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ORIGINAL PAPER

Relatively dominated representations from eigenvalue gaps and limit maps

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Abstract

Relatively dominated representations give a common generalization of geometrically finiteness in rank one on the one hand, and the Anosov condition which serves as a higher-rank analogue of convex cocompactness on the other. This note proves three results about these representations. Firstly, we remove the quadratic gaps assumption involved in the original definition. Secondly, we give a characterization using eigenvalue gaps, providing a relative analogue of a result of Kassel and Potrie for Anosov representations. Thirdly, we formulate characterizations in terms of singular value or eigenvalue gaps combined with limit maps, in the spirit of Guéritaud et al. for Anosov representations, and use them to show that inclusion representations of certain groups playing weak ping-pong are relatively dominated.

Keywords Discrete subgroups of Lie groups · Anosov representations · Relatively hyperbolic groups

Mathematics Subject Classification (2000) 22E40 (primary) · 20F67, 53C35 (secondary)

1 Introduction

Anosov representations were introduced by Labourie [21], and further developed by Guichard and Wienhard [14], as a generalization of convex cocompact representations into the isometry group of real hyperbolic space. Informally speaking, an Anosov representation is a representation of a word-hyperbolic group into a semisimple Lie group which still retains a certain amount of hyperbolicity in the image, which can be seen for instance in the form of an equivariant boundary map into a flag manifold with good dynamical properties. Since their initial introduction, there have been a number of different interpretations due to, among others, Kapovich et al. [17], Guéritaud et al. [9], Bochi et al. [2], and Kassel and Potrie [20].

These representations provide a rich class of discrete word-hyperbolic subgroups of semisimple Lie groups which are stably quasi-isometrically embedded, and come with associated geometric and dynamical structures which have features of negative curvature, see for instance [5, 14, 18].

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It is natural to wonder if the theory of Anosov representations can be extended to relatively hyperbolic groups. Such an extension would provide a common generalization of geometric finiteness in rank-one semisimple Lie groups and the Anosov condition in more general semisimple Lie groups.

In this direction, Kapovich and Leeb [16] developed relative versions of the characterizations in [17], and relatively dominated representations were introduced in [27] as relative versions of the characterization in [2]. These representations furnish classes of discrete relatively hyperbolic subgroups of semisimple Lie groups which are quasi-isometrically embedded modulo controlled distortion along their peripheral subgroups.

One definition of these representations is given in terms of singular value gaps, which may be interpreted in terms of the geometry of the associated symmetric spaces as distances from singular flats of specified type. The corresponding characterization of Anosov representations was given first by Kapovich et al. [17] under the name of URU subgroups, and subsequently reformulated, in language more closely resembling that used here, by Bochi et al. [2].

The key defining condition for relatively dominated representations asserts that the singular value gap $\frac{\sigma_1}{\sigma_2}(\rho(\gamma))$ grows uniformly exponentially in a notion of word-length $|\gamma|_c$ that has been modified to take into account the distortion along the peripheral subgroups.

The definition also involves additional technical conditions to control the images of the peripheral subgroups. In the first part of this note, we remove one of those technical conditions ("quadratic gaps"), by showing that its relevant consequences also follow from other parts of the definition. We refer the reader to Sect. 3, and specifically Proposition 3.7, for the full statement; here we present it slightly summarised as follows:

Proposition A *Let Γ be a finitely-generated group which is hyperbolic relative to a collection $\mathcal{P}$ of subgroups. Suppose we have a constant $C_0 > 0$ and a representation $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ such that for all $\gamma \in \Gamma$,*

$$C_0^{-1} \log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) - C_0 \le |\gamma|_c \le C_0 \log \frac{\sigma_1}{\sigma_2}(\rho(\gamma)) + C_0$$

where $|\gamma|_c := d_X(\mathrm{id}, \gamma)$ is distance from id *in a cusped space $X = X(\Gamma, \mathcal{P})$ (see Sect. 2.1).*

Then, given constants $\underline{\upsilon}, \bar{\upsilon} > 0$, there exists constants $C, \mu > 0$ such that for any bi-infinite sequence of elements $(\gamma_n)_{n\in\mathbb{Z}} \subset \Gamma$ satisfying

(i) $\gamma_0 = \mathrm{id}$, *and*
(ii) $\underline{\upsilon}^{-1}|n| - \underline{\upsilon} \le |\gamma_n|_c \le \bar{\upsilon}|n| + \bar{\upsilon}$ *for all n,*

and any $k \in \mathbb{Z}$,

$$d\left(U_1(\rho(\gamma_{k-1}\cdots\gamma_{k-n})), U_1(\rho(\gamma_{k-1}\cdots\gamma_{k-n-1}))\right) < Ce^{-\mu n}$$

for all $n > 0$.

Here $U_1(B)$, where defined, denotes the image of the 1-dimensional subspace of $\mathbb{R}^d$ most expanded by B. When ρ is P_1- dominated relative to $\mathcal{P}$, it admits continuous equivariant limit maps $\xi_\rho\colon \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ and $\xi_\rho^*\colon \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^{d*})$. Moreover, if $(g_n) \subset \Gamma$ is (the image of) a metric quasigeodesic path in $(\Gamma, \mathcal{P})$ converging to $x \in \partial(\Gamma, \mathcal{P})$, then

$$\xi_\rho(x) = \lim_{n\to\infty} U_1(\rho(g_n)) \qquad \text{and} \qquad \xi_\rho^*(x) = \lim_{n\to\infty} U_{d-1}(\rho(g_n))$$

(see the proof of [27, Th. 7.2]), and Proposition A allows us to obtain uniform convergence of the $U_1(\rho(g_n))$ (or $U_{d-1}(\rho(g_n))$, respectively) towards the limit points $\xi_\rho(x)$ (respectively $\xi_\rho^*(x)$, via the dual representation, see [27, Sect. 4.1]). The exponential convergence seen here is reminiscent of phenomena from hyperbolic dynamics, and is straightforward to obtain in the non-relative case.

In the proof of Proposition 3.7 we will find it useful to adopt elements of the point of view of Kapovich et al., which emphasizes the geometry of the symmetric space and the related geometry of its boundary and associated flag spaces.

More recently, Kassel and Potrie [20] have given a characterization of Anosov representations in terms of eigenvalue gaps $\frac{\lambda_1}{\lambda_2}$, which may be interpreted as asymptotic versions of singular value gaps $\frac{\sigma_1}{\sigma_2}$, i.e. distance to the Weyl chamber walls at infinity. In the second part of this note, we give an analogous characterization of relatively dominated representations:

Theorem B *(Corollary 5.3) Let Γ be finitely-generated and hyperbolic relative to $\mathcal{P}$. A semisimple representation $\rho\colon \Gamma \to \mathrm{SL}(d,\mathbb{R})$ is P_1-dominated relative to $\mathcal{P}$ if and only if the following four conditions hold:*

- *(D^{λ}_{-}) there exist constants $\underline{C}, \underline{\mu} > 0$ such that*

$$\frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \geq \underline{C} e^{\underline{\mu}|\gamma|_{c,\infty}}$$

  for all $\gamma \in \Gamma$,
- *(D^{λ}_{+}) there exist constants $\bar{C}, \bar{\mu} > 0$ such that*

$$\frac{\lambda_1}{\lambda_d}(\rho(\gamma)) \leq \bar{C} e^{\bar{\mu}|\gamma|_{c,\infty}}$$

  for all $\gamma \in \Gamma$,
- *(unique limits) for each $P \in \mathcal{P}$, there exists $\xi_\rho(P) \in \mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho(P) \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that for every sequence $(\eta_n) \subset P$ with $\eta_n \to \infty$, we have $\lim_{n\to\infty} U_1(\rho(\eta_n)) = \xi_\rho(P)$ and $\lim_{n\to\infty} U_{d-1}(\rho(\eta_n)) = \xi^*_\rho(P)$.*
- *(uniform transversality) for every $P, P' \in \mathcal{P}$ and $\gamma \in \Gamma$, $P \neq \gamma P' \gamma^{-1}$ implies $\xi_\rho(P) \neq \xi_\rho(\gamma P' \gamma^{-1})$. Moreover, for every $\underline{\upsilon}, \bar{\upsilon} > 0$, there exists $\delta_0 > 0$ such that for all $P, P' \in \mathcal{P}$ and $g, h \in \Gamma$ such that there exists a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path (see Definition 2.8) $\eta g h \eta'$ where η' is in P' and η is in P, we have*

$$\sin \angle(g^{-1}\xi_\rho(P), h\,\xi^*_\rho(P')) > \delta_0.$$

Here $|\gamma|_{c,\infty}$ is a stable version of the modified word-length $|\gamma|_c$ (see Sect. 2.4). By Proposition 2.11 the eigenvalue gap conditions below may be equivalently formulated in terms of the translation length $\ell_X(\gamma)$.

We remark that the fact that we are looking specifically at the first singular value gap or eigenvalue gap gives rise to the P_1 in "P_1-dominated" below. More precisely, P_1 refers to the parabolic subgroup of $\mathrm{SL}(d,\mathbb{R})$ corresponding to the first simple root; this subgroup is the stabilizer of a line.

Note there is an additional semisimplicity assumption in Theorem B: there are additional subtleties that arise in the relative case which make it tricky to remove this assumption. We recall that a representation into $\mathrm{SL}(d,\mathbb{R})$ is called semisimple if the Zariski closure of its image is a reductive group. Equivalently, semisimple representations may be written as direct sums of irreducible representations. The semisimplicity assumption helps us relate singular values and eigenvalues of $\rho(\gamma)$, using [26, Th. 2.6]. More generally, without additional assumptions such as semisimplicity, it is not possible to use only eigenvalue data, which is a sort of data "at infinity", to make desired conclusions about the representation, which requires "interior data" such as information about singular values.

The proof of Theorem B uses a recent result of Tsouvalas [26, Th. 5.3] stating that groups admitting non-trivial Floyd boundaries have property U: this property, roughly speaking, allows us to control stable translation lengths in terms of word-length. Relatively hyperbolic groups admit non-trivial Floyd boundaries ([8], see also Remark 2.10), and here we establish a modified version of property U adapted to the relatively hyperbolic case.

Finally, we present characterizations of relatively dominated representations which replace most of the additional conditions on the peripheral images with conditions about the existence of suitable limit maps. These are relative analogues of results due to Guéritaud et al. [9].

Theorem C *(Theorem 6.1 + Corollary 6.2) Given $(\Gamma, \mathcal{P})$ a finitely-generated and relatively hyperbolic group with Bowditch boundary $\partial(\Gamma, \mathcal{P})$, a representation $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is P_1-dominated relative to $\mathcal{P}$ if and only if*

- *There exist continuous, $\rho(\Gamma)$-equivariant, transverse, dynamics-preserving limit maps $\xi_\rho\colon \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho\colon \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^{d*})$,*

and one of the following sets of conditions holds:

- Either *there exist constants $\underline{C}, \underline{\mu} > 0$ and $\bar{C}, \bar{\mu} > 0$ such that*

  (D−) $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq \underline{C} e^{\underline{\mu}|\gamma|_c}$ *for all* $\gamma \in \Gamma$*, and*
  (D+) $\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) \leq \bar{C} e^{\bar{\mu}|\eta|_c}$ *for all* $\gamma \in \Gamma$*;*

- Or *ρ is semisimple, and there exist constants $\underline{C}, \underline{\mu} > 0$ and $\bar{C}, \bar{\mu} > 0$ such that*

  (D^λ_-) $\frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \geq \underline{C} e^{\underline{\mu}|\gamma|_{c,\infty}}$ *for all* $\gamma \in \Gamma$*, and*
  (D^λ_+) $\frac{\lambda_1}{\lambda_d}(\rho(\gamma)) \leq \bar{C} e^{\bar{\mu}|\gamma|_{c,\infty}}$ *for all* $\gamma \in \Gamma$*.*

Here, ξ and ξ^* are said to be **transverse** if $\xi(x) \oplus \xi^*(y) = \mathbb{R}^d$ for all $x \neq y$, and they are said to be **dynamics-preserving** if

(i) $\xi(\gamma^+) = (\rho(\gamma))^+$ and $\xi^*(\gamma^+)^\perp = (\rho^*(\gamma))^+$ for all nonperipheral $\gamma \in \Gamma$, where $\gamma^+ := \lim_{n\to\infty} \gamma^n \in \partial(\Gamma, \mathcal{P})$ and $\rho(\gamma)^+$ is the attracting eigenline for $\rho(\gamma)$, and
(ii) If $\partial P \in \partial(\Gamma, P)$ is the unique point associated to $P \in \mathcal{P}$, then $\xi^{(*)}(\partial P)$ is the parabolic fixed point in $\mathbf{P}(\mathbb{R}^{d(*)})$ associated to $\rho^{(*)}(P)$ (where $\rho^*\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is the dual representation defined by $\rho^*(\gamma) := (\rho(\gamma^{-1}))^T$). In particular, these fixed points exist and are well-defined.

