

$$\mathfrak{p}^\dagger \to \mathtt{Cont}(U) \to \mathtt{Cont}(f(U)) \to \mathfrak{p}^\dagger.$$

Let $L^f : P \to P$ be the corresponding Lie group automorphism, whose existence is assured by Lemma 3.1.

It is clear that, if f_1, f_2 are two such distribution-preserving diffeomorphisms then $L^{f_2 \circ f_1} = L^{f_2} \circ L^{f_1}$, whenever the composition is well defined. Moreover, notice that if $p \in P$, then seen as a diffeomorphism $p : M \to M$ the argument above shows that $\ell^p = \mathrm{Id}$, and so (4) holds with $L^p = \mathrm{Id}$ and $p_p = p$.

Back to the general case, let $p_0, p_1 \in P$ satisfy $\pi(p_0) \in U$ and $\pi(p_1) = f(\pi(p_0))$. Then $h(m) := p_1^{-1}.f(p_0.m)$ defines a distribution-preserving diffeomorphism $h : p_0^{-1}.U \to p_1.f(U)$. By the previous paragraph, we also have $L^h = L^f$. Since $h(\pi(e_P)) = \pi(e_P)$, integrating the contact vector fields, we get $h(\pi(x)) = \pi(L^h(x))$ whenever $\pi(x) \in U$. We conclude that (4) holds with $p^f := p_1 L(p_0^{-1})$.

Viceversa, if L^f and p^f satisfy (4), then the differential ℓ_f of L^f at e_P is a Lie algebra automorphism of $\mathfrak{p}^\dagger$ satisfying $\ell^f = \pi_*|_{f(U)}^{-1} \circ f_* \circ \pi_*|_U$. Therefore, L^f is unique. Moreover, since $f(\pi(x)) = \pi(p_f L^f(x)) = p_f.\pi(L^f(x))$, p_f is uniquely determined.

Finally, the fact that $x \mapsto p_f L^f(x)$ induces a global diffeomorphism $g : M \to M$, which extends f, is a consequence of Lemma 3.1. □

Proof of Theorem 3.3 Fix $x_0 \in U$ and a neighborhood $U' \subset U$ of x_0 such that f is a diffeomorphism $U' \to f(U')$. By Lemma 3.4, there is a distribution-preserving diffeomorphism $g : M \to M$ with $g|_{U'} = f|_{U'}$. Let $W \subset U$ be the largest open set where f and g are equal. If $x \in \partial W \cap U$ then continuity of the differential implies that $df(x) = dg(x)$ and it follows that $df(x)$ is nonsingular. Applying Lemma 3.4 again, there is a neighborhood U'' of x and a distribution-preserving diffeomorphism $g'' : M \to M$ such that $g''|_{U''} = f|_{U''}$. Notice that $U'' \cap W$ is a nonempty open set and that the restrictions of g'' and g are distribution-preserving diffeomorphisms on $U'' \cap W$ and $g''|_{U''\cap W} = g_{U''\cap W}$. Since Lemma 3.4 implies the uniqueness of smooth distribution-preserving extensions, we get $g'' = g$. In particular, we have that $x \in U'' \subset W$, in contradiction with $x \in \partial W$.

Therefore, we conclude that $\partial W \cap U = \emptyset$ and, since U is connected, $W = U$. □

Remark 3.5 Tanaka prolongation is usually stated in the C^∞ category. However, in Theorem 3.2, and consequently in Theorem 3.3, one can assume f to be only smooth of class C^2. The upgrade of the regularity works like in [13].

Finally, we prove that the group of $\mathtt{Aut}(\mathfrak{p}, \mathfrak{g})$ is the adjoint representation of Q.

Theorem 3.6 *Suppose that $\mathfrak{g}$ is rigid and that $\mathfrak{p}$ is the full Tanaka prolongation. The Lie algebra of* $\mathtt{Aut}(\mathfrak{p}, \mathfrak{g})$ *is* $\{\mathrm{ad}_X : X \in \mathfrak{q}\}$. *In particular, the connected component of the identity in* $\mathtt{Aut}(\mathfrak{p}, \mathfrak{g})$ *is* $\{\mathrm{Ad}_x : x \in Q\}$, *which is isomorphic to Q via the adjoint map $x \mapsto \mathrm{Ad}_x$.*

Proof We need to show that, if $D : \mathfrak{p} \to \mathfrak{p}$ is a derivation such that $D(\mathfrak{q}) \subset \mathfrak{q}$ and $D(\Delta_P|_e) \subset \Delta_P|_e$, then there is $X \in \mathfrak{q}$ such that $D = \mathrm{ad}_X$.

The one-parameter group of Lie algebra automorphisms $\ell_t := e^{tD}$ are such that $\ell_t(\mathfrak{q}) = \mathfrak{q}$ and $\ell_t(\Delta_P)_e = (\Delta_P)_e$. By Lemma 3.1, they induce a one-parameter group of Lie group automorphism L_t on P and a one-parameter group of distribution-preserving diffeomorphism $L_t^\pi : M \to M$.

Since L_t^π is a one-parameter group of distribution-preserving diffeomorphisms on M and by Theorem 3.2, there is $V \in \pi_*\mathfrak{p}^\dagger$ such that L_t^π is its flow. Let $X \in \mathfrak{p}$ be such that $\pi_*(X^\dagger) = V$. Since $L_t^\pi(o) = o$ and thus $V(o) = 0$, we have $X \in \mathfrak{q}$.

Notice that $L_t^\pi(m) = \exp(tX).m$ for all $m \in M$. Therefore, if $p \in P$ and $m = \pi(p)$, then

$$\begin{aligned}\pi(L_t(p)) &= L_t^\pi(\pi(p)) = \exp(tX).\pi(p)\\ &= \pi(\exp(tX)p) = \pi(\exp(tX)p\exp(-tX)) = \pi(C_{\exp(tX)}p),\end{aligned}$$

where $C_a p = apa^{-1}$ is the conjugation by $a \in P$. Since by Lemma 3.4 the lift of a distribution-preserving diffeomorphism from M to P is unique, we conclude that $C_{\exp(tX)} = L_t$.

Finally, for all $t \in \mathbb{R}$ we have

$$e^{t\,\mathrm{ad}_X} = \mathrm{Ad}_{\exp(tX)} = dC_{\exp(tX)}|_e = dL_t|_e = e^{tD}$$

and thus $D = \mathrm{ad}_X$.

