Geometriae Dedicata (2024) 218:16
https://doi.org/10.1007/s10711-023-00861-4

Rigidity of geometric structures

Ursula Hamenstädt[1] · Frieder Jäckel[1]

Received: 13 August 2023 / Accepted: 20 October 2023 / Published online: 20 November 2023

Abstract
Geometric structures on a manifold M arise from a covering of M by charts with values in a homogeneous space G/H, with chart transitions restrictions of elements of G. If M is aspherical, then such geometric structures are given by a homomorphism of the fundamental group of M into G. Rigidity of such structures means that the conjugacy class of the homomorphism can be reconstructed from topological or geometric information on M. We give an overview of such rigidity results, focusing on topological type and length functions.

Mathematics Subject Classification 53C30 · 57N16 · 57M50 · 37C15

1 Introduction

A *geometric structure* on a manifold M is defined by an atlas $\mathcal{A} = \{(U_\phi, \phi) \mid U_\phi \subseteq M \text{ open}\}$ of M, with chart maps ϕ taking values in a *homogeneous space* $X = G/H$ where G is a Lie group and $H < G$ is a closed subgroup. Furthermore, it is required that chart transitions $\phi \circ \psi^{-1} : \psi(U_\psi \cap U_\phi) \to \phi(U_\psi \cap U_\phi)$ are given by the restrictions of an element of G to $\psi(U_\psi \cap U_\phi)$. In this survey we are interested in geometric structures on *aspherical* manifolds, that is, manifolds M whose universal coverings are contractible (see [55] for more information). An example of a geometric structure on M is a *locally symmetric metric of non-positive curvature*, which is a geometric structure defined by a homogeneous space $X = G/K$ where G is a semi-simple Lie group of non-compact type and $K < G$ is a maximal compact subgroup. Namely, $X = G/K$ is just a symmetric space of non-compact type, and a geometric structure on M defines an atlas of charts into X whose chart transitions are isometries. As a consequence, the pull-back of the metric defined by the charts is globally defined on M. If the (G, X)-structure is complete then the Riemannian universal covering of M is isometric to X and hence $M = \Gamma\backslash G/K$ where $\Gamma < G$ is discrete and torsion-free.

In the sequel we assume for simplicity that the Lie group G is *simple*. This is equivalent to stating that the de Rham decomposition of the symmetric space X is trivial. We then

✉ Ursula Hamenstädt
ursula@math.uni-bonn.de

Frieder Jäckel
fjaeckel@math.uni-bonn.de

1 Mathematical Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

call a locally symmetric space $M = \Gamma\backslash G/K$ *irreducible*, that is, the (local) holonomy representation of M is irreducible.

Locally symmetric metrics on aspherical manifolds have various rigidity properties. The most fundamental rigidity result is *Mostow rigidity* which states that two *closed* irreducible aspherical locally symmetric manifolds of dimension at least 3 are homotopy equivalent if and only if they are homothetic.

On the other hand, locally symmetric manifolds form a rich class of aspherical manifolds. Uniformization shows that any closed oriented surface of genus at least 2 admits a *hyperbolic* metric, that is, a metric of constant curvature -1, modeled on the hyperbolic plane $\mathbb{H}^2 = PSL(2, \mathbb{R})/PSO(2)$. Perelman's solution to the geometrization conjecture yields a topological characterization of closed locally symmetric manifolds in dimension 3. Closed irreducible aspherical locally symmetric 3-manifolds are precisely the hyperbolic 3-manifolds, and this class of manifolds coincides with the class of closed aspherical *atoroidal* 3-manifolds. Here a 3-manifold is called atoroidal if its fundamental group does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Furthermore, this class coincides with the class of closed 3-manifolds which admit a Riemannian metric of negative sectional curvature.

In higher dimension, locally symmetric manifolds seem to be rare among all closed manifolds, even among those closed manifolds which admit a metric of negative curvature. Up to now this is mainly witnessed by constructions of closed negatively curved manifolds which do *not* admit locally symmetric metrics (see [34, 61]). Number theoretic tools allow for the construction of closed locally symmetric manifolds in any dimension. Hyperbolic manifolds of dimension at least 4 can be counted in dependence on their volumes (see [16] [29, Theorem 1.11]), but specific information on their geometry and topology is not available (see however [43]).

The search for characterizations of manifolds which admit locally symmetric metrics or, more generally, geometric structures, by some special geometric or topological properties is a fruitful line of research. Furthermore, whenever the Mostow rigidity theorem is not available, there may be a rich deformation space of such geometric structures. We survey some of the rigidity and flexibility results which can be described in terms of the topology of the manifold and length functions associated to geometric structures.

We begin with looking at geometric structures which enjoy the conclusion of the Mostow rigidity theorem, namely closed irreducible locally symmetric manifolds of dimension at least 3. The fundamental group of such a manifold is known to be residually finite and hence there are interesting towers of finite covers. In general, not much is known about these covering spaces in spite of substantial progress towards the understanding of towers of congruence covers (see [30, 54]). In Sect. 2 we point out that finite quotients of compact locally symmetric manifolds are homotopy locally symmetric. A special case of the following statement is due to Kapovich ([46, Theorem 8.36]), with a somewhat different proof. We expect that the full statement is known to the experts, however we did not find a reference in the literature.

Theorem (Manifolds covered by locally symmetric spaces) *Let M be a closed manifold of dimension $n \geq 2$. Assume that there is a finite sheeted cover $\hat{M} \to M$ of M where $\hat{M}$ is an irreducible locally symmetric space. Then M is homotopy equivalent to an irreducible locally symmetric space N. If $n \geq 5$ then M is homeomorphic to N, and if $n = 2, 3$ then M is diffeomorphic to N.*

Note that the case $n = 2$ is a classical consequence of uniformization. We expect that in all dimensions, closed manifolds which admit locally symmetric finite covers are homeomorphic to locally symmetric manifolds. However, we illustrate by example that they need not be diffeomorphic to a locally symmetric manifold.

In Sects. 3 and 4 we turn to a discussion of geometric rigidity which seeks to characterize non-positively curved locally symmetric metrics on closed manifolds by global geometric invariants. Namely, for a closed non-positively curved manifold M, every conjugacy class in the fundamental group of M can be represented by a closed geodesic. Any two such geodesics have the same length. The *marked length spectrum* of M is the function which associates to each such conjugacy class the length of such a geodesic. We explain the relation between the marked length spectrum and *cross ratios*, and we summarize some of the results regarding the so-called marked length spectrum rigidity question which asks to characterize such manifolds up to isometry by their marked length spectrum, a natural question which is interesting but challenging beyond the world of locally symmetric metrics.

In Sect. 5 we look at representations of surface groups, that is, fundamental groups of closed surfaces, into the isometry group $PSL(2,\mathbb{C})$ of hyperbolic 3-space. Such representations can be obtained by deforming homomorphisms arising from embeddings $\pi_1(S) \to PSL(2,\mathbb{R}) \to PSL(2,\mathbb{C})$. We describe what is known about the marked length spectrum, cross ratios and geometric and topological rigidity.

In Sect. 6 we turn to real projective structures, that is, geometric structures modeled on $PSL(n,\mathbb{R})/H$ where H is the stabilizer of a point for the transitive projective action of the group $PSL(n,\mathbb{R})$ on real projective space $\mathbb{R}P^{n-1}$. If the holonomy group for such a projective structure preserves an open convex subset Ω, contained in the complement of a projective hyperplane, then the usual cross ratio on S^{n-1} defines a so-called *Hilbert metric* on Ω and once again, we obtain a marked length spectrum. We summarize results related to marked length spectrum rigidity. We conclude by specializing to homomorphisms of surface groups into $PSL(n,\mathbb{R})$ contained in the so-called *Hitchin component*. Such representations are deformations of homomorphisms which factor through an embedding $PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ corresponding to an irreducible representation of $SL(2,\mathbb{R})$ on $\mathbb{R}^n$.

2 Topological rigidity

In this section we prove the theorem from the introduction. Since this is classical if the dimension n of M equals 2 we may assume that $n \geq 3$.

Let $\hat{M}$ be a closed irreducible locally symmetric space and let $\hat{M} \to M$ be a finite sheeted covering. Then $\hat{M}, M$ are aspherical, and the fundamental group $\pi_1(\hat{M})$ of $\hat{M}$ is a finite index subgroup of the fundamental group $\pi_1(M)$ of M. As a consequence, $\{\bigcap g\pi_1(\hat{M})g^{-1} \mid g \in \pi_1(M)\}$ is a finite index normal subgroup of $\pi_1(M)$. By replacing $\hat{M}$ by a finite sheeted cover, we therefore may assume that the covering $\pi : \hat{M} \to M$ is regular. This is equivalent to stating that the deck group $\mathrm{Deck}(\pi)$ acts simply transitively on the fibers $\pi^{-1}(x)$ over any point $x \in M$.