As an application of this, we show that certain free groups which contain unipotent generators and which play weak ping-pong in projective space are relatively P_1-dominated:

Proposition D *(Example 6.3) Suppose we have biproximal elements $t_1, \ldots, t_k \in \mathrm{PGL}(d, \mathbb{R})$ with attracting / repelling lines $t_i^\pm$ and attracting / repelling hyperplanes $H^\pm_{t_i}$, and unipotent elements $u_1, \ldots, u_{k'}$ with well-defined attracting lines u_j^+ and attracting hyperplanes $H^+_{u_j}$.*

Suppose the hyperplanes $H^\pm_{t_1}, \ldots, H^\pm_{t_k}, H^+_{u_1}, \ldots, H^+_{u_{k'}}$ are in sufficiently generic position, i.e. none of them contain any of the fixed points $t_1^\pm, \ldots, t_k^\pm, u_1^+, \ldots, u_{k'}^+$, except for the necessary containments $H^\pm_{t_i} \ni t_i^\pm$ and $H^+_{u_j} \ni u_j^+$.

Then there exists $N_0 \in \mathbb{Z}_{>0}$ such that for all $N \geq N_0$, the subgroup of $\mathrm{PGL}(d, \mathbb{R})$ generated by $t_1^N, \ldots, t_k^N, u_1^N, \ldots, u_{k'}^N$ is isomorphic to a non-abelian free group, and its inclusion into $\mathrm{PGL}(d, \mathbb{R})$ is P_1-dominated relative to $\left\{\langle u_1^N \rangle, \ldots, \langle u_{k'}^N \rangle\right\}$.

Organization

Section 2 collects the various preliminaries needed. Section 3 gives the definition of a relatively dominated representation, with the simplification allowed by Proposition A/3.7. Section 4 establishes a technical lemma which is central to the proof of Theorem B.

Section 5 contains the proof of the eigenvalue gaps + peripheral conditions characterization described in Theorem B, and Sect. 6 contains the proofs of the gaps + limit maps characterizations described in Theorem C, as well as their application to weak ping-pong groups.

The preliminaries in Sects. 2.1 and 2.2, about relative hyperbolicity and cusped spaces, and in Sect. 2.5, about singular value decompositions, are used in the definition of relatively dominated representations and throughout. The material in Sect. 2.3, about the Floyd boundary, and Sect. 2.4, about Gromov products and translation lengths in hyperbolic spaces, is used only in Sects. 4 and 5. The material in Sect. 2.6, regarding the visual boundary of the symmetric space, is used only in Sect. 3. Note also that Sects. 5 and 6 do not depend on Sect. 3 except for the definition of relatively dominated representations.

2 Preliminaries

2.1 Relatively hyperbolic groups and cusped spaces

Relative hyperbolicity is a group-theoretic notion of non-positive curvature inspired by the geometry of cusped hyperbolic manifolds and free products.

Consider a finite-volume cusped hyperbolic manifold with an open neighborhood of each cusp removed: call the resulting truncated manifold M. The universal cover $\tilde{M}$ of such an M is hyperbolic space with a countable set of horoballs removed. The universal cover $\tilde{M}$ is not Gromov-hyperbolic; distances along horospheres that bound removed horoballs are distorted. If we glue the removed horoballs back in to the universal cover, however, the resulting space will again be hyperbolic space.

Gromov generalized this in [12, Sect. 8.6] by defining a group Γ as hyperbolic relative to a conjugation-invariant collection of subgroups $\mathcal{P}$ if $(\Gamma, \mathcal{P})$ admits a **cusp-uniform action** on a (Gromov-)hyperbolic metric space X, meaning there exists some system $(\mathcal{H}_P)_{P\in\mathcal{P}}$ of disjoint horoballs of X, each preserved by a subgroup $P \in \mathcal{P}$, such that the group Γ acts on X discretely and isometrically, and the Γ-action on $X \smallsetminus \bigcup_P \mathcal{H}_P$ is cocompact.

The hyperbolic space X is sometimes called a Gromov model for $(\Gamma, \mathcal{P})$. There is in general no canonical Gromov model for a given relatively hyperbolic group, but there are systematic constructions one can give, one of which we describe here. The description below, as well as the material in the next Sect. 2.2, is taken from [27, Sect. 2] and is based on prior literature, in particular [10]; it is included here for completeness.

Definition 2.1 [10, Def. 3.1] Let Γ be a finitely-generated group and S be a symmetric finite generating set for Γ.

Given a subgraph Λ of the Cayley graph $\mathrm{Cay}(\Gamma, S)$, the **combinatorial horoball** based on Λ, denoted $\mathcal{H} = \mathcal{H}(\Lambda)$, is the 1-complex[1] formed as follows:

- The vertex set $\mathcal{H}^{(0)}$ is given by $\Lambda^{(0)} \times \mathbb{Z}_{\geq 0}$

[1] Groves-Manning combinatorial horoballs are actually defined as 2-complexes; the definition here is really of a 1-skeleton of a Groves-Manning horoball. For metric purposes only the 1-skeleton matters.

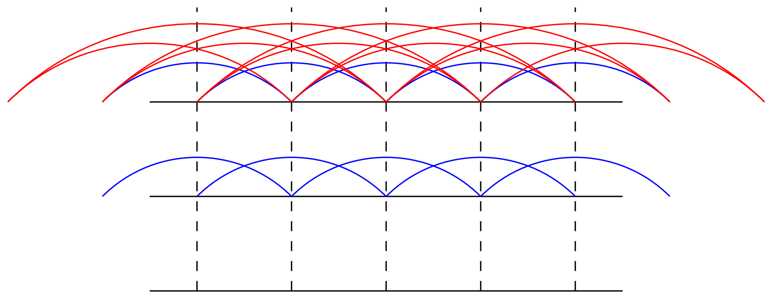

Fig. 1 Part of a combinatorial horoball over a graph that is an infinite line (e.g. corresponding to the Cayley subgraph for a $\mathbb{Z}$ subgroup of Γ). All edges have length 1, but there are exponentially more horizontal edges as we go deeper into the horoball. (The blue ones appear at levels 1 and up, the red ones at levels 2 and up, and so on.) The effect is that distances between points in the base graph shrink exponentially in the path metric for the combinatorial horoball

- The edge set $\mathcal{H}^{(1)}$ consists of the following two types of edges:
  (1) if $k \geq 0$ and v and $w \in \Lambda^{(0)}$ are such that $0 < d_\Lambda(v, w) \leq 2^k$, then there is a ("horizontal") edge connecting (v, k) to (w, k);
  (2) If $k \geq 0$ and $v \in \Lambda^{(0)}$, there is a ("vertical") edge joining (v, k) to $(v, k + 1)$.

$\mathcal{H}$ is metrized by assigning length 1 to all edges.

Next let $\mathcal{P}$ be a finite collection of finitely-generated subgroups of Γ, and suppose S is a **compatible generating set**, i.e. for each $P \in \mathcal{P}$, $S \cap P$ generates P.

Definition 2.2 [[10, Def. 3.12]] Given Γ, $\mathcal{P}$, S as above, the **cusped space** $X(\Gamma, \mathcal{P}, S)$ is the simplicial metric graph

$$\mathrm{Cay}(\Gamma, S) \cup \bigcup \mathcal{H}(\gamma P)$$

where the union is taken over all left cosets of elements of $\mathcal{P}$, i.e. over $P \in \mathcal{P}$ and (for each P) γP in a collection of representatives for left cosets of P.

Here the induced subgraph of $\mathcal{H}(\gamma P)$ on the $\gamma P \times \{0\}$ vertices is identified with (the induced subgraph of) $\gamma P \subset \mathrm{Cay}(\Gamma, S)$ in the natural way.

Definition 2.3 Γ is said to be hyperbolic relative to $\mathcal{P}$ if the cusped space $X(\Gamma, \mathcal{P}, S)$ is hyperbolic (for any compatible generating set S; the hyperbolicity constant may depend on S.)

We will also call $(\Gamma, \mathcal{P})$ a **relatively hyperbolic structure**.

It is a theorem of Groves and Manning that this definition is equivalent to other, older definitions of relative hyperbolicity [10, Th. 3.25].

We remark that for a fixed relatively hyperbolic structure $(\Gamma, \mathcal{P})$, any two cusped spaces, corresponding to different compatible generating sets S, are quasi-isometric [13, Cor. 6.7]: in particular, the notion above is well-defined independent of the choice of generating set S. There is a natural action of Γ on the cusped space $X = X(\Gamma, \mathcal{P}, S)$; with respect to this action, the quasi-isometry between two cusped spaces $X(\Gamma, \mathcal{P}, S_i)$ $(i = 1, 2)$ is Γ-equivariant.

In particular, this gives us a notion of a boundary associated to the data of a relatively hyperbolic group Γ and its peripheral subgroups $\mathcal{P}$:

Definition 2.4 For Γ hyperbolic relative to $\mathcal{P}$, the **Bowditch boundary** $\partial(\Gamma, \mathcal{P})$ is defined as the Gromov boundary $\partial_\infty X$ of any cusped space $X = X(\Gamma, \mathcal{P}, S)$.

This boundary is well-defined up to homeomorphism, independent of the choice of compatible generating set S [1, Sect. 9].

Below, with a fixed choice of Γ, $\mathcal{P}$ and S as above, for $\gamma, \gamma' \in \Gamma$, $d(\gamma, \gamma')$ will denote the distance between γ and γ' in the Cayley graph with the word metric, and $|\gamma| := d(\mathrm{id}, \gamma)$ denotes word length in this metric. Similarly, $d_c(\gamma, \gamma')$ denotes distance in the corresponding cusped space and $|\gamma|_c := d_c(\mathrm{id}, \gamma)$ is the **cusped word-length**.

2.2 Geodesics in the cusped space

Let Γ be a finitely-generated group, $\mathcal{P}$ be a malnormal finite collection of finitely-generated subgroups, and let $S = S^{-1}$ be a compatible finite generating set as above. Here malnormal means that given $P, P' \in \mathcal{P}$ and $\gamma \in \Gamma$, $\gamma P \gamma^{-1} \cap P' = \varnothing$ unless $P = P'$ *and* $\gamma \in P$.

Let $X = X(\Gamma, \mathcal{P}, S)$ be the cusped space, and $\mathrm{Cay}(\Gamma) = \mathrm{Cay}(\Gamma, S)$ the Cayley graph. Here we collect some technical results about geodesics in these spaces that will be useful below.

Lemma 2.5 [10, Lem. 3.10] *Let $\mathcal{H}(\Gamma)$ be a combinatorial horoball. Suppose that $x, y \in \mathcal{H}(\Gamma)$ are distinct vertices. Then there is a geodesic $\gamma(x, y) = \gamma(y, x)$ between x and y which consists of at most two vertical segments and a single horizontal segment of length at most 3.*

We will call any such geodesic a **preferred geodesic**.

Given a path $\gamma : I \to \mathrm{Cay}(\Gamma)$ in the Cayley graph such that $\gamma(I \cap \mathbb{Z}) \subset \Gamma$, we can consider γ as a **relative path** (γ, H), where H is a subset of I consisting of a disjoint union of finitely many subintervals $H_1, \ldots, H_n$ occurring in this order along I, such that each $\eta_i := \gamma|_{H_i}$ is a maximal subpath lying in (the Cayley subgraph corresponding to) a left coset $t_i P_i$ of a peripheral subgroup $P_i \in \mathcal{P}$, and $\gamma|_{I \smallsetminus H}$ contains no edges of $\mathrm{Cay}(\Gamma)$ labelled by a peripheral generator.