For the last part of the statement, we need to show that $x \mapsto \mathrm{Ad}_x$ is injective on Q. So, suppose that $x \in Q$ is such that Ad_x is the identity map on P. Since $Q = \exp(\mathfrak{q})$, there is $v \in \mathfrak{q}$ such that $x = \exp(v)$ and thus $\mathrm{Ad}_x = e^{\mathrm{ad}_v}$. The vector v can be decomposed as $v = \sum_{j \geq k} v_j$ with $v_j \in \mathfrak{g}_j$ and $v_k \neq 0$, where $k \geq 1$. If we denote by π_{k-1} the projection $\mathfrak{p} \to \mathfrak{g}_{k-1}$ given by the grading of $\mathfrak{p}$, then for every $w \in \mathfrak{g}_{-1}$ we have

$$\pi_{k-1}(e^{\mathrm{ad}_v}(w)) = [v, w]$$

Since $e^{\mathrm{ad}_v}(w) = w \in \mathfrak{g}_{-1}$, then $[v, w] = 0$. We obtain that $\mathrm{ad}_v|_{\mathfrak{g}_{-1}} = 0$ and thus, by definition of Tanaka prolongation, $v = 0$. We conclude that $x = e$ and thus Ad is injective. □

4 Modifications of stratified groups

A *polarised Lie algebra* is a pair $(\mathfrak{s}, \mathfrak{s}_{-1})$ where $\mathfrak{s}$ is a Lie algebra and $\mathfrak{s}_{-1}$ is a bracket-generating subspace. We say that two polarised Lie algebras $(\mathfrak{s}, \mathfrak{s}_{-1})$ and $(\mathfrak{s}', \mathfrak{s}'_{-1})$ are *isomorphic* if there is a Lie algebra isomorphism $\phi : \mathfrak{s} \to \mathfrak{s}'$ such that $\phi(\mathfrak{s}_{-1}) = \mathfrak{s}'_{-1}$. Given a stratified Lie algebra $\mathfrak{g}$ and a finite dimensional Tanaka prolongation $\mathfrak{p} = \mathrm{Prol}(\mathfrak{g}, \mathfrak{g}_0)$, a *modification of* $\mathfrak{g}$ *in* $\mathfrak{p}$ is a polarized algebra $(\mathfrak{s}, \mathfrak{s}_{-1})$ where $\mathfrak{s} \subset \mathfrak{p}$ is a subalgebra such that $\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{q}$ and $\mathfrak{s}_{-1} = (\mathfrak{g}_{-1} \oplus \mathfrak{q}) \cap \mathfrak{s}$. In other words, a modification of $\mathfrak{g}$ in $\mathfrak{p}$ is a subalgebra $\mathfrak{s} \subset \mathfrak{p}$ of the form

$$\mathfrak{s} := \{X + \sigma(X) : \ X \in \mathfrak{g}\},$$

for some $\sigma : \mathfrak{g} \to \mathfrak{q}$ linear, endowed with the polarization

$$\mathfrak{s}_{-1} := \{X + \sigma(X) : \ X \in \mathfrak{g}_{-1}\}.$$

Notice that $\mathfrak{s}_{-1}$ bracket generates $\mathfrak{s}$. Indeed, on the one hand, $\mathfrak{s}$ has the same dimension as $\mathfrak{g}$. On the other hand, one can easily check that, for iterated brackets of length $k \geq 0$, we have

$$\left([\mathfrak{s}_{-1}, \dots [\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] \dots] \bmod \bigoplus_{j \geq -k} \mathfrak{g}_j \right) = \left([\mathfrak{g}_{-1}, \dots [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \dots] \bmod \bigoplus_{j \geq -k} \mathfrak{g}_j \right).$$

If S is the connected Lie subgroup of P with $T_e S = \mathfrak{s}$, we call the pair (S, Δ_S) *modification of* G *in* P, where $(\Delta_S)_e = \mathfrak{s}_{-1}$. If $(\mathfrak{s}', \mathfrak{s}'_{-1})$ is a polarized Lie algebra that is isomorphic to a modification of $\mathfrak{g}$ in $\mathfrak{p}$, then we just say that $\mathfrak{s}$ is *a modification of* $\mathfrak{g}$. Similarly, *a modification of* G is any polarized group (S, Δ_S) whose Lie algebra is a modification of $\mathfrak{g}$.

Lemma 4.1 *Let S be a modification of G in P. The restriction $\pi|_S : S \to M$ is a distribution-preserving diffeomorphism when restricted from a neighbourhood of e_S to one of $\pi(e_S)$.*

Proof We denote by o the base point $\pi(e_S)$ in M. Observe that $d(\pi|_S)_{e_S} : T_{e_S} S = \mathfrak{s} \to T_o M$ is the restriction to $\mathfrak{s}$ of $d\pi_{e_P} : \mathfrak{p} \to T_o M$. Since the kernel of $d\pi_{e_P}$ is $\mathfrak{q}$, and since $\mathfrak{q} \cap \mathfrak{s} = \{0\}$, $d(\pi|_S)_{e_S}$ is injective. Moreover, $\dim \mathfrak{s} = \dim \mathfrak{g} = \dim M$, so that $d(\pi|_S)_{e_S}$ is a linear isomorphism. In particular, $\pi|_S$ is a diffeomorphism between two open neighbourhoods of e_S and o, respectively. Finally, on the one hand

$$d(\pi|_S)(\Delta_S) = d\pi(\Delta_P \cap TS) \subseteq \Delta_M,$$

while on the other hand $\dim(\Delta_S)_s = \dim(\mathfrak{g}_{-1}) = \dim(\Delta_M)_{\pi(s)}$ for all $s \in S$. So, at all points s where $d(\pi|_S)_s$ is injective we have $d(\pi|_S)_s(\Delta_S)_s = (\Delta_M)_{\pi(s)}$. □

By Lemma 4.1, both maps

$$\psi_S^G = \pi|_S^{-1} \circ \pi|_G : U_G \to U_S \qquad \psi_G^S = \pi|_G^{-1} \circ \pi|_S : U_S \to U_G$$

are distribution-preserving diffeomorphisms between a neighborhood U_G of e_G in G and a neighborhood U_S of e_S in S. One can also easily prove that the differential $d\psi_S^G|_{e_G} : \mathfrak{g} \to \mathfrak{s}$ is the map $X \mapsto X + \sigma(X)$.

Using the maps ψ_S^G and ψ_G^S, the following theorem is a direct consequence of Lemma 4.1.

Theorem 4.2 *Modifications of a stratified Lie group G are equivalent to G.*

Remark 4.3 If we are given G and S in P (for instance as matrix groups), then for any $s \in S$ the image $\psi_G^S(s)$ is the only element g of G, if it exists, such that $(sQ) \cap G = \{g\}$. Such an element is unique because $\pi : G \to P/Q$ is injective.

The following theorem is a converse of Theorem 4.2 in the rigid case.

Theorem 4.4 *Suppose that G is a rigid stratified group and that (S, Δ_S) is a polarized Lie group that is equivalent to G. Then (S, Δ_S) is a modification of G.*

Proof Let $\mathfrak{p}$ be the full Tanaka prolongation of $\mathfrak{g}$. Let $\psi : U_S \to U_G$ be a distribution-preserving diffeomorphism from an open subset $U_S \subset S$ to $U_G \subset G$. Up to composing ψ with left translations on S and on G, we may assume $\psi(e_S) = e_G$.

