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ORIGINAL PAPER

Total torsion of three-dimensional lines of curvature

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Abstract
A curve γ in a Riemannian manifold M is *three-dimensional* if its torsion (signed second curvature function) is well-defined and all higher-order curvatures vanish identically. In particular, when γ lies on an oriented hypersurface S of M, we say that γ is *well positioned* if the curve's principal normal, its torsion vector, and the surface normal are everywhere coplanar. Suppose that γ is three-dimensional and closed. We show that if γ is a well-positioned line of curvature of S, then its total torsion is an integer multiple of 2π; and that, conversely, if the total torsion of γ is an integer multiple of 2π, then there exists an oriented hypersurface of M in which γ is a well-positioned line of curvature. Moreover, under the same assumptions, we prove that the total torsion of γ vanishes when S is convex. This extends the classical total torsion theorem for spherical curves.

Keywords Darboux curvatures · Parallel rotation · Three-dimensional curve · Total geodesic torsion

Mathematics Subject Classification (2020) Primary 53A04; Secondary 53A07 · 53C40

Contents

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1 Introduction and main result

In classical differential geometry, the *total torsion theorem* states that the total torsion of a closed spherical curve vanishes; see [3, 4, 10, 11] and [7, p. 170].

Theorem 1.1 *Let* $I = [0, \ell]$*, and let* $\gamma : I \to \mathbb{R}^3$ *be a smooth regular curve. If* γ *is closed and* $\gamma(I) \in \mathbb{S}^2$*, then*

$$\int_0^\ell \tau \, dt = 0.$$

Theorem 1.1 manifests the fact that "the torsion of a closed curve lying on a surface in $\mathbb{R}^3$ is somehow constrained by the geometry of [the] surface" [8, p. 111]; see, e.g., [1, 5, 6, 12] for further evidence of the same fact.

Closely related to Theorem 1.1 is the following result of Qin and Li.

Theorem 1.2 [9] *Let S be a (smooth) oriented surface in* $\mathbb{R}^3$*. If* γ *is a closed line of curvature of S, then the total torsion is an integer multiple of* 2π*. Conversely, if the total torsion of a closed curve in* $\mathbb{R}^3$ *is an integer multiple of* 2π*, then it can appear as a line of curvature of an oriented surface.*

Theorem 1.1 and the first part of Theorem 1.2 have been generalized to three-dimensional orientable Riemannian manifolds of constant curvature M_c^3 [8]; see also [2, 15] for related results. In the present note we shall see that, under suitable assumptions, both theorems remain valid when M_c^3 is replaced by an arbitrary Riemannian manifold $M^m \equiv M$, provided one restricts the attention to *three-dimensional* curves; roughly speaking, a curve in M is three-dimensional if it has one curvature and one "torsion", all other curvature functions being zero. As we explain below, in that case one should interpret "torsion" as a signed version of Spivak's "second curvature function" [13, p. 22].

Let γ be a unit-speed curve $I \to M$, let N be a unit normal vector field along γ, and let $\pi_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H} = (\gamma' \oplus N)^\perp$. We say that N is *torsion-defining* if there exists a smooth unit vector field $W(N)$ along γ that is everywhere parallel to $T_g = -\pi_{\mathcal{H}} D_t N$. If N is torsion-defining, then the function $\tau_g = \langle T_g, W(N) \rangle$ is called the *(first) geodesic torsion of* γ *with respect to* N. In particular, if $D_t \gamma'$ is never zero, then the geodesic torsion of γ with respect to the principal normal $P = D_t \gamma' / \kappa$ is called the *(first) torsion of* γ, and γ is said to be a *Frenet curve*.

The logic behind our terminology is the following. In the same way a generic curve in $\mathbb{R}^3$ has one (unsigned) curvature plus one (signed) torsion, a generic curve in M may have one (unsigned) curvature plus $m - 2$ (signed) torsions; cf. [13]. On the other hand, since we never deal with higher-order torsions, we typically speak of "torsion" as a shorthand for "first torsion".

Now, to state our generalization of Theorems 1.1 and 1.2, let S be an oriented hypersurface of M, and let N_S be its unit normal. A Frenet curve on S is said to be *well positioned* if N_S, P, and $W(P)$ are everywhere coplanar.