For any homotopy self-equivalence of $\hat{M}$ there exists, due to Mostow-rigidity, a unique isometry homotopic to it (see [59, Theorem 24.1′]). In particular, there is a group homomorphism

$$\Psi : \mathrm{Deck}(\pi) \to \mathrm{Isom}(\hat{M}), \quad \sigma \mapsto \Psi(\sigma) \tag{2.1}$$

sending any deck transformation $\sigma : \hat{M} \to \hat{M}$ to the unique isometry $\Psi(\sigma)$ homotopic to it. Note that Ψ is a homomorphism due to the uniqueness of the isometry. The following result is the key ingredient for the proof of the theorem from the introduction.

Lemma 2.1 *For every* $\sigma \in \mathrm{Deck}(\pi)\backslash\{\mathrm{id}_{\hat{M}}\}$ *the isometry* $\Psi(\sigma)$ *has no fixed point.*

In particular, $\Psi : \mathrm{Deck}(\pi) \to \mathrm{Isom}(\hat{M})$ is injective, so that $\mathrm{Deck}(\pi) \cong \Psi(\mathrm{Deck}(\pi))$. Lemma 2.1 will follow immediately from the following general result.

Lemma 2.2 *Let X be a finite dimensional aspherical CW-complex such that $\pi_1(X)$ has trivial center, and let $\sigma : X \to X$ be a homeomorphism of finite order $m \geq 2$ such that the projection $X \to X/\langle\sigma\rangle$ is a covering projection and $X/\langle\sigma\rangle$ has the homotopy type of a finite dimensional CW-complex. Then σ is not homotopic to a map $\psi : X \to X$ with non-empty fixed point set and $\psi^m = \mathrm{id}_X$.*

The proof of Lemma 2.2 only uses standard group homological arguments.

Proof Arguing by contradiction, assume that σ is homotopic to some map ψ with $\mathrm{Fix}(\psi) \neq \emptyset$ and $\psi^m = \mathrm{id}_X$. Fix $x_0 \in \mathrm{Fix}(\psi)$ and set $\bar{X} = X/\langle\sigma\rangle$. As $X \to \bar{X}$ is a covering map of degree m, there is a short exact sequence

$$1 \to \pi_1(X, x_0) \to \pi_1(\bar{X}, \bar{x}_0) \to \mathbb{Z}/m\mathbb{Z} \to 1. \tag{2.2}$$

It follows from the properties of ψ that the short exact sequence (2.2) splits.

Indeed, (2.2) induces an action $\bar{\rho} : \mathbb{Z}/m\mathbb{Z} \to \mathrm{Out}(\pi_1(X, x_0))$. As $x_0 \in \mathrm{Fix}(\psi)$ and $\psi^m = \mathrm{id}_X$, the homomorphism $\rho : \mathbb{Z}/m\mathbb{Z} \to \mathrm{Aut}(\pi_1(X, x_0))$, $\bar{1} \mapsto \psi_*$ is well-defined, and it is a lift of $\bar{\rho} : \mathbb{Z}/m\mathbb{Z} \to \mathrm{Out}(\pi_1(X, x_0))$ because ψ is homotopic to σ.

Consider the short exact sequence

$$1 \to \pi_1(X, x_0) \to \pi_1(X, x_0) \rtimes_\rho \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 1. \tag{2.3}$$

The action $\mathbb{Z}/m\mathbb{Z} \to \mathrm{Out}(\pi_1(X, x_0))$ induced by (2.3) agrees with the induced action $\bar{\rho} : \mathbb{Z}/m\mathbb{Z} \to \mathrm{Out}(\pi_1(X, x_0))$ of (2.2) because ρ is a lift of $\bar{\rho}$. Hence, as $\pi_1(X, x_0)$ has trivial center, [15, Corollary 6.8 in Chapter IV] implies that the short exact sequences (2.2) and (2.3) are equivalent. Therefore, (2.2) splits because (2.3) splits.

Because the short exact sequence (2.2) splits, there is an injection $\mathbb{Z}/m\mathbb{Z} \hookrightarrow \pi_1(\bar{X})$. However, it is well-known (and follows from standard group homological arguments) that the fundamental group $\pi_1(\bar{X})$ of a topological space $\bar{X}$ with the homotopy type of an aspherical finite dimensional CW-complex is torsion-free (see [15]). Namely, as $\bar{X}$ has the homotopy type of an aspherical and finite dimensional CW-complex, there is a finite dimensional CW-model for the classifying space $E\pi_1(\bar{X})$. Restricting the action of $\pi_1(\bar{X})$ on $E\pi_1(\bar{X})$ to $\mathbb{Z}/m\mathbb{Z} \hookrightarrow \pi_1(\bar{X})$, there is a finite dimensional model for $E\mathbb{Z}/m\mathbb{Z}$. Hence $B\mathbb{Z}/m\mathbb{Z}$ has a finite dimensional CW-model, and thus $H_k(B\mathbb{Z}/m\mathbb{Z})$ can only be non-zero for finitely many k. But it is well-known (see [15, (3.1) on page 35]) that the group homology $H_k(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z}) \cong H_k(B\mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$ of $\mathbb{Z}/m\mathbb{Z}$ with $\mathbb{Z}$-coefficients is non-zero for all odd k. This is a contradiction. □

As an easy consequnce we obtain Lemma 2.1.

Proof of Lemma 2.1 Since $\hat{M}$ is a closed irreducible non-positively curved manifold, $\hat{M}$ has the structure of a finite dimensional aspherical CW-complex and $\pi_1(\hat{M})$ has trivial center. Fix any $\sigma \in \mathrm{Deck}(\pi)\backslash\{\mathrm{id}_{\hat{M}}\}$, and set $m := \mathrm{ord}(\sigma) \geq 2$. It holds $\Psi(\sigma)^m = \mathrm{id}_{\hat{M}}$ because the map $\Psi : \mathrm{Deck}(\pi) \to \mathrm{Isom}(\hat{M})$ defined in (2.1) is a group homomorphism. Therefore, as $\Psi(\sigma)$ is homotopic to σ, Lemma 2.2 yields that $\Psi(\sigma)$ has no fixed points. This completes the proof. □

By Lemma 2.1, the isometries $\Psi(\sigma) : \hat{M} \to \hat{M}$ $(\sigma \neq \mathrm{id}_{\hat{M}})$ have no fixed points. As a consequence, the action of $\Psi(\mathrm{Deck}(\pi))$ on $\hat{M}$ is free and properly discontinuous. Thus the quotient

$$N := \hat{M}/\Psi(\mathrm{Deck}(\pi))$$

is a manifold, and the quotient map $\hat{M} \to N$ is a covering map. As $\Psi(\mathrm{Deck}(\pi))$ acts via isometries on $\hat{M}$, N admits a metric that turns it into a non-positively curved locally symmetric space.

Lemma 2.3 *The fundamental groups of M and N are isomorphic.*

Proof The regular covering maps $\hat{M} \to M$ and $\hat{M} \to N$ induce short exact sequences

$$1 \to \pi_1(\hat{M}) \to \pi_1(M) \to \mathrm{Deck}(\pi) \to 1$$

and

$$1 \to \pi_1(\hat{M}) \to \pi_1(N) \to \Psi(\mathrm{Deck}(\pi)) \to 1.$$

Because the maps σ and $\Psi(\sigma)$ are homotopic for every $\sigma \in \mathrm{Deck}(\pi)$, it follows that, under the identification $\mathrm{Deck}(\pi) \cong \Psi(\mathrm{Deck}(\pi))$, the induced actions $\mathrm{Deck}(\pi) \to \mathrm{Out}(\pi_1(\hat{M}))$ of these short exact sequences agree. Recall that $\pi_1(\hat{M})$ has trivial center. So [15, Corollary 6.8 in Chapter IV] implies $\pi_1(N) \cong \pi_1(M)$. □

Corollary 2.4 *If the closed manifold M has a finite sheeted cover which is an irreducible locally symmetric manifold of non-compact type, then M is homotopy equivalent to an irreducible locally symmetric manifold (the manifold N defined above).*

Proof As M has a finite sheeted cover which is aspherical, M is aspherical. Thus the homotopy type of M is determined by its fundamental group (see [55, Theorem 1.1 (i)] or [40, Theorem 1B.8]). By Lemma 2.3, the fundamental group of M equals the fundamental group of an irreducible locally symmetric manifold. This shows the corollary. □

As a consequence, for the proof of the theorem in the introduction, it remains to promote the homotopy equivalence given in Corollary 2.4 to a homeomorphism provided that $n \neq 4$.

A closed aspherical manifold M is called *topologically rigid* if every homotopy equivalence to another closed manifold is homotopic to a homeomorphism. The *Borel conjecture* asserts that closed aspherical manifolds are topologically rigid. This conjecture is open in general, but it is known for aspherical manifolds of dimension at least 5 which admit a non-positively curved Riemannian metric.

Proof of the theorem in the introduction Since M and N are both aspherical, it follows from Corollary 2.4 that there is a homotopy equivalence $M \xrightarrow{\simeq} N$. Since N admits a non-positively curved metric, if $\dim(M) \geq 5$ it follows from the solution of the Borel conjecture [25, Theorem 1], [2, Theorem A] that the homotopy equivalence $M \xrightarrow{\simeq} N$ is homotopic to a homeomorphism $M \xrightarrow{\cong} N$.

