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ORIGINAL PAPER

New lower bounds for optimal horoball packing density in hyperbolic n-space for $6 \leq n \leq 9$

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Abstract

Koszul type Coxeter simplex tilings exist in hyperbolic n-space $\mathbb{H}^n$ up to $n = 9$, and their horoball packings achieve the highest known regular ball packing densities for $n = 3, 4, 5$. In this paper we determine the optimal horoball packing densities of Koszul simplex tilings in dimensions $6 \leq n \leq 9$, which give new lower bounds for optimal packing density in each dimension. The symmetries of the packings are given by Coxeter simplex groups, and a parameter related to the Busemann function gives an isometry invariant description of different optimal horoball packing configurations.

Keywords Busemann function · Coxeter group · Horoball · Hyperbolic geometry · Packing · Tiling

1 Introduction

This is the fourth paper in a series we determine the optimal horoball packing densities of Koszul-type noncompact Coxeter simplex tilings that exist in $\mathbb{H}^n$ for $2 \leq n \leq 9$. In [13–15] we considered dimensions $3 \leq n \leq 5$ respectively and in the present paper we consider dimensions $6 \leq n \leq 9$.

First, in [13], we showed that the classical example of the horoball packing in $\overline{\mathbb{H}}^3$ that achieves the Böröczky-type simplicial packing density upper bound $d_3(\infty)$ (cf. Theorem 2) by tiling by a regular ideal simplex is not unique, and gave several new examples using horoballs of different types. Second, in [14], we found seven horoball packings of Coxeter simplex tilings in $\overline{\mathbb{H}}^4$ that yield densities of $5\sqrt{2}/\pi^2 \approx 0.71645$, counterexamples to L. Fejes

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Tóth's conjecture for the maximal packing density of $\frac{5-\sqrt{5}}{4} \approx 0.69098$ in his foundational book *Regular Figures* [7, p. 323]. Finally, in [15] we constructed the densest known ball packing in $\overline{\mathbb{H}}^5$ with a density of $\frac{5}{7\zeta(3)}$ where $\zeta(\cdot)$ is the Riemann Zeta function, and the closed-form value for the $\overline{\mathbb{H}}^4$ case appears first in this paper.

We summarize the results of this paper stated in Theorems 3–6 as follows.

Theorem 1 *The optimal horoball packing densities for noncompact Coxeter simplex tilings in* $\overline{\mathbb{H}}^n$ *are given by*

n	Closed-form Density Value	Approximate Value
6	$81/\left(4\sqrt{2}\pi^3\right)$	0.46180…
7	$28/(81\mathrm{L}(4,3))$	0.36773…
8	$225/\left(8\pi^4\right)$	0.28873…
9	$1/(4\zeta(5))$	0.24109…

where $L(\cdot,\cdot)$ *is the Dirichlet* L*-Series.*

Upper bounds for the packing density were published by Kellerhals [11] using the simplicial density function $d_n(\infty)$. This bound is strict for $n = 3$, and Table 1 summarizes our main results where Δ is the gap between the packing density upper bound and our effective lower bounds, cf. Corollaries 3–6.

New to this paper, the notion of 'horoball type' with respect to a fundamental domain is strengthened using isometry invariant Busemann functions. We use Busemann functions to parameterize horoballs centered at $\xi \in \partial\mathbb{H}^n$ with respect to a marked reference point $o \in \mathbb{H}^n$ (alternatively a reference horoball through ξ and o) in the model of $\overline{\mathbb{H}}^n$, see Sect. 3.3. This new point of view shows that the optimal packings cannot be made equivalent by repartitioning, a nontrivial hyperbolic isometry, or some paradoxical construction, and clarifies our earlier results. Our method for computing densities in the projective Cayley-Klein model is largely similar to the earlier lower dimensional cases, see Sect. 4, although the computations in coordinates are more involved. Hence the procedure was imporved to obtain exact closed-form expressions for packing densities in arithmetic lattices in Sect. 5.

Table 1 Packing density upper and lower bounds for $\mathbb{H}^n$

n	Optimal Coxeter simplex packing density	Approximate value	$d_n(\infty)$	Δ
3	$\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\dots\right)^{-1}$	0.85328…	0.85328…	0
4	$5\sqrt{2}/\pi^2$	0.71644…	0.73046…	0.0140…
5	$5/(7\zeta(3))$	0.59421…	0.60695…	0.0127…
6	$81/\left(4\sqrt{2}\pi^3\right)$	0.46180…	0.49339…	0.0315…
7	$28/(81\mathrm{L}(4,3))$	0.36773…	0.39441…	0.0266…
8	$225/\left(8\pi^4\right)$	0.28873…	0.31114…	0.0223…
9	$1/(4\zeta(5))$	0.24109…	0.24285…	0.0017…

2 Background

Let X denote a space of constant curvature, either the n-dimensional sphere $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, or hyperbolic space $\mathbb{H}^n$ with $n \geq 2$. An important question of discrete geometry is to find the highest possible packing density in X by congruent non-overlapping balls of a given radius [6]. The definition of packing density is critical in hyperbolic space as shown by Böröczky [4], for the standard paradoxical construction see [6] or [21]. The most widely accepted notion of packing density considers the local densities of balls with respect to their Dirichlet-Voronoi cells (cf. [4] and [11]). In order to study horoball packings in $\overline{\mathbb{H}}^n$, we use an extended notion of such local density.

Let B be a horoball of packing $\mathcal{B}$, and $P \in \overline{\mathbb{H}}^n$ an arbitrary point. Define $d(P, B)$ to be the shortest distance from point P to the horosphere $S = \partial B$, where $d(P, B) \leq 0$ if $P \in B$. The Dirichlet-Voronoi cell $\mathcal{D}(B, \mathcal{B})$ of horoball B is the convex body

$$\mathcal{D}(B, \mathcal{B}) = \{P \in \mathbb{H}^n | d(P, B) \leq d(P, B'), \forall B' \in \mathcal{B}\}.$$

Both B and $\mathcal{D}$ have infinite volume, so the standard notion of local density is modified. Let $Q \in \partial\mathbb{H}^n$ denote the ideal center of B, and take its boundary S to be the one-point compactification of Euclidean $(n-1)$-space. Let $B_C^{n-1}(r) \subset S$ be the Euclidean $(n-1)$-ball with center $C \in S \setminus \{Q\}$. Then Q and $B_C^{n-1}(r)$ determine a convex cone $\mathcal{C}^n(r) = cone_Q\left(B_C^{n-1}(r)\right) \in \overline{\mathbb{H}}^n$ with apex Q consisting of all hyperbolic geodesics passing through $B_C^{n-1}(r)$ with limit point Q. The local density $\delta_n(B, \mathcal{B})$ of B to $\mathcal{D}$ is defined as

$$\delta_n(\mathcal{B}, B) = \overline{\lim_{r \to \infty}} \frac{vol(B \cap \mathcal{C}^n(r))}{vol(\mathcal{D} \cap \mathcal{C}^n(r))}.$$

This limit is independent of the choice of center C for $B_C^{n-1}(r)$.

In the case of periodic ball or horoball packings, this local density defined above extends to the entire hyperbolic space via its symmetry group, and is related to the simplicial density function (defined below) that we generalized in [25] and [26]. In this paper we shall use such definition of packing density (cf. Sect. 4).

A Coxeter simplex is a top dimensional simplex in X with dihedral angles either integral submultiples of π or zero. The group generated by reflections on the sides of a Coxeter simplex is a Coxeter simplex reflection group. Such reflections generate a discrete group of isometries of X with the Coxeter simplex as the fundamental domain; hence the groups give regular tessellations of X if the fundamental simplex is characteristic. The Coxeter groups are finite for $\mathbb{S}^n$, and infinite for $\mathbb{E}^n$ or $\overline{\mathbb{H}}^n$.

There are non-compact Coxeter simplices in $\overline{\mathbb{H}}^n$ with ideal vertices on $\partial\mathbb{H}^n$, however only for dimensions $2 \leq n \leq 9$; furthermore, only a finite number exist in dimensions $n \geq 3$. Johnson et al. [8] found the volumes of all Coxeter simplices in hyperbolic n-space. Such simplices are the most elementary building blocks of hyperbolic manifolds, the volume of which is an important topological invariant.

In n-dimensional space X of constant curvature $(n \geq 2)$, define the simplicial density function $d_n(r)$ to be the density of $n+1$ mutually tangent balls of radius r in the simplex spanned by their centers. Fejes Tóth and Coxeter conjectured that the packing density of balls of radius r in X cannot exceed $d_n(r)$. Rogers [22] proved this conjecture in Euclidean space $\mathbb{E}^n$. The 2-dimensional spherical case was settled by L. Fejes Tóth [7], and Böröczky [4] gave a proof for the extension:

Theorem 2 *(K. Böröczky) In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. In the case of spherical space, assume that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet-Voronoi cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centers.*

In hyperbolic 3-space, the monotonicity of $d_3(r)$ was proved by Böröczky and Florian in [5]; in [16] Marshall showed that for sufficiently large n, function $d_n(r)$ is strictly increasing in variable r. Kellerhals [11] showed $d_n(r) < d_{n-1}(r)$, and that in cases considered by Marshall the local density of each ball in its Dirichlet-Voronoi cell is bounded above by the simplicial horoball density $d_n(\infty)$. Theorem 2 is extended to the horoball case in [4, §6] as a remark.

The simplicial packing density upper bound $d_3(\infty) = (1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - - + + \dots)^{-1} = 0.85327\dots$ cannot be achieved by packing regular balls, instead it is realized by horoball packings of $\overline{\mathbb{H}}^3$, the regular ideal simplex tiles $\overline{\mathbb{H}}^3$. More precisely, the centers of horoballs in $\partial\overline{\mathbb{H}}^3$ lie at the vertices of the ideal regular Coxeter simplex tiling with Schläfli symbol [3, 3, 6].

In three dimensions the Böröczky-type bound for horoball packings are used for volume estimates of cusped hyperbolic manifolds [1, 18], more recently [2, 17]. Lifts of horoball neighborhoods of cusps give horoball packings in the universal cover $\mathbb{H}^n$, and for some discrete torsion free subgroup of isometries $\mathbb{H}^n/\Gamma$ is a cusped hyperbolic manifold where the cusps lift to ideal vertices of the fundamental domain. In this setting a manifold with a single cusp has a well defined maximal cusp neighborhood, while manifolds with multiple cusps have a range of non-overlapping cusp neighborhoods with boundaries with nonempty tangential intersection, these lift to different horoball types in the universal cover. An important application is Adams' proof that the Geiseking manifold is the noncompact hyperbolic 3-manifold of minimal volume [1]. Kellerhals then used the Böröczky-type bounds to estimate volumes of higher dimensional hyperbolic manifolds [12].

In [13] we proved that the classical horoball packing configuration in $\mathbb{H}^3$ realizing the Böröczky-type upper bound is not unique. We gave several examples of different regular horoball packings using horoballs of different types, that is horoballs that have different relative densities with respect to the fundamental domain, that yield the Böröczky-Florian-type simplicial upper bound [5].

Furthermore, in [25, 26] we found that by allowing horoballs of different types at each vertex of a totally asymptotic simplex and generalizing the simplicial density function to $\overline{\mathbb{H}}^n$ for $n \geq 2$, the Böröczky-type density upper bound is not valid for the fully asymptotic simplices for $n \geq 4$. In $\overline{\mathbb{H}}^4$ the locally optimal simplicial packing density is $0.77038\dots$, higher than the Böröczky-type density upper bound of $d_4(\infty) = 0.73046\dots$ using horoballs of a single type. However these ball packing configurations are only locally optimal and cannot be extended to the entirety of $\overline{\mathbb{H}}^n$. Finally, we mention the second-named author's preliminary results on horoball packings that motivated our collaboration [23, 24].