Let $\mathfrak{s}^\dagger \subset \Gamma(TS)$ be the Lie algebra of right-invariant vector fields on S. Since $\mathfrak{s}^\dagger$ is made of contact vector fields on S and since the Tanaka prolongation of $\mathfrak{g}$ coincides canonically with the Lie algebra of germs of contact vector fields on G, $\psi_* : \Gamma(TU_S) \to \Gamma(TU_G)$ gives an injective Lie algebra morphism $\psi_* : \mathfrak{s}^\dagger \hookrightarrow \mathfrak{p}$.

Notice that if $X \in \mathfrak{s}^\dagger$ is such that $\psi_* X(e_G) = 0$, then $X = 0$. Therefore, $\psi_*(\mathfrak{s}^\dagger) \cap \mathfrak{q} = \{0\}$. Since S and G have the same dimension, we obtain that

$$\psi_*(\mathfrak{s}^\dagger) = \{X + \sigma X : X \in \mathfrak{g}\}$$

for some linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$.

Finally, since $d\psi((\Delta_S)_{e_S}) = (\Delta_G)_{e_G}$, we obtain that

$$\psi_*\{X \in \mathfrak{s}^\dagger : X(e_G) \in (\Delta_S)_{e_S}\} = \{X + \sigma X : X \in \mathfrak{g}_{-1}\}.$$

We conclude that $(\psi_*(\mathfrak{s}^\dagger), d\psi((\Delta_S)_{e_S}))$ is a modification of $\mathfrak{g}$ in $\mathfrak{p}$. □

Remark 4.5 In the case G is not rigid, i.e., the full Tanaka prolongation of $\mathfrak{g}$ is infinite dimensional, then the argument in the proof of Theorem 4.4 does not work. However, the example of the Heisenberg group, which is not rigid, shows that it may still be possible to obtain as modifications all Lie groups that are equivalent to G. See Sect. 5.1.

In the rigid case, isomorphisms of modifications are all elements of $\mathtt{Aut}(\mathfrak{p}, \mathfrak{g})$:

Theorem 4.6 *Suppose* $\mathfrak{g}$ *is rigid. If* $\mathfrak{s}, \mathfrak{s}'$ *are two modifications of* $\mathfrak{g}$ *in* $\mathfrak{p}$*, and if there is an isomorphism* $\phi : \mathfrak{s} \to \mathfrak{s}'$ *such that* $\phi(\mathfrak{s} \cap (\mathfrak{g}_{-1} \oplus \mathfrak{q})) = \mathfrak{s}' \cap (\mathfrak{g}_{-1} \oplus \mathfrak{q})$*, then there is a unique* $\ell \in \mathtt{Aut}(\mathfrak{p}, \mathfrak{g})$ *such that* $\phi = \ell|_{\mathfrak{s}}$.

Proof. Let $S, S' < P$ be the subgroups of P whose Lie algebra are $\mathfrak{s}$ and $\mathfrak{s}'$ respectively, endowed with the polarizations induced by P, e.g., $\Delta_S = \Delta_P \cap TS$. The map ϕ defines a local distribution-preserving diffeomorphism $\Phi : \Omega \to \Phi(\Omega)$, $\Omega \subset S$ open with $e \in \Omega$. We may assume that $\pi|_\Omega : \Omega \to \pi(\Omega) \subset M$ and $\pi|_{\Phi\Omega} : \Phi(\Omega) \to \pi(\Phi\Omega) \subset M$ are distribution-preserving diffeomorphisms, see Lemma 4.1. Define $U := \pi(\Omega)$ and $f := \pi \circ \Phi \circ \pi|_\Omega^{-1} : U \to f(U) = \pi(\Phi\Omega)$. The map f is then a distribution-preserving diffeomorphism. By Theorem 3.3, there is a Lie group automorphism $L : P \to P$ such that $f(\pi(p)) = \pi(L(p))$ for all $p \in \pi^{-1}(U)$. Now, we claim that the map $L_* : \mathfrak{p} \to \mathfrak{p}$ restricted to $\mathfrak{s}$ is equal to ϕ. Indeed, if $X \in \mathfrak{s}$, then

$$\begin{aligned} L_*[X] &= L_*[X^\dagger]\big|_e = (\pi_*|_{\mathfrak{p}^\dagger})^{-1} \circ \pi_* \circ L_* \circ (\pi_*|_{\mathfrak{p}^\dagger})^{-1} \circ \pi_*[X^\dagger]\big|_e \\ &= (\pi_*|_{\mathfrak{p}^\dagger})^{-1} \circ f_* \circ \pi_*[X^\dagger]\big|_e = (\pi_*|_{\mathfrak{p}^\dagger})^{-1} \circ (\pi \circ \Phi \circ \pi|_\Omega^{-1})_* \circ \pi_*[X^\dagger]\Big|_e \\ &= \Phi_*[X^\dagger|_S]\big|_e = \phi[X]. \end{aligned}$$

□

5 Examples

In this section we consider a few applications of our main results. First, we observe that every three-dimensional polarized Lie group is equivalent either to $\mathbb{R}^3$ or to the Heisenberg group. Although this is of course a consequence of the more general Darboux Theorem, we believe it is a good example for presenting our techniques. Second, we study some modifications of the free nilpotent Lie algebra $\mathfrak{f}_{24}$. In this case we are able to find a nilpotent modification (N, Δ_N) of the stratified group F_{24} corresponding to $\mathfrak{f}_{24}$ that, although not isomorphic, admits a global distribution-preserving diffeomorphism to F_{24}. In particular, if we endow N and F_{24} with left-invariant sub-Riemannian distances, our example shows two nilpotent Lie groups that are bi-Lipschitz on every compact set but not isomorphic.

5.1 Modifications of the Heisenberg group

We study the consequences of the results of the previous section in the case where $\mathfrak{g}$ is the three-dimensional Heisenberg algebra. It is well known that the full Tanaka prolongation of the Heisenberg Lie algebra $\mathfrak{g}$ is infinite. However, there is a number a different choices of subalgebras $\mathfrak{g}_0 \subset \mathrm{Der}(\mathfrak{g})$ that generate finite dimensional prolongations. We shall show that these finite prolongations are enough to recover all polarized Lie groups that are equivalent to the Heisenberg group, i.e., all three dimensional Lie groups with a non-trivial polarization:

Theorem 5.1 *Let* $(\mathfrak{s}, \mathfrak{s}_{-1})$ *be a three dimensional polarised Lie algebra such that* $\dim(\mathfrak{s}_{-1}) = 2$*. Then there is a finite-dimensional prolongation* $\mathfrak{p}$ *of the Heisenberg Lie algebra* $\mathfrak{h}$ *so that* $(\mathfrak{s}, \mathfrak{s}_{-1})$ *is isomorphic to a modification in* $\mathfrak{p}$.