Theorem 1.3 *Suppose that* γ *is three-dimensional, i.e., that* γ *is a Frenet curve such that* $W(P)$ *is parallel in* $\mathcal{H}(P)$*; see Definition 6.1. If* γ *is a well-positioned closed line of curvature of S, then the total torsion of* γ *is an integer multiple of* 2π*; in particular, the total torsion vanishes when S is convex, i.e., when the second fundamental form of S is positive definite. Conversely, if* γ *is open, then there exists an orientable hypersurface in which* γ *is a well-positioned line of curvature; if* γ *is closed, then the same holds provided the total torsion of* γ *is an integer multiple of* 2π*.*

Clearly, when $\dim M = 3$, every Frenet curve is three-dimensional, and every Frenet curve on S is well positioned. Specializing the theorem to that case, we may state the following result.

Corollary 1.4 *Suppose that* $\dim M = 3$ *and that* γ *is a closed Frenet curve. If* γ *is a line of curvature of S, then the total torsion of* γ *is an integer multiple of* 2π*; in particular, the total torsion vanishes when S is convex. Conversely, if the total torsion of* γ *is an integer multiple of* 2π*, then there exists an orientable surface in which* γ *is a line of curvature.*

Remark 1.5 If $\dim M = 3$, then every regular curve with nonvanishing curvature is a Frenet curve.

We will obtain Theorem 1.3 as a corollary of a more general statement involving the geodesic torsion of γ with respect to an arbitrary unit normal vector field N along γ, in which the assumption that γ is three-dimensional is replaced by the condition that N is a *parallel rotation* of N_S.

Let N and Z be unit normal vector fields along γ. We say that Z is a *rotation of* N if there exists a continuous unit vector field $H(N, Z) \equiv H$ such that

(1) $\langle H, \gamma' \rangle = \langle H, N \rangle = 0$, i.e., $H \in \Gamma(\mathcal{H})$;
(2) H, N, and Z are everywhere linearly dependent.

Clearly, if $N \wedge Z$ is nowhere zero, then the vector field H is defined up to a sign.

Now, suppose that Z is a rotation of N. Then we can write

$$Z \equiv N(\theta) = -\sin(\theta) H + \cos(\theta) N$$

for some continuous function $\theta : I \to \mathbb{R}$.

Definition 1.6 A rotation of N is said to be *parallel* if H is parallel with respect to the induced connection on $\mathcal{H}$, and *closed* if $\theta(\ell) - \theta(0) = 2n\pi$ for some $n \in \mathbb{Z}$.

Remark 1.7

(1) If $\dim M = 3$, then any unit normal vector field along γ is a parallel rotation of N.
(2) If γ is closed, then so is any rotation of N.

Theorem 1.8 *If* γ *is a line of curvature of S, then the total geodesic torsion of* γ *with respect to any closed parallel rotation of* N_S *is an integer multiple of* 2π*. Conversely, suppose that N is torsion-defining and that* $W(N)$ *is parallel in* $\mathcal{H}$*. If* γ *is open, then there exists an orientable hypersurface of M in which* γ *is a line of curvature; if* γ *is closed, then the same holds provided the total geodesic torsion of* γ *with respect to N is an integer multiple of* 2π*.*

Remark 1.9 It follows from Sect. 4 that, if γ is a line of curvature of S and P is a parallel rotation of N_S, then γ is three-dimensional.

The remainder of the paper is organized a follows. In Sect. 2 we set up some notations. In Sect. 3 we generalize the well-known concepts of geodesic curvature, normal curvature, and geodesic torsion of a curve on a surface in $\mathbb{R}^3$ to a curve on a hypersurface of M; although, under reasonable assumptions, one may define $m - 2$ geodesic curvatures and geodesic torsions, for the sake of simplicity we shall limit ourselves to first-order curvatures. In Sect. 4 we obtain formulas expressing the curvature vectors of γ with respect to a rotation of N in terms of the rotation angle. Finally, in Sects. 5 and 6 we prove Theorems 1.8 and 1.3, respectively.

2 Preliminaries

In this section we discuss some preliminaries.

Let M be an m-dimensional Riemannian manifold, let γ be a smooth unit-speed curve $I \to M$, and let $TM|_\gamma$ be the ambient tangent bundle over γ. Recall that

$$TM|_\gamma = \bigsqcup_{t\in I} T_{\gamma(t)}M.$$

We define a *distribution of rank r along γ* to be a rank-r subbundle of $TM|_\gamma$.