3 Preliminaries

We use the projective Cayley-Klein model of hyperbolic geometry to preserves lines and convexity for the packing of simplex tilings with convex fundamental domains. Hyperbolic symmetries are modeled as Euclidean projective transformations using the projective linear group $\mathrm{PGL}(n+1, \mathbb{R})$. In this section we review some key concepts, for a general discussion of the projective models of Thurston geometries see [19, 20].

3.1 The projective model of $\overline{\mathbb{H}}^n$

Let $\mathbb{E}^{1,n}$ denote $\mathbb{R}^{n+1}$ with the Lorentzian inner product $\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + \cdots + x^ny^n$ where non-zero real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$ represent points in projective space $\mathbb{P}^n = \mathbb{P}(\mathbb{E}^{n+1})$, equipped with the quotient topology of the natural projection $\Pi : \mathbb{E}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$. Partitioning $\mathbb{E}^{1,n}$ into $Q_+ = \{\mathbf{v} \in \mathbb{R}^{n+1} | \langle \mathbf{v}, \mathbf{v} \rangle > 0\}$, $Q_0 = \{\mathbf{v} | \langle \mathbf{v}, \mathbf{v} \rangle = 0\}$, and $Q_- = \{\mathbf{v} | \langle \mathbf{v}, \mathbf{v} \rangle < 0\}$, the proper points of hyperbolic n-space are $\mathbb{H}^n = \Pi(Q_-)$, $\partial\mathbb{H}^n = \Pi(Q_0)$ are the boundary or ideal points, we will refer to points in $\Pi(Q_+)$ as outer points, and $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial\mathbb{H}^n$ as extended hyperbolic space.

Points $[\mathbf{x}], [\mathbf{y}] \in \mathbb{P}^n$ are conjugate when $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. The set of all points conjugate to $[\mathbf{x}]$ form a projective (polar) hyperplane $pol([\mathbf{x}]) = \{[\mathbf{y}] \in \mathbb{P}^n | \langle \mathbf{x}, \mathbf{y} \rangle = 0\}$. Hence Q_0 induces a duality $\mathbb{R}^{n+1} \leftrightarrow \mathbb{R}_{n+1}$ between the points and hyperplanes of $\mathbb{P}^n$. Point $[\mathbf{x}]$ and hyperplane $[\boldsymbol{a}]$ are incident if the value of the linear form $\boldsymbol{a}$ evaluated on vector $\mathbf{x}$ is zero, i.e. $\mathbf{x}\boldsymbol{a} = 0$ where $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$, and $\boldsymbol{a} \in \mathbb{R}_{n+1} \setminus \{\mathbf{0}\}$. Similarly, the lines in $\mathbb{P}^n$ are given by 2-subspaces of $\mathbb{R}^{n+1}$ or dual $(n-1)$-subspaces of $\mathbb{R}_{n+1}$ [19].

Let $P \subset \overline{\mathbb{H}}^n$ be a polyhedron bounded by a finite set of hyperplanes H^i with unit normals $\boldsymbol{b}^i \in \mathbb{R}_{n+1}$ directed towards the interior of P:

$$H^i = \{\mathbf{x} \in \mathbb{H}^n | \mathbf{x}\boldsymbol{b}^i = 0\} \text{ with } \langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1. \tag{1}$$

In this paper P is assumed to be an acute-angled polyhedron with proper or ideal vertices. The Gram matrix of P is $G(P) = (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle)_{i,j}$, $i, j \in \{0, 1, 2, \ldots, n\}$ is symmetric with signature $(1, n)$, its entries satisfy $\langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1$ and $\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle \leq 0$ for $i \neq j$ where

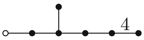
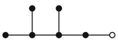
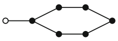
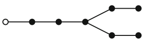
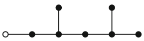
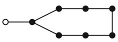
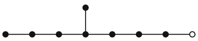
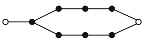
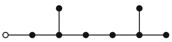
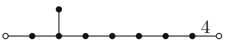
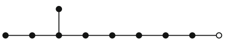
$$\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{cases} 0 & \text{if } H^i \perp H^j, \\ -\cos\alpha^{ij} & \text{if } H^i, H^j \text{intersect along an edge of } P \text{ at angle } \alpha^{ij}, \\ -1 & \text{if } H^i, H^j \text{are parallel in the hyperbolic sense}, \\ -\cosh l^{ij} & \text{if } H^i, H^j \text{admit a common perpendicular of length } l^{ij}. \end{cases}$$

This is summarized in the Coxeter graph of the polytope $\sum(P)$. The graph nodes correspond to the hyperplanes H^i and are connected if H^i and H^j are not perpendicular ($i \neq j$). If connected the positive weight k where $\alpha_{ij} = \pi/k$ is indicated on the edge, unlabeled edges denote an angle of $\pi/3$. Coxeter diagrams appear in Table 2.

In this paper we set the sectional curvature of $\mathbb{H}^n$, $K = -k^2$, to be $k = 1$. The distance d between two proper points $[\mathbf{x}]$ and $[\mathbf{y}]$ is given by

$$\cosh d = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \tag{2}$$

Table 2 Notation and volumes for the 14 asymptotic Coxeter Simplices in $\mathbb{H}^n$ for $6 \le n \le 9$, empty circles in the Coxeter diagram denote reflection planes opposite an ideal vertex

Coxeter Diagram	Notation	Witt Symbol	Simplex Volume	Optimal Packing density
$n = 6$ dimensions				
	$[4, 3^2, 3^{2,1}]$	$\overline{S}_6$	$\pi^3/777600$	$\frac{81}{4\sqrt{2}\pi^3} \approx 0.46180\ldots$
	$[3^{1,1}, 3, 3^{2,1}]$	$\overline{Q}_6$	$\pi^3/388800$	"
	$[3, 3^{[6]}]$	$\overline{P}_6$	$13\pi^3/1360800$	$\frac{189\sqrt{3}}{26\pi^3} \approx 0.40609\ldots$
$n = 7$ dimensions				
	$[3^{2,2,2}]$	$\overline{T}_7$	$\sqrt{3}\mathrm{L}(4,3)/860160$	$\frac{28}{81\mathrm{L}(4,3)} \approx 0.36773\ldots$
	$[4, 3^3, 3^{2,1}]$	$\overline{S}_7$	$\mathrm{L}(4)/362880$	$\frac{21}{64\mathrm{L}(4)} \approx 0.331793\ldots$
	$[3^{1,1}, 3^2, 3^{2,1}]$	$\overline{Q}_7$	$\mathrm{L}(4)/181440$	"
	$[3, 3^{[7]}]$	$\overline{P}_7$	$7^{5/2}\mathrm{L}(4,7)/3317760$	$\frac{96}{343\mathrm{L}(4,7)} \approx 0.26605\ldots$
$n = 8$ dimensions				
	$[3^{4,3,1}]$	$\overline{T}_8$	$\pi^4/4572288000$	$\frac{225}{8\pi^4} \approx 0.28873\ldots$
	$[3, 3^{[8]}]$	$\overline{P}_8$	$17\pi^3/285768000$	"
	$[4, 3^4, 3^{2,1}]$	$\overline{S}_8$	$17\pi^4/9144576000$	$\frac{2025}{68\sqrt{2}\pi^4} \approx 0.21617\ldots$
	$[4, 3, 3^{1,1,1}]$	$\overline{Q}_8$	$17\pi^4/4572288000$	"
$n = 9$ dimensions				
	$[4, 3^5, 3^{2,1}]$	$\overline{S}_9$	$527\zeta(5)/44590694400$	$\frac{151}{1054\zeta(5)} \approx 0.13816\ldots$
	$[3^{6,2,1}]$	$\overline{T}_9$	$\zeta(5)/222953472000$	$\frac{1}{4\zeta(5)} \approx 0.24109\ldots$
	$[3^{1,1}, 3^4, 3^{2,1}]$	$\overline{Q}_9$	$527\zeta(5)/222953472000$	"

The projection $[\mathbf{y}]$ of point $[\mathbf{x}]$ on plane $[\boldsymbol{u}]$ is given by

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \boldsymbol{u}, \boldsymbol{u} \rangle}\mathbf{u}, \tag{3}$$

where $[\mathbf{u}]$ is the pole of the plane $[\boldsymbol{u}]$.

3.2 Horospheres and horoballs in $\overline{\mathbb{H}}^n$

A horosphere in $\overline{\mathbb{H}}^n$ ($n \ge 2$) is as hyperbolic n-sphere with infinite radius centered at an ideal point $\xi \in \partial\mathbb{H}^n$ obtained as a limit of spheres through $x \in \mathbb{H}^n$ as its center $c \to \xi$.

Equivalently, a horosphere is an $(n-1)$-surface orthogonal to the set of parallel straight lines passing through $\xi \in \partial\mathbb{H}^n$. A horoball is a horosphere together with its interior.

To derive the equation of a horosphere, fix a projective coordinate system for $\mathbb{P}^n$ with standard basis $\boldsymbol{a}_i, 0 \le i \le n$ so that the Cayley-Klein ball model of $\overline{\mathbb{H}}^n$ is centered at $O = (1, 0, 0, \dots, 0)$, and orient it by setting point $\xi \in \partial\mathbb{H}^n$ to lie at $A_0 = (1, 0, \dots, 0, 1)$. The equation of a horosphere with center $\xi = A_0$ passing through interior point $S = (1, 0, \dots, 0, s)$ is derived from the equation of the the boundary sphere $-x^0x^0 + x^1x^1 + x^2x^2 + \dots + x^nx^n = 0$, and the plane $x^0 - x^n = 0$ tangent to the boundary sphere at $\xi = A_0$. The general equation of the horosphere is

$$0 = \lambda(-x^0x^0 + x^1x^1 + x^2x^2 + \dots + x^nx^n) + \mu(x^0 - x^n)^2. \tag{4}$$

Evaluating at S obtain

$$\lambda(-1 + s^2) + \mu(-1 + s)^2 = 0 \text{ and } \frac{\lambda}{\mu} = \frac{1 - s}{1 + s}.$$

For $s \neq \pm 1$, the equation of a horosphere in projective coordinates is

$$(s - 1)\left(-x^0x^0 + \sum_{i=1}^{n}(x^i)^2\right) - (1 + s)(x^0 - x^n)^2 = 0. \tag{5}$$

In $\overline{\mathbb{H}}^n$ there exists an isometry $g \in \mathrm{Isom}(\mathbb{H}^n)$ for any two horoballs B and B' such that $g.B = B'$. However, it is often useful to distinguish between certain horoballs of a packing; we shall use the notion of horoball type with respect to the fundamental domain of a tiling (lattice) as introduced in [26]. In Sect. 3.3 we show that this coincides with the Busemann function up to scaling, hence is isometry invariant.

Two horoballs of a horoball packing are said to be of the *same type* or *equipacked* if and only if their local packing densities with respect to a particular cell (in our case a Coxeter simplex) are equal, otherwise the two horoballs are of *different type*. For example, the horoballs centered at A_0 passing through S with different values for the final coordinate $s \in (-1, 1)$ are of different type relative to a given cell, and the set of all horoballs centered at vertex A_0 is a one-parameter family.