Similarly, a path $\hat{\gamma} : \hat{I} \to X$ in the cusped space with endpoints in $\mathrm{Cay}(\Gamma) \subset X$ may be considered as a relative path $(\hat{\gamma}, \hat{H})$, where $\hat{H} = \coprod_{i=1}^{n} \hat{H}_i$, $\hat{H}_1, \ldots, \hat{H}_n$ occur in this order along $\hat{I}$, each $\hat{\eta}_i := \hat{\gamma}|_{\hat{H}_i}$ is a maximal subpath in a closed combinatorial horoball B_i, and $\hat{\gamma}|_{\hat{I} \smallsetminus \hat{H}}$ lies inside the Cayley graph. Below, we will consider only geodesics and quasigeodesic paths $\hat{\gamma} : \hat{I} \to X$ where all of the $\hat{\eta}_i$ are preferred geodesics (in the sense of Lemma 2.5.)

We will refer to the η_i and $\hat{\eta}_i$ as **peripheral excursions**. We remark that the η_i, or any other subpath of γ in the Cayley graph, may be considered as a word and hence a group element in Γ; this will be used without further comment below.

Given a path $\hat{\gamma} : \hat{I} \to X$ whose peripheral excursions are all preferred geodesics, we may replace each excursion $\hat{\eta}_i = \hat{\gamma}|_{\hat{H}_i}$ into a combinatorial horoball with a geodesic path (or, more precisely, a path with geodesic image) $\eta_i = \pi \circ \hat{\eta}_i$ in the Cayley (sub)graph of the corresponding peripheral subgroup connecting the same endpoints, by omitting the vertical segments of the preferred geodesic $\hat{\eta}_i$ and replacing the horizontal segment with the corresponding segment at level 0, i.e. in the Cayley graph.[2] We call this the “project” operation, since it involves “projecting” paths inside combinatorial horoballs onto the boundaries of those horoballs. This produces a path $\gamma = \pi \circ \hat{\gamma} : \hat{I} \to \mathrm{Cay}(\Gamma)$.

Given any path α in the Cayley graph with endpoints $g, h \in \Gamma$, we write $\ell(\alpha)$ to denote $d(g, h)$, i.e. distance measured according to the word metric in $\mathrm{Cay}(\Gamma)$.

[2] As a parametrized path this has constant image on the subintervals of $\hat{H}_i$ corresponding to the vertical segments, and travels along the projected horizontal segment at constant speed.

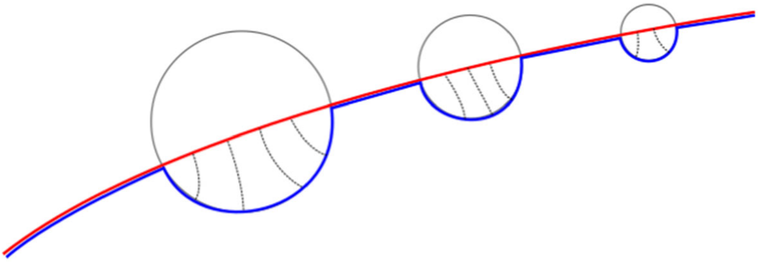

Fig. 2 Schematic illustration of a path $\hat{\gamma}\colon \hat{I} \to X$ and its projection. In red, the (image of the) path $\hat{\gamma}$, which travels through some combinatorial horoballs (grey circles and their interiors). The parts of this path inside these combinatorial horoballs are the peripheral excursions. In blue, the projected path. The dotted lines descending from the red to the blue path inside the horoballs indicate (roughly) the parametrization of the projected path

We have the following biLipschitz equivalence between cusped distances and suitably-modified distances in the Cayley graph:

Proposition 2.6 [27, Prop. 2.12] *Given a geodesic $\hat{\gamma}\colon \hat{I} \to X$ with endpoints in* $\mathrm{Cay}(\Gamma) \subset X$ *and whose peripheral excursions are all preferred geodesics, let $\gamma = \pi \circ \hat{\gamma}\colon \hat{I} \to \mathrm{Cay}(\Gamma)$ be its projected image.*

Given any subinterval $[a, b] \subset \hat{I}$, consider the subpath $\gamma|_{[a,b]}$ as a relative path $(\gamma|_{[a,b]}, H)$ where $H = (H_1, \ldots, H_n)$, and write $\eta_i := \gamma|_{H_i}$; then we have

$$\frac{1}{3} \leq \frac{d_c(\gamma(a), \gamma(b))}{\ell(\gamma|_{[a,b]}) - \sum_{i=1}^{n} \ell(\eta_i) + \sum_{i=1}^{n} \hat{\ell}(\eta_i)} \leq \frac{2}{\log 2} + 1 < 4$$

where $\hat{\ell}(\eta_i) := \max\{\log(\ell(\eta_i)), 1\}$.

Below we will occasionally find it useful to consider paths in $\mathrm{Cay}(\Gamma)$ that "behave metrically like quasi-geodesics in the relative Cayley graph", in the following sense:

Definition 2.7 Given any path $\gamma\colon I \to \mathrm{Cay}(\Gamma)$ such that I has integer endpoints and $\gamma(I \cap \mathbb{Z}) \subset \Gamma$, define the **depth** $\delta(n) = \delta_\gamma(n)$ of a point $\gamma(n)$ in $(\Gamma, \mathcal{P})$ for any $n \in I \cap \mathbb{Z})$ as

(a) the smallest integer $d \geq 0$ such that at least one of $\gamma(n - d)$, $\gamma(n + d)$ is well-defined (i.e. $\{n - d, n + d\} \cap I \neq \varnothing$) and not in the same peripheral coset as $\gamma(n)$, **or**
(b) if no such integer exists, $\min\{\sup I - n, n - \inf I\}$.

Definition 2.8 Given constants $\underline{\upsilon}, \bar{\upsilon} > 0$, an $(\underline{\upsilon}, \bar{\upsilon})$-**metric quasigeodesic path** in $(\Gamma, \mathcal{P})$ is a path $\gamma\colon I \to \mathrm{Cay}(\Gamma)$ with $\gamma(I \cap \mathbb{Z}) \subset \Gamma$ such that for all integers $m, n \in I$,

(i) $|\gamma(n)^{-1}\gamma(m)|_c \geq \underline{\upsilon}^{-1}|m - n| - \underline{\upsilon}$,
(ii) $|\gamma(n)^{-1}\gamma(m)|_c \leq \bar{\upsilon}(|m - n| + \min\{\delta(m), \delta(n)\}) + \bar{\upsilon}$, and
(iii) if $\gamma(n)^{-1}\gamma(n + 1) \in P$ for some $P \in \mathcal{P}$, we have $\gamma(n)^{-1}\gamma(n + 1) = p_{n,1} \cdots p_{n,\ell(n)}$ where each $p_{n,i}$ is a peripheral generator of P, and

$$2^{\delta(n)-1} \leq \ell(n) := |\gamma(n)^{-1}\gamma(n + 1)| \leq 2^{\delta(n)+1}.$$

The terminology comes from the following fact: given a geodesic segment $\hat{\gamma}$ in the cusped space with endpoints in $\mathrm{Cay}(\Gamma)$, we can project the entire segment to the Cayley graph and reparametrize the projected image to be a metric quasigeodesic path γ — the idea being that in such a reparametrization, the increments $|\gamma(n)^{-1}\gamma(n + 1)|$ correspond, approximately, to linear increments in cusped distance: see the discussion in [27, Sect. 2.3], and in particular Prop. 2.16 there for more details.

2.3 Floyd boundaries

Let Γ be a finitely-generated group, and S a finite generating set giving a word metric $|\cdot|$.

A **Floyd boundary** $\partial_f \Gamma$ for Γ is a boundary for Γ meant to generalize the ideal boundary of a Kleinian group. Its construction uses the auxiliary data of a **Floyd function**, which is a function $f : \mathbb{N} \to \mathbb{R}_{>0}$ satisfying

(i) $\sum_{n=1}^{\infty} f(n) < \infty$, and
(ii) There exists $m > 0$ such that $\frac{1}{m} \leq \frac{f(k+1)}{f(k)} \leq 1$ for all $k \in \mathbb{N}$.

Given such a function, there exists a metric d_f on Γ defined by setting $d_f(g, h) = f(\max\{|g|, |h|\})$ if g, h are adjacent vertices in $\mathrm{Cay}(\Gamma, S)$, and considering the resulting path metric. Then the Floyd boundary $\partial_f \Gamma$ with respect to f is given by

$$\partial_f \Gamma := \bar{\Gamma} \smallsetminus \Gamma$$

where $\bar{\Gamma}$ is the metric completion of Γ with respect to the metric d_f.

Below, the Floyd boundary, in particular the ability of the Floyd function to serve as a sort of "distance to infinity", will be useful as a tool in the proof of Theorem 5.1.

The Floyd boundary $\partial_f \Gamma$ is called **non-trivial** if it has at least three points. Gerasimov and Potyagailo have studied Floyd boundaries of relatively hyperbolic groups:

Theorem 2.9 [11, Th. A], see also [8] *Suppose we have a non-elementary relatively hyperbolic group Γ which is hyperbolic relative to $\mathcal{P}$.*

Then there exists a Floyd function f such that $\partial_f \Gamma$ is non-trivial, and moreover

(a) There exists a continuous equivariant map $F : \partial_f \Gamma \to \partial(\Gamma, \mathcal{P})$ which is injective on the set of conical limit points, and
(b) For any parabolic point $p \in \partial(\Gamma, \mathcal{P})$, we have $F^{-1}(p) = \partial_f(\mathrm{Stab}_\Gamma\, p)$, and if there exist $a \neq b$ such that $F(a) = F(b) = p$, then p is parabolic.

Remark 2.10 It is an open question whether every group with a non-trivial Floyd boundary is relatively hyperbolic—see e.g. [22].

For more details, including justifications for some of the assertions above, we refer the reader to [7] and [15].

2.4 Gromov products and translation lengths in hyperbolic spaces

We collect here, for the reader's convenience, assorted facts about Gromov products and translation lengths in Gromov-hyperbolic spaces that we use below, in particular in and around the statement and proof of Theorem 5.1.

Given X a proper geodesic metric space, $x_0 \in X$ a fixed basepoint, and γ an isometry of X, we define the **translation length** of γ as

$$\ell_X(\gamma) := \inf_{x \in X} d_X(\gamma x, x)$$

and the **stable translation length** of γ as

$$|\gamma|_{X,\infty} := \lim_{n \to \infty} \frac{d_X(\gamma^n x_0, x_0)}{n}.$$

When X is δ-hyperbolic space, these two quantities are coarsely equivalent:

Proposition 2.11 [3, Chap. 10, Prop. 6.4] *If X is hyperbolic metric space, the quantities $\ell_X(\gamma)$ and $|\gamma|_{X,\infty}$ defined above satisfy*

$$\ell_X(\gamma) - 16\delta \le |\gamma|_{X,\infty} \le \ell_X(\gamma).$$

The **Gromov product** with respect to x_0 is the function $\langle \cdot, \cdot \rangle_{x_0} \colon X \times X \to \mathbb{R}$ defined by

$$\langle x, y \rangle_{x_0} := \frac{1}{2} \left(d_X(x, x_0) + d_X(y, x_0) - d_X(x, y) \right).$$

There is a relation between the Gromov product, the stable translation length $|\gamma|_{X,\infty}$, and the quantity $|\gamma|_X = d_X(\gamma x_0, x_0)$, given by

Lemma 2.12 *Given X a proper geodesic metric space, $x_0 \in X$ a basepoint, and γ an isometry of X, we can find a sequence of integers $(m_i)_{i \in \mathbb{N}}$*

$$2 \lim_{i \to \infty} \langle \gamma^{m_i} x_0, \gamma^{-1} x_0 \rangle_{x_0} \ge |\gamma|_X - |\gamma|_{X,\infty}.$$

Proof By the definition of the stable translation length, we can find a sequence $(m_i)_{i \in \mathbb{N}}$ such that

$$\lim_{i \to \infty} \left(|\gamma^{m_i+1}|_X - |\gamma^{m_i}|_X \right) \le |\gamma|_{X,\infty}.$$

By the definition of the Gromov product,

$$2\langle \gamma^{m_i} x_0, \gamma^{-1} x_0 \rangle_{x_0} := |\gamma^{m_i}|_X + d_X(\gamma^{-1} x_0, x_0) - d_X(\gamma^{m_i} x_0, \gamma^{-1} x_0).$$

Since γ acts isometrically on X, $d_X(\gamma^{m_i} x_0, \gamma^{-1} x_0) = |\gamma^{m_i+1}|_X$ and $d_X(\gamma^{-1} x_0, x_0) = |\gamma|_X$. Then we have

$$\lim_{i \to \infty} 2\langle \gamma^{m_i} x_0, \gamma^{-1} x_0 \rangle_{x_0} = \lim_{i \to \infty} |\gamma^{m_i}|_X + |\gamma|_X - |\gamma^{m_i+1}|_X \ge |\gamma|_X - |\gamma|_{X,\infty}$$

as desired. □

2.5 Singular value decompositions

We collect here facts about singular values and Cartan decomposition in $\mathrm{SL}(d, \mathbb{R})$. The defining conditions for our representations will be phrased, in the first instance, in terms of these, and more generally they will be helpful for understanding the geometry associated to our representations.