Let $\mathcal{D}$ be a distribution of rank r along γ, and let $\mathcal{D}^\perp$ be the distribution of rank $m - r$ along γ whose fiber at t is the orthogonal complement $\mathcal{D}_t^\perp$ of $\mathcal{D}_t$ in $T_{\gamma(t)}M$, so that $TM|_\gamma$ splits as

$$TM|_\gamma = \mathcal{D} \oplus \mathcal{D}^\perp;$$

accordingly, we write

$$X = X^v + X^h$$

for any vector field X along γ.

In this setting, the *tangential projection* is the map $\pi_\mathcal{D} \colon \Gamma(TM|_\gamma) \to \Gamma(\mathcal{D})$ given by

$$X \mapsto X^v.$$

Likewise, the *normal projection* is the map $\pi_\mathcal{D}^\perp \colon \Gamma(TM|_\gamma) \to \Gamma(\mathcal{D}^\perp)$ sending each X to the corresponding X^h.

3 Darboux curvatures and curvature vectors

The purpose of this section is to extend the classical notions of geodesic curvature, normal curvature, and geodesic torsion of a curve on a surface in $\mathbb{R}^3$ to a curve on a hypersurface of M.

Let γ be a (smooth) unit-speed curve $I \to M$, let N be a unit normal vector field along γ, and let $\mathcal{H}(N) \equiv \mathcal{H}$ be the distribution of rank $m-2$ along γ whose fiber at t is the orthogonal complement of $E(t) = \gamma'(t)$ and $N(t)$ in $T_{\gamma(t)}M$. Denoting by D_t the covariant derivative along γ, we define

- the *(first) geodesic curvature vector K_g of γ with respect to N* by

$$K_g = \pi_\mathcal{H} D_t E;$$

- the *normal curvature vector K_n of γ with respect to N* by

$$K_n = \pi_\mathcal{N} D_t E,$$

  where $\mathcal{N} = \operatorname{span} N$;
- the *(first) geodesic torsion vector T_g of γ with respect to N* by

$$T_g = -\pi_\mathcal{H} D_t N.$$

To express these vector fields in coordinates, let $(H_1, \dots, H_{m-2})$ be a smooth orthonormal frame for $\mathcal{H}$. Then there are functions $\kappa_g^1, \dots, \kappa_g^{m-2}$, κ_n, and $\tau_g^1, \dots, \tau_g^{m-2}$ such that

$$K_g = \kappa_g^1 H_1 + \cdots + \kappa_g^{m-2} H_{m-2},$$

$$K_n = \kappa_n N,$$
$$T_g = \tau_g^1 H_1 + \cdots + \tau_g^{m-2} H_{m-2}.$$

Note that, since $(E, H_1, \ldots, H_{m-2}, N)$ is orthonormal, the following equations hold for all $j = 1, \ldots, m-2$:

$$D_t E = \kappa_g^1 H_1 + \cdots + \kappa_g^{m-2} H_{m-2} + \kappa_n N,$$
$$D_t H_j = -\kappa_g^j E + \tau_g^j N + \pi_{\mathcal{H}} D_t H_j,$$
$$D_t N = -\kappa_n E - \tau_g^1 H_1 - \cdots - \tau_g^{m-2} H_{m-2}.$$

The curvature vectors allow us to define corresponding curvature *functions*. In one case, the definition is trivial: the function $\kappa_n = \langle D_t E, N\rangle$ is called the *normal curvature of γ with respect to N*. For the remaining two cases, we proceed as follows.

We say that N is *curvature-defining* if there exists a smooth unit vector field $V(N) \equiv V$ along γ that is everywhere parallel to K_g. If N is curvature-defining, then the function $\kappa_g = \langle K_g, V\rangle$ is called the *(first) geodesic curvature of γ with respect to N.*

Similarly, we say that N is *torsion-defining* if there exists a smooth unit vector field $W(N) \equiv W$ along γ that is everywhere parallel to T_g. If N is torsion-defining, then the function $\tau_g = \langle T_g, W\rangle$ is called the *(first) geodesic torsion of γ with respect to N.*

It is clear that both κ_g and τ_g are defined up to a sign.

Armed with the notion of geodesic torsion, we may now define torsion. Suppose that the curvature $\kappa = \|D_t E\|$ of γ is nowhere zero, so that the *principal normal $P = D_t E/\kappa$* is well-defined. The geodesic torsion vector of γ with respect to P is called the *(first) torsion vector of γ*. In particular, if P is torsion-defining, then the geodesic torsion of γ with respect to P is called the *(first) torsion of γ*.