Volumes of horoball pieces are given by János Bolyai's classical formulas from the mid 19-th century. The hyperbolic length $L(x)$ of a horospherical arc contained in a chord segment of length x is

$$L(x) = 2\sinh\left(\tfrac{x}{2}\right). \tag{6}$$

The intrinsic geometry of a horosphere is Euclidean, so the $(n-1)$-dimensional volume $\mathcal{A}$ of a polyhedron A on the surface of the horosphere can be calculated as in $\mathbb{E}^{n-1}$. The volume of the horoball piece $\mathcal{H}(A)$ bounded by A, the set consisting of the union of geodesic segments joining A to the center of the horoball, is

$$vol(\mathcal{H}(A)) = \tfrac{1}{n-1}\mathcal{A}. \tag{7}$$

3.3 The busemann function in $\overline{\mathbb{H}}^n$

Define the Busemann function on $\overline{\mathbb{H}}^n$ as the map $\beta : \mathbb{H}^n \times \mathbb{H}^n \times \partial\mathbb{H}^n \to \mathbb{R}$ with $\beta(x, y, \xi) = \lim_{z \to \xi} (d(x, z) - d(y, z))$, where the limit $z \to \xi$ is taken along any geodesic in $\mathbb{H}^n$ ending at boundary point ξ. The Busemann function satisfies $\beta(x, x, \xi) = 0$, antisymmetry $\beta(x, y, \xi) = -\beta(y, x, \xi)$, the cocycle property $\beta(x, y, \xi) + \beta(y, z, \xi) = \beta(x, z, \xi)$ for all $x, y, z \in \mathbb{H}^n$, and is invariant under actions of $\mathrm{Isom}(\mathbb{H}^n)$. A horosphere centered at ξ through o is the level set of the Busemann function $\mathrm{Hor}_\xi(o) = \{x \in \mathbb{H}^n | \beta(x, o, \xi) = 0\}$, while a horoball is the sublevel set $\mathrm{Hor}_\xi(o) = \{x \in \mathbb{H}^n | \beta(x, o, \xi) \leq 0\}$. The space of all horospheres $\mathrm{Hor}(\mathbb{H}^n)$ gives an $\mathbb{R}$-fibration $h : \mathrm{Hor}(\mathbb{H}^n) \to \partial\mathbb{H}^n$ where $\mathrm{Hor}_\xi(o) \mapsto \xi$. The Busemann function then is an oriented distance between two concentric horospheres $\mathrm{Hor}_\xi(o_1)$ and $\mathrm{Hor}_\xi(o_2)$. For Busemann functions in Hadamard spaces defined by various authors cf. [3], we adopt [10].

Set reference point $o \in \mathbb{H}^n$ for the model at $o = (1, 0, \dots, 0)$ and reference horosphere $\mathrm{Hor}_\xi(o)$ at $\xi = (1, 0, \dots, 0, 1)$. The s-parameter of horosphere $\mathrm{Hor}_\xi(x)$ is $s = \mathrm{th}(\beta(o, x, \xi))$ where $\mathrm{th}(\cdot)$ is the hyperbolic tangent function. A choice of reference point $o \in \mathbb{H}^n$ gives a trivialization of the fibration according to diagram

$$\begin{array}{ccc} \mathrm{Hor}(\mathbb{H}^n) & \xrightarrow{\varphi_o} & \partial\mathbb{H}^n \times \mathbb{R} \\ h\big\downarrow & \swarrow_{\pi} & \\ \partial\mathbb{H}^n & & \end{array}$$

where $\mathrm{Hor}_\xi(x) \mapsto (\xi, \beta(o, x, \xi))$. An element $g \in \mathrm{Isom}(\mathbb{H}^n)$ acts on a horosphere as an additive cocycle

$$\begin{aligned} g.\mathrm{Hor}_\xi(x) = \mathrm{Hor}_{g.\xi}(gx) \mapsto (g\xi, \beta(o, gx, g\xi)) &= \left(g\xi, \beta(g^{-1}o, x, \xi)\right) \\ &= \left(g\xi, \beta(o, x, \xi) + \beta(g^{-1}o, o, \xi)\right). \end{aligned}$$

Let $\hat{s} = \mathrm{arcth}(s)$ then g acts on the trivialization by

$$g(\xi, \hat{s}) = (g\xi, \hat{s} + \beta(g^{-1}o, o, \xi).$$

In summary Busemann functions are related to the s-parameters by scaling and describe packing configurations relative to a marked point o in an isometry invariantly.

4 Packing density in the projective model

In this section we define packing density and collect three Lemmas used in Section 5 to find the optimal packing densities for the Koszul simplex tilings.

Let $\mathcal{T}$ be a Coxeter tiling of $\overline{\mathbb{H}}^n$ [9]. The symmetry group of a Coxeter tiling $\Gamma_\mathcal{T}$ contains its Coxeter group, and isometric mapping between two cells in $\mathcal{T}$ preserves the tiling. Any simplex cell of $\mathcal{T}$ acts as a fundamental domain $\mathcal{F}_\mathcal{T}$ of $\Gamma_\mathcal{T}$, and the Coxeter group is generated by reflections on the $(n-1)$-dimensional facets of $\mathcal{F}_\mathcal{T}$. In this paper we consider only noncompact or Koszul-type Coxeter simplices, that is simplices with one or more ideal vertex, then the orbifold $\mathbb{H}^n / \Gamma_\mathcal{T}$ has at least one cusp. In Table 2 we list the 14 Koszul-type Coxeter simplices in $\overline{\mathbb{H}}^n$ for $6 \leq n \leq 9$, and their volumes. For a detailed discussion of the volume formulae for the other hyperbolic Coxeter simplices of dimensions $n \geq 3$, see Johnson *et al.* [8].

Define the density of a regular horoball packing $\mathcal{B}_\mathcal{T}$ of Coxeter simplex tiling $\mathcal{T}$ as

$$\delta(\mathcal{B}_\mathcal{T}) = \frac{\sum_{i=1}^{m} vol(B_i \cap \mathcal{F}_\mathcal{T})}{vol(\mathcal{F}_\mathcal{T})}. \tag{8}$$

$\mathcal{F}_\mathcal{T}$ denotes the simplex fundamental domain of tiling $\mathcal{T}$, m the number of ideal vertices of $\mathcal{F}_\mathcal{T}$, and B_i the horoball centered at the i-th ideal vertex. We allow horoballs of different types at each asymptotic vertex of the tiling. A particular set of horoballs $\{B_i\}_{i=1}^m$ with different horoball types is allowed if it gives a packing: no two horoballs may have an interior point in common, and we require that no horoball extend beyond the facet opposite to the vertex where it is centered. The second condition ensures that the packing remains invariant under the actions of $\Gamma_{\mathcal{T}}$ with $\mathcal{F}_{\mathcal{T}}$. With these conditions satisfied, the packing density in $\mathcal{F}_{\mathcal{T}}$ extends to the entire $\overline{\mathbb{H}}^5$ by actions of Γ_τ. In the case of Coxeter simplex tilings, Dirichlet-Voronoi cells coincide with the Coxeter simplices. We denote the optimal horoball packing density as

$$\delta_{opt}(\mathcal{T}) = \sup_{\mathcal{B}_{\mathcal{T}}\ \text{packing}} \delta(\mathcal{B}_\tau). \tag{9}$$

Let $\mathcal{F}_\Gamma$ denote the simplicial fundamental domain of Coxeter tiling $\mathcal{T}_\Gamma$ with vertex set $\{A_i\}_{i=0}^n \in \mathbb{P}(E^{1,n})$, where $A_0 = (1, 0, \dots, 0, 1)$ is ideal and $A_1 = (1, 0, \dots, 0)$ is the center of the model O. Vertex coordinates $A_2, \dots, A_n$ then are set according to the dihedral angles of $\mathcal{F}_\Gamma$ indicated in the Coxeter diagrams in Table 2, see Tables 3–6 for a choice of vertices, here $\boldsymbol{u}_i$ denote the hyperplane opposite to vertex A_i.

Lemma 1 describes a procedure for finding the optimal horoball packing density in the fundamental domain $\mathcal{F}_\Gamma$ with a single ideal vertex A_0. Packing density is maximized by the largest horoball type admissible in cell $\mathcal{F}_\Gamma$ centered at A_0. Let $\mathcal{B}_0(s)$ denote the 1-parameter family of horoballs centered at A_0 where s-parameter related to the Busemann function measures the “radius" of the horoball, the minimal Euclidean signed distance between the horoball and the center of the model O, taken negative if the horoball contains the model center.

Lemma 1 *(Local horoball density) The local optimal horoball packing density of simply asymptotic Coxeter simplex $\mathcal{F}_\Gamma$ is $\delta_{opt}(\Gamma) = \frac{vol(\mathcal{B}_0 \cap \mathcal{F}_\Gamma)}{vol(\mathcal{F}_\Gamma)}$.*

Proof The maximal horoball $\mathcal{B}_0(s)$ opposite A_0 with fundamental domain $\mathcal{F}_\Gamma$ is tangent to the hyperface of the simplex given by $\boldsymbol{u}_0$. This tangent point of $\mathcal{B}_0(s)$ and hyperface $\boldsymbol{u}_0$ is $[\mathbf{f}_0]$ the projection of vertex A_0 on plane $\boldsymbol{u}_0$ given by,

$$\mathbf{f}_0 = \mathbf{a}_0 - \frac{\langle \mathbf{a}_0, \mathbf{u}_0 \rangle}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle} \mathbf{u}_0. \tag{10}$$

The value of the s-parameter for the maximal horoball can be read from the equation of the horosphere through A_0 and $\mathbf{f}_0$. The intersections $[\mathbf{h}_i]$ of horosphere $\partial\mathcal{B}_0$ and the edges of the simplex $\mathcal{F}_\Gamma$ are found by parameterizing the edges $\mathbf{h}_i(\lambda) = \lambda \mathbf{a}_0 + \mathbf{a}_i$ $(1 \le i \le 5)$ then finding their intersections with $\partial\mathcal{B}_0$. The volume of the horospherical $(n-1)$-simplex determines the volume of the horoball piece by equation (7). The data for the horospheric $(n-1)$-simplex is obtained by finding hyperbolic distances l_{ij} via equation (2), $l_{ij} = d(H_i, H_j)$ where $d(\mathbf{h}_i, \mathbf{h}_j) = \arccos\left(\frac{-\langle \mathbf{h}_i, \mathbf{h}_j \rangle}{\sqrt{\langle \mathbf{h}_i, \mathbf{h}_i \rangle \langle \mathbf{h}_j, \mathbf{h}_j \rangle}}\right)$. Moreover, the horospherical distances L_{ij} are found by formula (6). The intrinsic geometry of a horosphere is Euclidean, so the Cayley-Menger determinant gives the volume $\mathcal{A}$ of the horospheric $(n-1)$-simplex $\mathcal{A}$,

$$\mathcal{A}^2 = \frac{1}{(n!)^2 2^n} \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & L_{1,2}^2 & L_{1,3}^2 & \dots & L_{1,n}^2 \\ 1 & L_{1,2}^2 & 0 & L_{2,3}^2 & \dots & L_{2,n}^2 \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ 1 & L_{1,n}^2 & L_{2,n}^2 & \dots & L_{n-1,n}^2 & 0 \end{vmatrix}. \tag{11}$$

Table 3 Data for asymptotic Coxeter tilings of $\mathbb{H}^6$ in the Cayley–Klein ball model centered at $O = (1, 0, 0, 0, 0, 0, 0)$