Given a matrix $g \in \mathrm{GL}(d, \mathbb{R})$, let $\sigma_i(g)$ (for $1 \le i \le d$) denote its i^{th} singular value, and write $U_i(g)$ to denote the span of the i largest axes in the image of the unit sphere in $\mathbb{R}^d$ under g, and $S_i(g) := U_i(g^{-1})$. Note $U_i(g)$ is well-defined if and only if we have a singular-value gap $\sigma_i(g) > \sigma_{i+1}(g)$.

More algebraically, given $g \in \mathrm{GL}(d, \mathbb{R})$, we may write $g = KAL$, where K and L are orthogonal matrices and A is a diagonal matrix with nonincreasing positive entries down the diagonal. The diagonal matrix A is uniquely determined, and we may define $\sigma_i(g) = A_{ii}$; $U_i(g)$ is given by the span of the first i columns of K.

For $g \in \mathrm{SL}(d, \mathbb{R})$, this singular-value decomposition is a concrete manifestation of a more general Lie-theoretic object, a (particular choice of) Cartan decomposition $\mathrm{SL}(d, \mathbb{R}) = \mathrm{SO}(d) \cdot \exp(\mathfrak{a}^+) \cdot \mathrm{SO}(d)$, where $\mathrm{SO}(d)$ is the maximal compact subgroup of $\mathrm{SL}(d, \mathbb{R})$, and $\mathfrak{a}^+$ is a positive Weyl chamber.

We recall that there is an adjoint action Ad of $\mathrm{SL}(d, \mathbb{R})$ on $\mathfrak{sl}(d, \mathbb{R})$.

We will occasionally write (given $g = KAL$ as above)

$$a(g) := (\log A_{11}, \dots, \log A_{dd}) = (\log \sigma_1(g), \dots, \log \sigma_d(g));$$

we note that the norm $\|a(g)\| = \sqrt{(\log \sigma_1(g))^2 + \cdots + (\log \sigma_d(g))^2}$ is equal to the distance $d(o, g \cdot o)$ in the associated symmetric space $\mathrm{SL}(d, \mathbb{R})/\,\mathrm{SO}(d)$, where $o := [\mathrm{SO}(d)] \in \mathrm{SL}(d, \mathbb{R})/\,\mathrm{SO}(d)$ (see e.g. formula (7.3) in [2]).

2.6 Regular ideal points and the projective space

Finally, we collect here some remarks about a subset of the visual boundary of the symmetric space which will be relevant to us, and its relation to the projective space as a flag space boundary.

Given fixed constants $C_r, c_r > 0$, a matrix $g \in \mathrm{SL}(d, \mathbb{R})$ will be called (P_1, C_r, c_r)-**regular** if it satisfies

$$\log \frac{\sigma_1}{\sigma_2}(g) \geq C_r \log \frac{\sigma_1}{\sigma_d}(g) - c_r. \tag{1}$$

Recall that the visual boundary of the symmetric space $\mathrm{SL}(d, \mathbb{R})/\,\mathrm{SO}(d)$ consists of equivalence classes of geodesic rays, where two rays are equivalent if they remain bounded distance apart. For any complete simply-connected non-positively curved Riemannian manifold X, such as our symmetric space, the visual boundary is homeomorphic to a sphere, and may be identified with the unit sphere around any basepoint o by taking geodesic rays $\xi : [0, \infty) \to X$ based at o and identifying $\xi(1)$ on the unit sphere with $\lim_{t\to\infty} \xi(t)$ in the visual boundary.

The set of all points in this visual boundary which are accumulation points of sequences $(B_n \cdot o)$, where o varies over all possible basepoints in the symmetric space and (B_n) over all divergent sequences of (P_1, C_r, c_r)-regular matrices with all $c_r > 0$, will be called the (P_1, C_r)-**regular ideal points**.

For fixed C_r, the set of (P_1, C_r)-regular ideal points is compact. Indeed, it has the structure of a fiber bundle over the projective space $\mathbf{P}(\mathbb{R}^d)$ with compact fibers which can be identified with compact subsets of the Weyl chamber at infinity: the fibration π from the set of (P_1, C_r)-regular ideal points to $\mathbf{P}(\mathbb{R}^d)$ is given by taking $\lim_n g_n \cdot o$ to $\lim_{n\to\infty} U_1(g_n)$ (see [17, Subsection 2.5.1 & 4.6], where this is stated in slightly different language, or [27, Th. 7.2]). The map π is Lipschitz, with Lipschitz constant C_{Lip} depending only on the regularity constant C_r and the choice of basepoint o implicit in the measurement of the singular values [25, Sect. 4.4].

Throughout the paper, we will use the *angle distance* on $\mathbf{P}(\mathbb{R}^d)$, defined as follows: if $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product on $\mathbb{R}^d$, then

$$d([v], [w]) := \cos^{-1}\left(\frac{|\langle v, w\rangle|}{\sqrt{\langle v, v\rangle}\sqrt{\langle w, w\rangle}}\right)$$

for all non-zero $v, w \in \mathbb{R}^d$.

3 Relatively dominated representations

In this section we introduce the central notion of study, relatively dominated representations, and prove that one of the hypotheses in the original definition [27, Def. 4.2] can be removed.

The following is the key definition of the paper.

Definition 3.1 Let Γ be a finitely-generated torsion-free group which is hyperbolic relative to a collection $\mathcal{P}$ of proper infinite subgroups.

Let S be a compatible generating set, and let $X = X(\Gamma, \mathcal{P}, S)$ be the corresponding cusped space (see Definitions 2.1 and 2.2 above.) As above, let d_c denote the metric on X, and $|\cdot|_c := d_c(\mathrm{id}, \cdot)$ denote the cusped word-length.

A representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is P_1-**dominated relative to** $\mathcal{P}$, with lower domination constants $\underline{C}, \underline{\mu} > 0$, if it satisfies

– (D−) for all $\gamma \in \Gamma$, $\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) \geq \underline{C} e^{\underline{\mu}|\gamma|_c}$,

and the images of peripheral subgroups under ρ are well-behaved, meaning that the following three conditions are satisfied:

– (D+) there exist constants $\bar{C}, \bar{\mu} > 0$ such that $\frac{\sigma_1}{\sigma_d}(\rho(\eta)) \leq \bar{C} e^{\bar{\mu}|\eta|_c}$ for every $\eta \in \bigcup_{P \in \mathcal{P}} P$;
– (Unique limits) for each $P \in \mathcal{P}$, there exists $\xi_\rho(P) \in \mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho(P) \in \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ such that for every sequence $(\eta_n) \subset P$ with $\eta_n \to \infty$, we have $\lim_{n\to\infty} U_1(\rho(\eta_n)) = \xi_\rho(P)$ and $\lim_{n\to\infty} U_{d-1}(\rho(\eta_n)) = \xi^*_\rho(P)$;
– (Uniform transversality) for every $P, P' \in \mathcal{P}$ and $\gamma \in \Gamma$, $P \neq \gamma P' \gamma^{-1}$ implies $\xi_\rho(P) \neq \xi_\rho(\gamma P' \gamma^{-1})$. Moreover, for every $\underline{\upsilon}, \bar{\upsilon} > 0$, there exists $\delta_0 > 0$ such that for all $P, P' \in \mathcal{P}$ and $g, h \in \Gamma$ such that there exists a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path $\eta g h \eta'$ where η' is in P' and η is in P, we have
$$\sin \angle(g^{-1}\xi_\rho(P), h\,\xi^*_\rho(P')) > \delta_0.$$

Remark 3.2 It follows from the (D+) hypothesis above that there exist constants $\bar{C}, \bar{\mu} > 0$ such that $\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) \leq \bar{C} e^{\bar{\mu}|\gamma|_c}$ for all $\gamma \in \Gamma$ [27, Cor. 4.8]. In particular, the bound is automatically satisfied when $\gamma \in \Gamma$ is a non-peripheral element because Γ is finitely-generated. Below, we will refer to this *a priori* stronger (but in fact equivalent) statement as (D+) as well.

Remark 3.3 Since Γ is finitely-generated, so are its peripheral subgroups, by [6, Prop. 4.28 & Cor. 4.32].

Remark 3.4 It is also possible to formulate the definition without assuming relative hyperbolicity, if one imposes additional hypotheses (RH) (see below) on the peripheral subgroups $\mathcal{P}$; it is then possible to show that any group admitting such a representation must be hyperbolic relative to $\mathcal{P}$: see [27] for details.

Definition 3.5 [27, Def. 4.1] Given Γ a finitely-generated group, we say that a collection $\mathcal{P}$ of finitely-generated subgroups satisfies (RH) if

– (Malnormality) $\mathcal{P}$ is malnormal, i.e. for all $\gamma \in \Gamma$ and $P, P' \in \mathcal{P}$, $\gamma P \gamma^{-1} \cap P' = 1$ unless $\gamma \in P = P'$;
– (Non-distortion) there exists $\nu > 0$ such that for any infinite-order non-peripheral element $\gamma \in \Gamma$, $|\gamma^n|_c \geq \nu|n|$;
– (Local-to-global) a sufficient long peripheral word p' with sufficiently long overlap with a geodesic word γp combine to form a uniform quasigeodesic $\gamma p'$ (we refer the reader to [27] for the precise formulation.)

All of these conditions hold when Γ is hyperbolic relative to $\mathcal{P}$ (see e.g. [23]).

The original definition of a relatively dominated representation in [27] also had an additional "quadratic gaps" hypothesis, as part of the definition of the peripheral subgroups having well-behaved images. The only input of this assumption into the subsequent results there was in [27, Lem. 5.4]; the next proposition obtains the conclusion of that lemma from the (D±) (and (RH)) hypotheses, without using the quadratic gaps hypothesis.

Definition 3.6 Let $\alpha : \mathbb{Z} \to \mathrm{Cay}(\Gamma)$ be a bi-infinite path with $\alpha(\mathbb{Z}) \subset \Gamma$.
We define the sequence

$$\begin{aligned} x_\alpha &= (\ldots A_{a-1}, \ldots, A_{-1}, A_0, \ldots, A_{b-1}, \ldots) \\ &:= (\ldots, \rho(\alpha(a)^{-1}\alpha(a-1)), \ldots, \rho(\alpha(0)^{-1}\alpha(-1)), \\ &\qquad \rho(\alpha(1)^{-1}\alpha(0)), \ldots, \rho(\alpha(b)^{-1}\alpha(b-1)), \ldots) \end{aligned}$$

and call this the **matrix sequence associated to** α.

Proposition 3.7 *Given a representation $\rho : (\Gamma, \mathcal{P}) \to \mathrm{SL}(d, \mathbb{R})$ satisfying (D±) (so that $\mathcal{P}$ implicitly satisfies (RH), and we can define a cusped space $X(\Gamma, \mathcal{P})$), and given $\underline{\upsilon}, \bar{\upsilon} > 0$, there exist constants $C \geq 1$ and $\mu > 0$, depending only on the representation ρ and $\underline{\upsilon}, \bar{\upsilon}$, such that for any matrix sequence $x = x_\gamma$ associated to a bi-infinite $(\underline{\upsilon}, \bar{\upsilon})$-metric quasigeodesic path γ with $\gamma(0) = \mathrm{id}$, we have*

$$\begin{aligned} d(U_1(A_{k-1} \cdots A_{k-n}), U_1(A_{k-1} \cdots A_{k-(n+1)})) &\leq Ce^{-n\mu} \\ d(S_{d-1}(A_{k+n-1} \cdots A_k), S_{d-1}(A_{k+n} \cdots A_k)) &\leq Ce^{-n\mu} \end{aligned}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$.