Our study is based on a classification of three-dimensional Lie algebras due to several authors. We summarise the results we need in the following theorem.

Proposition 5.2 *Let* $(\mathfrak{s}, \mathfrak{s}_{-1})$ *be a three-dimensional polarised Lie algebra such that* $\dim(\mathfrak{s}_{-1}) = 2$*. Then there is a basis* (f_1, f_2, f_3) *of* $\mathfrak{s}$ *with* $\mathfrak{s}_{-1} = \mathrm{span}\{f_1, f_2\}$ *such that exactly one of the following cases occurs:*

(A) $[f_1, f_2] = f_3$, $[f_1, f_3] = \alpha f_2 + \beta f_3$ *and* $[f_2, f_3] = 0$, *for some* $\alpha \in \mathbb{R}$ *and* $\beta \in \{0, 1\}$. *In this case* $\mathfrak{s}$ *is solvable and the non-isomorphic cases are exactly the following four:*

- *(A.1)* $[f_1, f_2] = f_3$, $[f_1, f_3] = 0$ *and* $[f_2, f_3] = 0$*;*
- *(A.2)* $[f_1, f_2] = f_3$, $[f_1, f_3] = f_2$ *and* $[f_2, f_3] = 0$*;*
- *(A.3)* $[f_1, f_2] = f_3$, $[f_1, f_3] = -f_2$ *and* $[f_2, f_3] = 0$*;*
- *(A.4)* $[f_1, f_2] = f_3$, $[f_1, f_3] = \alpha f_2 + f_3$ *and* $[f_2, f_3] = 0$.

(B) $[f_1, f_2] = f_3$, $[f_1, f_3] = -f_2$, $[f_2, f_3] = f_1$. *In this case* $\mathfrak{s} = \mathfrak{su}(2)$ *is simple.*
(C) $[f_1, f_2] = f_3$, $[f_1, f_3] = -f_1$, $[f_2, f_3] = f_2$. *In this case* $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ *is simple.*
(D) $[f_1, f_2] = f_3$, $[f_1, f_3] = f_2$, $[f_2, f_3] = -f_1$. *In this case* $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ *is simple.*

Part of the proof of Proposition 5.2 is based on the following lemma, see [2]. Proposition 5.2 is also a consequence of Winternitz classification [16].

Lemma 5.3 (Baudoin–Cecil) *Let S be a three-dimensional solvable Lie group endowed with a left-invariant sub-Riemannian structure (Δ_S, g). There exist vectors e_1, e_2, e_3 linearly independent in $\mathfrak{s}$, $\alpha \in \mathbb{R}$ and $\beta \geq 0$ such that e_1, e_2 is an orthonormal basis of $(\Delta_S)_e$ and*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = \alpha e_2 + \beta e_3, \quad [e_2, e_3] = 0. \tag{5}$$

Proof of Proposition 5.2 If $\mathfrak{s}$ is a three dimensional Lie algebra, then it is either solvable or simple. Indeed, the claim follows from the Levi decomposition and the fact that there are no simple Lie groups of dimension 1 or 2. If $\mathfrak{s}$ is simple, then the $(\mathfrak{s}, \mathfrak{s}_{-1})$ falls into one the cases (B), (C) or (D), see [1]. Notice that the cases (C) and (D) are not isomorphic as polarised Lie algebras because ad_{f_3} is a reflexion of $\mathfrak{s}_{-1}$ in case (C), while in case (D) it is a rotation.

If $\mathfrak{s}$ is solvable, then we apply Lemma 5.3 and obtain case (A). However, since the classification in Lemma 5.3 is up to isometry, we have to further discriminate to obtain non-isomorphic subcases. So, if f_1, f_2, f_3 is a basis of $\mathfrak{s}$ with $\mathfrak{s}_{-1} = \mathrm{span}\{f_1, f_2\}$, $[f_1, f_2] = f_3$ and $[f_2, f_3] = 0$, then we must have

$$\begin{cases} f_1 &= a_1^1 e_1 + a_1^2 e_2 \\ f_2 &= a_2^2 e_2 \\ f_3 &= a_1^1 a_2^2 e_3, \end{cases}$$

for some real coefficients. The third bracket relation is

$$[f_1, f_3] = \alpha (a_1^1)^2 f_2 + \beta a_1^1 f_3.$$

Since $\alpha \in \mathbb{R}$ and $\beta \geq 0$, in each case we can choose a_i^j in the following way:

$$\begin{array}{lll} \alpha = \beta = 0 & a_1^1 = 1,\ a_1^2 = 0,\ a_2^2 = 1: & [f_1, f_3] = 0 \\ \beta > 0,\ \alpha \in \mathbb{R} & a_1^1 = \dfrac{1}{\beta},\ a_1^2 = 0,\ a_2^2 = 1: & [f_1, f_3] = \dfrac{\alpha}{\beta^2} f_2 + f_3 \\ \beta = 0,\ \alpha > 0 & a_1^1 = \dfrac{1}{\sqrt{\alpha}},\ a_1^2 = 0,\ a_2^2 = 1: & [f_1, f_3] = f_2 \\ \beta = 0,\ \alpha < 0 & a_1^1 = \dfrac{1}{\sqrt{|\alpha|}},\ a_1^2 = 0,\ a_2^2 = 1: & [f_1, f_3] = -f_2 \end{array}$$

Now, we want to show that cases (A.1), (A.2), (A.3) and (A.4) are not isomorphic to each other. Notice that $\ell := \mathrm{span}\{f_3\} = [\mathfrak{s}_{-1}, \mathfrak{s}_{-1}]$ and $\mathfrak{s}^{(2)} := [\mathfrak{s}, \mathfrak{s}]$ are invariant under isomorphisms of polarised Lie algebras.

First, case (A.1) is not isomorphic to the others because in case (A.1) we have $\mathfrak{s}^{(2)} = \operatorname{span}\{f_3\}$ while in all other three cases we have $\mathfrak{s}^{(2)} = \operatorname{span}\{f_2, f_3\}$.

Second, case (A.4) is not isomorphic to the others because in case (A.4) we have $[\ell, \mathfrak{s}_{-1}] \not\subset \mathfrak{s}_{-1}$ while in all other cases we have $[\ell, \mathfrak{s}_{-1}] \subset \mathfrak{s}_{-1}$.

Third, for different choices of $\alpha \in \mathbb{R}$ in case (A.4) we get non-isomorphic polarised Lie algebras: To prove this, we shall show that the parameter α is independent of the choice of the basis. So, suppose that $g_1, g_2, g_3 \in \mathfrak{s}$ form another basis with $\mathfrak{s}_{-1} = \operatorname{span}\{g_1, g_2\}$, $[g_1, g_2] = g_3$, $[g_2, g_3] = 0$ and $[g_1, g_3] = \alpha' g_2 + g_3$. Then one easily shows that $g_1 = x f_1 + y f_2$, $g_2 = \mu f_2$ and $g_3 = \lambda f_3$, for some $x, y, \lambda, \mu \in \mathbb{R}$ with $\frac{x\mu}{\lambda} = 1$. Moreover, $[g_1, g_3] = \alpha \frac{x\mu}{\lambda} g_2 + x g_3$, which implies $x = 1$ and $\alpha = \alpha'$.