The 6 dimensional coxeter simplex tilings			
Witt Symb.	$\overline{S}_6$	$\overline{Q}_6$	$\overline{P}_6$
A_0	$(1, 0, 0, 0, 0, 0, 1)$	$(1, 0, 0, 0, 0, 0, 1)$	$(1, 0, 0, 0, 0, 0, 1)$
A_1	$(1, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0)$
A_2	$(1, 0, 0, 0, 0, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, \frac{\sqrt{15}}{6}, 0)$
A_3	$(1, 0, 0, 0, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, \frac{\sqrt{10}}{5}, \frac{2\sqrt{15}}{15}, 0)$
A_4	$(1, 0, 0, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, 0, \frac{1}{2}, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, 0, \frac{\sqrt{6}}{4}, \frac{3\sqrt{10}}{20}, \frac{\sqrt{15}}{10}, 0)$
A_5	$(1, 0, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, \frac{1}{2}, 0, \frac{\sqrt{2}}{4}, \frac{1}{2}, 0)$	$(1, 0, -\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{10}}{10}, \frac{\sqrt{15}}{15}, 0)$
A_6	$(1, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0)$	$(1, -\frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0)$	$(1, -\frac{1}{2}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{12}, \frac{\sqrt{10}}{20}, \frac{\sqrt{15}}{30}, 0)$
The form $\boldsymbol{u}_i$ of sides opposite A_i			
$\boldsymbol{u}_0$	$(0, 0, 0, 0, 0, 0, 1)$	$(0, 0, 0, 0, 0, 0, 1)$	$(0, 0, 0, 0, 0, 0, 1)$
$\boldsymbol{u}_1$	$(1, 0, 0, 0, 0, -2, -1)$	$(1, 0, 0, 0, 0, -2, -1)$	$(1, 1, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{10}}, -2\sqrt{\frac{3}{5}}, -1)$
$\boldsymbol{u}_2$	$(0, -1, 0, 0, -\sqrt{2}, 1, 0)$	$(0, 1, 0, 0, -\sqrt{2}, 1, 0)$	$(0, 0, 0, 0, -\sqrt{\frac{2}{3}}, 1, 0)$
$\boldsymbol{u}_3$	$(0, 0, 0, -1, 1, 0, 0)$	$(0, 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, 0, 0)$	$(0, 0, 0, -\sqrt{\frac{3}{5}}, 1, 0, 0)$
$\boldsymbol{u}_4$	$(0, 0, -1, 1, 0, 0, 0)$	$(0, 0, 0, 1, 0, 0, 0)$	$(0, 0, \frac{1}{\sqrt{2}}, 1, 0, 0, 0)$
$\boldsymbol{u}_5$	$(0, 0, 1, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 0)$	$(0, -\frac{\sqrt{3}}{3}, 1, 0, 0, 0, 0)$
$\boldsymbol{u}_6$	$(0, 1, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0)$

Table 3 continued

The 6 dimensional coxeter simplex tilings			
Witt Symb.	$\overline{S}_6$	$\overline{Q}_6$	$\overline{P}_6$
Maximal horoball parameter s_0			
s_0	0	0	0
Intersections $H_i = \mathcal{B}(A_0, s_0) \cap A_0 A_i$ **of horoballs with simplex edges**			
H_1	$(1, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0)$
H_2	$(1, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, \frac{4\sqrt{15}}{29}, \frac{5}{29})$
H_3	$(1, 0, 0, 0, \frac{4\sqrt{2}}{19}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, \frac{4\sqrt{2}}{19}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, \frac{3}{2\sqrt{10}}, \frac{\sqrt{\frac{3}{5}}}{2}, \frac{1}{4})$
H_4	$(1, 0, 0, \frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}, \frac{2}{5}, \frac{1}{5})$	$(1, 0, 0, \frac{8}{21}, \frac{4\sqrt{2}}{21}, \frac{8}{21}, \frac{5}{21})$	$(1, 0, 0, \frac{2\sqrt{6}}{11}, \frac{6\sqrt{\frac{2}{5}}}{11}, \frac{4\sqrt{\frac{3}{5}}}{11}, \frac{3}{11})$
H_5	$(1, 0, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21}, \frac{8}{21}, \frac{5}{21})$	$(1, 0, \frac{8}{21}, 0, \frac{4\sqrt{2}}{21}, \frac{8}{21}, \frac{5}{21})$	$(1, 0, -\frac{\sqrt{3}}{4}, \frac{\sqrt{\frac{3}{2}}}{4}, \frac{3}{4\sqrt{10}}, \frac{\sqrt{\frac{3}{5}}}{4}, \frac{1}{4})$
H_6	$(1, \frac{2}{5}, 0, 0, 0, \frac{2}{5}, \frac{1}{5})$	$(1, -\frac{2}{5}, 0, 0, 0, \frac{2}{5}, \frac{1}{5})$	$(1, -\frac{12}{29}, -\frac{4\sqrt{3}}{29}, \frac{2\sqrt{6}}{29}, \frac{6\sqrt{\frac{2}{5}}}{29}, \frac{4\sqrt{\frac{3}{5}}}{29}, \frac{5}{29})$
Volume of maximal horoball piece			
$vol(\mathcal{B}_0 \cap \mathcal{F}_\Gamma)$	$(38400\sqrt{2})^{-1}$	$(19200\sqrt{2})^{-1}$	$(4800\sqrt{3})^{-1}$
Optimal Packing Density			
δ_{opt}	$\frac{81}{4\sqrt{2}\pi^3} \approx 0.46180\ldots$	$\frac{81}{4\sqrt{2}\pi^3}$	$\frac{189\sqrt{3}}{26\pi^3} \approx 0.40606\ldots$

Table 4 Data for asymptotic Coxeter tilings of $\mathbb{H}^7$ in the Cayley–Klein ball model centered at $O = (1, 0, 0, 0, 0, 0, 0, 0)$

The 7 Dimensional Coxeter Simplex Tilings				
Witt Symb.	$\overline{S}_7$	$\overline{Q}_7$	$\overline{T}_7$	$\overline{P}_7$
A_0	$(1,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,1)$
A_1	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$
A_2	$(1,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,\frac{\sqrt{21}}{7},0)$
A_3	$(1,0,0,0,0,\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{15}}{6},\frac{5\sqrt{21}}{42},0)$
A_4	$(1,0,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,\frac{\sqrt{3}}{6},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,0,\frac{\sqrt{10}}{5},\frac{2\sqrt{15}}{15},\frac{2\sqrt{21}}{21},0)$
A_5	$(1,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,\frac{1}{2},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,\frac{\sqrt{6}}{4},\frac{3\sqrt{10}}{20},\frac{\sqrt{15}}{10},\frac{\sqrt{21}}{14},0)$
A_6	$(1,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,\frac{1}{2},0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,-\frac{\sqrt{3}}{6},0,0,\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,\frac{\sqrt{3}}{3},\frac{\sqrt{6}}{6},\frac{\sqrt{10}}{10},\frac{\sqrt{15}}{15},\frac{\sqrt{21}}{21},0)$
A_7	$(1,\frac{1}{2},0,0,0,0,\frac{1}{2},0)$	$(1,\frac{1}{2},0,0,0,0,\frac{1}{2},0)$	$(1,-\frac{1}{2},-\frac{\sqrt{3}}{6},0,0,\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{6}}{12},\frac{\sqrt{10}}{20},\frac{\sqrt{15}}{30},\frac{\sqrt{21}}{42},0)$
The form $\boldsymbol{u}_i$ of sides opposite A_i				
$\boldsymbol{u}_0$	$(0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,1)$
$\boldsymbol{u}_1$	$(1,0,0,0,0,0,-2,-1)$	$(1,0,0,0,0,0,-2,-1)$	$(1,0,0,0,0,0,-2,-1)$	$(1,-1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{10}},-\frac{1}{\sqrt{15}},-\sqrt{\frac{7}{3}},-1)$
$\boldsymbol{u}_2$	$(0,-1,0,0,0,-\sqrt{2},1,0)$	$(0,-1,0,0,0,-\sqrt{2},1,0)$	$(0,0,0,0,0,-\sqrt{3},1,0)$	$(0,0,0,0,0,-\sqrt{\frac{5}{7}},1,0)$
$\boldsymbol{u}_3$	$(0,0,0,0,-1,1,0,0)$	$(0,0,0,0,-1,1,0,0)$	$(0,0,1,0,-1,1,0,0)$	$(0,0,0,0,-\sqrt{\frac{2}{3}},1,0,0)$
$\boldsymbol{u}_4$	$(0,0,0,-1,1,0,0,0)$	$(0,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},1,0,0,0)$	$(0,0,0,-\frac{1}{\sqrt{3}},1,0,0,0)$	$(0,0,0,-\sqrt{\frac{3}{5}},1,0,0,0)$
$\boldsymbol{u}_5$	$(0,0,-1,1,0,0,0,0)$	$(0,0,0,1,0,0,0,0)$	$(0,0,0,1,0,0,0,0)$	$(0,0,-\frac{1}{\sqrt{2}},1,0,0,0,0)$
$\boldsymbol{u}_6$	$(0,0,1,0,0,0,0,0)$	$(0,0,1,0,0,0,0,0)$	$(0,-\frac{1}{\sqrt{3}},1,0,0,0,0,0)$	$(0,-\frac{1}{\sqrt{3}},1,0,0,0,0,0)$
$\boldsymbol{u}_7$	$(0,1,0,0,0,0,0,0)$	$(0,1,0,0,0,0,0,0)$	$(0,1,0,0,0,0,0,0)$	$(0,1,0,0,0,0,0,0)$