Proof Given (D±), there exists $C_r, c_r > 0$ such that inequality (1) is satisfied for all $\gamma \in \Gamma$. Specifically, we can take $C_r = \underline{\mu}/\bar{\mu}$ and $c_r = (\underline{\mu}/\bar{\mu}) \log \bar{C} - \log \underline{C}$, where $\underline{C}, \underline{\mu}, \bar{C}, \bar{\mu}$ are the constants coming from the (D±) conditions. In the language of Kapovich–Leeb–Porti — see [17], or [16] for the relative case; we adapt the relevant parts of this language and framework here — $\rho(\Gamma)$ is a uniformly regular subgroup of $\mathrm{SL}(d, \mathbb{R})$.

Hence $\rho(\gamma)$ is (P_1, C_r, c_r)-regular, in the sense of Sect. 2.6, for all $\gamma \in \Gamma$, and given a divergent sequence (γ_n), $\rho(\gamma_n) \cdot o$ converges to a (P_1, C_r, c_r)-regular ideal point in the visual boundary.

Roughly speaking, geodesics converging to (P_1, C_r)-regular ideal points stay uniformly away from intersections of maximal flats, and hence "have as many hyperbolic directions as possible" in the symmetric space, in the sense that variations of geodesics parametrized by a large-dimensional subspace of the tangent space behave like families of geodesics in a hyperbolic space. Because of this, the convergence of the U_1 and S_{d-1} spaces along these geodesics, which can be seen as a coarser version of convergence in the symmetric space towards the visual boundary, occurs exponentially quickly, just as in the hyperbolic case. This intuition can be made precise with more work, which occupies the rest of the proof.

Recall that we have a Lipschitz map π from the set of (P_1, C_r)-regular ideal points to $\mathbf{P}(\mathbb{R}^d)$, with Lipschitz constant depending only on the regularity constant C_r and the choice of basepoint o implicit in the measurement of the singular values.

Moreover, since $\rho(\gamma)$ is (P_1, C_r, c_r)-regular for any $\gamma \in \Gamma$, given the Cartan decomposition $\rho(\gamma) = K_\gamma \cdot \exp(a(\rho(\gamma))) \cdot L_\gamma$, we have

$$U_1(\rho(\gamma)) = \pi \left(\lim_{n \to \infty} K_\gamma \cdot \exp(na(\rho(\gamma))) \cdot L_\gamma \cdot o \right).$$

Thus, given any sequence $(\gamma_n) \subset \Gamma$, we have

$$d(U_1(\rho(\gamma_n)), U_1(\rho(\gamma_m))) \leq C_{Lip} \cdot \sin \angle \left(\mathrm{Ad}(K_{\gamma_n}) \cdot a(\rho(\gamma_n)), \mathrm{Ad}(K_{\gamma_m}) \cdot a(\rho(\gamma_m)) \right)$$

where the angle is taken with respect to the Riemannian metric on $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$, which restricts to a Euclidean metric on the Cartan subalgebra $\mathfrak{a}$.

Now, if $x = x_\gamma = (A_n)_{n\in\mathbb{N}}$ is a matrix sequence associated to a bi-infinite $(\underline{\upsilon},\bar{\upsilon})$-metric quasigeodesic path γ with $\gamma(0) = \mathrm{id}$, then

$$\left(A_{k-1}\dots A_{k-n}\cdot o = \rho(\gamma(k)^{-1}\gamma(k-n))\cdot o\right)_n$$

(where $o := [\mathrm{SO}(d)] \in \mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$) gives a quasigeodesic in $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$ by (D$\pm$). Write $\rho(\gamma(k)^{-1}\gamma(k-n)) = K_{k,n}\cdot\exp(a(k,n))\cdot L_{k,n}$ to denote the parts of the Cartan decomposition.

By (P_1, C_r, c_r)-regularity and the higher-rank Morse lemma [19, Th. 1.3], the limit

$$\lim_{n\to\infty} U_1(A_{k-1}\cdots A_{k-n}) = \lim_{n\to\infty} K_{k,n}\langle e_1\rangle = \lim_{n\to\infty}\langle \mathrm{Ad}(K_{k,n})\cdot a(k,n)\rangle$$

exists,[3] and we have a bound C_a on the distance[4] from $A_{k-1}\cdots A_{k-n}\cdot o$ to a nearest point on any (P_1, C_r)-regular ray $(g_n\cdot o)$ starting at o such that $\lim_{n\to\infty} U_1(g_n) = \lim_n K_{k,n}\langle e_1\rangle$ (below, we refer to any such point as $\pi_{\lim} A_{k-1}\cdots A_{k-n}\cdot o$), where C_a depends only on C_r, c_r and $\underline{\upsilon},\bar{\upsilon}$.

Then, by [25, Lem. 4.9] applied with $p = o$ our basepoint, $\alpha_0 = C_r$, τ a model Weyl chamber corresponding to the first singular value gap, $q = A_{k-1}\cdots A_{k-n}\cdot o$, the point $r = \pi_{\lim} q$, the constant $2l = \|a(k,n)\| \geq \underline{\upsilon}^{-1}n - \underline{\upsilon}$ and $D = C_a$, we have

$$\begin{aligned}\sin\angle\left(\mathrm{Ad}(K_{k,n})\,a(k,n), \lim_n K_{k,n}\langle e_1\rangle\right) &= \sin\angle\left(\frac{1}{2}\,\mathrm{Ad}(K_{k,n})\,a(k,n), \lim_n K_{k,n}\langle e_1\rangle\right)\\ &\leq \frac{d(q/2, \pi_{\lim} q/2)}{d(o, \pi_{\lim} q/2)}\\ &\leq \frac{2C_a e^{C_a/\sqrt{d}+\underline{\upsilon}/2} e^{-(C_r/2\underline{\upsilon})n}}{d(o, \pi_{\lim}\, q/2)}\\ &\leq 2C_a e^{C_a/\sqrt{d}+\underline{\upsilon}/2} e^{-(C_r/2\underline{\upsilon})n}\end{aligned}$$

once n is sufficiently large, where "sufficiently large" depends only on the dimension d, our constants C_r, C_a and choice of basepoint o; here $q/2$ denotes the midpoint of oq, which can be written as

$$K_{k,n}\cdot\exp\left(\frac{1}{2}a(k,n)\right)\cdot L_{k,n}\cdot o.$$

Hence we can find $\hat{C} \geq 2C_a e^{C_a/\sqrt{d}+\underline{\upsilon}/2}$ such that

$$\sin\angle\left(\mathrm{Ad}(K_{k,n})\,a(k,n), \lim_n K_{k,n}\langle e_1\rangle\right) \leq \hat{C} e^{-(C_r/2\underline{\upsilon})n}$$

for all n, and so $d\left(U_1(A_{k-1}\cdots A_{k-n}), U_1(A_{k-1}\cdots A_{k-n-1})\right)$ is bounded above by

$$\begin{aligned}&C_{Lip}\sin\angle\left(\mathrm{Ad}(K_{k,n})\,a(k,n), \mathrm{Ad}(K_{k,n+1})\,a(k,n+1)\right)\\ &\quad\leq C_{Lip}\hat{C}\left(1 + e^{-C_r/2\underline{\upsilon}}\right)e^{-(C_r/2\underline{\upsilon})n}.\end{aligned}$$

[3] In the language of Kapovich–Leeb–Porti: this limit is the unique simplex τ such that our uniformly regular quasigeodesic is close to the Weyl cone over τ.

[4] For readers more acquainted with the language of Kapovich–Leeb–Porti: this is the distance to the Weyl cone over the C_r-regular open star of $\lim_n K_{k,n}\langle e_1\rangle$.

This gives us the desired bound with

$$\mu = \frac{1}{2} C_r \underline{\upsilon}^{-1} = \frac{1}{2} \underline{\mu} (\bar{\mu} \underline{\upsilon})^{-1} \qquad \text{and} \qquad C = C_{Lip} \hat{C} \left(1 + e^{-\mu}\right).$$

The analogous bound for $d\left(S_{d-1}(A_{k+n-1} \cdots A_k), S_{d-1}(A_{k+n} \cdots A_k)\right)$ can be obtained by arguing similarly, or by working with the dual representation — for the details of this part we refer the interested reader to the end of the proof of [27, Lem. 5.4]. □

4 Towards a relative property U

Suppose $(\Gamma, \mathcal{P})$ is a relatively hyperbolic group. By [26, Th. 5.3] together with Theorem 2.9, Γ satisfies property U, i.e. there exist a finite subset $F \subset \Gamma$ and a constant $L > 0$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ with

$$|f\gamma|_\infty \geq |f\gamma| - L. \tag{2}$$

We observe that this means that given any $\gamma \in \Gamma$ and $\epsilon > 0$, there exists $n_0 > 0$ such that $|(f\gamma)^n| \geq n|f\gamma| - (1+\epsilon)Ln$ for all $n \geq n_0$. In other words, we have a bound on cancellation between each pair of adjacent copies of $f\gamma$ in $(f\gamma)^n$.

We will now obtain a version of this statement where we impose some additional requirements on the finite set F. This statement will be useful in the proof of Theorem 5.1 below.

To describe these requirements, and to prove our relative inequality, we will use the framework and terminology described in Sect. 2.2. Abusing notation slightly, write $f\gamma$ to denote a geodesic path from id to $f\gamma$ in the Cayley graph. Consider this $f\gamma$ as a relative path $(f\gamma, H)$ with $H = H_1 \cup \cdots \cup H_k$, and write $\eta_i = f\gamma|_{H_i}$, so each η_i is a peripheral excursion.

Lemma 4.1 *Given Γ a non-elementary relatively hyperbolic group, there exists a finite subset $F \in \Gamma$ and a constant $L > 0$ such that for every $\gamma \in \Gamma$ there exists $f \in F$ such that*

$$|f\gamma|_\infty \geq |f\gamma| - L$$

and the peripheral excursions of $(f\gamma)^n$ are precisely n copies of the peripheral excursions of $f\gamma$.

Proof We adapt the proof of [26, Th. 5.3] to show that we can choose F to satisfy the additional requirements we have imposed here.

Let f be a Floyd function $f\colon \mathbb{N} \to \mathbb{R}^+$ for which the Floyd boundary $\partial_f \Gamma$ of Γ is non-trivial. By Theorem 2.9, there is a map from $\partial_f \Gamma$ to the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ which is injective on the set of conical limit points; hence, by [15, Prop. 5], we can find *non-peripheral* f_1, f_2 such $\{f_1^+, f_1^-\} \cap \{f_2^+, f_2^-\} = \varnothing$. We will use sufficiently high powers of these to form our set F; the north–south dynamics of the convergence group action of Γ on $\partial_f \Gamma$ will do the rest.

To specify what "sufficiently high" means it will be useful to define an auxiliary function $G\colon \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$, which gives a measure of "distance to infinity" as measured by the Floyd function: concretely, take $G(x) := 10 \sum_{k=\lfloor x/2 \rfloor}^{\infty} f(k)$. Since f is a Floyd function, $G(x)$ is non-increasing and $G(x) \to 0$ as $x \to \infty$. By [15, Lem. 1],[5] we have

$$d_f(g, h) \leq G\left(\langle g, h\rangle_e\right) \qquad\qquad d_f(g, g^+) \leq G\left(|g|/2\right) \tag{3}$$

[5] By the monotonicity and positivity of f and because $x \in \mathbb{Z}_{>0}$, our choice of G bounds from above the function $4xf(x) + 2\sum_{k=x}^{\infty} f(k)$ appearing in Karlsson's proof.

for all $g, h \in \Gamma$. (Notice that the second distance makes sense and is finite given the definition of the Floyd boundary.) Let $\epsilon = \frac{1}{6}\min\{d_f(f_1^+, f_2^\pm), d_f(f_1^-, f_2^\pm)\}$. Fix $R > 0$ such that $G(x) \geq \frac{\epsilon}{10}$ if and only if $x \leq R$, and N such that $\min\{|f_1^{N'}|, |f_2^{N'}|\} \geq 4R$ for all $N' \geq N$.