Remark 3.1 If P is well-defined, then the normal curvature of γ with respect to P coincides with the curvature of γ, while the geodesic curvature with respect to P vanishes.

To see that our curvature functions naturally extend the classical Darboux curvatures, consider an oriented hypersurface S of M, and let N_S be its unit normal. If γ is a curve on S, then the geodesic (resp., normal) curvature vector of γ (with respect to N_S) is the projection onto TS (resp., NS) of the ambient acceleration $D_t E$ of γ; and if γ is not a geodesic of M, then the geodesic torsion vector of γ at $\gamma(t)$ is nothing but the torsion vector of the S-geodesic passing from $\gamma(t)$ with tangent vector $\gamma'(t)$ [14, p. 193].

Yet another indication of the naturality of our definition of geodesic torsion is provided by the following lemma, which will play a key role in the proof of Theorem 1.8.

Lemma 3.2 *A curve on S is a line of curvature if and only if its geodesic torsion vector with respect to N_S vanishes.*

Remark 3.3 Under suitable assumptions, one may define $m-2$ geodesic curvature and (geodesic) torsion functions. For instance, the second geodesic torsion is defined as follows. Let $\mathcal{H}_2 = (T \oplus N \oplus T_g)^\perp$, let $\pi_{\mathcal{H}_2}$ be the orthogonal projection onto $\mathcal{H}_2$, and let

$$T_{g,2} = -\pi_{\mathcal{H}_2} D_t T_g.$$

If T_g is itself torsion-defining, i.e., there exists a smooth unit vector field W_2 along γ that is everywhere parallel to $T_{g,2}$, then the function $\tau_{g,2} = \langle T_{g,2}, W_2\rangle$ is called the *second geodesic torsion of γ with respect to N*. Higher-order geodesic torsions are defined similarly.

4 Rotating the normal

Suppose that the normal vector N along γ rotates about the curve's tangent. Then how do the curvature vectors change? The purpose of this section is to answer such question.

Let Z be a rotation of N. Then, by definition, there exists a unit normal vector field $H(N, Z) \equiv H \in \Gamma(\mathcal{H})$ along γ such that N, Z, and H are everywhere linearly dependent; besides, there is a continuous function $\theta\colon I \to \mathbb{R}$ such that

$$Z = -\sin(\theta)H + \cos(\theta)N.$$

Denoting Z by $N(\theta)$, we call the function θ the *rotation angle of* $N(\theta)$ *with respect to* H.

Now, let $(H_1, \ldots, H_{m-2})$ be a smooth orthonormal frame for $\mathcal{H} = (E \oplus N)^{\perp}$, with $H_1 = H$. It follows that

$$N(\theta) = -\sin(\theta)H_1 + \cos(\theta)N,$$

while the vector fields

$$\begin{aligned} H_1(\theta) &= \cos(\theta)H_1 + \sin(\theta)N, \\ H_2(\theta) &= H_2, \\ &\vdots \\ H_{m-2}(\theta) &= H_{m-2} \end{aligned}$$

span $\mathcal{H}(N(\theta)) = (E \oplus N(\theta))^{\perp}$.

Lemma 4.1 *The curvature vectors of* γ *with respect to* $N(\theta)$ *are given by*

$$\begin{aligned} K_g(\theta) &= \left(\kappa_g^1 c + \kappa_n s\right) H_1(\theta) + \kappa_g^2 H_2(\theta) + \cdots + \kappa_g^{m-2} H_{m-2}(\theta), \\ K_n(\theta) &= \left(-\kappa_g^1 s + \kappa_n c\right) N(\theta), \\ T_g(\theta) &= \left(\theta' + \tau_g^1\right) H_1(\theta) + \left(\tau_g^2 c - \mu_2 s\right) H_2(\theta) + \cdots + \left(\tau_g^{m-2} c - \mu_{m-2} s\right) H_{m-2}(\theta), \end{aligned}$$

where $\mu_j = \langle D_t H_j, H_1 \rangle$, *and where* c *and* s *are shorthands for* $\cos(\theta)$ *and* $\sin(\theta)$, *respectively.*

5 Proof of Theorem 1.8

Here we prove our most general result, Theorem 1.8 in the introduction.