Finally, cases (A.2) and (A.3) are not isomorphic to each other, because in case (A.2) it holds $\mathrm{ad}_{f_1}|^2_{\mathfrak{s}^{(2)}} = \mathrm{Id}|_{\mathfrak{s}^{(2)}}$, while while in case (A.3) it holds $\mathrm{ad}_{f_1}|^2_{\mathfrak{s}^{(2)}} = -\mathrm{Id}|_{\mathfrak{s}^{(2)}}$. □

Proof of Theorem 5.1 Let us fix the notation for the Heisenberg Lie algebra. Fix a basis e_1, e_2, e_3 so that $[e_1, e_2] = e_3$, and choose $\mathfrak{g}_{-1} = \operatorname{span}\{e_1, e_2\}$. The space $\operatorname{Der}(\mathfrak{g})$ of the strata preserving derivations of $\mathfrak{g}$ may be identified with $\mathfrak{gl}(2, \mathbb{R})$.

First, we consider

$$\mathfrak{g}_0 := \{D \in \operatorname{Der}(\mathfrak{g}) : \ D(e_1) \subseteq \mathbb{R}e_1 \text{ and } D(e_2) \subseteq \mathbb{R}e_2\}.$$

In this case, $\operatorname{Prol}(\mathfrak{g}, \mathfrak{g}_0) = \mathfrak{sl}(3, \mathbb{R}) = \mathfrak{g} \oplus \mathfrak{q}$ (see, e.g., [3]), where $\mathfrak{g}$ is identified with the Lie algebra generated by

$$e_1 = \begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}. \tag{6}$$

and $\mathfrak{q}$ is the set of matrices in $\mathfrak{sl}(3, \mathbb{R})$ of the form

$$\begin{pmatrix} *&0&0\\ *&*&0\\ *&*&* \end{pmatrix}$$

The modifications of $\mathfrak{g}$ in $\mathfrak{sl}(3, \mathbb{R})$ are the subalgebras of $\mathfrak{sl}(3, \mathbb{R})$ of the form $\{X + \sigma(X) : X \in \mathfrak{g}\}$, for some linear map $\sigma : \mathfrak{g} \to \mathfrak{q}$. We show that all three dimensional Lie algebras with a bracket generating plane are graphs of such a σ:

Case (A): If $\mathfrak{s}$ is solvable, then define σ by the assignments:

$$\sigma(e_1) = \begin{pmatrix} \frac{2\beta}{3} & 0 & 0\\ \alpha & -\frac{\beta}{3} & 0\\ 0 & 0 & -\frac{\beta}{3} \end{pmatrix}, \quad \sigma(e_2) = \sigma(e_3) = 0.$$

It is easy to check that vectors $f_i := e_i + \sigma(e_i)$, $i = 1, 2, 3$, satisfy the bracket relations of case (A) in Proposition 5.2.

Case (B):

For this case, we choose

$$\sigma(e_1) = \begin{pmatrix} 0&0&0\\ -1&0&0\\ 0&0&0 \end{pmatrix}, \quad \sigma(e_2) = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&-1&0 \end{pmatrix}, \quad \sigma(e_3) = \begin{pmatrix} 0&0&0\\ 0&0&0\\ -1&0&0 \end{pmatrix}.$$

Case (C): we obtain the brackets in (C) by choosing

$$\sigma(e_1) = 0, \quad \sigma(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma(e_3) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (D): In this case we use the finite prolongation $\mathfrak{su}(2, 1)$ of the Heisenberg algebra, as in [8, p313]. Let

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{su}(2, 1)$ is given by 3×3 complex matrices A with zero trace and such that $A^*J + JA = 0$, where A^* is the hermitian transpose of A. Define the Lie algebra automorphism $\theta : \mathfrak{su}(2, 1) \to \mathfrak{su}(2, 1)$, $\theta A := JAJ$. Define

$$X = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 2i & 0 & 2i \\ 0 & 0 & 0 \\ -2i & 0 & -2i \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad U = \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix}$$

$$\theta X = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & -i & 0 \end{pmatrix} \qquad \theta Y = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \theta Z = \begin{pmatrix} 2i & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & -2i \end{pmatrix}$$

The grading of $\mathfrak{su}(2, 1)$ is

$$\begin{aligned} \mathfrak{g}_{-2}(\mathfrak{g}) &= \operatorname{span}\{Z\} \\ \mathfrak{g}_{-1}(\mathfrak{g}) &= \operatorname{span}\{X, Y\} \\ \mathfrak{g}_{0}(\mathfrak{g}) &= \operatorname{span}\{H, U\} \\ \mathfrak{g}_{1}(\mathfrak{g}) &= \operatorname{span}\{\theta X, \theta Y\} \\ \mathfrak{g}_{2}(\mathfrak{g}) &= \operatorname{span}\{\theta Z\}, \end{aligned}$$

where $\mathfrak{g}_{-2}(\mathfrak{g}) \oplus \mathfrak{g}_{-2}(\mathfrak{g}) = \mathfrak{g}$ is the Heisenberg Lie algebra: notice that $[X, Y] = Z$ while $[X, Z] = [Y, Z] = 0$. So, $\mathfrak{q} = \operatorname{span}\{H, U, \theta X, \theta Y, \theta Z\}$. Define $\sigma : \mathfrak{g} \to \mathfrak{q}$ by setting

$$\sigma X := -\frac{1}{16}\theta X + i\frac{9}{16}\theta Y = \begin{pmatrix} 0 & -i\frac{1}{2} & 0 \\ -i\frac{5}{8} & 0 & i\frac{5}{8} \\ 0 & -i\frac{1}{2} & 0 \end{pmatrix},$$

$$\sigma Y := -i\frac{9}{16}\theta X - \frac{1}{16}\theta Y = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{5}{8} & 0 & -\frac{5}{8} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix},$$

$$\sigma Z := -i\frac{9}{4}H + \frac{1}{4}U - \frac{5}{16}\theta Z = \begin{pmatrix} -i\frac{3}{8} & 0 & -i\frac{13}{8} \\ 0 & -i\frac{1}{2} & 0 \\ -i\frac{23}{8} & 0 & i\frac{7}{8} \end{pmatrix}.$$

One can easily check that $f_1 = X + \sigma X$, $f_2 = Y + \sigma Y$ and $f_3 = Z + \sigma Z$ form a basis of a Lie subalgebra of $\mathfrak{su}(2, 1)$ satisfying the relations of Case (D). $\square$

Remark 5.4 The map σ above can easily be found using the software Maple and it is not unique.