Claim For every non-trivial $\gamma \in \Gamma$ such that $d_f(\gamma^+, \gamma^-) \leq \epsilon$, there exists $i \in \{1, 2\}$ such that $d_f(f_i^{N'}\gamma^+, \gamma^-) \geq \epsilon$ for all $N' \geq N$.

Proof of claim By our choice of ϵ, we can find $i \in \{1, 2\}$ such that $d_f(\gamma^+, f_i^\pm) \geq 3\epsilon$: if $d_f(\gamma^+, f_1^\pm) < 3\epsilon$, then $d_f(\gamma^+, f_2^\pm) \geq \min\{d(f_2^\pm, f_1^+), d(f_2^\pm, f_1^-)\} - 3\epsilon = 3\epsilon$. Without loss of generality suppose $i = 1$.

There exists n_0 such that $G\left(\frac{1}{2}|\gamma^n|\right) < \epsilon$ for all $n \geq n_0$. For $n \geq n_0$ and $N' \geq N$, by our choice of N, we have

$$\begin{aligned} d_f(\gamma^n, f_1^{-N'}) &\geq d_f(\gamma^+, f_1^-) - d_f\left(f_1^-, f_1^{-N'}\right) - d_f(\gamma^+, \gamma^n) \\ &\geq 3\epsilon - G\left(\frac{1}{2}|f_1^{N'}|\right) - G\left(\frac{1}{2}|\gamma^n|\right) > \epsilon. \end{aligned}$$

Hence, for all $n \geq n_0$ and $N' \geq N$, we have $G(\langle\gamma^n, f_1^{-N'}\rangle_e) \geq d_f(\gamma^n, f_1^{-N'}) > \epsilon$, and $\langle\gamma^n, f_1^{-N'}\rangle_e \leq R$ by our choice of R. Now choose a sequence $(k_i)_{i\in\mathbb{N}}$ such that $|f_1^{k_i - N}| < |f_1^{k_i}|$ for all $i \in \mathbb{N}$. For $n \geq n_0$ and $N' \geq N$, we have, by the definition of the Gromov product and the inequalities above,

$$\begin{aligned} 2\langle f_1^{N'}\gamma^n, f_1^{k_n}\rangle_e &= |f_1^{N'}\gamma^n| + |f_1^{k_n}| - |f_1^{N'-k_n}\gamma^n| \\ &= |\gamma^n| + |f_1^{N'}| - 2\langle\gamma^n, f_1^{-N'}\rangle_e + |f_1^{k_n}| - |f_1^{N'-k_n}\gamma^n| \\ &\geq |f_1^{N'}| - 2R + |f_1^{k_n}| - (|f_1^{N'-k_n}\gamma^n| - |\gamma^n|) \\ &\geq |f_1^{N'}| - 2R + |f_1^{k_n}| - |f_1^{N'-k_n}| \\ &\geq |f_1^{N'}| - 2R \geq 2R. \end{aligned}$$

Then by our choice of R we have

$$d_f(f_1^{N'}\gamma^+, f_1^+) \leq \lim_{n\to\infty} G(\langle f_1^{N'}\gamma^n, f_1^{k_n}\rangle_e) \leq \epsilon/10$$

whenever $n \geq n_0$ and $N' \geq N$; thus

$$d_f(f_1^{N'}\gamma^+, \gamma^-) \geq d_f(\gamma^+, f_1^+) - d_f(f_1^{N'}\gamma^+, f_1^+) - d_f(\gamma^+, \gamma^-) \geq \epsilon$$

whence the claim.

Now, with f_1, f_2 and N as above, fix $F_0 = \{f_1^N, f_1^{N+1}, f_2^N, f_2^{N+1}, e\}$. Then there exists $g \in F_0$ such that $d_f(g\gamma^+, \gamma^-) \geq \epsilon$: if $d_f(\gamma^+, \gamma^-) \geq \epsilon$, choose $g = e$. Otherwise, from the above argument, either $g = f_1^N$ or $g = f_2^N$ works, and then so does $g = f_1^{N+1}$ or $g = f_2^{N+1}$ respectively.

Next fix $L = 2\max_{g\in F_0}|g| + 2R + 1$. Without loss of generality suppose $|\gamma| > L - 1$; otherwise $|\gamma| - |\gamma|_\infty \leq L$ and we have our desired inequality with $g = e$. We will show that the desired result holds with $F := F_0 \cup S$ and this L. Otherwise choose $g \in F_0$ such that $d_f(g\gamma^+, \gamma^-) \geq \epsilon$. To use this to obtain an inequality between $|g\gamma|$ and $|g\gamma|_\infty$, we use Lemma 2.12 with $g\gamma$ in the place of γ, the Cayley graph in the place of X, and $x_0 = e$ to obtain a sequence $(m_i)_{i\in\mathbb{N}}$ such that

$$2\lim_{i\to\infty}\langle(g\gamma)^{m_i}, (g\gamma)^{-1}\rangle_e \geq |g\gamma| - |g\gamma|_\infty, \tag{4}$$

so it suffices to obtain an upper bound on the Gromov products $\langle (g\gamma)^{m_i}, (g\gamma)^{-1}\rangle_e$.

To obtain this bound, we start by noting that $g\gamma^+ = (g\gamma g^{-1})^+$, and using this, the triangle inequality, and the inequalities in (3) to observe that

$$\begin{aligned} d_f\left(g\gamma^+, (g\gamma)^+\right) &\le d_f\left(g\gamma^+, g\gamma g^{-1}\right) + d_f\left(g\gamma g^{-1}, g\gamma\right) + d_f\left((g\gamma)^+, g\gamma\right) \\ &\le G\left(\frac{1}{2}|g\gamma g^{-1}|\right) + G\left(\langle g\gamma g^{-1}, g\gamma\rangle_e\right) + G\left(\frac{1}{2}|g\gamma|\right) \end{aligned}$$

and using liberally the monotonicity of G on the last right-hand side, we obtain the further upper bound

$$d_f\left(g\gamma^+, (g\gamma)^+\right) \le 3G\left(\frac{1}{2}|\gamma| - |g|\right)$$

which, finally, because $\frac{1}{2}|\gamma| - |g| \ge R$, is bounded above by $\frac{3\epsilon}{10}$. Arguing similarly, we have

$$\begin{aligned} d_f\left(\gamma^{-1}, \gamma^{-1}g^{-1}\right) &\le d_f\left(\gamma^-, \gamma^{-1}\right) + d_f\left(\gamma^{-1}, \gamma^{-1}g^{-1}\right) \\ &\le G\left(\frac{1}{2}|g\gamma|\right) + G\left(\langle \gamma^{-1}, \gamma^{-1}g^{-1}\rangle_e\right) \\ &\le 2G\left(\frac{1}{2}|\gamma| - |g|\right) \le \frac{\epsilon}{5} \end{aligned}$$

and hence we have

$$\begin{aligned} d_f\left((g\gamma)^+, \gamma^{-1}g^{-1}\right) &\ge d_f\left(g\gamma^+, \gamma^-\right) - d_f\left(g\gamma^+, (g\gamma)^+\right) - d_f\left(\gamma^-, \gamma^{-1}g^{-1}\right) \\ &\ge \epsilon - \frac{3\epsilon}{10} - \frac{\epsilon}{5} = \frac{\epsilon}{2}. \end{aligned}$$

Thus we have $n_1 > 0$ such that $G\left(\langle (g\gamma)^n, (g\gamma)^{-1}\rangle_e\right) \ge d_f\left((g\gamma)^n, (g\gamma)^{-1}\right) \ge \frac{\epsilon}{3}$ and so $\langle (g\gamma)^n, (g\gamma)^{-1}\rangle_e \le R$ for all $n \ge n_1$. This is the bound we feed into (4) to obtain $|g\gamma| - |g\gamma|_\infty \le 2R \le L$, which was the inequality to be shown.

Finally, we prove the statement about the peripheral excursions. We may also assume, without loss of generality, that $g\gamma$ contains at least one peripheral excursion, otherwise there is nothing left to prove.

If we have a relation $\alpha\eta\beta$ with $\eta \in P \setminus \{\mathrm{id}\}$ peripheral and $\alpha, \beta \notin P$ (and α not ending in any letter of P and β not starting in any letter of P), then $\alpha\eta\alpha^{-1} = \beta^{-1}\eta\beta$, and by malnormality this implies $\alpha = \beta^{-1}$, which is not possible since $\eta \neq \mathrm{id}$. Since we are assuming $g\gamma$ has peripheral excursions, we may thus assume that in $(g\gamma)^n$ there is no cancellation across more than two copies of $g\gamma$, i.e. it suffices to look at cancellation between adjacent copies.

The peripheral excursions of $(g\gamma)^n$ are exactly n copies of that of $g\gamma$ precisely when cancellation between adjacent copies of $g\gamma$ does not reach any of the peripheral excursions.

Suppose now that this is not the case, i.e. cancellation between adjacent copies does reach the peripheral excursions. If $g = f_i^N$ (resp. $g = f_i^{N+1}$), then we may take $g = f_i^{N+1}$ (resp. $g = f_i^N$) instead; the desired inequalities still hold from the arguments above, and now cancellation between adjacent copies no longer reaches the peripheral excursions.

Suppose instead $g = e$; then we may assume, from the argument above, that $|\gamma| \le L - 1$. We will instead take g to be a non-peripheral generator s; then, while we had cancellation between adjacent copies before with $g = e$, we can no longer have it with $g = s$. Then $|s\gamma| \le |\gamma| + 1 \le L$, and we are done. □

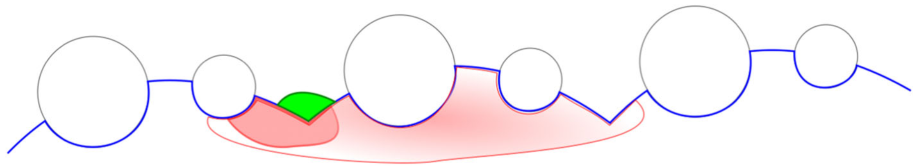

Fig. 3 Schematic illustration of cancellation that can happen in a word $(g\gamma)^n$. As in Fig. 2, grey circles and their interiors indicate combinatorial horoballs. Blue indicates the word $(g\gamma)^3$. Green loops indicate relations in Γ, which induce cancellation within the word. By the argument in the text, we cannot have relations like the red loops, which include part of a peripheral excursion or all of one copy of $g\gamma$ within $(g\gamma)^3$

5 A characterisation using eigenvalue gaps

Suppose Γ is hyperbolic relative to $\mathcal{P}$. We have, as above, the cusped space $X = X(\Gamma, \mathcal{P}, S)$, which is a δ-hyperbolic space on which Γ acts isometrically and properly. We define $|\cdot|_{c,\infty}$ to be the stable translation length on this space, i.e.

$$|\gamma|_{c,\infty} := \lim_{n\to\infty} \frac{|\gamma^n|_c}{n}$$

where $|\cdot|_c := d_X(\mathrm{id}, \cdot)$ as above.

Given $A \in \mathrm{GL}(d, \mathbb{R})$, let $\lambda_i(A)$ denote the magnitude of the i^{th} largest eigenvalue of A. We will prove the following theorem. We remind the reader that the (D$\pm$) and ($\mathrm{D}^\lambda_\pm$) conditions referred to in the theorem statement were defined in Definition 3.1 and in the statements of Theorems B and C.

Theorem 5.1 *Let Γ be hyperbolic relative to $\mathcal{P}$ and $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation. If ρ satisfies (D$\pm$), then it satisfies ($\mathrm{D}^\lambda_\pm$).*

Conversely, if ρ is semisimple and satisfies ($\mathrm{D}^\lambda_\pm$), then ρ also satisfies (D$\pm$).

Before proving the theorem, we pause to note that the (D^λ_+) condition, although formulated as a condition for all elements $\gamma \in \Gamma$, is in fact (equivalent to) a condition on only peripheral elements.