If $\dim(M) = 3$, then the manifold N is hyperbolic by irreducibility. As N is hyperbolic and M has a finite cover diffeomorphic to a finite cover of N, the universal covering of M is diffeomorphic to $\mathbb{R}^3$. Thus M is irreducible, and [28, Theorem 0.1] shows that the homotopy equivalence $M \xrightarrow{\simeq} N$ is again homotopic to a homeomorphism. On the other hand, homeomorphic 3-manifolds are also diffeomorphic (see [7, Theorem 8], [27, Section 8.3], [60, Corollary]). This completes the proof. □

The following example shows that in general, we can not expect that the manifold M in the statement of the theorem in the introduction is diffeomorphic to an irreducible locally symmetric manifold, even if M admits a negatively curved Riemannian metric.

Example 2.5 An *exotic* n*-sphere* is a smooth manifold which is homeomorphic but not diffeomorphic to the sphere S^n of dimension n. Such exotic spheres form a finite group Θ_n under connected sum (see [48, Theorems 1.1 and 1.2],[56, Theorem 12.1 and Section 12.2]). The orders of these groups can be computed in certain cases, see for example the table (12.2) of Ref. [56].

Farrel and Jones [24, Theorem 1.1] proved the following: Let M_0 be a closed hyperbolic manifold whose dimension $n \geq 7$ is such that there exists an exotic n-sphere Σ. Then for every $\varepsilon > 0$ there exists a finite cover $M \to M_0$ such that the connected sum $M\#\Sigma$ is *not* diffeomorphic to M (and hence $M\#\Sigma$ is *not* diffeomorphic to any locally symmetric space of rank one due to Mostow-rigidity), but $M\#\Sigma$ admits a Riemannian metric whose sectional curvature is contained in $(-1-\varepsilon, -1+\varepsilon)$.

Let us now assume that the first Betti number $\dim(H^1(M_0, \mathbb{Z}))$ of M_0 is at least 1. Then the same holds true for M, and there exists a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$. Such a homomorphism induces for any $p \geq 1$ a surjective homomorphism $\pi_1(M) \to \mathbb{Z}/p\mathbb{Z}$. The kernel of this homomorphism is an index p normal subgroup of $\pi_1(M)$ which defines a p-sheeted cover $\hat{M} \to M$.

Choose p in such a way that p is a multiple of $|\Theta_n|$, the order of the group of exotic spheres in dimension n. Let Σ be any nontrivial exotic sphere of dimension n. Since taking the connected sum with a simply connected space commutes with coverings, the p-fold connected sum $\hat{M}\#\Sigma\#...\#\Sigma$ is a p-sheeted smooth cover of $M\#\Sigma$. As p is a multiple of $|\Theta_n|$, the p-fold connected sum of Σ is the standard sphere, and thus the manifold $\hat{M}\#\Sigma\#...\#\Sigma$ is diffeomorphic to the hyperbolic manifold $\hat{M}$. Hence, $M\#\Sigma$ admits a smooth cover by the hyperbolic manifold $\hat{M}$, but $M\#\Sigma$ is *not* diffeomorphic to any locally symmetric space.

Since the Borel conjecture is not known in dimension $n = 4$, we are unable to promote the homotopy equivalence obtained in Corollary 2.4 to a homeomorphism, even though we have control on some finite coverings. Note to this end that the case of the lens spaces shows that there are non-homeomorphic manifolds N_1, N_2 obtained from the same manifold N by taking the quotient by two finite groups of homotopic diffeomorphisms (see [56, Corollary 3.70 and Example 3.71]). These manifolds are however not aspherical.

3 Cross ratios and length functions

In this section we begin with considering an arbitrary n-dimensional Riemannian manifold M of negative sectional curvature. The *geodesic flow* Φ^t is a smooth dynamical system acting on the unit tangent bundle T^1M of M. The image $\Phi^t v$ of a unit tangent vector v is the tangent at t of the geodesic with initial velocity v at $t = 0$. Periodic orbits of Φ^t correspond to conjugacy classes of the fundamental group Γ of M.

The universal covering $\tilde{M}$ of M is a simply connected manifold of negative sectional curvature which can be compactified by adding the *boundary at infinity* $\partial\tilde{M}$. This boundary is homeomorphic to the $(n-1)$-sphere. The fundamental group Γ of M acts as a group of transformations on $\partial\tilde{M}$.

A Hölder continuous *additive cocycle* for the geodesic flow Φ^t is a Hölder continuous function $\zeta : T^1M \times \mathbb{R} \to \mathbb{R}$ which satisfies $\zeta(v, s+t) = \zeta(v, s) + \zeta(\Phi^s v, t)$ for all $v \in T^1M, s, t \in \mathbb{R}$. Two such cocycles ζ, η are *cohomologous* if there exists a Hölder function $f : T^1M \to \mathbb{R}$ such that

$$\eta(v,t) + f(\Phi^t v) = \zeta(v,t) + f(v)$$

for all $v \in T^1M$, $t \in \mathbb{R}$. If the manifold M is closed then by the Livshitz theorem for Hölder cocycles (see [39, Theorem 19.2.4] and [37, pp. 94 and 95]), this is equivalent to stating that for every periodic point $v \in T^1M$ of period $\tau > 0$ we have $\zeta(v,\tau) = \eta(v,\tau)$. In other words, the *periods* of ζ coincide with the periods of η. The cocycle ζ is called *quasi-invariant under the flip* $\mathcal{F} : v \to -v$ if the cocycle $\mathcal{F}\zeta : (v,t) \to \zeta(\mathcal{F}\Phi^t v, t)$ is cohomologous to ζ.

Definition 3.1 A *multiplicative cross ratio* is a $(0,\infty)$-valued Hölder function Cr on the space

$$(\partial\tilde{M})^{4,*} = \{(\xi,\xi',\eta,\eta') \mid \xi,\xi',\eta,\eta' \in \partial\tilde{M}, \{\xi,\xi'\} \cap \{\eta,\eta'\} = \emptyset\}$$

with the following properties.

1. $\mathrm{Cr}(\xi,\xi',\eta,\eta') = \mathrm{Cr}(\xi,\xi',\eta',\eta)^{-1}$.
2. $\mathrm{Cr}(\xi,\xi',\eta,\eta') = \mathrm{Cr}(\eta,\eta',\xi,\xi')$.
3. $\mathrm{Cr}(\xi,\xi',\eta,\eta')\mathrm{Cr}(\xi',\xi'',\eta,\eta') = \mathrm{Cr}(\xi,\xi'',\eta,\eta')$.
4. $\mathrm{Cr}(\xi,\xi,\eta,\eta') = 1 = \mathrm{Cr}(\xi,\xi',\eta,\eta)$.
5. Cocycle identity: $\mathrm{Cr}(\xi,\xi',\eta,\eta')\mathrm{Cr}(\xi',\eta,\xi,\eta')\mathrm{Cr}(\eta,\xi,\xi',\eta') = 1$.

Note that Property (4) is a consequence of Properties (1)–(3). For a group Γ of isometries of $\tilde{M}$ we call Cr a Γ-*cross ratio* if it is invariant under the action of Γ.

An *additive cross ratio* $[\cdot,\cdot,\cdot,\cdot]$ is obtained from a multiplicative cross ratio by taking the logarithm.

Example 3.2 Consider the unit circle S^1, viewed as the ideal boundary of the hyperbolic plane $\mathbb{H}^2$. For a fixed point $x \in \mathbb{H}^2$ we can define the *Gromov product* $(\xi,\eta)_x$ of $\xi \neq \eta \in S^1$ based at x as

$$(\xi,\eta)_x = \lim_{t\to\infty} \frac{1}{2}(2t - d(\gamma_\xi(t),\gamma_\eta(t)))$$

where γ_ξ, γ_η are the geodesic rays from x to ξ, η, respectively. Then

$$[\xi,\xi',\eta,\eta'] = (\xi,\eta)_x + (\xi',\eta')_x - (\xi,\eta')_x - (\xi',\eta)_x$$

defines an additive cross ratio $[\cdot,\cdot,\cdot,\cdot]$ on S^1 which does not depend on the choice of x (see for example [10, p. 96] for this independence). This cross ratio is invariant under the action of the full group $PSL(2,\mathbb{R})$ of orientation preserving isometries of $\mathbb{H}^2$.

The corresponding multiplicative cross ratio $\mathrm{Cr} = e^{[\cdot,\cdot,\cdot,\cdot]}$ satisfies the additional identity

$$\mathrm{Cr}(a,b,c,d) + \mathrm{Cr}(b,c,d,a) = 1 \tag{3.1}$$

for every *ordered* quadruple (a,b,c,d). It is due to Bonahon [9, Theorem 13] that this identity characterizes completely the multiplicative cross ratio defined by the length function of a hyperbolic metric on a closed orientable surface S among all $\pi_1(S)$-cross ratios on S^1.