5.1.1 Rigid motions of the plane as a modification of the Heisenberg group

We conclude this section discussing more in detail the case of the group of rigid motions of the plane as a modification of the Heisenberg group. At a group level, we may represent points in the Heisenberg group $\mathbb{H}$ as matrices in $SL(3,\mathbb{R})$ by

$$H(x_1,x_2,x_3) := \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

for $x_1, x_2, x_3 \in \mathbb{R}$.

The Lie algebra of the the group of rigid motions of the plane $E(2)$ corresponds to the case (A) with $\alpha = -1$ and $\beta = 0$. The corresponding representation in $\mathfrak{sl}(3,\mathbb{R})$ given in the previous theorem is the span of the vectors

$$f_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At the group level, the points of $E(2)$ inside $SL(3,\mathbb{R})$ are parametrized by

$$R(y_1,y_2,y_3) := \begin{pmatrix} \cos y_1 & \sin y_1 & y_3 \\ -\sin y_1 & \cos y_1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $y_1 \in \mathbb{R}/(2\pi\mathbb{Z})$ and $y_2, y_3 \in \mathbb{R}$.

With the procedure described in Remark 4.3, we find the mapping $E(2) \to \mathbb{H}$:

$$R(y_1,y_2,y_3) \mapsto H(\tan y_1, y_2, y_3),$$

which is defined on the domain $(-\pi/2, \pi/2) \times \mathbb{R}^2$.

5.2 Modifications of the free nilpotent Lie group F_{24}

We consider the free nilpotent Lie algebra $\mathfrak{f}_{24} = \operatorname{span}\{e_i \,:\, i = 1,\dots,8\}$ of rank 2 and step 4 and the corresponding simply connected Lie group $F_{2,4}$. We will prove the following result

Theorem 5.5 *There exists a nilpotent Lie group S, not isomorphic to $F_{2,4}$, that is a modification of $F_{2,4}$ and is globally equivalent to $F_{2,4}$.*

Proof The Lie brackets in $\mathfrak{f}_{24}$ are

$$\begin{aligned} &[e_2,e_1] = e_3, \quad [e_3,e_1] = e_4, \quad [e_3,e_2] = e_5, \\ &[e_4,e_1] = e_6, \quad [e_5,e_1] = e_7, \quad [e_4,e_2] = e_7, \quad [e_5,e_2] = e_8. \end{aligned}$$

It is known that the full Tanaka prolongation of $\mathfrak{f}_{24}$ is $\mathfrak{p} = \mathfrak{f}_{24} \oplus \mathrm{Der}(\mathfrak{g})$, with $\mathrm{Der}(\mathfrak{g}) \simeq \mathfrak{gl}(2,\mathbb{R})$ (see [17]). Therefore, the modifications of $\mathfrak{f}_{24}$ are subalgebras of $\mathfrak{p}$ that are graphs of some linear map $\sigma : \mathfrak{f}_{24} \to \mathfrak{gl}(2, \mathbb{R})$. Here we only consider σ that on the basis of $\mathfrak{f}_{24}$ is zero except for $\sigma(e_1)$. Imposing that the graph is a Lie algebra, a direct computation shows that

$$\sigma(e_1) = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. We obtain a three parameter family $\mathfrak{s}(a, b, c)$ of Lie algebras with basis $f_1, \dots, f_8$, where $f_1 = e_1 + \sigma(e_1)$ and $f_i = e_i$ for $i = 2, \dots, 8$, and brackets

$$\begin{aligned}
&[f_2, f_1] = f_3 - bf_2, \quad [f_3, f_1] = f_4 - (a+b)f_3, \quad [f_3, f_2] = f_5,\\
&[f_4, f_1] = f_6 - cf_5 - (2a+b)f_4, \quad [f_4, f_2] = f_7, \quad [f_5, f_1] = f_7 - (a+2b)f_5,\\
&[f_1, f_6] = 2cf_7 + (3a+b)f_6, \quad [f_1, f_7] = cf_8 + 2(a+b)f_7,\\
&[f_1, f_8] = (a+3b)f_8, \quad [f_5, f_2] = f_8.
\end{aligned}$$

In particular, setting $a = b = 0$ gives a one parameter family of nilpotent Lie algebras $\mathfrak{s}(c)$. We now find the distribution-preserving diffeomorphism Ψ from $S(c)$ to F_{24} when $c = 1$, as in Remark 4.3. Every point in $S(1)$ is of the form $\exp_P(\sum x_i f_i)$. Following [12], $\exp_P(\sum x_i f_i) = (\mathcal{E}_{F_{24}}(x_1\sigma(e_1); \sum x_i e_i), \exp_{GL}(x_1\sigma(e_1)) \in F_{24} \rtimes GL(2, \mathbb{R})$, where $\mathcal{E}_{F_{24}}(x_1 e_1; \sum x_i e_i) = \gamma(1)$ and $\gamma : [0, 1] \to F_{24}$ is the solution of

$$\begin{cases} \gamma'(t) &= dL_{\gamma(t)} \exp_{GL}(tx_1\sigma(e_1))(\sum x_i e_i) \\ \gamma(0) &= e_{F_{24}}. \end{cases}$$

The image of this point via Ψ is going to be that element $p \in F_{24}$ such that $gQ = \exp_P(\sum x_i f_i)Q$, i.e.,

$$\Psi\left(\exp_P\left(\sum x_i f_i\right)\right) = \mathcal{E}_{F_{24}}(x_1 e_1; \sum x_i e_i).$$

To compute this, we first observe that

$$\begin{aligned}
v &:= \exp_{GL}(tx_1\sigma(e_1))\left(\sum x_i e_i\right)\\
&= \left(x_1, x_1^2 t + x_2, x_3, x_4, x_5 + tx_1x_4, x_6, x_7 + 2tx_1x_6, x_8 + tx_1x_7 + t^2x_1^2x_6\right).
\end{aligned}$$

Second, we need to compute $dL_\gamma v$ using the Baker–Campbell–Hausdorff formula:

$$dL_\gamma v = \left.\frac{d}{dh}\right|_{h=0} \exp^{-1}(\exp(\gamma)\exp(hv)) = v + \frac{1}{2}[\gamma, v] + \frac{1}{12}[\gamma, [\gamma, v]].$$