Proposition 5.2 *Let Γ be hyperbolic relative to $\mathcal{P}$ and $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation.*

Then ρ satisfies the (D^λ_+) condition if and only if for every peripheral element $\eta \in \bigcup_{P\in\mathcal{P}} \subset \Gamma$, all the eigenvalues of $\rho(\eta)$ have magnitude 1.

Proof Suppose ρ satisfies the (D^λ_+) condition. Then, since every peripheral element $\eta \in \Gamma$ satisfies $|\eta|_{c,\infty} = 0$, then $\frac{\sigma_1}{\sigma_d}(\rho(\eta^n))$ is bounded for all n, and so the eigenvalues of $\rho(\eta)$ must have magnitude 1.

Conversely, suppose all the eigenvalues of $\rho(\eta)$ have magnitude 1. By a computation involving the Jordan normal form, there exist constants $C > 1, \mu > 0$ such that for any peripheral element $\eta \in \bigcup_{P\in\mathcal{P}} P \subset \Gamma$, we have $\sigma_1(\rho(\eta)) \leq Ce^{\mu|\eta|_c}$, i.e. the (D+) condition holds, and hence there exist constants $\bar{C}, \bar{\mu} > 0$ such that $\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) \leq \bar{C}e^{\bar{\mu}|\gamma|_c}$ for all $\gamma \in \Gamma$ (see Remark 3.2). Then we have

$$\begin{aligned}(\log\lambda_1 - \log\lambda_d)(\rho(\gamma)) &= \lim_{n\to\infty} \frac{1}{n}(\log\sigma_1 - \log\sigma_d)(\rho(\gamma^n)) \\ &\leq \lim_{n\to\infty} \frac{1}{n}(\log\bar{C} + \bar{\mu}|\gamma^n|_c) = \bar{\mu}|\gamma|_{c,\infty}\end{aligned}$$

and so

$$\frac{\lambda_1}{\lambda_d}(\rho(\gamma)) \leq e^{\bar{\mu}|\gamma|_{c,\infty}}.$$

□

We also remark that the next statement follows immediately from the theorem and the definition of relatively dominated representations presented in Sect. 3.

Corollary 5.3 *[Theorem B] Let* Γ *be hyperbolic relative to* $\mathcal{P}$. *A semisimple representation* $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ *is* P_1*-dominated relative to* $\mathcal{P}$ *if and only if it satisfies* ($\mathrm{D}^{\lambda}_{\pm}$) *as well as the unique limits and uniform transversality conditions from Definition 3.1.*

Proof *(Proof of Theorem 5.1)* We recall the identity $\log \lambda_i(A) = \lim_{n\to\infty} \frac{\log \sigma(A^n)}{n}$. Given (D−), we have

$$\begin{aligned}(\log \lambda_1 - \log \lambda_2)(\rho(\gamma)) &= \lim_{n\to\infty} \frac{1}{n}(\log \sigma_1 - \log \sigma_2)(\rho(\gamma^n)) \\ &\geq \lim_{n\to\infty} \frac{1}{n}(\log \underline{C} + \underline{\mu}|\gamma^n|_c) = \underline{\mu}|\gamma|_{c,\infty}\end{aligned}$$

and so

$$\frac{\lambda_1}{\lambda_2}(\rho(\gamma)) \geq e^{\underline{\mu}|\gamma|_{c,\infty}}.$$

Given (D+) (see Remark 3.2), we obtain

$$\frac{\lambda_1}{\lambda_d}(\rho(\gamma)) \leq e^{\bar{\mu}|\gamma|_{c,\infty}}$$

by the argument at the end of the proof of Proposition 5.2.

Hence (D±) implies ($\mathrm{D}^{\lambda}_{\pm}$).

In the other direction, we will use Lemma 4.1 to obtain a relative version of (2): for any given $\epsilon > 0$, there exists $n_1 > 0$ such that

$$|(f\gamma)|_{c,\infty} \geq \frac{1}{12}|f\gamma|_c - L.$$

As observed in Sect. 4, this gives us a bound on cancellation between each pair of adjacent copies of $f\gamma$ in $(f\gamma)^n$; the relative version will give us some further control over peripheral letters in any such cancellation. We make this more precise below.

By Proposition 2.6,

$$|f\gamma|_c \leq 4\left(\ell(f\gamma) - \sum_{i=1}^{k}\ell(\eta_i) + \sum_{i=1}^{k}\hat{\ell}(\eta_i)\right).$$

By Lemma 4.1, $\ell((f\gamma)^n) \geq n|f\gamma| - (1+\epsilon)Ln$ for all sufficiently large n (recall that $\ell(\gamma) := |\gamma|$). Crucially, by the part of the lemma on the peripheral excursions of $(f\gamma)^n$, the total length of peripheral excursions for $(f\gamma)^n$ remains $n\sum_{i=1}^{k}\ell(\eta_i)$, and the sum of the resulting $\hat{\ell}$ remains $n\sum_{i=1}^{k}\hat{\ell}(\eta_i)$.

Now we may use Proposition 2.6 to conclude that

$$|(f\gamma)^n|_c \geq \frac{1}{3}\left(n\ell(f\gamma) - n\sum_{i=1}^{k}\ell(\eta_i) + n\sum_{i=1}^{k}\hat{\ell}(\eta_i) - (1+\epsilon)Ln\right).$$

But this implies

$$\begin{aligned}|f\gamma|_{c,\infty} &= \lim_{n\to\infty}\frac{1}{n}|(f\gamma)^n|_c \\ &\geq \frac{1}{3}\left(\ell(f\gamma) - \sum_{i=1}^{k}\ell(\eta_i) + \sum_{i=1}^{k}\hat{\ell}(\eta_i) - (1+\epsilon)L\right) \\ &> \frac{1}{12}|f\gamma|_c - (1+\epsilon)L.\end{aligned}$$

We then obtain the desired inequality by taking ϵ to 0.

On the other hand it is clear from the definition of the stable translation length that $|f\gamma|_{c,\infty} \leq |f\gamma|_c$.

Now, for semisimple ρ, there exists a finite $F' \subset \Gamma$ and $C > 0$ such that for every $\gamma \in \Gamma$ there exists $f' \in F'$ such that for every i,

$$|\log\lambda_i(\rho(\gamma f')) - \log\sigma_i(\rho(\gamma))| \leq C.$$

This follows from [26, Th. 2.6].

Then, given (D^{λ}_{+}), we have

$$\begin{aligned}\frac{\sigma_1}{\sigma_d}(\rho(\gamma)) &\leq e^{2C}\cdot\frac{\lambda_1}{\lambda_d}(\rho(\gamma f')) \\ &\leq e^{2C}e^{\bar{\mu}|\gamma f'|_{c,\infty}} \leq e^{2C}e^{\bar{\mu}|\gamma f'|_c} \\ &\leq e^{2C}(C_{F'})^{\bar{\mu}}\cdot e^{\bar{\mu}|\gamma|_c}\end{aligned}$$

where $C_{F'} := \max_{f'\in F'} e^{|f'|_c}$ and so (D+) holds. Given (D^{λ}_{-}), we have

$$\begin{aligned}\frac{\sigma_1}{\sigma_2}(\rho(\gamma)) &\geq e^{-2C}\cdot\frac{\lambda_1}{\lambda_2}(\rho(\gamma f')) \\ &\geq e^{-2C}\underline{C}e^{\underline{\mu}|\gamma f'|_{c,\infty}} \geq e^{-2C}\underline{C}e^{-\underline{\mu}L}e^{\frac{1}{12}\underline{\mu}|f\gamma f'|_c} \\ &\geq e^{-2C}\underline{C}e^{-\underline{\mu}L}(C_F C'_F)^{-\frac{1}{12}\underline{\mu}}\cdot e^{\underline{\mu}|\gamma|_c}\end{aligned}$$

where $C_{F'}$ is as above and $C_F := \max_{f\in F} e^{|f|_c}$, and hence (D−) holds. □

6 Limit maps imply well-behaved peripherals

If we assume that our group Γ is hyperbolic relative to $\mathcal{P}$, then the additional conditions of unique limits and uniform transversality which appear in either of the definitions of relatively dominated representations so far may also be replaced by a condition stipulating the existence of suitable limit maps from the Bowditch boundary $\partial(\Gamma,\mathcal{P})$. As noted above, this gives us relative analogues of some of the characterizations of Anosov representations due to Guéritaud et al. [9, Th. 1.3 and 1.7 (1),(3)].

Theorem 6.1 *[Theorem C] Let Γ be hyperbolic relative to $\mathcal{P}$. A representation $\rho\colon\Gamma\to\mathrm{SL}(d,\mathbb{R})$ is P_1-dominated relative to $\mathcal{P}$ if and only if (D±) (as in Definition 3.1) are satisfied and there exist continuous, ρ-equivariant, transverse, dynamics-preserving limit maps $\xi_\rho\colon\partial(\Gamma,\mathcal{P})\to\mathbf{P}(\mathbb{R}^d)$ and $\xi^*_\rho\colon\partial(\Gamma,\mathcal{P})\to\mathbf{P}(\mathbb{R}^{d*})$.*

Proof If ρ is P_1-dominated relative to $\mathcal{P}$, then it satisfies (D$\pm$), and admits continuous, equivariant, transverse, dynamics-preserving limit maps [27, Th. 7.2].

Conversely, if suffices to show that the unique limits and uniform transversality conditions must hold once we have continuous, equivariant, transverse, dynamics-preserving limit maps, and (D$\pm$) hold.

Unique limits follows from the limit maps being well-defined and dynamics-preserving. There is a single limit point $x_P \in \partial(\Gamma, \mathcal{P})$ for each peripheral subgroup $P \in \mathcal{P}$, and the dynamics-preserving property says that ξ_ρ sends x_P to the parabolic fixed point in $\mathbf{P}(\mathbb{R}^d)$ corresponding to $\rho(P)$. That parabolic fixed point should coincide with $\lim_{n\to\infty} U_1(\rho(\eta^n))$ for any $\eta \in P$, or more generally with any $\lim_{n\to\infty} U_1(\rho(\eta_n))$ for any divergent sequence $(\eta_n) \subset P$, and hence furnishes the unique limit $\xi_\rho(P)$. We may argue similarly with ξ^*_ρ and its image in $\mathbf{P}(\mathbb{R}^{d*})$.

Uniform transversality follows from [27, Prop. 8.5]: briefly, if we did not have uniform transversality, we would be able to find sequences $(\gamma_n), (\eta_n) \subset \Gamma$ and peripheral subgroups P, P' such that $\angle(\gamma_n^{-1}\xi_\rho(P'), \eta_n\xi_\rho(P))$ goes to zero. Up to subsequence, the γ_n^{-1} and η_n converge to infinite (projected quasi-)geodesic rays asymptotic to different forward endpoints, and $\angle\left(\xi_\rho(\lim_n \gamma_n^{-1}), \xi^*_\rho(\lim_n \eta_n)\right) = 0$; but this contradicts transversality. □

We remind the reader that the (D$^\lambda_\pm$) conditions which appear in the corollaries below were defined in the statement of Theorem C.

Corollary 6.2 *Let Γ be hyperbolic relative to $\mathcal{P}$. A semisimple representation $\rho\colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ is P_1-dominated relative to $\mathcal{P}$ if and only if (D$^\lambda_\pm$) are satisfied and there exist continuous, ρ-equivariant, transverse, dynamics-preserving limit maps $(\xi_\rho, \xi^*_\rho)\colon \partial(\Gamma, \mathcal{P}) \to \mathbf{P}(\mathbb{R}^d) \times \mathbf{P}(\mathbb{R}^{d*})$.*

Proof This follows immediately from Theorems 6.1 and 5.1. □

As an application of Theorem 6.1, we can show that certain groups that play weak ping-pong on flag spaces are relatively dominated. We remark that these examples have previously been claimed in [16].

Example 6.3 [Proposition D] Fix biproximal elements $t_1, \ldots, t_k \in \mathrm{PGL}(d, \mathbb{R})$. Write $t_i^\pm$ to denote the attracting lines and $H^\pm_{t_i}$ to denote the repelling hyperplanes of $t_i^{\pm 1}$.