Alternatively, this cross ratio can naturally be expressed in terms of projective geometry by passing from the disk model of the hyperbolic plane to the upper half space model $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. The corresponding transformation maps the unit circle S^1 onto $\mathbb{R} \cup \{\infty\}$. Now for any quadruple (x,y,z,w) of pairwise distinct points in $\mathbb{R}$, the classical cross ratio is defined by

$$b(x,y,z,w) = \frac{|x-z|\cdot|y-w|}{|x-w|\cdot|y-z|}. \tag{3.2}$$

This function extends continuously to 1 as $x \to y$ or as $z \to w$ and has all the above properties. Furthermore, it clearly is invariant under the action of the group $PSL(2,\mathbb{R})$ by linear fractional transformations $z \to \frac{az+b}{cz+d}$ on the upper half-plane and hence it equals the cross ratio given above.

The following is [36, Theorem A].

Theorem 3.3 *For any closed negatively curved manifold M with fundamental group Γ, flip invariant cohomology classes of Hölder cocycles are in one-to-one correspondence with additive Γ-cross-ratios on $\partial\tilde{M}$.*

Assume for the remainder of this section that M is compact. Then the correspondence between cohomology classes of Hölder cocycles and cross ratios has the following property. Let $\phi \in \pi_1(M)$ be an arbitrary element. Then ϕ acts on $\partial\tilde{M}$ as a homeomorphism, and there are precisely two fixed points p_+, p_- for this action. The fixed point p_+ is attracting. More precisely, for any two neighborhoods U_+ of p_+, U_- of p_- there exists a number $k > 0$ such that $\phi^k(\partial\tilde{M}\backslash U_-) \subset U_+$.

Let ζ be a Hölder cocycle. Choose any point $z \in \partial\tilde{M} - \{p_\pm\}$ and put

$$[p_+, p_-, z, \phi(z)]_\zeta = \zeta(v, T)$$

where v is a point on the periodic orbit of Φ^t on T^1M defined by ϕ and T is the period of this orbit. This extends to a cross ratio on $\partial\tilde{M}$ which only depends on the periods of the cocycle ζ and hence it only depends on its cohomology class (see [36, Lemma 1.5]).

The length function defined by the Riemannian metric determines a flip invariant Hölder *length cocycle* ζ by defining $\zeta(v,t) = t$ for all v, t. The periods of ζ over the periodic orbits of Φ^t are precisely the periods of Φ^t.

Theorem 3.3 shows that this Hölder cocycle determines an additive cross ratio $[\cdot,\cdot,\cdot,\cdot]$ on $\partial\tilde{M}$ which only depends on the lengths of the *marked* closed geodesics on M, that is, on the function which associates to a free homotopy class in $\pi_1(M) = \Gamma$ the length of a shortest representative.

The geodesic flow Φ^t on T^1M lifts to the geodesic flow on the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$. The *space of oriented geodesics* in the universal covering $\tilde{M}$ of M is the space of oriented orbits of this flow. Since a flow line for this flow as a set is just the set of unit tangent vectors of an oriented geodesic and since any oriented pair of distinct points in $\partial\tilde{M}$ is the oriented pair of endpoints of precisely one oriented geodesic, and every oriented geodesic arises in this way, the space of oriented geodesics on $\tilde{M}$ can naturally be identified with $\partial\tilde{M} \times \partial\tilde{M} - \Delta$ where Δ is the diagonal. Every flow invariant Borel measure on T^1M then induces a flow invariant Borel measure on $T^1\tilde{M}$ which is moreover invariant under the action of the fundamental group Γ of M. This measure determines a *geodesic current*, that is, a Γ-invariant Radon measure on $\partial\tilde{M} \times \partial\tilde{M} - \Delta$.

The Riemannian metric on M determines a particular geodesic current as follows. There is a natural induced Riemannian metric on T^1M called the *Sasaki metric* (see [63, Definition 1.17] or [50, p. 457]). The fibers of the fibration $T^1M \to M$ are isometric to the round sphere of radius 1. The volume element of this metric defines a measure on T^1M which is invariant under the geodesic flow Φ^t, of total mass $\omega_{n-1}\mathrm{vol}(M)$ where ω_{n-1} is the volume of the standard $(n-1)$-sphere (see [63, Corollary 1.31 and Exercises 1.32 and 1.33] or [50, Lemma 1.3 on p. 457]). This measure is called the *Lebesgue Liouville measure*. The geodesic current defined by this measure is called the *Liouville current*.

The measure class of the Liouville current can easily be described. Namely, for any point $x \in \tilde{M}$, there is a natural homeomorphism $T^1_x\tilde{M} \to \partial\tilde{M}$ which associates to a unit tangent vector v the equivalence class of the geodesic ray with initial velocity v. Via this homeomorphism, the standard volume element on the unit sphere $T_x^1\tilde{M}$ pushes forward to a finite measure λ_x on $\partial\tilde{M}$. It turns out that the measure class of this measure, called the *Lebesgue measure class*, does not depend on x, in particular it is invariant under the action of $\pi_1(M)$. The Liouville current defines the measure class of the product measure $\lambda_x \times \lambda_x$.

The Liouville current and the length cocycle carry substantial geometric information. To explain what this means let N denote another negatively curved Riemannian manifold whose fundamental group is isomorphic to the fundamental group of M. Since M, N are $K(\pi, 1)$-spaces, there then exists a homotopy equivalence $F : M \to N$ (in fact, a homeomorphism if $\dim(M) \neq 4$), determined by the choice of an isomorphism $\rho : \pi_1(M) \to \pi_1(N)$. This homotopy equivalence then lifts to a ρ-equivariant quasi-isometry $\tilde{M} \to \tilde{N}$ between the universal covers of M, N [59]. This quasi-isometry in turn defines a ρ-equivariant homeomorphism $u : \partial\tilde{M} \to \partial\tilde{N}$. If F is homotopic to an isometry, then the map u preserves the measure class of the Liouville measures, that is, the push-forward of the Lebesgue measure class on $\partial\tilde{M}$ equals the Lebesgue measure class on $\partial\tilde{N}$.

Part of the following conjecture can for example be found in [17]. For its formulation, note that any conjugacy class in $\pi_1(M)$ or $\pi_1(N)$ can be represented by a unique closed geodesic. We define the length of the geodesic to be the *length* of the conjugacy class.

Conjecture 3.4 *The following are equivalent.*

1. *The isomorphism ρ preserves the length of all conjugacy classes up to a universal multiplicative constant.*
2. *The push-forward of the Lebesgue measure class on $\partial\tilde{M}$ by the equivariant map $u : \partial\tilde{M} \to \partial\tilde{N}$ is the Lebesgue measure class on $\partial\tilde{N}$.*
3. *The map F is homotopic to an homothety (an isometry up to rescaling).*

In the case that one of the manifolds is rank one locally symmetric, the equivalence of (1) and (3) was established in [37]building on [3]. It is also known locally, that is, for negatively curved metrics which are sufficiently close to each other [32]. The equivalence of (1) and (2) is known if the ideal boundaries $\tilde{M}, \tilde{N}$ admit a C^1-structure, that is, if there exists a C^1-structure on $\partial M, \partial N$ which is invariant under the action of the group Γ (we refer to [37] for more information).

As a consequence, rank one locally symmetric structures on closed manifolds are geometrically rigid among *all* negatively curved metrics.

4 Cross ratios and rigidity of geometries on surfaces

In this section we specialize to cross ratios for the fundamental group of a closed orientable surface of genus at least two, acting on the ideal boundary S^1 of its universal covering. An orientation of the surface determines an orientation of S^1.

Any (ordered) quadruple (a, b, c, d) of pairwise distinct points in S^1 defines a closed subset $[a, b] \times [c, d] \subset S^1 \times S^1 - \Delta$ where $[a, b]$ is the oriented subsegment of S^1 connecting a to b. A Radon measure η on $S^1 \times S^1 - \Delta$ without atoms associates to each such set a measure $\eta([a, b] \times [c, d]) \in [0, \infty)$ with the property that

$$\eta([a, b'] \times [c, d]) = \eta([a, b] \times [c, d]) + \eta([b, b'] \times [c, d])$$

provided that the ordered 5-tuple (a, b, b', c, d) is compatible with the orientation of S^1. If η is invariant under the flip exchanging the two factors, then $\eta([a, b]\times[c, d]) = \eta([c, d]\times[a, b])$.

Thus such a measure which is in addition invariant under the action of Γ has the equivalent of properties (2)–(5) of an additive cross ratio, but viewed as a function on $(S^1)^{4,*}$, it need not be Hölder continuous.

Vice versa, an additive cross ratio with the additional property that its value on any ordered quadruple (a, b, c, d) of points defining the orientation of S^1 is non-negative defines a finitely additive non-negative function on quadrangles in $S^1 \times S^1 - \Delta$, that is, products of disjoint closed invervals. Since such quadrangles generate the Borel σ-algebra of $S^1 \times S^1 - \Delta$, the cross ratio determines in fact a Γ-invariant Radon measure on $S^1 \times S^1 - \Delta$.

In this vein, the following is the second part of Lemma 2.6 of [37]. It is essentially due to Otal [62, Proposition 3], and to Bonahon [9, Proposition 14] in the case of hyperbolic surfaces.

Proposition 4.1 *If (S, g) is a closed negatively curved surface, then the cross ratio of the length cocycle of S equals the Liouville current of the metric g.*

Note that there is a small inconsistency here as we define a current to be a Radon measure on oriented geodesics. Other articles work with unoriented geodesics.