The system of differential equations $\dot{\gamma} = dL_\gamma v$ that we obtain is

$$
\begin{aligned}
\dot{\gamma}_1 &= x_1\\
\dot{\gamma}_2 &= tx_1^2 + x_2\\
\dot{\gamma}_3 &= -\frac{1}{2}tx_1^2\gamma_1 - \frac{1}{2}x_2\gamma_1 + \frac{1}{2}x_1\gamma_2 + x_3\\
\dot{\gamma}_4 &= \frac{1}{12}tx_1^2\gamma_1^2 + \frac{1}{12}x_2\gamma_1^2 - \frac{1}{12}(\gamma_1\gamma_2 - 6\gamma_3)x_1 - \frac{1}{2}x_3\gamma_1 + x_4\\
\dot{\gamma}_5 &= \frac{1}{12}x_2\gamma_1\gamma_2 - \frac{1}{12}x_1\gamma_2^2 + \frac{1}{12}\big(x_1^2\gamma_1\gamma_2 + 6x_1^2\gamma_3 + 12x_1x_4\big)t - \frac{1}{2}x_3\gamma_2\\
&\quad + \frac{1}{2}x_2\gamma_3 + x_5\\
\dot{\gamma}_6 &= \frac{1}{12}x_3\gamma_1^2 - \frac{1}{12}(\gamma_1\gamma_3 - 6\gamma_4)x_1 - \frac{1}{2}x_4\gamma_1 + x_6\\
\dot{\gamma}_7 &= \frac{1}{6}x_3\gamma_1\gamma_2 - \frac{1}{12}x_2\gamma_1\gamma_3 - \frac{1}{12}\left(x_1^2\gamma_1\gamma_3 + 6x_1x_4\gamma_1 - 6x_1^2\gamma_4 - 24x_1x_6\right)t\\
&\quad - \frac{1}{12}\left(\gamma_2\gamma_3 - 6\gamma_5\right)x_1 - \frac{1}{2}x_5\gamma_1 - \frac{1}{2}x_4\gamma_2 + \frac{1}{2}x_2\gamma_4 + x_7\\
\dot{\gamma}_8 &= t^2x_1^2x_6 + \frac{1}{12}x_3\gamma_2^2 - \frac{1}{12}x_2\gamma_2\gamma_3 - \frac{1}{12}\left(x_1^2\gamma_2\gamma_3 + 6x_1x_4\gamma_2 - 6x_1^2\gamma_5 - 12x_1x_7\right)t\\
&\quad - \frac{1}{2}x_5\gamma_2 + \frac{1}{2}x_2\gamma_5 + x_8.
\end{aligned}
$$

Third, we need to integrate this system of ODEs with initial conditions $\gamma_i(0) = 0$ for every $i = 1, \dots, 8$. The solution is

$$
\begin{aligned}
\gamma_1(t) &= tx_1\\
\gamma_2(t) &= \frac{1}{2}t^2x_1^2 + tx_2\\
\gamma_3(t) &= -\frac{1}{12}t^3x_1^3 + tx_3\\
\gamma_4(t) &= tx_4\\
\gamma_5(t) &= -\frac{1}{240}t^5x_1^5 + \frac{1}{12}t^3x_1^2x_3 + \frac{1}{2}t^2x_1x_4 + tx_5\\
\gamma_6(t) &= \frac{1}{720}t^5x_1^5 + tx_6\\
\gamma_7(t) &= \frac{1}{720}t^6x_1^6 + \frac{1}{360}t^5x_1^4x_2 + t^2x_1x_6 + tx_7\\
\gamma_8(t) &= \frac{1}{5040}t^7x_1^7 + \frac{1}{720}t^6x_1^5x_2 + \frac{1}{720}\left(x_1^3x_2^2 + 3x_1^4x_3\right)t^5\\
&\quad - \frac{1}{12}\left(x_1x_2x_4 - x_1^2x_5 - 4x_1^2x_6\right)t^3 + \frac{1}{2}t^2x_1x_7 + tx_8.
\end{aligned}
$$

Therefore, the mapping from $S(1)$ to G is $\Psi : \exp_P(\sum x_i f_i) \mapsto \gamma(1)$, which is a global, surjective smooth distribution-preserving diffeomorphism.

Finally, $S(1)$ is not isomorphic to $F_{2,4}$ because $S(1)$ has nilpotency step 5 instead of 4, as one can easily see from the expression of the Lie brackets in $\mathfrak{s}(1)$. □

Remark 5.6 The mapping from $S(1)$ to G described above is in particular bi-Lipschitz on every compact set, when the groups are endowed with left-invariant sub-Riemannian distances. Notice, however, that this is not a global quasiconformal mapping.

5.3 Modifications of ultra-rigid stratified groups

A stratified Lie algebra $\mathfrak{g}$ is called *ultra-rigid* if the only automorphisms of $\mathfrak{g}$ preserving the stratifications are dilations, see [10]. In particular, the full Tanaka prolongation of such $\mathfrak{g}$ is $\mathfrak{p} = \mathfrak{g} \rtimes \mathbb{R}$, as semi-direct product of Lie algebras. In this section we describe all modifications in $\mathfrak{g} \rtimes \mathbb{R}$ and their equivalence relation. Many results do not need the assumption of $\mathfrak{g}$ being ultra-rigid, so we assume this hypothesis only when needed.

Let $\mathfrak{g} = \bigoplus_{j=-s}^{-1} \mathfrak{g}_j$ be a stratified Lie algebra. Let $D : \mathfrak{g} \to \mathfrak{g}$ be the linear map with $Dv = jv$ for $v \in \mathfrak{g}_{-j}$. Notice that D is a derivation of $\mathfrak{g}$ that preserves the layers and that $\delta_{e^t} = e^{tD} : \mathfrak{g} \to \mathfrak{g}$ are the dilations.

The semi-direct product $\mathfrak{p} := \mathfrak{g} \rtimes \mathbb{R}$ is the Lie algebra whose Lie brackets are

$$[(0, a), (Y, 0)] = (aDY, 0) \quad \text{hence} \quad [(X, a), (Y, b)] = ([X, Y] + aDY - bDX, 0).$$

Proposition 5.7 *Let $\sigma : \mathfrak{g} \to \mathbb{R}$ be a linear map and set $\mathfrak{s} := \{(X, \sigma X) : X \in \mathfrak{g}\}$. The vector space $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g} \rtimes \mathbb{R}$ if and only if $\bigoplus_{j=-2}^{-s} \mathfrak{g}_j \subset \ker \sigma$.*

Proof First, we note that $\mathfrak{s}$ is a Lie algebra if and only if, for all $X, Y \in \mathfrak{g}$,

$$\sigma([X, Y]) + (\sigma X)(\sigma DY) - (\sigma Y)(\sigma DX) = 0. \tag{7}$$

$\boxed{\Rightarrow}$ Suppose $\mathfrak{s}$ is a Lie algebra, i.e., (7) holds for all $X, Y \in \mathfrak{g}$. We prove $\bigoplus_{j=-2}^{-s} \mathfrak{g}_j \subset \ker \sigma$ by induction on j. If $X, Y \in \mathfrak{g}_{-1}$, then $DX = X$ and $DY = Y$, thus (7) implies $\sigma([X, Y]) = 0$. Since $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, it follows that $\mathfrak{g}_{-2} \subset \ker \sigma$. Now, suppose that $\mathfrak{g}_{-k} \subset \ker \sigma$ for $k \geq 2$. If $X \in \mathfrak{g}_{-1}$ and $Y \in \mathfrak{g}_{-k}$, then (7) implies that $\sigma([X, Y]) = 0$. Since $\mathfrak{g}_{-k-1} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-k}]$, it follows that $\mathfrak{g}_{-k-1} \subset \ker \sigma$. We conclude that $\bigoplus_{j=-2}^{-s} \mathfrak{g}_j \subset \ker \sigma$.