Assuming $t_i^+ \neq t_j^+$ for $i \neq j$ and $t_i^\pm \not\subset H^\mp_{t_j}$ for all i, j, and replacing the t_i with sufficiently high powers if needed, we have open neighborhoods $A_i^\pm \subset \mathbf{P}(\mathbb{R}^d) =: X$ of $t_i^\pm$, and $B_i^\pm \subset X$ of $H^\pm_{t_i}$ such that

- $A_i^\pm \subset B_i^\pm$ for $i = 1 \ldots, k$, and $A_i^\sigma \cap B_j^{\sigma'} = \varnothing$ unless $i = j$ and $\sigma = \sigma'$,
- $t_i^{\pm 1}\left(X \smallsetminus B_i^\pm\right) \subset A_i^\pm$ for $i = 1, \ldots, k$, and moreover
- there exists $\epsilon > 0$ such that $t_i^{\pm 1}$ is ϵ-Lipschitz on $X \smallsetminus B_i^\pm$ for all i (see [4, Lem. A.8]).

Suppose we have, in addition, unipotent elements $u_1, \ldots, u_{k'} \in \mathrm{PGL}(d, \mathbb{R})$ which each have well-defined attracting lines u_j^+ and attracting hyperplanes $H^+_{u_j}$ (equivalently, well-defined largest Jordan blocks). Suppose, again passing to sufficiently high powers of the $u_1, \ldots, u_{k'}$ if need be, there exist open neighborhoods C_j^+ of u_j^+ and C_j^- of $H^+_{u_j}$ in $X = \mathbf{P}(\mathbb{R}^d)$, such that

- $C_j^+ \subset C_j^-$ for $j = 1, \ldots, k'$, and the $\overline{C_1^+}, \ldots, \overline{C_{k'}^+}$ are pairwise disjoint and also disjoint from the the closures of all of the $B_i^\pm$,

- $u_j^{\pm n}(X \smallsetminus C_j^-) \subset C_j^+$ for all non-zero n, and moreover
- there exists $c > 0$ such that $u_j^{\pm n}$ is $\frac{c}{n}$-Lipschitz on $X \smallsetminus C_j^-$ for all $n \in \mathbb{Z}_{>0}$.

To see that we may assume the last hypothesis to hold: fix $u = u_j$. Let $v_1, \ldots, v_d$ be a basis for $\mathbb{R}^d$ with respect to which u may be written in Jordan normal form, where v_1 spans u^+ and $v_1, \ldots, v_{d-1}$ span H_u^+.

Up to introducing a biLipschitz error, we can choose a metric on $\mathbf{P}(\mathbb{R}^d)$ given by pushing forward the suitable spherical metric obtained by viewing u^+ as the north pole and $\mathbf{P}\langle v_2, \ldots, v_d\rangle$ as the (projectivization of the) equator. In the affine chart given by taking $\langle v_2, \ldots, v_d\rangle$ to be the hyperplane at infinity, if we consider polar coordinates (r, θ) with origin u^+, the spherical metric satisfies

$$d\left((r,\theta),(r',\theta')\right) \leq |\phi - \phi'| + \min\{\phi, \phi'\} \cdot |\theta - \theta'|$$

where $\phi := \sin \arctan r$.

Then, given two points $\xi_1, \xi_2 \in \mathbf{P}(\mathbb{R}^d) \smallsetminus C_j^-$, with $\xi_i = (\theta_i, \phi_i)$ for $i = 1, 2$ in our coordinates, and abusing notation slightly to write $u^{\pm n}\ell_i = (u^{\pm n}\theta_i, u^{\pm n}\phi_i)$ for $i = 1, 2$, we have some constants $L, L' > 0$ such that

$$\begin{aligned} u^{\pm n}\phi_2 - u^{\pm n}\phi_1| &\leq L \cdot \frac{\sigma_2}{\sigma_1}(u^{\pm n}) \cdot |\phi_2 - \phi_1| \leq \frac{L'}{n} \cdot |\phi_2 - \phi_1| \\ |u^{\pm n}\theta_2 - u^{\pm n}\theta_1| &\leq |\theta_2 - \theta_1| \end{aligned}$$

and so we have

$$d(u^{\pm n}\ell_1, u^{\pm n}\ell_2) \leq \frac{L'}{n}\left(|\phi_2 - \phi_1| + \min(\phi_2, \phi_1)|\theta_2 - \theta_1|\right) \leq \frac{2L'}{n} \cdot d(\ell_1, \ell_2)$$

for all $n > 0$. Hence we have the Lipschitz constants we seek.

Then, by a ping-pong argument, the group $\Gamma := \langle t_1, \ldots, t_k, u_1, \ldots, u_{k'}\rangle$ is isomorphic to a non-abelian free group $F_{k+k'}$.

Since we have finitely many generators, we can pick $\epsilon_0 > 0$ such that

- For all $i = 1, \ldots, k$ and for any $n > 0$ (resp. $n < 0$), $U_1(t_i^n)$ is within ϵ_0 of t_i^+ (resp. t_i^-),
- For all $i = 1, \ldots, k$ and for any $n < 0$ (resp. $n > 0$), $U_{d-1}(t_i^n)$ is within ϵ_0 of $H_{t_i}^+$ (resp. $H_{t_i}^-$), and
- For all $j = 1, \ldots, k'$ and for any $n \neq 0$, $U_1(u_j^{\pm n})$ are within ϵ_0 of u_j^+.

By taking powers of the generators and slightly expanding the ping-pong neighborhoods if needed, we may assume that ϵ_0 is sufficiently small so that the $A_i^\pm$ and $B_i^\pm$ contain the $2\epsilon_0$-neighborhoods of the $t_i^\pm$ and $H_{t_i}^\pm$ respectively, and the C_j^+ and C_j^- contain the $2\epsilon_0$-neighborhoods of the u_j^+ and $H_{u_j}^+$ respectively. This slight strengthening of ping-pong will be useful for establishing the transversality of our limit maps below.

Below, we replace Γ by the free subgroup generated by these powers.

Let $\mathcal{P} = \{\langle u_1\rangle, \ldots, \langle u_{k'}\rangle\}$. Then Γ is hyperbolic relative to $\mathcal{P}$ and there are continuous Γ-equivariant homeomorphisms ξ, ξ^* from the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ to the limit set $\Lambda_\Gamma \subset \mathbf{P}(\mathbb{R}^d)$ and the dual limit set $\Lambda_\Gamma^* \subset \mathbf{P}(\mathbb{R}^{d*})$ given by

$$\lim_n \gamma_n \mapsto \lim_n U_1(\gamma_n) \qquad \text{and} \qquad \lim_n \gamma_n \mapsto \lim_n U_{d-1}(\gamma_n)$$

respectively; the limits exist by the Lipschitz behavior of the generators (cf. [4, Prop. A.5], see also [24]).

By definition, ξ and ξ^* are dynamics-preserving.

To prove that ξ and ξ^* are transverse, we will need that the inclusion $\iota\colon \Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ satisfies (D$-$).

To obtain (D$-$), one can use the following

Lemma 6.4 [2, Lem. A.7] *If $A, B \in \mathrm{GL}(d, \mathbb{R})$ are such that $\sigma_p(A) > \sigma_{p+1}(A)$ and $\sigma_p(B) > \sigma_{p+1}(B)$, then*

$$\sigma_p(AB) \geq (\sin\alpha)\cdot\sigma_p(A)\,\sigma_p(B)$$
$$\sigma_{p+1}(AB) \leq (\sin\alpha)^{-1}\sigma_{p+1}(A)\,\sigma_{p+1}(B)$$

where $\alpha := \angle\big(U_p(B), S_{d-p}(A)\big)$.

Note that the generators $t_1^{\pm}, \ldots, t_k^{\pm}, u_1, \ldots, u_{k'}$ must satisfy the hypotheses (with $p = 1$) by assumption in order to have well-defined attracting lines.

To use this lemma here, we show that there exists a uniform constant $\alpha_0 > 0$ such that whenever $(\gamma_n = g_1 \cdots g_n)_{n\in\mathbb{N}} \subset \Gamma$ is a sequence converging to a point in $\partial(\Gamma, \mathcal{P})$, where each g_i is a power of a generator and g_i and g_j are not powers of a a common generator whenever $|i - j| = 1$, then $\angle\big(U_p(g_1 \cdots g_{i-1}), S_{d-p}(g_i)\big) \geq \alpha_0$ for $p \in \{1, d-1\}$ and for all n.

Suppose this were not true, so that there exist

- A generator s,
- A divergent sequence (k_n) of integers, and
- A divergent sequence (w_n) of words in Γ not starting in $s^{\pm 1}$, which without loss of generality — passing to a subsequence if needed — converges to some point in $\partial(\Gamma, \mathcal{P})$,

such that

$$\angle(U_1(\rho(w_n)), S_{d-1}(\rho(s^{k_n}))) \leq 2^{-n};$$

then, in the limit, we obtain

$$\angle\Big(\lim_{n\to\infty} U_1(\rho(w_n)), \lim_{n\to\infty} S_{d-1}(\rho(s^{k_n}))\Big) = 0$$

but this contradicts transversality, since, by our hypothesis that none of the words w_n starts with s, we must have $\lim w_n \neq \lim s^{k_n}$ as $n \to \infty$.

Thus we do have a uniform lower bound $\alpha_0 \leq \alpha$ as desired. Then Lemma 6.4, together with the existence of a proper polynomial $\underline{q}$ such that

$$\frac{\sigma_1}{\sigma_2}(u_j^n) \geq \underline{q}(n)$$

for all j (which follows from a computation involving the Jordan normal form, since the u_j are unipotent), tells us that $\log\frac{\sigma_1}{\sigma_2}(\gamma)$ grows at least linearly in $|\gamma|_c$, which gives us (D$-$).

We now claim that ξ and ξ^* are transverse: given two distinct points $x = \lim\gamma_n$ and $y = \lim\eta_n$ in $\partial(\Gamma, \mathcal{P})$, we have $\xi(x) \notin \xi^*(y)$ — the latter considered as a projective hyperplane in $\mathbf{P}(\mathbb{R}^d)$ — using ping-pong and the following

Lemma 6.5 [9, Lem. 5.8]; [2, Lem. A.5] *If $A, B \in \mathrm{GL}(d, \mathbb{R})$ are such that $\sigma_p(A) > \sigma_{p+1}(A)$ and $\sigma_p(AB) > \sigma_{p+1}(AB)$, then*

$$d\left(B\cdot U_p(A), U_p(BA)\right) \leq \frac{\sigma_1}{\sigma_d}(B)\cdot\frac{\sigma_{p+1}}{\sigma_p}(A).$$

To establish the claim: write $\gamma_n = g_1 \cdots g_n$ and $\eta_n = h_1 \ldots h_n$. Pick n_0 minimal such that $U_1(\gamma_{n_0})$ and $U_1(\eta_{n_0})$ are in different ping-pong sets. Since $\gamma_n, \eta_n \to \infty$, by (D−), as long as n_0 is sufficiently large, $\sigma_1(\gamma_n) > \sigma_2(\gamma_n)$ and $\sigma_{d-1}(\eta_n) > \sigma_d(\eta_n)$ for all $n \geq n_0$. Hence the lemma above implies that for any given $\epsilon > 0$, there exists some n_1 so that for all $n \geq n_1$, $U_1(\gamma_n) = U_1(g_1 \cdots g_n)$ is ϵ-close to $\gamma_{n_0} \cdot U_1(g_{n_0+1} \cdots g_n)$, and $U_{d-1}(\eta_n)$ is ϵ-close to $\eta_{n_0} \cdot U_{d-1}(h_{n_0+1} \cdots h_n)$. By our ping-pong setup, for sufficiently small ϵ these are uniformly close to $U_1(\gamma_{n_0})$ and $U_{d-1}(\eta_{n_0})$ respectively, and in particular they are transverse to each other.

Finally, the inclusion $\iota \colon \Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ satisfies (D+), because Γ is finitely-generated, there exists a polynomial $\bar{q}$ of degree $d-1$ such that $\frac{\sigma_1}{\sigma_d}(u) \leq \bar{q}(|u|)$ for every unipotent element $u \in \Gamma$ (by a computation involving the Jordan normal form), and the first singular value σ_1 is sub-multiplicative.

We then conclude, by Theorem 6.1, that $\iota \colon \Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is P_1-dominated relative to $\mathcal{P}$.

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Declarations

Conflict of interest The author declares that they have no conflict of interest.

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