As a consequence, we obtain

Corollary 4.2 *The marked length spectrum of a closed negatively curved surface S determines the Liouville current of T^1S and hence the volume of S.*

Proof By Proposition 4.1, the marked length spectrum of S determines the Liouville current and hence the Lebesgue Liouville measure on T^1S. The volume of S equals the total mass of T^1S for this measure divided by 2π. □

In fact much more is true. The following result summarizes work of Otal [62, Théorème 1], Croke [20, Theorem B] (also using that the geodesic flow as a dynamical system can be reconstructed from the marked length spectrum—see [17, (3.5) and (10.3)], [35]) and Guillarmou, Lefeuvre, Paternain [33, Theorem 1.1].

Theorem 4.3 *(Otal, Croke, Guillarmou-Lefeuvre-Paternain) Let Σ be a closed oriented surface of genus $g \geq 2$. Then two metrics on Σ with the same marked length spectrum which are either non-positively curved or whose geodesic flow is Anosov are isometric.*

An additive cross ratio $[\cdot, \cdot, \cdot, \cdot]$ for a closed oriented surface S is called *positive* if for every *ordered* quadruple (a, b, c, d) of pairwise distinct points in $\partial\tilde{S}$ we have $[a, b, c, d] > 0$. The following is now immediate from Theorem 4.3 since the marked length spectrum can be read off from the cross ratio associated to the length cocycle.

Corollary 4.4 *The length cocycle of a negatively curved metric on a closed oriented surface S defines a positive cross ratio which determines the metric up to isometry.*

The space of all smooth negatively curved metrics on S up to diffeomorphisms which are isotopic to the identity can be equipped with the structure of an infinite dimensional Frechet manifold. Namely, such a metric is a smooth positive definite section of the tensor bundle Σ^2T^*S of symmetric bilinear forms on TS. Since being positive definite is an open condition, the space of such metrics is an open subset of the infinite dimensional vector space of smooth sections of Σ^2T^*S, which can naturally be equipped with a topology (a Frechet topology). The group of diffeomorphisms of S acts by pull-back on such metrics. The quotient of the space of metrics by this action can locally be identified with a slice transverse to the orbit of this action. That locally near a given metric g, divergence free sections of $\Sigma^2 T^*M$ with respect to g determine such a slice is a fairly standard fact. We refer to Ref. [21] for more information.

Corollary 4.4 shows that there is a continuous embedding of this space into the space of cross ratios, equipped with the topology as a direct limit of the space of functions of Hölder class C^α for some $\alpha \in (0, 1)$. Namely, each negatively curved metric on S determines a Hölder structure on $S^1 = \partial\tilde{S}$. That is, start with a given identification $T^1_x\tilde{S} \to S^1$ defined by associating to a unit tangent vector v at x the equivalence class of the geodesic ray with initial velocity v. The same construction at a different point $y \in \tilde{S}$ defines a homeomorphism $T^1_x\tilde{S} \to T^1_y\tilde{S}$. Since the dimension of S^1 equals 1 and this map is absolutely continuous with respect to the Lebesgue measure, this homeomorphism is in fact of class C^1. Hence S^1 has a natural invariant C^1-structure, so Hölder continuity with exponent $\alpha \in (0, 1)$ can be defined without ambiguity. A cross ratio defined by a different metric is then Hölder continuous with exponent α for some $\alpha \in (0, 1)$ depending on the metric.

Definition 4.5 An *order preserving orbit equivalence* between two flows (M, Φ^t), (N, Ψ^t) is given by a continuous map $F : M \to N$ and a continuous function $\sigma : M \times \mathbb{R} \to \mathbb{R}$ whose restriction to each set $\{v\} \times \mathbb{R}$ is an increasing homeomorphism and such that $F(\Phi^t v) = \Psi^{\sigma(v,t)}F(v)$ for all $v \in M, t \in \mathbb{R}$. A *reparameterization* of the flow Φ^t is a flow Ψ^t on M such that the identity is an order preserving orbit equivalence between Φ^t and Ψ^t.

Note that a reparameterization of an Anosov flow maps periodic orbits to periodic orbits, so we can talk about the periods of the reparameterized flow. That a negatively curved metric g_1 on a closed manifold which is connected by a smooth path of such metrics to another metric g_0 gives rise to a reparameterization of the geodesic flow for g_0 with controlled reparameterization function was worked out in [47, Proposition 2.2], [22, Theorem A.1].

Reparameterizations of the geodesic flow can also more abstractly be obtained from positive cross ratios, as was observed in [37, Beginning of Section 2], where it was formulated in terms of positive flip invariant Hölder cocycles.

Proposition 4.6 *Let S be a closed hyperbolic surface with unit tangent bundle T^1S and geodesic flow Φ^t. A positive additive cross ratio* $[\,,\,,\,,\,]$ *determines a Hölder reparametrization Ψ^t of Φ^t.*

Proof An ordered triple $(a, b, c) \in (S^1)^3$ of pairwise distinct points determines a point $v \in T^1\tilde{S}$ by the requirement that v is tangent to the oriented geodesic γ connecting a to b and that the footpoint of $v = \gamma'(0)$ is the shortest distance projection of c into γ. For $t > 0$ define $\sigma(v, t) = [a, b, c, u(t)]$ where $u(t)$ is contained in the component of $S^1 - \{a, b\}$ containing c and is such that the projection of $u(t)$ into γ equals $\gamma(t)$. This clearly does not depend on choices and determines the required reparameterization by equivariance. ◻

5 Cross ratios and Kleinian surface groups

For any closed oriented surface S of genus $g \geq 2$, the set of marked hyperbolic metrics on S has naturally the structure of a manifold of dimension $6g - 6$, the so-called *Teichmüller space* of S. Thus the set of all $\pi_1(S)$-invariant cross ratios arising from negatively curved metrics on the surface S contains as a finite dimensional subspace the space of all cross ratios of hyperbolic metrics.

Teichmüller space can be thought of as a connected component of the *character variety* of homomorphisms $\pi_1(S) \to PSL(2,\mathbb{R})$ up to conjugation. This identification arises from the fact that a hyperbolic metric on S defines an isometric action of $\pi_1(S)$ on the universal covering of S which is isometric to the hyperbolic plane, and the thus defined embedding of $\pi_1(S)$ into the group $PSL(2,\mathbb{R})$ of orientation preserving isometries of $\mathbb{H}^2$ is uniquely determined up to conjugation by the hyperbolic metric and a choice of a marking of S. Such a marking is the data of an isomorphism of $\pi_1(S)$ with its image in $PSL(2,\mathbb{R})$. The component of the character variety defining Teichmüller space consists entirely of discrete and faithful representations.

The topology of the character variety for surface group homomorphisms $\pi_1(S) \to G$ (where G is any simple Lie group) is the *algebraic topology* which is defined as follows.

Let $\alpha_1, \dots, \alpha_{2g}$ be a generating set of $\pi_1(S)$. The image in G^{2g} of the tuple $(\alpha_1, \dots, \alpha_{2g})$ under a homomorphism ρ determines ρ completely. Thus the topology of G^{2g} induces a topology on the space of all homomorphisms $\pi_1(S) \to G$. Since $\pi_1(S)$ is a one-relator group, the images of the elements α_i are solutions of a single algebraic equation which defines a subvariety of G^{2g}, the so-called representation variety. The Lie group G acts continuously on the representation variety by conjugation, and the character variety is the geometric invariant theoretic quotient.

As we discussed in Sect. 4 (see Corollary 4.4), a point in Teichmüller space, which is a component of character variety for $PSL(2,\mathbb{R})$, is uniquely determined by a cross ratio, and the cross ratios arising in this way form a finite dimensional family, characterized by special symmetries. Replacing the group $PSL(2,\mathbb{R})$ by another simple Lie group of non-compact type gives rise to other finite dimensional spaces of length functions and cross ratios for $\pi_1(S)$ which are geometrically significant but less well understood. The first difficulty is of topological nature. Namely, let G be a simple Lie group of non-compact type not locally isomorphic to $PSL(2,\mathbb{R})$ and let $K < G$ be a maximal compact subgroup. Then $X = G/K$ is a *symmetric space of non-compact type*, which is a simply connected manifold of non-positive curvature. The image of $\pi_1(S)$ under a discrete and faithful homomorphism $\rho : \pi_1(S) \to G$ is torsion-free and hence acts freely on X. Then $\rho(\pi_1(S))\backslash X$ is a manifold whose fundamental group is isomorphic to $\pi_1(S)$. However, this quotient manifold has infinite volume, and it is a priori unclear how to describe the diffeomorphism type of this manifold.

We next discuss the case $G = PSL(2,\mathbb{C})$, the isometry group of hyperbolic 3-space. Then the space of discrete and faithful representations $\rho : \pi_1(S) \to G$ up to conjugation is precisely the space of complete hyperbolic 3-manifolds whose fundamental group is marked isomorphic to $\pi_1(S)$. Via the natural inclusion $PSL(2,\mathbb{R}) \to PSL(2,\mathbb{C})$, the space of such manifolds, equipped with the algebraic topology, contains Teichmüller space as the subspace of hyperbolic 3-manifolds which deformation retract onto a closed totally geodesic hyperbolic surface diffeomorphic to S.