Table 4 continued

The 7 Dimensional Coxeter Simplex Tilings				
Witt Symb.	$\overline{S}_7$	$\overline{Q}_7$	$\overline{T}_7$	$\overline{P}_7$
Maximal horoball parameter s_0				
s_0	0	0	0	0
Intersections $H_i = \mathcal{B}(A_0, s_0) \cap A_0A_i$ **of horoballs with simplex edges**				
H_1	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$
H_2	$(1,0,0,0,0,0,\frac{4}{9},\frac{1}{9})$	$(1,0,0,0,0,0,\frac{4}{9},\frac{1}{9})$	$(1,0,0,0,0,0,\frac{4}{9},\frac{1}{9})$	$(1,0,0,0,0,0,\frac{2\sqrt{21}}{17},\frac{3}{17})$
H_3	$(1,0,0,0,0,\frac{4\sqrt{2}}{19},\frac{8}{19},\frac{3}{19})$	$(1,0,0,0,0,\frac{4\sqrt{2}}{19},\frac{8}{19},\frac{3}{19})$	$(1,0,0,0,0,\frac{\sqrt{3}}{7},\frac{3}{7},\frac{1}{7})$	$(1,0,0,0,0,\frac{7\sqrt{\frac{5}{3}}}{19},\frac{5\sqrt{\frac{7}{3}}}{19},\frac{5}{19})$
H_4	$(1,0,0,0,\frac{\sqrt{2}}{5},\frac{\sqrt{2}}{5},\frac{2}{5},\frac{1}{5})$	$(1,0,0,0,\frac{\sqrt{2}}{5},\frac{\sqrt{2}}{5},\frac{2}{5},\frac{1}{5})$	$(1,0,0,0,\frac{4\sqrt{3}}{29},\frac{4\sqrt{3}}{29},\frac{12}{29},\frac{5}{29})$	$(1,0,0,0,\frac{7}{5\sqrt{10}},\frac{7}{5\sqrt{15}},\frac{\sqrt{\frac{7}{3}}}{5},\frac{3}{10})$
H_5	$(1,0,0,\frac{4\sqrt{2}}{21},\frac{4\sqrt{2}}{21},\frac{4\sqrt{2}}{21},\frac{8}{21},\frac{5}{21})$	$(1,0,0,\frac{4}{11},\frac{2\sqrt{2}}{11},\frac{2\sqrt{2}}{11},\frac{4}{11},\frac{3}{11})$	$(1,0,0,\frac{3}{8},\frac{\sqrt{3}}{8},\frac{\sqrt{3}}{8},\frac{3}{8},\frac{1}{4})$	$(1,0,0,\frac{7\sqrt{\frac{3}{2}}}{20},\frac{21}{20\sqrt{10}},\frac{7\sqrt{\frac{3}{5}}}{20},\frac{\sqrt{21}}{20},\frac{3}{10})$
H_6	$(1,0,\frac{2\sqrt{2}}{11},\frac{2\sqrt{2}}{11},\frac{2\sqrt{2}}{11},\frac{2\sqrt{2}}{11},\frac{4}{11},\frac{3}{11})$	$(1,0,\frac{4}{11},0,\frac{2\sqrt{2}}{11},\frac{2\sqrt{2}}{11},\frac{4}{11},\frac{3}{11})$	$(1,0,-\frac{4\sqrt{3}}{29},0,0,\frac{4\sqrt{3}}{29},\frac{12}{29},\frac{5}{29})$	$(1,0,\frac{14}{19\sqrt{3}},\frac{7\sqrt{\frac{2}{3}}}{19},\frac{7\sqrt{\frac{2}{5}}}{19},\frac{14}{19\sqrt{15}},\frac{2\sqrt{\frac{7}{3}}}{19},\frac{5}{19})$
H_7	$(1,\frac{2}{5},0,0,0,0,\frac{2}{5},\frac{1}{5})$	$(1,\frac{2}{5},0,0,0,0,\frac{2}{5},\frac{1}{5})$	$(1,-\frac{3}{8},-\frac{\sqrt{3}}{8},0,0,\frac{\sqrt{3}}{8},\frac{3}{8},\frac{1}{4})$	$(1,\frac{7}{17},\frac{7}{17\sqrt{3}},\frac{7}{17\sqrt{6}},\frac{7}{17\sqrt{10}},\frac{7}{17\sqrt{15}},\frac{\sqrt{\frac{7}{3}}}{17},\frac{3}{17})$
Volume of maximal horoball piece				
$vol(\mathcal{B}_0 \cap \mathcal{F})$	1105920^{-1}	552960^{-1}	$(829440\sqrt{3})^{-1}$	$(34560\sqrt{7})^{-1}$
Optimal Packing Density				
δ_{opt}	$\frac{21}{64\mathrm{L}(4)} \approx 0.33179\ldots$	$\frac{21}{64\mathrm{L}(4)}$	$\frac{28}{81\mathrm{L}(4,3)} 0.36773\ldots$	$\frac{96}{343\mathrm{L}(4,7)} \approx 0.26605\ldots$

The volume of the horoball piece contained in the fundamental simplex is

$$vol(\mathcal{B}_0 \cap \mathcal{F}_\Gamma) = \frac{1}{n-1}\mathcal{A}. \tag{12}$$

The locally optimal horoball packing density of Coxeter Simplex $\mathcal{F}_\Gamma$ is

$$\delta_{opt}(\mathcal{F}_\Gamma) = \frac{vol(\mathcal{B}_0 \cap \mathcal{F}_\Gamma)}{vol(\mathcal{F}_\Gamma)}. \tag{13}$$

□

Lemma 2 *The optimal horoball packing density $\delta_{opt}(\Gamma)$ of tiling $\mathcal{T}_\Gamma$ and the local horoball packings density $\delta_{opt}(\mathcal{F}_\Gamma)$ are equal.*

Proof The local construction the the proof of Lemma 1 is preserved by the isometric actions of $g \in \Gamma$. The Coxeter group Γ extends the optimal local horoball packing density from the fundamental domain $\mathcal{F}_\Gamma$ to the entire tiling $\mathcal{T}_\Gamma$ of $\overline{\mathbb{H}}^n$, that is $\delta_{opt}(\Gamma) = \delta_{opt}(\mathcal{F}_\Gamma) = \frac{vol(\mathcal{B}_0 \cap \mathcal{F}_\Gamma)}{vol(\mathcal{F}_\Gamma)}$. □

The volumes of two tangent horoball pieces centered at two distinct ideal vertices of the fundamental domain as the horoball type is continuously varied are related in the Lemma 3.

In $\overline{\mathbb{H}}^n$ with $n \geq 2$ let τ_1 and τ_2 be two congruent n-dimensional convex cones with vertices at $C_1, C_2 \in \partial\overline{\mathbb{H}}^n$ that share a common geodesic edge $\overline{C_1C_2}$. Let $B_1(x)$ and $B_2(x)$ denote two horoballs centered at C_1 and C_2 respectively, mutually tangent at $I(x) \in \overline{C_1C_2}$. Define $I(0)$ as the point with $V(0) = 2vol(B_1(0) \cap \tau_1) = 2vol(B_2(0) \cap \tau_2)$ for the volumes of the horoball sectors.

Lemma 3 *([25]) Let x be the hyperbolic distance between $I(0)$ and $I(x)$, then*

$$\begin{aligned} V(x) =& vol(B_1(x) \cap \tau_1) + vol(B_2(x) \cap \tau_2) \\ =& V(0)\frac{e^{(n-1)x} + e^{-(n-1)x}}{2} = V(0)\cosh((n-1)x) \end{aligned} \tag{14}$$

is strictly convex and strictly increasing as $x \to \pm\infty$.

Proof See our paper [25] for a proof. □

5 The optimal packing densities of the Koszul simplex tilings

In this section we determine the optimal horoball packing densities of the fourteen Koszul type Coxeter simplex tilings in dimensions $n = 6, 7, 8, 9$. Table 2 summarizes the data and optimal packing density of each tiling. Fig. 1 gives the commensurability relations of the groups in each dimension. We shall use the Witt symbols to denote each possible Γ.

5.1 Case $n = 6$ dimensions

Theorem 3 *The optimal horoball packing density of Coxeter simplex tilings $\mathcal{T}_\Gamma$, $\Gamma \in \left\{\overline{S}_6, \overline{Q}_6\right\}$ is $\delta_{opt}(\Gamma) = \frac{81}{4\sqrt{2}\pi^3}$, and for $\mathcal{T}_{\overline{P}_6}$ is $\delta_{opt}(\overline{P}_6) = \frac{189\sqrt{3}}{26\pi^3}$.*

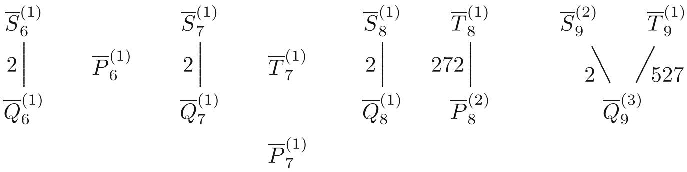


Fig. 1 Lattice of subgroups for each commensurability class of cocompact Coxeter groups. The subscript indicates the dimension, the superscript the number of ideal vertices of the fundamental simplex, and the index is indicated along edges

Proof Each Coxeter simplex $\mathcal{F}_\Gamma$ in $\overline{\mathbb{H}}^6$ has a single ideal vertex (see Table 2), so the local optimal packing densities follow from Lemma 1, and extend to the entire space by Lemma 2. Our choice of vertices A_i, forms of hyperplanes $\boldsymbol{u}_i$ opposite to vertices A_i, optimal horoball parameters s, and horoball intersection points are given in Table 3. □

The following Corollary relates Theorem 3 to the simplicial packing density upper bound, recall Table 1.

Corollary 1 *The optimal congruent ball packing density in $\mathbb{H}^6$ up to horoballs of the same type is bounded by* $\frac{81}{4\sqrt{2}\pi^3} \leq \delta_{opt}(\overline{\mathbb{H}}^6) \leq 0.49339\ldots$.

5.2 Case $n = 7$ dimensions

Theorem 4 *The optimal horoball packing density of Coxeter simplex tilings $\mathcal{T}_\Gamma$, $\Gamma \in \left\{\overline{S}_7, \overline{Q}_7\right\}$ is $\delta_{opt}(\Gamma) = \frac{21}{64L(4)}$. The Coxeter simplex tiling $\mathcal{T}_{\overline{P}_7}$ is $\delta_{opt}(\overline{P}_7) = \frac{96}{343L(4,7)}$, and $\mathcal{T}_{\overline{T}_7}$ is $\delta_{opt}(\overline{T}_7) = \frac{28}{81L(4,3)}$.*

Proof Each Coxeter simplex $\mathcal{F}_\Gamma$ in $\overline{\mathbb{H}}^7$ has one ideal vertex (see Table 2), so the locally optimal packing densities follow from Lemma 1, and extend to the entire space by Lemma 2. Our choice of vertices A_i, forms of hyperplanes $\boldsymbol{u}_i$ opposite to vertices A_i, optimal horoball parameters s, and horoball intersection points are given in Table 4. Here we used the Dirichlet L-function $\mathrm{L}(s, d) = \sum_{n=1}^{\infty} \left(\frac{n}{d}\right) n^{-s}$, where (n/d) is the Legendre symbol. □

Corollary 2 *The optimal congruent ball packing density in $\mathbb{H}^7$ up to horoballs of the same type is bounded by* $\frac{28}{81L(4,3)} \leq \delta_{opt}(\overline{\mathbb{H}}^7) \leq 0.39441\ldots$.

5.3 Case $n = 8$ dimensions

Theorem 5 *The optimal horoball packing density of Coxeter simplex tilings $\mathcal{T}_\Gamma$, $\Gamma \in \left\{\overline{S}_8, \overline{Q}_8\right\}$ is $\delta_{opt}(\Gamma) = \frac{2025}{68\sqrt{2}\pi^4}$, and for $\Gamma \in \left\{\overline{T}_8, \overline{P}_8\right\}$, $\delta_{opt}(\Gamma) = \frac{225}{8\pi^4}$.*

Proof There are two cases, the fundamental domain has one or two ideal vertices.

Case 1: Coxeter simplices $\mathcal{F}_\Gamma$ for $\Gamma \in \left\{\overline{S}_8, \overline{Q}_8, \overline{T}_8\right\}$ in $\overline{\mathbb{H}}^8$ have one ideal vertex and the local optimal packing densities follow from Lemma 1, and extends to the entire space by Lemma 2. Our choice of coordinates for vertices A_i, forms of hyperplanes $\boldsymbol{u}_i$ opposite to vertices A_i, and the computed optimal horoball s parameters, horoball intersection points are given in Table 5.