$\boxed{\Leftarrow}$ Suppose $\bigoplus_{j=-2}^{-s} \mathfrak{g}_j \subset \ker \sigma$. By the bilinearity of the expression, we need to show that (7) holds only when $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ for some i and j. Since σ is non-zero only on the first layer, the only non-trivial instance of (7) is for $X, Y \in \mathfrak{g}_{-1}$. In this case, $\sigma([X, Y]) = 0$, and $(\sigma X)(\sigma DY) - (\sigma Y)(\sigma DX) = (\sigma X)(\sigma Y) - (\sigma Y)(\sigma X) = 0$. Therefore, (7) is satisfied and $\mathfrak{s}$ is a Lie algebra. □

Lemma 5.8 *The Lie algebra automorphisms $\phi : \mathfrak{p} \to \mathfrak{p}$ such that $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(\mathfrak{g}_{-1} \times \mathbb{R}) = \mathfrak{g}_{-1} \times \mathbb{R}$ are exactly those of the form $\phi(X, a) = (\phi_1 X, a)$ for some Lie algebra automorphism $\phi_1 : \mathfrak{g} \to \mathfrak{g}$ that preserves the layers.*

Proof On the one hand, if $\phi_1 : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism that preserves the layers, then $\phi(X, a) = (\phi_1 X, a)$ is clearly a Lie algebra automorphism $\phi : \mathfrak{p} \to \mathfrak{p}$ with $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(\mathfrak{g}_{-1} \times \mathbb{R}) = \mathfrak{g}_{-1} \times \mathbb{R}$, because $\phi_1 D = D\phi_1$.

On the other hand, if $\phi : \mathfrak{p} \to \mathfrak{p}$ is a Lie algebra automorphism, then $\phi(\mathfrak{g} \times \{0\}) = \mathfrak{g} \times \{0\}$ because $\mathfrak{g} \times \{0\} = [\mathfrak{p}, \mathfrak{p}]$. Suppose also that $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(\mathfrak{g}_{-1} \times \mathbb{R}) = \mathfrak{g}_{-1} \times \mathbb{R}$. Then $\phi(X, a) = \phi(X, 0) + \phi(0, a) = (\phi_1(X), 0) + (0, \phi_2(a))$ and $\phi_1(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$. This implies that $\phi_1(\mathfrak{g}_j) = \mathfrak{g}_j$ for all j, as one can prove by induction on j. Notice that, for all $X \in \mathfrak{g}$ and all $a \in \mathbb{R}$,

$$\phi_2(a) D\phi_1 X = [(0, \phi_2(a)), (\phi_1 X, 0)] = \phi([(0, a), (X, 0)]) = \phi(aDX, 0) = a\phi_1 DX.$$

For every $X \in \mathfrak{g}_{-1}$, $DX = X$ and $D\phi_1 X = \phi_1 X$, hence $\phi_2(a)\phi_1 X = a\phi_1 X$, i.e., $\phi_2(a) = a$. □

Theorem 5.9 *Suppose that $\mathfrak{g}$ is ultra-rigid, i.e., $\mathfrak{p} = \mathfrak{g} \rtimes \mathbb{R}$ is its full Tanaka prolongation. The set of all non-isomorphic modifications of $\mathfrak{g}$ is parametrized by $\mathfrak{g}_{-1}^*/\mathbb{R}_{>0}$. Moreover, all modifications of $\mathfrak{g}$ in $\mathfrak{p}$ are solvable and the only nilpotent one is $\mathfrak{g}$ itself.*

Proof The set of all modifications of $\mathfrak{g}$ in $\mathfrak{p}$ can be identified with $\mathfrak{g}_{-1}^*$ by Proposition 5.7, where $\sigma \in \mathfrak{g}_{-1}^*$ is identified with $\sigma(\sum_j v_j) = \sigma(v_{-1})$ for $\sum_j v_j \in \mathfrak{g}$ and the modification $\mathfrak{s}_\sigma := \{(X, \sigma X) : X \in \mathfrak{g}\} \subset \mathfrak{p}$. Since $\mathfrak{g}$ is rigid, by Theorem 4.6 two modifications $\sigma, \tau \in \mathfrak{g}_{-1}^*$ are isomorphic if and only if there is a Lie algebra automorphism $\phi : \mathfrak{p} \to \mathfrak{p}$ with $\phi(\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R}$ and $\phi(\mathfrak{g}_{-1} \times \mathbb{R}) = \mathfrak{g}_{-1} \times \mathbb{R}$ such that $\phi(\mathfrak{s}_\sigma) = \mathfrak{s}_\tau$. Therefore, by Lemma 5.8, two modifications $\sigma, \tau \in \mathfrak{g}_{-1}^*$ are isomorphic if and only if there is a Lie algebra automorphism $\phi_1 : \mathfrak{g} \to \mathfrak{g}$ such that, for all $X \in \mathfrak{g}$,

$$(\phi_1 X, \sigma X) = (\phi_1 X, \tau\phi_1 X),$$

i.e., $\sigma X = \tau\phi_1 X$ for all $X \in \mathfrak{g}_{-1}$. Now, since $\mathfrak{g}$ is ultrarigid, $\phi_1 = \delta_\lambda$ for some $\lambda > 0$. Therefore, two modifications $\sigma, \tau \in \mathfrak{g}_{-1}^*$ are isomorphic if and only if there is $\lambda > 0$ such that $\sigma = \lambda\tau$.

Finally, notice that all modifications of $\mathfrak{g}$ in $\mathfrak{p}$ are solvable, because $\mathfrak{p}$ itself is solvable. Moreover, the only nilpotent modification is $\mathfrak{g}$ itself. Indeed, if $\mathfrak{s} \neq \mathfrak{g}$, then there is $X \in \mathfrak{g}_{-1}$ with $\sigma X \neq 0$, so that, if $Y \in \mathfrak{g}_{-s}$ is nonzero, then $[(X, \sigma X), (Y, 0)] = s\sigma X(Y, 0)$, where s is the step of $\mathfrak{g}$. Therefore, we obtain that $(Y, 0) \in [\mathfrak{s}, [\dots, [\mathfrak{s}, \mathfrak{s}] \dots]]$ for any order of brackets, that is, $\mathfrak{s}$ is not nilpotent. □

Author Contributions Sebastiano Nicolussi Golo and Alessandro Ottazzi were equally involved in the research that led to this manuscript and in writing it.

Funding Open Access funding enabled and organized by CAUL and its Member Institutions.

Declarations

Conflict of interest The authors declare no competing interests.

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