That the diffeomorphism type of the manifold M defined by the image does not depend on the representation provided that the representation is discrete and faithful is a consequence of a result of Bonahon [8, Theoreme A], see also [1] and [18, Theorem 0.4] for a more general result, and earlier work in 3-dimensional topology [41]. Namely, Bonahon's work implies that M is homeomorphic to the interior of a compact 3-manifold $\bar{M}$ with boundary, and classical tools in 3–manifold topology can be used to show that $\bar{M}$ is homeomorphic to an I-bundle over S (recall that we require S to be orientable).

Proposition 5.1 *Any hyperbolic* 3*-manifold whose fundamental group is isomorphic to the fundamental group* $\pi_1(S)$ *of a closed surface* S *is diffeomorphic to* $S \times \mathbb{R}$.

A *Kleinian group* is a discrete subgroup of $PSL(2, \mathbb{C})$. The *limit set* Λ of a Kleinian group Γ is the set of accumulation points of a Γ-orbit in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. It is the smallest non-empty closed Γ-invariant subset of the ideal boundary $\mathbb{C}P^1 = S^2$ of $\mathbb{H}^3$. If Γ is isomorphic to $\pi_1(S)$ then Γ is called *quasi-Fuchsian* if its limit set Λ is homeomorphic to a circle and if there exists an equivariant Hölder homeomorphism $\partial\tilde{S} \to \Lambda$

A quasi-Fuchsian group Γ acts on the two components of $\mathbb{C}P^1 - \Lambda$ as a group of conformal transformations. Since each of these components is biholomorphic to a disk by the Riemann mapping theorem, this action determines two (in general distinct) points in $\mathcal{T}(S)$ which in turn determine Γ completely. Quasi-Fuchsian groups form an open connected subset of the smooth locus of character variety, but they are strictly contained in their proper component of character variety. We refer to [57, Chapter 4] for more information.

Now on $\partial\mathbb{H}^3 = S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ there exists a natural $PSL(2, \mathbb{C})$-invariant multiplicative cross ratio Cr, defined by the formula (3.2) for points in $\partial\mathbb{H}^3 - \{\infty\} = \mathbb{C}$. This cross ratio then extends to a $PSL(2, \mathbb{C})$-invariant cross ratio on $\partial\mathbb{H}^3$. The following was pointed out in [38, Section 3]. For its formulation, note that the space of cross ratios on S^1 is naturally equipped with the compact open topology.

Proposition 5.2 *The map which associates to a quasi-Fuchsian representation ρ the cross ratio on $S^1 = \partial\mathbb{H}^2$ which is the pull-back of the restriction of the cross ratio of $\mathbb{C}P^1$ to the limit set of ρ is an embedding.*

As discussed in Sect. 3, Proposition 5.2 is equivalent to stating that quasi-Fuchsian manifolds enjoy marked length spectrum rigidity: a quasi-Fuchsian manifold is determined by its marked length spectrum among all quasi-Fuchsian manifolds. This does *not* imply that a quasi-Fuchsian manifold is determined by its marked length spectrum among other classes of geometric structures.

The quasi-Fuchsian group Γ acts properly, freely and cocompactly on the set $\mathcal{G}$ of unit tangent vectors of oriented geodesic lines in $\mathbb{H}^3$ with both endpoints in the limit set Λ of Γ. The quotient manifold N is homeomorphic to T^1S, and the geodesic flow on N, which is the projection of the geodesic flow on $\mathcal{G}$, is order preserving orbit equivalent to the geodesic flow on T^1S (see [38, Lemma 3.1]). However, we have

Proposition 5.3 *There are quasi-Fuchsian manifolds whose associated additive cross ratio is not positive. For such a quasi-Fuchsian manifold there is no negatively curved metric on S which gives rise to the same length function.*

Proof The group $PSL(2, \mathbb{C})$ acts triply transitively on $\mathbb{C}P^1$, and it preserves the cross ratio Cr. By invariance, an ordered triple of distinct points in the limit set Λ of the quasi-Fuchsian group Γ may be moved with an element of $PSL(2, \mathbb{C})$ to the triple $(0, 1, \infty)$. The set $U = \{z \in \mathbb{C}P^1 \mid \mathrm{Cr}(0, z, 1, \infty) < 1\}$ is open and not empty. We claim that there are quasi-Fuchsian groups Γ so that the oriented arc in the limit set Λ connecting 0 to 1 intersects U.

To this end note that there exists a sequence of quasi-Fuchsian groups Γ_i which converge algebraically to a *doubly degenerate* Kleinian group isomorphic to $\pi_1(S)$. An example of such a group arises from the natural infinite cyclic covering of a hyperbolic 3-manifold which fibers over the circle. We refer to the survey [53] for details and references. By a result of Thurston (see [57, Theorems 7.35 and 7.41]), the limit sets Λ_i of the groups Γ_i converge in the Hausdorff topology to $\mathbb{C}P^1$, which is the limit set of a doubly degenerate Kleinian group. Furthermore, the corresponding equivariant maps $S^1 = \partial\mathbb{H}^2 \to \Lambda_i$ converge in the C^0-topology to a space filling curve. We refer to [58] for details and a historical account. By equivariance, that is, north–south dynamics of the action of all nontrivial elements of Γ, the image of any open subset of S^1 under the limiting map is space filling.

But this means that for sufficiently large i, the image of the arc in Λ_i connecting 0 to 1 meets the region U and hence there exists an ordered quadruple of distinct points in S^1 which is mapped to a quadruple on which $\log \mathrm{Cr}$ takes on a negative value. Since on the other hand the cross ratio of the length function of a negatively curved metric on S is positive, this shows that the length function of this quasi-Fuchsian manifold is not the length funtion of any negatively curved metric on S. □

Since the quasi-Fuchsian space for S is a complex manifold of complex dimension $6g-6$ it makes sense to ask the following question, which seeks an answer similar to Bonahon's characterization [9, Theorem 13] in the Fuchsian case (see (3.1)).

Question 5.4 *Can cross ratios defined by quasi-Fuchsian groups be characterized by additional symmetries?*

Bridgeman and Canary [11, Theorem 1.1] established a strengthening of Proposition 5.2 which is valid for all hyperbolic 3-manifolds whose fundamental group is isomorphic to a surface group. Due to the fact that the limit set of a Kleinian surface group may be the entire sphere $\mathbb{C}P^1$, this result does not seem to have an interpretation via cross ratios.

Theorem 5.5 *(Bridgeman-Canary) A hyperbolic 3-manifold whose fundamental group is isomorphic to $\pi_1(S)$ is determined up to isometry by the lengths of all closed geodesics corresponding to simple closed curves on S.*

A result of Jörgensen [45, Theorem 1] shows that for $G = PSL(2, \mathbb{C})$, discreteness of a homomorphism is a closed condition with respect to the algebraic topology. On the other hand, we saw above that quasi-Fuchsian space is a proper subset of a component of the character variety.

The structure of the space of all conjugacy classes of discrete and faithful representations of surface groups into $PSL(2, \mathbb{C})$, equipped with the algebraic topology, was uncovered by the solution to the so-called *ending lamination conjecture* [14]. As a consequence, the set of quasi-Fuchsian representations of a surface group forms a dense subset of this space.

6 Cross ratios, Hilbert metrics and real projective structures

In this final section we consider subgroups of the simple Lie group $PSL(n+1, \mathbb{R})$ acting as a group of projective automorphisms on real projective space $\mathbb{R}P^n$.

An open subset Ω of $\mathbb{R}P^n$ which is contained in the complement of a projective hyperplane has two disjoint preimages Ω_+, Ω_- in the sphere S^n which are exchanged by the isometric involution $x \to -x$. The set Ω is called *properly convex* if one of its preimages in S^n is convex, and it is called *strictly convex* if any projective line intersects the boundary $\partial\Omega$ of Ω in at most two points. An example of such a set is an *ellipsoid* which is defined as follows. Let q be a quadratic form on $\mathbb{R}^{n+1}$ of signature $(1, n)$ and put

$$\Omega_q = \{[v] \mid q(v) > 0\}.$$

Clearly an ellipsoid is invariant under the stabilizer $PSO(1, n) \subset PSL(n+1, \mathbb{R})$ of the quadratic form q. Furthermore, all ellipsoids are projectively equivalent.

To each properly convex set Ω is associated its dual Ω^* which is the convex subset of the projectivization of the dual defined by

$$\Omega^* = \{\mathbb{R}f \mid f(v) > 0 \text{ for all } v \in \Omega_+ \subset S^n\}$$

which also is properly convex. If Ω is an ellipsoid then so is Ω^*.

If $\Omega \subset \mathbb{R}P^n$ is a strictly convex set, then one can use the invariant additive cross ratio $[\,,\,,\,]$, defined as before using projection into $\mathbb{R}^n$, to define a so-called *Hilbert metric* on Ω as follows. For $x \neq y \in \Omega$ there exists a unique projective line L passing through x, y. This line intersects the boundary $\partial\Omega$ of Ω in two points $p_\pm$. The *Hilbert distance* between x, y is then defined as $\frac{1}{2}[p_-, x, y, p_+]$ where we assume that the points (p_-, x, y, p_+) are ordered on L.