Table 5 Data for asymptotic Coxeter tilings of $\mathbb{H}^8$ in the Cayley–Klein ball model centered at $O=(1,0,0,0,0,0,0,0,0)$

Coxeter Simplex Tilings				
Witt Symb.	$\overline{S}_8$	$\overline{Q}_8$	$\overline{T}_8$	$\overline{P}_8$
A_0	$(1,0,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,0,1)$
A_1	$(1,0,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0,0)$
A_2	$(1,0,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,0,\frac{1}{2},0)$	$(1,0,0,0,0,0,0,\frac{\sqrt{7}}{4},0)$
A_3	$(1,0,0,0,0,0,\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,0,\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,0,\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,0,0,0,\sqrt{\frac{3}{7}},\frac{3}{2\sqrt{7}},0)$
A_4	$(1,0,0,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{6}}{12},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,0,0,\frac{\sqrt{\frac{5}{3}}}{2},\frac{5}{2\sqrt{21}},\frac{5}{4\sqrt{7}},0)$
A_5	$(1,0,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,\frac{1}{2},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,0,\frac{\sqrt{6}}{12},\frac{\sqrt{6}}{12},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,0,\sqrt{\frac{2}{5}},\frac{2}{\sqrt{15}},\frac{2}{\sqrt{21}},\frac{1}{\sqrt{7}},0)$
A_6	$(1,0,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,\frac{1}{2},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,0,\frac{\sqrt{3}}{6},\frac{\sqrt{6}}{12},\frac{\sqrt{6}}{12},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,0,\frac{\sqrt{\frac{3}{2}}}{2},\frac{3}{2\sqrt{10}},\frac{\sqrt{\frac{3}{5}}}{2},\frac{\sqrt{\frac{3}{7}}}{2},\frac{3}{4\sqrt{7}},0)$
A_7	$(1,0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,\frac{1}{2},0,\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{2}}{4},\frac{1}{2},0)$	$(1,0,\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{6}}{12},\frac{\sqrt{6}}{12},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{10}},\frac{1}{\sqrt{15}},\frac{1}{\sqrt{21}},\frac{1}{2\sqrt{7}},0)$
A_8	$(1,\frac{1}{2},0,0,0,0,0,\frac{1}{2},0)$	$(1,\frac{1}{2},0,0,0,0,0,\frac{1}{2},0)$	$(1,\frac{1}{4},0,0,0,\frac{\sqrt{6}}{12},\frac{\sqrt{3}}{6},\frac{1}{2},0)$	$(1,\frac{1}{2},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{6}},\frac{1}{2\sqrt{10}},\frac{1}{2\sqrt{15}},\frac{1}{2\sqrt{21}},\frac{1}{4\sqrt{7}},0)$
The form u_i of sides opposite A_i				
u_0	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0,1)$
u_1	$(1,0,0,0,0,0,0,-2,-1)$	$(1,0,0,0,0,0,0,-2,-1)$	$(1,0,0,0,0,0,0,-2,-1)$	$(1,-1,-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{10}},-\frac{1}{\sqrt{15}},-\frac{1}{\sqrt{21}},-\frac{4}{\sqrt{7}},-1)$
u_2	$(0,-1,0,0,0,0,-\sqrt{2},1,0)$	$(0,-1,0,0,0,0,-\sqrt{2},1,0)$	$(0,0,0,0,0,0,-\sqrt{3},1,0)$	$(0,0,0,0,0,0,-\frac{\sqrt{3}}{2},1,0)$
u_3	$(0,0,0,0,0,-1,1,0,0)$	$(0,0,0,0,0,-1,1,0,0)$	$(0,0,0,0,0,-\sqrt{2},1,0,0)$	$(0,0,0,0,0,-\sqrt{\frac{5}{7}},1,0,0)$
u_4	$(0,0,0,0,-1,1,0,0,0)$	$(0,0,0,0,-1,1,0,0,0)$	$(0,-\sqrt{\frac{2}{3}},0,0,-1,1,0,0,0)$	$(0,0,0,0,-\sqrt{\frac{2}{3}},1,0,0,0)$
u_5	$(0,0,0,-1,1,0,0,0,0)$	$(0,0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},1,0,0,0,0)$	$(0,0,0,-\frac{1}{\sqrt{2}},1,0,0,0,0)$	$(0,0,0,-\sqrt{\frac{3}{5}},1,0,0,0,0)$

Table 5 continued

Coxeter Simplex Tilings				
Witt Symb.	$\overline{S}_8$	$\overline{Q}_8$	$\overline{T}_8$	$\overline{P}_8$
$\boldsymbol{u}_6$	$(0, 0, -1, 1, 0, 0, 0, 0, 0)$	$(0, 0, 0, 1, 0, 0, 0, 0, 0)$	$(0, 0, -\frac{1}{\sqrt{3}}, 1, 0, 0, 0, 0, 0)$	$(0, 0, -\frac{1}{\sqrt{2}}, 1, 0, 0, 0, 0, 0)$
$\boldsymbol{u}_7$	$(0, 0, 1, 0, 0, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 0, 0, 0)$	$(0, -\frac{1}{\sqrt{3}}, 1, 0, 0, 0, 0, 0, 0)$
$\boldsymbol{u}_8$	$(0, 1, 0, 0, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0, 0, 0)$
Maximal horoball parameter s_0				
s_0	0	0	0	$s_0 = 0, s_5 = \frac{3}{5}$
Intersections $H_i = \mathcal{B}(A_0, s_0) \cap A_0A_i$ **of horoballs with simplex edges**				
H_1	$(1, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0)$
H_2	$(1, 0, 0, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, 0, 0, \frac{8\sqrt{7}}{39}, \frac{7}{39})$
H_3	$(1, 0, 0, 0, 0, 0, \frac{4\sqrt{2}}{19}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, 0, 0, \frac{4\sqrt{2}}{19}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{7}, \frac{3}{7}, \frac{1}{7})$	$(1, 0, 0, 0, 0, 0, \frac{8\sqrt{\frac{3}{7}}}{11}, \frac{12}{11\sqrt{7}}, \frac{3}{11})$
H_4	$(1, 0, 0, 0, 0, \frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}, \frac{2}{5}, \frac{1}{5})$	$(1, 0, 0, 0, 0, \frac{\sqrt{2}}{5}, \frac{\sqrt{2}}{5}, \frac{2}{5}, \frac{1}{5})$	$(1, 0, 0, 0, 0, \frac{4\sqrt{\frac{2}{3}}}{19}, \frac{8}{19\sqrt{3}}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, 0, \frac{16\sqrt{\frac{5}{3}}}{47}, \frac{80}{47\sqrt{21}}, \frac{40}{47\sqrt{7}}, \frac{15}{47})$
H_5	$(1, 0, 0, 0, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21},$ $\frac{8}{21}, \frac{5}{21})$	$(1, 0, 0, 0, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21}, \frac{4\sqrt{2}}{21},$ $\frac{8}{21}, \frac{5}{21})$	$(1, 0, 0, 0, \frac{2\sqrt{6}}{29}, \frac{2\sqrt{6}}{29}, \frac{4\sqrt{3}}{29},$ $\frac{12}{29}, \frac{5}{29})$	$(1, 0, 0, 0, \frac{2\sqrt{\frac{2}{5}}}{3}, \frac{4}{3\sqrt{15}}, \frac{4}{3\sqrt{21}},$ $\frac{2}{3\sqrt{7}}, \frac{1}{3})$

Table 5 continued

Coxeter Simplex Tilings				
Witt Symb.	$\overline{S}_8$	$\overline{Q}_8$	$\overline{T}_8$	$\overline{P}_8$
H_6	$(1, 0, 0, \frac{2\sqrt{2}}{11}, \frac{2\sqrt{2}}{11}, \frac{2\sqrt{2}}{11}, \frac{2\sqrt{2}}{11}, \frac{4}{11}, \frac{3}{11})$	$(1, 0, 0, \frac{8}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{8}{23}, \frac{7}{23})$	$(1, 0, 0, \frac{2}{5\sqrt{3}}, \frac{\sqrt{\frac{2}{3}}}{5}, \frac{\sqrt{\frac{2}{3}}}{5}, \frac{2}{5\sqrt{3}}, \frac{2}{5}, \frac{1}{5})$	$(1, 0, 0, \frac{8\sqrt{6}}{47}, \frac{24\sqrt{\frac{2}{5}}}{47}, \frac{16\sqrt{\frac{3}{5}}}{47}, \frac{16\sqrt{\frac{3}{7}}}{47}, \frac{24}{47\sqrt{7}}, \frac{15}{47})$
H_7	$(1, 0, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{8}{23}, \frac{7}{23})$	$(1, 0, \frac{8}{23}, 0, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{4\sqrt{2}}{23}, \frac{8}{23}, \frac{7}{23})$	$(1, 0, \frac{4}{11}, \frac{4}{11\sqrt{3}}, \frac{2\sqrt{\frac{2}{3}}}{11}, \frac{2\sqrt{\frac{2}{3}}}{11}, \frac{4}{11\sqrt{3}}, \frac{4}{11}, \frac{3}{11})$	$(1, 0, \frac{8}{11\sqrt{3}}, \frac{4\sqrt{\frac{2}{3}}}{11}, \frac{4\sqrt{\frac{2}{5}}}{11}, \frac{8}{11\sqrt{15}}, \frac{8}{11\sqrt{21}}, \frac{4}{11\sqrt{7}}, \frac{3}{11})$
H_8	$(1, \frac{2}{5}, 0, 0, 0, 0, 0, \frac{2}{5}, \frac{1}{5})$	$(1, \frac{2}{5}, 0, 0, 0, 0, 0, \frac{2}{5}, \frac{1}{5})$	$(1, \frac{8}{39}, 0, 0, 0, \frac{8\sqrt{\frac{2}{3}}}{39}, \frac{16}{39\sqrt{3}}, \frac{16}{39}, \frac{7}{39})$	$(1, \frac{16}{39}, \frac{16}{39\sqrt{3}}, \frac{8\sqrt{\frac{2}{3}}}{39}, \frac{8\sqrt{\frac{2}{5}}}{39}, \frac{16}{39\sqrt{15}}, \frac{16}{39\sqrt{21}}, \frac{8}{39\sqrt{7}}, \frac{7}{39})$
Volume of maximal horoball piece				
$vol(\mathcal{B}_0 \cap \mathcal{F})$	$(18063360\sqrt{2})^{-1}$	$(9031680\sqrt{2})^{-1}$	162570240^{-1}	1128960^{-1}
Optimal Packing Density				
δ_{opt}	$\frac{2025}{68\sqrt{2}\pi^4} \approx 0.21617\ldots$	$\frac{2025}{68\sqrt{2}\pi^4}$	$\frac{225}{8\pi^4} \approx 0.28873\ldots$	$(\frac{9}{17} + \frac{8}{17})\frac{225}{8\pi^4}$

Table 6 Data for asymptotic Coxeter tilings of $\mathbb{H}^9$ in the Cayley–Klein ball model centered at $O = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