To begin with, suppose that γ is a line of curvature of S and that $N_S(\theta)$ is a parallel rotation of N_S. Then the geodesic torsion of γ with respect to N_S vanishes and the vector field $H(N_S, N_S(\theta))$ is parallel in $\mathcal{H}$.

Let $(H_1, \ldots, H_{m-2})$ be a smooth orthonormal frame for $(E \oplus N_S)^{\perp}$ such that $H_1 = H$. Applying Lemma 4.1, we deduce that $T_g(\theta) = \theta' H_1(\theta)$, which implies that $N_S(\theta)$ is torsion-defining and that $\theta' = \pm\tau_g(\theta)$.

Since

$$\int_0^{\ell} \theta' \, dt = \theta(\ell) - \theta(0),$$

it follows that, when $N_S(\theta)$ is a closed rotation of N_S,

$$\int_0^\ell \tau_g(\theta) = 2n\pi \quad \text{for some } n \in \mathbb{Z},$$

as desired.

Conversely, given any (torsion-defining) unit normal vector field N along γ, suppose that $W(N) \equiv W$ is parallel in $\mathcal{H}$. Choose an orthonormal frame $(H_1, \ldots, H_{m-2})$ for $\mathcal{H}$, with $H_1 = W$, and let

$$N(\theta) = -\sin(\theta)H_1 + \cos(\theta)N,$$
$$H_1(\theta) = \cos(\theta)H_1 + \sin(\theta)N,$$

where

$$\theta(t) = -\int_0^t \tau_g(s)\,ds. \tag{1}$$

(Note that $N(\theta)$ is a parallel rotation of N.)

Define a map $\sigma : [0, \ell] \times \mathbb{R}^{m-1} \to M$ by

$$\sigma(t, u) = \exp_{\gamma(t)}\left(u^1 H_1(\theta)(t) + u^2 H_2(t) + \cdots + u^{m-2} H_{m-2}(t)\right).$$

It is clear that σ is a smooth immersion in a neighborhood of $[0, \ell] \times \{0\}$; besides, its image is normal to $N(\theta)$ along γ.

It remains to show that γ is a line of curvature of σ, i.e., that the geodesic torsion $\tau_g(\theta)$ of γ with respect to $N(\theta)$ vanishes. Differentiating (1), we have

$$\theta' = -\tau_g^1,$$

which implies $\tau_g^1(\theta) = 0$, as desired. Since $\tau_g^2 = \cdots = \tau_g^{m-1} = 0$ and H_1 is parallel in $\mathcal{H}$, we conclude that $\tau_g(\theta) = 0$ by Lemma 4.1.

6 Three-dimensional curves

Let $\gamma : I \to M$ be a Frenet curve, let $H_1 = W(P)$, and let $(H_2, \ldots, H_{m-2})$ be a parallel frame for the orthogonal complement of H_1 in $\mathcal{H}(P)$.

Definition 6.1 We say that γ is *three-dimensional* if the following equations hold:

$$D_t E = \kappa P,$$
$$D_t H_1 = \tau P,$$
$$D_t H_2 = \cdots = D_t H_{m-2} = 0,$$
$$D_t P = -\kappa E - \tau H_1.$$

It is clear that γ is three-dimensional if and only if $W(P)$ is parallel in $\mathcal{H}(P)$.

The purpose of this section is to prove Theorem 1.3 in the introduction.

Proof of Theorem 1.3 Suppose that P is a parallel rotation of N_S, and let θ be the rotation angle of P with respect to $W(P)$. We know from the proof of Theorem 1.8 that if γ is a line of curvature and P is a *closed* rotation, then

$$\pm \int_0^\ell \tau\,dt = \theta(\ell) - \theta(0) = 2n\pi \quad \text{for some } n \in \mathbb{Z}. \tag{2}$$

On the other hand, applying Lemma 4.1, we observe that the normal curvature of γ with respect to N_S is related to the curvature κ by the relation

$$\kappa_n = \kappa(\theta) = \kappa \cos(\theta).$$

Suppose that M is convex, so that $\kappa_n > 0$. Since $\kappa > 0$, we have $\cos(\theta) > 0$, from which we conclude that

$$\theta(\ell) - \theta(0) \in (-\pi, \pi).$$

Together with (2), this implies $n = 0$, as desired. □

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Declarations

Conflict of interest The author declares no competing interests.

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