If Ω is an ellipsoid then its Hilbert metric is just the standard hyperbolic metric. In general the Hilbert metric is only a Finsler metric. Nevertheless it makes sense to ask about conditions which guarantee that the Hilbert metric is hyperbolic in the sense of Gromov, and one can ask for discrete subgroups of $PSL(n+1, \mathbb{R})$ acting properly and cocompactly on Ω. The first question was completely settled by Benoist [4, Théorème 1.4] and can be expressed once again in terms of cross ratios. We next explain his result.

A properly convex subset Ω of $\mathbb{R}P^n$ is called *divisible* if it admits a proper cocompact action by a subgroup Γ of $PSL(n+1, \mathbb{R})$. We then say that Γ divides Ω. Among the ellipsoids (which can be identified with hyperbolic space), there are other *symmetric* divisible properly convex sets. These are precisely the convex sets Ω on which the automorphism group $\mathrm{Aut}(\Omega)$ acts transitively and such that $\mathrm{Aut}(\Omega)$ is reductive. Here the automorphism group $\mathrm{Aut}(\Omega)$ is the subgroup of $PSL(n+1, \mathbb{R})$ which stabilizes Ω. They arise from symmetric convex cones and are completely classified. We refer to [23] for more information.

The following result of Benoist [5, Théorème 1.1] characterizes divisible strictly convex sets.

Theorem 6.1 *(Benoist)*

1. *If Γ divides some divisible properly convex set Ω then Γ is hyperbolic in the sense of Gromov if and only if Ω is strictly convex.*
2. *A divisible properly convex open set has a boundary of class C^1 if and only if it is strictly convex.*

As in the case of Fuchsian groups or quasi-Fuchsian groups, it makes sense to study the deformation space of convex projective structures for a given finitely generated group Γ. More precisely, we are interested in

$$\mathcal{F}_\Gamma = \{\rho \in \mathrm{Hom}(\Gamma, PSL(n+1, \mathbb{R})) \mid \rho \text{ is faithful with discrete image}$$
$$\text{dividing a properly convex open set } \Omega_\rho \subset S^n\}$$

as well as its quotient $X_\Gamma = \mathcal{F}_\Gamma / PSL(n+1, \mathbb{R})$ under the action of $PSL(n+1, \mathbb{R})$ by conjugation.

A convex set $\Omega \subset S^n$ is *irreducible* if the cone C over Ω can not be written as the sum $C = C_1 + C_2$ of two convex cones contained in proper subspaces V_i of $\mathbb{R}^{n+1}$. Call a homomorphism $\rho : \Gamma \to PSL(n+1, \mathbb{R})$ *strongly irreducible* if no finite index subgroup stabilizes a reducible properly convex subset of $\mathbb{R}P^n$. The following combines results of Koszul [51, Corollaire] and Benoist [6, Théorème 1.1, Corollaires 1.2 and 2.13(i'),(iii)] and contrasts the case of surface group representations into $PSL(2, \mathbb{C})$. Recall that a quasi-Fuchsian group acts properly and cocompactly on a component Ω of its domain of continuity in $\mathbb{C}P^1$.

Theorem 6.2 *(Koszul, Benoist) If $\mathcal{F}_\Gamma$ contains a strongly irreducible representation, then the set $\mathcal{F}_\Gamma$ is open and closed in* $\mathrm{Hom}(\Gamma, PSL(m+1,\mathbb{R}))$ *and hence it is a union of connected components.*

In the case $n = 2$, a group Γ dividing a properly convex subset of $\mathbb{R}P^2$ is necessarily a surface group. It is due to Goldman [31, Corollary after Theorem 1] that the moduli space of such real projective structures on a surface of genus $g \geq 2$ is diffeomorphic to $\mathbb{R}^{16g-16}$. Perhaps unexpectedly, the moduli space of such structures can also be interesting in higher dimensions. The following is due to Johnson and Millson [44, Theorem 1].

Theorem 6.3 *(Johnson-Millson) For any $n \geq 2$ there exist discrete cocompact subgroups Γ of $PO(1,n)$ so that the component $\mathcal{F}_\Gamma$ containing the inclusion $\Gamma \to PSL(n+1,\mathbb{R})$ has positive dimension.*

Since the Borel conjecture is known for hyperbolic groups by the work of Barthels-Lück [2, Theorem A], these deformations give rise to mutually homeomorphic quotient manifolds if $n \neq 4$.

For any subgroup $\Gamma < PSL(n+1,\mathbb{R})$ dividing a properly convex subset of $\mathbb{R}P^n$, the marked length spectrum is defined as before in terms of the Hilbert metric. The translation length of an element $\gamma \in \Gamma$ for this metric equals

$$\ell(\gamma) = \log \frac{\lambda_1(\gamma)}{\lambda_{n+1}(\gamma)}$$

where $\lambda_1(\gamma)$ is the spectral radius of γ as a matrix in $SL(n+1,\mathbb{R})$ and $\lambda_{n+1}(\gamma)^{-1}$ is the spectral radius of γ^{-1}.

The following rigidity result is due to Cooper and Delp [19, Theorem 1.1], extending (and correcting) earlier work of Kim [49, Theorem I]. For its formulation, let $\Omega \subset \mathbb{R}P^n$ be a strictly convex set. Choose a component Ω_+ of its preimage in S^n, with dual Ω^*.

Theorem 6.4 *(Cooper-Delp) Two distinct strictly convex projective structures on Γ have the same marked Hilbert length spectrum if and only if each structure is the projective dual of the other.*

Assume that Γ acts properly and cocompactly on the stricly convex subset Ω of $\mathbb{R}P^n$. Then Γ is a finitely generated subgroup of $PSL(n+1,\mathbb{R})$. By Selberg's lemma (see [64, Corollary 4 on page 331]) Γ has a finite index normal subgroup which is torsion-free. Thus let us assume that Γ is torsion-free for simplicity. Then Ω/Γ is a closed manifold M. As Γ is hyperbolic in the sense of Gromov by Theorem 6.1, the homeomorphism type of M does not depend on Ω if $n \neq 4$. As a consequence, by considering a non-self-dual strictly convex projective structure (see [19, Theorem 1.2]), Theorem 6.4 yields non-isometric real projective structures on a fixed manifold M with the same marked length spectrum. However, this ambiguity is completely understood.

More precisely, there is an asymmetry for an element $A \in SL(n+1,\mathbb{R})$ with pairwise distinct real eigenvalues $\lambda_1 > \cdots > \lambda_{n+1}$ and its inverse A^{-1}. Namely, the largest eigenvalue of A^{-1} equals λ_{n+1}^{-1}, which is in general different from λ_1. The Hilbert length gives the same translation length to A and A^{-1}, and it turns out that this identifies the translations lengths on a convex set and its dual.

This ambiguity can be resolved by replacing the Hilbert translation length by other natural length functions. For example, the *spectral length* $\ell(A)$ of an element $A \in SL(n+1,\mathbb{R})$ which is diagonalizable over $\mathbb{R}$ is the logarithm of the maximal eigenvalue of A. Associating to a representation $\rho : \Gamma \to PSL(n+1, \mathbb{R})$ and $\gamma \in \Gamma$ the spectral length of its image $\rho(\gamma)$ defines a length function for the representation which turns out to be rigid.

Namely, given a surface S of genus at least two, there is a distinguished component of the character variety for surface group homomorphisms $\pi_1(S) \to PSL(n, \mathbb{R})$, the so-called *Hitchin component* (see [42, Theorem A]). This is the component containing the *Fuchsian locus*, the set of all homomorphisms which factor through an embedding $PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ defined by an irreducible representation of $PSL(2, \mathbb{R})$ on $\mathbb{R}^n$. Such a representation is unique up to conjugation. The Hitchin component consists entirely of discrete and faithful representations (see [52, Theorem 1.5], [26]). For such a representation ρ and any element $\gamma \in \Gamma$, the element $\rho(\gamma)$ is diagonalizable over $\mathbb{R}$ and hence its spectral length is defined. The resulting function on Γ is called the *spectral marked length spectrum*. The following rigidity result is due to Bridgeman, Canary, Labourie and Sambarino ([13, Theorem 1.2 and Corollary 11.6]).

Theorem 6.5 *(Spectral length rigidity) Two surface group representations in the Hitchin component with the same spectral marked length spectrum are conjugate.*

In fact, as for quasi-Fuchsian groups, a Hitchin representation is already determined by the spectral lengths of the images of all non-separating simple closed curves on S [12, Theorem 1.1].