Coxeter Simplex Tilings			
Witt Symb.	$\overline{T}_9$	$\overline{S}_9$	$\overline{Q}_9$
A_0	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
A_1	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
A_2	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{3}, 0)$
A_3	$(1, 0, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, 0, 0, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, 0, 0, 0, 0, \frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, 0)$
A_4	$(1, 0, 0, 0, 0, 0, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, 0, 0, 0, \frac{\sqrt{10}}{10}, \frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, 0)$
A_5	$(1, 0, 0, 0, 0, \frac{\sqrt{10}}{20}, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, 0, \frac{\sqrt{5}}{10}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, 0, 0, \frac{\sqrt{15}}{15}, \frac{\sqrt{10}}{10}, \frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, 0)$
A_6	$(1, 0, 0, 0, \frac{\sqrt{15}}{30}, \frac{\sqrt{10}}{20}, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, 0, 0, \frac{\sqrt{30}}{30}, \frac{\sqrt{5}}{10}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, 0, \frac{\sqrt{3}}{6}, \frac{\sqrt{15}}{15}, \frac{\sqrt{10}}{10}, \frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, 0)$
A_7	$(1, 0, 0, \frac{\sqrt{3}}{12}, \frac{\sqrt{15}}{30}, \frac{\sqrt{10}}{20}, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, 0, \frac{\sqrt{6}}{12}, \frac{\sqrt{30}}{30}, \frac{\sqrt{5}}{10}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 0, \frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{15}}{15}, \frac{\sqrt{10}}{10}, \frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{3}, 0)$
A_8	$(1, 0, \frac{1}{4}, \frac{\sqrt{3}}{12}, \frac{\sqrt{15}}{30}, \frac{\sqrt{10}}{20}, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, 0, \frac{\sqrt{2}}{4}, \frac{\sqrt{6}}{12}, \frac{\sqrt{30}}{30}, \frac{\sqrt{5}}{10}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
A_9	$(1, \frac{1}{6}, 0, 0, \frac{\sqrt{15}}{30}, \frac{\sqrt{10}}{20}, \frac{\sqrt{6}}{12}, \frac{\sqrt{3}}{6}, \frac{1}{2}, 0)$	$(1, \frac{\sqrt{2}}{6}, 0, 0, \frac{\sqrt{30}}{30}, \frac{\sqrt{5}}{10}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}, 0)$	$(1, \frac{1}{3}, 0, 0, 0, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, 0)$
The form $\boldsymbol{u}_i$ **of sides opposite** A_i			
$\boldsymbol{u}_0$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
$\boldsymbol{u}_1$	$(1, 0, 0, 0, 0, 0, 0, 0, -2, -1)$	$(1, 0, 0, 0, 0, 0, 0, 0, -\sqrt{2}, -1)$	$(1, 0, -1, 0, 0, 0, 0, 0, -\sqrt{3}, -1)$
$\boldsymbol{u}_2$	$(0, 0, 0, 0, 0, 0, 0, -\sqrt{3}, 1, 0)$	$(0, 0, 0, 0, 0, 0, 0, -\sqrt{3}, 1, 0)$	$(0, 0, 0, 0, 0, 0, 0, -\sqrt{2}, 1, 0)$
$\boldsymbol{u}_3$	$(0, 0, 0, 0, 0, 0, -\sqrt{2}, 1, 0, 0)$	$(0, 0, 0, 0, 0, 0, -\sqrt{2}, 1, 0, 0)$	$(0, 0, 0, 0, 0, 0, -\sqrt{\frac{5}{3}}, 1, 0, 0)$
$\boldsymbol{u}_4$	$(0, 0, 0, 0, 0, -\sqrt{\frac{5}{3}}, 1, 0, 0, 0)$	$(0, 0, 0, 0, 0, -\sqrt{\frac{5}{3}}, 1, 0, 0, 0)$	$(0, 0, 0, 0, 0, -\sqrt{\frac{3}{2}}, 1, 0, 0, 0)$
$\boldsymbol{u}_5$	$(0, 0, 0, 0, -\sqrt{\frac{3}{2}}, 1, 0, 0, 0, 0)$	$(0, 0, 0, 0, -\sqrt{\frac{3}{2}}, 1, 0, 0, 0, 0)$	$(0, -\sqrt{\frac{3}{5}}, 0, 0, -\frac{2}{\sqrt{5}}, 1, 0, 0, 0, 0)$
$\boldsymbol{u}_6$	$(0, -\sqrt{\frac{3}{5}}, 0, -\frac{2}{\sqrt{5}}, 1, 0, 0, 0, 0, 0)$	$(0, -\sqrt{\frac{3}{5}}, 0, -\frac{2}{\sqrt{5}}, 1, 0, 0, 0, 0, 0)$	$(0, 0, 0, -\frac{1}{\sqrt{3}}, 1, 0, 0, 0, 0, 0)$
$\boldsymbol{u}_7$	$(0, 0, -\frac{1}{\sqrt{3}}, 1, 0, 0, 0, 0, 0, 0)$	$(0, 0, -\frac{1}{\sqrt{3}}, 1, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$

Table 6 continued

Coxeter Simplex Tilings			
Witt Symb.	$\overline{T}_9$	$\overline{S}_9$	$\overline{Q}_9$
$\boldsymbol{u}_8$	$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 0, 0, 0, 0)$
$\boldsymbol{u}_9$	$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$	$(0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$
Maximal horoball parameters s_i			
s_i	$s_0 = 0$	$s_0 = 0, s_8 = 7/9$	$s_0 = 0, s_7 = 3/5, s_8 = 0$
Intersections $H_i = \mathcal{B}(A_0, s_0) \cap A_0A_i$ **of horoballs with simplex edges**			
H_1	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$
H_2	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{4}{9}, \frac{1}{9})$	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{2\sqrt{2}}{5}, \frac{1}{5})$	$(1, 0, 0, 0, 0, 0, 0, 0, \frac{2\sqrt{3}}{7}, \frac{1}{7})$
H_3	$(1, 0, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{7}, \frac{3}{7}, \frac{1}{7})$	$(1, 0, 0, 0, 0, 0, 0, \frac{\sqrt{\frac{3}{2}}}{4}, \frac{3}{4\sqrt{2}}, \frac{1}{4})$	$(1, 0, 0, 0, 0, 0, 0, \frac{2\sqrt{\frac{2}{3}}}{5}, \frac{4}{5\sqrt{3}}, \frac{1}{5})$
H_4	$(1, 0, 0, 0, 0, 0, \frac{4\sqrt{\frac{2}{3}}}{19}, \frac{8}{19\sqrt{3}}, \frac{8}{19}, \frac{3}{19})$	$(1, 0, 0, 0, 0, 0, \frac{4}{11\sqrt{3}}, \frac{4\sqrt{\frac{2}{3}}}{11}, \frac{4\sqrt{2}}{11}, \frac{3}{11})$	$(1, 0, 0, 0, 0, 0, \frac{\sqrt{10}}{13}, \frac{5\sqrt{\frac{2}{3}}}{13}, \frac{10}{13\sqrt{3}}, \frac{3}{13})$
H_5	$(1, 0, 0, 0, 0, \frac{\sqrt{\frac{5}{2}}}{12}, \frac{5}{12\sqrt{6}}, \frac{5}{12\sqrt{3}}, \frac{5}{12}, \frac{1}{6})$	$(1, 0, 0, 0, 0, \frac{\sqrt{5}}{14}, \frac{5}{14\sqrt{3}}, \frac{5}{7\sqrt{6}}, \frac{5}{7\sqrt{2}}, \frac{2}{7})$	$(1, 0, 0, 0, 0, \frac{\sqrt{\frac{3}{5}}}{4}, \frac{3}{4\sqrt{10}}, \frac{\sqrt{\frac{3}{2}}}{4}, \frac{\sqrt{3}}{4}, \frac{1}{4})$
H_6	$(1, 0, 0, 0, \frac{4\sqrt{\frac{3}{5}}}{29}, \frac{6\sqrt{\frac{2}{5}}}{29}, \frac{2\sqrt{6}}{29}, \frac{4\sqrt{3}}{29}, \frac{12}{29}, \frac{5}{29})$	$(1, 0, 0, 0, \frac{2\sqrt{\frac{6}{5}}}{17}, \frac{6}{17\sqrt{5}}, \frac{2\sqrt{3}}{17}, \frac{2\sqrt{6}}{17}, \frac{6\sqrt{2}}{17}, \frac{5}{17})$	$(1, 0, 0, 0, \frac{4}{11\sqrt{3}}, \frac{8}{11\sqrt{15}}, \frac{4\sqrt{\frac{2}{5}}}{11}, \frac{4\sqrt{\frac{2}{3}}}{11}, \frac{8}{11\sqrt{3}}, \frac{3}{11})$
H_7	$(1, 0, 0, \frac{8}{39\sqrt{3}}, \frac{16}{39\sqrt{15}}, \frac{8\sqrt{\frac{2}{5}}}{39}, \frac{8\sqrt{\frac{2}{3}}}{39}, \frac{16}{39\sqrt{3}}, \frac{16}{39}, \frac{7}{39})$	$(1, 0, 0, \frac{4\sqrt{\frac{2}{3}}}{23}, \frac{8\sqrt{\frac{2}{15}}}{23}, \frac{8}{23\sqrt{5}}, \frac{8}{23\sqrt{3}}, \frac{8\sqrt{\frac{2}{3}}}{23}, \frac{8\sqrt{2}}{23}, \frac{7}{23})$	$(1, 0, 0, \frac{1}{3}, \frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{15}}, \frac{\sqrt{\frac{2}{5}}}{3}, \frac{\sqrt{\frac{2}{3}}}{3}, \frac{2}{3\sqrt{3}}, \frac{1}{3})$

Table 6 continued

Coxeter Simplex Tilings			
Witt Symb.	$\overline{T}_9$	$\overline{S}_9$	$\overline{Q}_9$
H_8	$(1, 0, \frac{1}{5}, \frac{1}{5\sqrt{3}}, \frac{2}{5\sqrt{15}}, \frac{\sqrt{\frac{2}{5}}}{5}, \frac{\sqrt{\frac{2}{3}}}{5}, \frac{2}{5\sqrt{3}}, \frac{2}{5}, \frac{1}{5})$	$(1, 0, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{6}}, \frac{\sqrt{\frac{2}{15}}}{3}, \frac{1}{3\sqrt{5}}, \frac{1}{3\sqrt{3}}, \frac{\sqrt{\frac{2}{3}}}{3}, \frac{\sqrt{2}}{3}, \frac{1}{3})$	$(1, 0, \frac{2}{3}, 0, 0, 0, 0, 0, 0, \frac{1}{3})$
H_9	$(1, \frac{3}{22}, 0, 0, \frac{3\sqrt{\frac{3}{5}}}{22}, \frac{9}{22\sqrt{10}}, \frac{3\sqrt{\frac{3}{2}}}{22}, \frac{3\sqrt{3}}{22}, \frac{9}{22}, \frac{2}{11})$	$(1, \frac{3}{13\sqrt{2}}, 0, 0, \frac{3\sqrt{\frac{3}{10}}}{13}, \frac{9}{26\sqrt{5}}, \frac{3\sqrt{3}}{26}, \frac{3\sqrt{\frac{3}{2}}}{13}, \frac{9}{13\sqrt{2}}, \frac{4}{13})$	$(1, \frac{6}{25}, 0, 0, 0, \frac{6\sqrt{\frac{3}{5}}}{25}, \frac{9\sqrt{\frac{2}{5}}}{25}, \frac{3\sqrt{6}}{25}, \frac{6\sqrt{3}}{25}, \frac{7}{25})$
Volume of maximal horoball piece			
$vol(\mathcal{B}_0 \cap \mathcal{F})$	89181388800^{-1}	5573836800^{-1}	348364800^{-1}
		Optimal packing density	
δ_{opt}	$\frac{1}{4\zeta(5)} \approx 0.24109\ldots$	$(\frac{135}{151} + \frac{16}{151})\frac{151}{1054\zeta(5)} \approx 0.138162\ldots$	$(\frac{256}{527} + \frac{270}{527} + \frac{1}{527})\frac{1}{4\zeta(5)} = \frac{1}{4\zeta(5)} \approx 0.24109\ldots$