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Agol, I.: Tameness of hyperbolic 3-manifolds. arXiv:math/0405568, (2004)
2. Barthels, A., Lück, W.: The Borel Conjecture for hyperbolic and CAT(0)-groups. Ann. Math. **175**(2), 631–689 (2012)
3. Besson, G., Courtois, G., Gallot, S.: Entrop ies et rigidités des espaces localement symétriques de courbure strictement négative. Geom. Funct. Anal. **5**, 731–799 (1995)
4. Benoist, Y.: Convexes hyperboliques et fonctions quasisymétriques. Publications Mathématiques de l'Institut des Hautes Études Scientifiques **97**, 181–237 (2003)
5. Benoist, Y.: Convexes divisibles I. Tata Inst. Fundam. Res. Stud. Math. **17**, 339–374 (2004)
6. Benoist, Y.: Convexes divisibles III. Annales Scientifiques de l'Ecole Normale Supérieure **38**(5), 793–832 (2005)
7. Bing, R.H.: An alternative proof that 3-manifolds can be triangulated. Ann. Math. **69**(1), 37–65 (1959)
8. Bonahon, F.: Bouts des variétés hyperboliques de dimension 3. Ann. Math. **124**(2), 71–158 (1986)
9. Bonahon, F.: The geometry of Teichmüller space via geodesic currents. Inventiones Mathematicae **92**, 139–162 (1988)
10. Bourdon, M.: Sur le birapport au bord des CAT(-1)-espaces. Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques **83**, 95–104 (1996)
11. Bridgeman, M., Canary, R.: Simple length rigidity for Kleinian surface groups and applications. Commentarii Mathematici Helvetici **92**(4), 715–750 (2017)
12. Bridgeman, M., Canary, R., Labourie, F.: Simple length rigidity for Hitchin representations. Adv. Math. **360**, 106901 (2020)

13. Bridgeman, M., Canary, R., Labourie, F., Sambarino, A.: The pressure metric for Anosov representations. Geom. Funct. Anal. **25**, 1089–1179 (2015)
14. Brock, J., Canary, R., Minsky, Y.: The classification of Kleinian surface groups II: The ending lamination conjecture. Ann. Math. **176**(1), 1–149 (2012)
15. Brown, K.S.: *Cohomology of Groups*. Graduate Texts in Mathematics. Springer, Berlin, (1982)
16. Burger, M., Gelander, T., Lubotzky, A., Mozes, S.: Counting hyperbolic manifolds. Geom. Funct. Anal. **12**(6), 1161–1173 (2002)
17. Burns, K., Katok, A.: Manifolds with non-positive curvature. Ergodic Theor. Dyn. Syst. **5**(2), 307–317 (1985)
18. Calegari, D., Gabai, D.: Shrinkwrapping and the Taming of Hyperbolic 3-Manifolds. J. Am. Math. Soc. **19**(2), 385–446 (2005)
19. Cooper, D., Delp, K.: The marked length spectrum of a projective manifold or orbifold. Proc. Am. Math. Soc. **138**(9), 3361–3376 (2010)
20. Croke, C.: Rigidity for surfaces of nonpositive curvature. Commentarii Mathematici Helvetici **65**(1), 150–169 (1990)
21. Croke, C., Sharafutinov, V.A.: Spectral rigidity of a compact negatively curved manifold. Topology **37**, 1265–1273 (1998)
22. de la Llave, R., Marco, J.M., Moriyòn, R.: Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation. Ann. Math. **123**(3), 537–611 (1986)
23. Farault, J., Koranyi, A.: Analysis on symmetric cones. Oxford Mathematics. Monographs. Clarendon Press, Oxford (1994)
24. Farrel, F.T., Jones, L.E.: Negatively curved manifolds with exotic smooth structures. J. Am. Math. Soc. **2**(4), 899–908 (1989)
25. Farrel, F.T., Jones, L.E.: Rigidity and other topological aspects of compact nonpositively curved manifolds. Bull. Am. Math. Soc. **22**(1), 59–64 (1990)
26. Fock, V., Goncharov, A.: Moduli spaces of local systems and higher Teichmüller theory. Publications Mathématiques de l'Institut des Hautes Etudes Scientifiques **103**, 1–211 (2006)
27. Freedman, M.H., Quinn, F.: Topology of 4-Manifolds (PMS-39). Princeton University Press, Princeton (1990)
28. Gabai, D., Meyerhoff, R., Thurston, N.: Homotopy hyperbolic 3-manifolds are hyperbolic. Ann. Math. **157**(2), 335–431 (2003)
29. Gelander, T.: Homotopy type and volume of locally symmetric manifolds. Duke Math. J. **124**(3), 459–515 (2004)
30. Goldfeld, D., Lubotzky, A., Pyber, L.: Counting congruence subgroups. Acta Math. **193**, 73–104 (2004)
31. Goldman, W.: Convex real projective structures on compact surfaces. J. Differ. Geom. **31**, 791–845 (1990)
32. Guillarmou, C., Lefeuvre, T.: The marked length spectrum of Anosov manifolds. Ann. Math. **190**, 321–344 (2019)
33. Gouillarmou, C., Lefeuvre, T., Paternain, G.: Marked length spectrum rigidity for Anosov surfaces. arXiv:2303.12007 (2023)
34. Gromov, M., Thurston, W.: Pinching constants for hyperbolic manifolds. Inventiones Mathematicae **89**, 1–12 (1987)
35. Hamenstädt, U.: Time-preserving conjugacies of geodesic flows. Ergodic Theor. Dyn. Syst. **12**(1), 67–74 (1992)
36. Hamenstädt, U.: Cocycles, Hausdorff measures and cross ratios. Ergodic Theor. Dyn. Syst. **17**, 1061–1081 (1997)
37. Hamenstädt, U.: Cocycles, symplectic structures and interesection. Geom. Funct. Anal. **9**(1), 90–140 (1999)
38. Hamenstädt, U.: Ergodic properties of function groups. Geometriae Dedicata **93**, 163–176 (2002)
39. Hasselblatt, B., Katok, A.: Introduction to the Modern Theory of Dynamical Systems. Cambridge University Press, Cambridge (1995)
40. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
41. Hempel, J.: 3-Manifolds. Annals of Mathematics Studies. Princeton Univ. Press, Princeton (1976)
42. Hitchin, N.J.: Lie groups and Teichmüller space. Topology **31**(3), 449–473 (1992)
43. Italiano, G., Martelli, B., Migliorini, M.: Hyperbolic 5-manifolds that fiber over S^1. Inventiones Mathematicae **231**, 1–38 (2023)
44. Johnson, D., Millson, J.: Deformation spaces associated to compact hyperbolic manifolds. Bull. Am. Math. Soc. (N.S.) **14**(2), 99–102 (1986)
45. Jorgensen, T.: On discrete groups of Möbius transformations. Am. J. Math. **98**(3), 739–749 (1976)
46. Kapovich, M.: Hyperbolic Manifolds and Discrete Groups. Progress in Mathematics, Birkhäuser, Boston (2001)

47. Katok, A., Knieper, G., Pollicott, M., Weiss, H.: Differentiability and analyticity of topological entropy for Anosov and geodesic flows. Inventiones Mathematicae **98**, 581–597 (1989)
48. Kervaire, M., Milnor, J.: Groups of Homotopy Spheres: I. Ann. Math. **77**(2), 504–537 (1963)
49. Kim, I.: Rigidity and deformation spaces of strictly convex real projective structures on compact manifolds. J. Differ. Geom. **58**, 189–218 (2001)
50. Knieper, G.: *Hyperbolic Dynamics and Riemannian Geometry*. Chapter 6 in the Handbook of Dynamical Systems, (Vol. 1(A), pp. 453–545), (2002)
51. Koszul, J.L.: Déformation des connexions localement plates. Annales de l' Institut Fourier **18**, 103–114 (1968)
52. Labourie, F.: Anosov flows, surface groups and curves in projective space. Inventiones Mathematicae **165**, 51–114 (2006)
53. Lecuire, C.: The double limit theorem and its legacy. In: *The tradition of Thurston-Geometry and Topology*, pp. 263–290, Springer, Cham, (2020)
54. Lubotzky, A.: Subgroup growth and congruence subgroups. Inventiones Mathematicae **119**, 267–295 (1995)
55. Lück, W.: Survey on aspherical manifolds. arXiv:0902.2480, (2009)
56. Lück, W.: *Surgery Theory: Foundations*. Unfinished Monograph, Available on the Author's Webpage, Version from June 12, (2023)
57. Matsuzaki, K., Taniguchi, M.: Hyperbolic Manifolds and Kleinian Groups. Oxford Mathematical Monographs. Clarendon Press, Oxford (1998)
58. Mj, M.: Cannon Thurston maps for surface groups. Ann. Math. **179**(1), 1–80 (2014)
59. Mostow, G.D.: Strong Rigidity of Locally Symmetric Spaces. Annals of Mathematics Studies, vol. 78. Princeton University Press, Princeton (1973)
60. Munkres, J.: Obstructions to the smoothing of piecewise-differentiable homeomorphisms. Bull. Am. Math. Soc. **65**(5), 332–334 (1959)
61. Ontaneda, P.: Riemannian hyperbolization. Publications Mathématiques de l'Institut des Hautes Études Scientifiques **131**, 1–72 (2020)
62. Otal, J.P.: Le Spectre Marque des Longueurs des Surfaces a Courbure Negative. Ann. Math. **131**(1), 151–162 (1990)
63. Paternain, G.: Geodesic Flows. Springer, New York (1999)
64. Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds, 3rd edn. Springer, New York (2019)