Case 2: $\mathcal{F}_{\overline{P}_8}$ has two ideal vertices A_0 and A_5, see Table 5. Let B_0 (arctanh s_0) and B_5(arctanh s_5) be horoballs with parameters s_0 and s_5 centered at A_0 and A_5. To find the horosphere equation for horoball B_5, we transform the model and rotate A_5 to A_0 by $\mathrm{Rot}_{A_5A_0} \in \mathrm{PGL}(n+1,\mathbb{R})$ in coordinates represented by matrix

$$\mathrm{Rot}_{A_5A_0} = \begin{pmatrix} 1\,0\,0\,0 & 0 & 0 & 0 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 & 0 & 0 & 0 \\ 0\,0\,1\,0 & 0 & 0 & 0 & 0 & 0 \\ 0\,0\,0\,1 & 0 & 0 & 0 & 0 & 0 \\ 0\,0\,0\,0 & \frac{3}{5} & -\frac{1}{5}\left(2\sqrt{\frac{2}{3}}\right) & -2\sqrt{\frac{2}{105}} & -\sqrt{\frac{2}{35}} & \sqrt{\frac{2}{5}} \\ 0\,0\,0\,0 & -\frac{1}{5}\left(2\sqrt{\frac{2}{3}}\right) & \frac{11}{15} & -\frac{4}{3\sqrt{35}} & -\frac{2}{\sqrt{105}} & \frac{2}{\sqrt{15}} \\ 0\,0\,0\,0 & -2\sqrt{\frac{2}{105}} & -\frac{4}{3\sqrt{35}} & \frac{17}{21} & -\frac{2}{7\sqrt{3}} & \frac{2}{\sqrt{21}} \\ 0\,0\,0\,0 & -\sqrt{\frac{2}{35}} & -\frac{2}{\sqrt{105}} & -\frac{2}{7\sqrt{3}} & \frac{6}{7} & \frac{1}{\sqrt{7}} \\ 0\,0\,0\,0 & -\sqrt{\frac{2}{5}} & -\frac{2}{\sqrt{15}} & -\frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{7}} & 0 \end{pmatrix}. \tag{15}$$

Let $x_i = \operatorname{arctanh} s_i = \beta(S_i, O, A_i)$ denote the hyperbolic distance of center of the model $A_1 = (1, 0, \dots, 0)$ to $S_i = (1, 0, \dots, 0, s_i)$ for $i \in \{0, 5\}$, rotated in the case of A_5. If horoball B_0 is maximal $s_0 = 0$. If horoball B_5 is maximal then $s_5 = \frac{3}{5}$. These two maximal horoballs $B_0(\operatorname{arctanh} 0)$ and $B_5(\operatorname{arctanh} \frac{3}{5})$ are tangent to hyperfaces $[\boldsymbol{u}_0]$ and $[\boldsymbol{u}_5]$ respectively, and to each other at H_5. By two applications of Lemma 1, and Lemma 2 the optimal backing packing density is $\delta_{opt}(\Gamma) = \frac{225}{8\pi^4}$. □

Corollary 3 *The optimal congruent ball packing density in* $\mathbb{H}^8$ *up to horoballs of the same type is bounded by* $\frac{225}{8\pi^4} \leq \delta_{opt}(\overline{\mathbb{H}}^8) \leq 0.31114\ldots$

5.4 Case $n = 9$ dimensions

Theorem 6 *The optimal horoball packing density of Coxeter simplex tilings* $\mathcal{T}_\Gamma$, $\Gamma \in \left\{\overline{T}_9, \overline{Q}_9\right\}$ *is* $\delta_{opt}(\Gamma) = \frac{1}{4\zeta(5)}$, *and for* $\mathcal{T}_{\overline{S}_9}$ *is* $\delta_{opt}(\overline{S}_9) = \frac{151}{1054\zeta(5)}$.

Proof There are three cases for when $\mathcal{F}_\Gamma$ has one, two, or three ideal vertices.

Case 1: Coxeter simplex $\mathcal{F}_{\overline{T}_9}$ in $\overline{\mathbb{H}}^9$ has one ideal vertex, the local optimal packing density follows from Lemma 1, and extends to the entire space by Lemma 2. Our choice of vertices A_i, hyperplanes $\boldsymbol{u}_i$ opposite to A_i, optimal the horoball parameter s, horoball intersection points, and horoball piece volumes are given in Table 6.

Case 2: $\mathcal{F}_{\overline{S}_9}$ has two ideal vertices, Table 6 assignes coordinates, with ideal vertices at A_0 and A_8. We use two horoballs B_0 (arctanh s_0) and B_8(arctanh s_8) with parameters s_0 and s_8 at centered at A_0 and A_8 respectively. Let $x_i = \operatorname{arctanh} s_i = \beta(S_i, O, A_i)$ denote the hyperbolic distance from the center of the model A_1 to $S_i = (1, 0, \dots, 0, s_i)$ for $i \in \{0, 8\}$ (after rotation of B_8 as in Theorem 4). If horoball B_0 is maximal then $s_0 = 0$. If horoball B_8 is maximal the $s_8 = \frac{7}{9}$. One can check that the two maximal type horoballs do not intersect, so with two applications of Lemma 1, and then Lemma 2 yields the optimal packing density $\delta_{opt}(\overline{S}_9) = \frac{151}{1054\zeta(5)}$.

Case 3: Assign coordinates to the fundamental domain $\mathcal{F}_{\overline{Q}_9}$ as in Table 6. The ideal vertices are A_0, A_7, and A_8. Place horoballs $B_i(\operatorname{arctanh} s_i)$ with parameters s_i at A_i for $i \in \{0, 7, 8\}$.

Let $x_i = \operatorname{arctanh} s_i = \beta(S_i, O, A_i)$ denote the hyperbolic distance from the center of the model A_1 to point $S_i = (1, 0, \dots, 0, s_i)$. $S_i \in B_i$ after the rotation of A_i to A_0.

If horoball B_0 is maximal then $s_0 = 0$, and the maximal tangent horoballs B_7 and B_8 have $s_7 = \frac{3}{5}$ and $s_8 = \frac{3}{5}$. If horoball B_8 is maximal type it is the same case up to symmetry, so it suffices to find the densities up to the midpoint of the allowed s_i parameter range. If horoball B_7 is maximal its parameter is $s_7 = \frac{3}{5}$ and the tangent maximal horoballs at B_0 and B_8 are respectively $s_0 = 0$ and $s_8 = 0$. Horoballs $B_0(\operatorname{arctanh} 0)$ and $B_8(\operatorname{arctanh} \frac{3}{5})$ are tangent to hyperfaces $\boldsymbol{u}_0$ and $\boldsymbol{u}_8$ respectively. The densities of the extremal horoball arrangements are $\Theta = \frac{1}{4\zeta(5)}$, in particular

$$\begin{aligned}
\Theta &= \delta_{s_0=0, s_7=\frac{3}{5}, s_8=\frac{3}{5}}(\overline{Q}_9) \\
&= \frac{vol(\mathcal{B}_0(\operatorname{arctanh} 0) \cap \mathcal{F}_{\overline{Q}_9}) + \sum_{i\in\{7,8\}} vol(\mathcal{B}_i(\operatorname{arctanh} \frac{3}{5})) \cap \mathcal{F}_{\overline{Q}_9})}{vol(\mathcal{F}_{\overline{Q}_9})}, \\
\Theta &= \delta_{s_0=\frac{3}{5}, s_7=\frac{3}{5}, s_8=0}(\overline{Q}_9) \\
&= \frac{vol(\mathcal{B}_8(\operatorname{arctanh} 0) \cap \mathcal{F}_{\overline{Q}_9}) + \sum_{i\in\{0,7\}} vol(\mathcal{B}_i(\operatorname{arctanh} \frac{3}{5}) \cap \mathcal{F}_{\overline{Q}_9})}{vol(\mathcal{F}_{\overline{Q}_9})}.
\end{aligned} \tag{16}$$

Next consider the horoball arrangements that continuously transition between the two extremal cases. Begin with the horoball arrangement with parameters $s_0 = 0$ and $s_8 = \frac{3}{5}$, the horoballs $B_i(\operatorname{arctanh} s_i)$ where $i \in \{0, 8\}$ are tangent. Define volumes $V_i(x) = vol(B_i(\operatorname{arctanh} s_i - x) \cap \mathcal{F}_{\overline{Q}_9})$ for $i \in \{0, 8\}$ with $x \in [0, \operatorname{arctanh} \frac{3}{5}]$ where $\operatorname{arctanh} \frac{3}{5}$ is the hyperbolic distance of A_1 and $S_i = (1, 0, \dots, 0, \frac{3}{5})$. By formulas (2), (5), (6), and (7), $V_0(\operatorname{arctanh} 0) = \frac{1}{348364800}$, $V_7(\operatorname{arctanh} \frac{3}{5}) = \frac{1}{330301440}$ and $V_8(\operatorname{arctanh} \frac{3}{5}) = \frac{1}{89181388800}$. By a weighted modification of Lemma 3,

$$\begin{aligned}
V(x) &= V_0(0)e^{-8x} + V_2\left(\operatorname{arctanh} \tfrac{3}{5}\right) + V_8\left(\operatorname{arctanh} \tfrac{3}{5}\right) e^{8x} \\
&= \frac{256e^{-8x} + 270 + e^{8x}}{89181388800}.
\end{aligned} \tag{17}$$

The densities of the intermediate cases between of the two extremal arrangements are given by

$$\begin{aligned}
\delta_x(\overline{Q}_9) &= \frac{vol(B_0(x) \cap \mathcal{F}_{\overline{Q}_9}) + vol(B_7(\operatorname{arctanh} \frac{3}{5}) \cap \mathcal{F}_{\overline{Q}_9}) + vol(B_8(\operatorname{arctanh} \frac{3}{5} - x) \cap \mathcal{F}_{\overline{Q}_9})}{vol(\mathcal{F}_{\overline{Q}_9})} \\
&= \left(\tfrac{256}{527}e^{-8x} + \tfrac{270}{527} + \tfrac{1}{527}e^{8x}\right)\Theta.
\end{aligned} \tag{18}$$

where $x \in [0, \text{arctanh}\, \frac{3}{5}]$. Analysis of $\delta_x(\overline{Q}_9)$ shows that its maxima are attained at the endpoints of the interval $[0, \text{arctanh}\, \frac{3}{5}]$. In particular

$$\begin{aligned}\delta_{x=\text{arctanh}\,\frac{3}{5}}(\overline{Q}_9) &= \left(\tfrac{256}{527}e^{-8\,\text{arctanh}\,\frac{3}{5}} + \tfrac{270}{527} + \tfrac{1}{527}e^{8\,\text{arctanh}\,\frac{3}{5}}\right)\Theta \\ &= \left(\frac{256}{527}\left(\frac{1-\frac{3}{5}}{1+\frac{3}{5}}\right)^4 + \frac{270}{527} + \frac{1}{527}\left(\frac{1+\frac{3}{5}}{1-\frac{3}{5}}\right)^4\right)\Theta \\ &= \left(\left(\tfrac{1}{4}\right)^4\tfrac{256}{527} + \tfrac{270}{527} + 4^4\tfrac{1}{527}\right)\Theta \\ &= \left(\tfrac{1}{527} + \tfrac{270}{527} + \tfrac{256}{527}\right)\Theta = \Theta.\end{aligned} \tag{19}$$

The numeric data of the optimal horoball packings are summarized in Table 6. The symmetry group $\Gamma_{\overline{Q}_9}$ extends the density from $\mathcal{F}_{\overline{Q}_9}$ to the entire tiling. □

Corollary 4 *The optimal congruent ball packing density in* $\mathbb{H}^9$ *up to horoballs of the same type is bounded by* $\frac{1}{4\zeta(5)} \leq \delta_{opt}(\overline{\mathbb{H}}^9) \leq 0.24285\ldots$.

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