ORIGINAL PAPER

Extra-special quotients of surface braid groups and double Kodaira fibrations with small signature

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Abstract

We study some special systems of generators on finite groups, introduced in previous work by the first author and called *diagonal double Kodaira structures*, in order to investigate finite non-abelian quotients of the pure braid group on two strands $P_2(\Sigma_b)$, where Σ_b is a closed Riemann surface of genus b. In particular, we prove that, if a finite group G admits a diagonal double Kodaira structure, then $|G| \geq 32$, and equality holds if and only if G is extra-special. In the last section, as a geometrical application of our algebraic results, we construct two 3-dimensional families of double Kodaira fibrations having signature 16. Such surfaces are different from the ones recently constructed by Lee, Lönne and Rollenske and, as far as we know, they provide the first examples of positive-dimensional families of double Kodaira fibrations with small signature.

1 Introduction

A *Kodaira fibration* is a smooth, connected holomorphic fibration $f_1 \colon S \longrightarrow B_1$, where S is a compact complex surface and B_1 is a compact closed curve, which is not isotrivial (this means that not all fibres are biholomorphic each other). The genus $b_1 := g(B_1)$ is called the *base genus* of the fibration, and the genus $g := g(F)$, where F is any fibre, is called the *fibre genus*. A surface S that is the total space of a Kodaira fibration is called a *Kodaira fibred surface*. For every Kodaira fibration, we have $b_1 \geq 2$ and $g \geq 3$, see [19, Theorem

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1.1]. Since the fibration is smooth, the condition on the base genus implies that S contains no rational or elliptic curves; hence S is minimal and, by the sub-additivity of the Kodaira dimension, it is of general type, hence algebraic.

An important topological invariant of a Kodaira fibred surface S is its *signature* $\sigma(S)$, namely the signature of the intersection form on the middle cohomology group $H^2(S, \mathbb{R})$. Actually, the first examples of Kodaira fibrations (see [21]) were constructed in order to show that σ is not multiplicative for fibre bundles. In fact, $\sigma(S) > 0$ for every Kodaira fibration (see the introduction to [24]), whereas $\sigma(B_1) = \sigma(F) = 0$, hence $\sigma(S) \neq \sigma(B_1)\sigma(F)$; by [10], this in turn means that the monodromy action of $\pi_1(B)$ on the rational cohomology ring $H^*(S, \mathbb{Q})$ is non-trivial.

Every Kodaira fibred surface S has the structure of a real surface bundle over a smooth real surface, and so $\sigma(S)$ is divisible by 4, see [27]. If, in addition, S has a spin structure, i.e. its canonical class is 2-divisible in $\mathrm{Pic}(S)$, then $\sigma(S)$ is a positive multiple of 16 by Rokhlin's theorem, and examples with $\sigma(S) = 16$ are constructed in [24]. It is not known whether there exists a Kodaira fibred surface with $\sigma(S) \leq 12$.

Kodaira fibred surfaces are a source of fascinating and deep questions at the cross-road between the algebro-geometric properties of a compact, complex surface and the topological properties of the underlying closed, oriented 4-manifold. In fact, they can be studied by using, besides the usual algebro-geometric methods, techniques borrowed from geometric topology such as the Meyer signature formula, the Birman-Hilden relations in the mapping class group and the subtraction of Lefchetz fibrations, see [12, 13, 23, 36]. We refer the reader to the survey paper [8] and the references contained therein for further details.

The original examples by Kodaira (see for instance [4, Chapter V, Section 14]) and its variants described in [2, 17] are obtained by taking cyclic covers of a product of curves $C \times D$, branched over a smooth divisor which is the disjoint union of a finite number of graphs of regular maps $C \longrightarrow D$. Thus, they come with two distinct Kodaira fibrations, namely the pull-backs of the two natural fibrations in $C \times D$ (followed by a Stein factorization, if necessary). This leads to the following definition of "double" Kodaira fibration, see [5–7, 22, 24, 32, 40]:

Definition 1.1 A *double Kodaira surface* is a compact, complex surface S, endowed with a *double Kodaira fibration*, namely a surjective, holomorphic map $f\colon S \longrightarrow B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i\colon S \longrightarrow B_i, i = 1, 2$.

In the sequel, we will describe our approach to the construction of double Kodaira fibrations based on the techniques introduced in [9, 28], and present our results. The main step is to "detopologize" the problem, by transforming it into a purely algebraic one. This will be done in the particular case of *diagonal* double Kodaira fibrations, namely, Stein factorizations of finite Galois covers

$$\mathbf{f}\colon S \longrightarrow \Sigma_b \times \Sigma_b, \qquad (1)$$

branched with order $n \geq 2$ over the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$, where Σ_b is a closed Riemann surface of genus b. By Grauert-Remmert's extension theorem and Serre's GAGA, the existence of a G-cover $\mathbf{f}$ as in (1), up to cover isomorphisms, is equivalent to the existence of a group epimorphism

$$\varphi\colon \pi_1(\Sigma_b \times \Sigma_b - \Delta) \longrightarrow G, \qquad (2)$$

up to automorphisms of G. Furthermore, the condition that $\mathbf{f}$ is branched of order n over Δ is rephrased by asking that $\varphi(\gamma_\Delta)$ has order n in G, where γ_Δ is the homotopy class in

$\Sigma_b \times \Sigma_b - \Delta$ of a loop in $\Sigma_b \times \Sigma_b$ that "winds once" around Δ. The requirement $n \geq 2$ means that φ does not factor through $\pi_1(\Sigma_b \times \Sigma_b)$; it also implies that G is non-abelian, because γ_Δ is a non-trivial commutator in $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$. An epimorphism (or quotient) of type (2) such that $\varphi(\gamma_\Delta)$ is non-trivial will be called *admissible*.

Recall now that the group $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$ is isomorphic to $P_2(\Sigma_b)$, the pure braid group of genus b on two strands, which admits a finite geometric presentation with $4b + 1$ generators, see [14, Theorem 7]. Taking the images of these generators via an admissible group epimorphism, we get an ordered set

$$\mathfrak{S} = (r_{11}, t_{11}, \ldots, r_{1b}, t_{1b}, r_{21}, t_{21}, \ldots, r_{2b}, t_{2b}, z)$$

of $4b + 1$ generators of G, such that $o(z) = n$ and subject to a suitable finite set of relations. This will be called a *diagonal double Kodaira structure* of type (b, n) on G, see Definition 3.1. Thus, the geometric problem of constructing an admissible G-cover is translated into the combinatorial-algebraic problem of finding a diagonal double Kodaira structure of type (b, n) in G.

It turns out that the G-cover $\mathbf{f}$ is a diagonal double Kodaira fibration (namely, the two surjective maps $f_i \colon S \longrightarrow \Sigma_b$, obtained as composition with the natural projections, have connected fibres) if and only if the related structure $\mathfrak{S}$ is *strong*, an additional condition introduced in Definition 3.8; furthermore, the algebraic signature $\sigma(\mathfrak{S})$, introduced in Definition 3.7, equals the geometric signature $\sigma(S)$.

Note that not every double Kodaira fibration is of diagonal type. In fact, one proves that if S is of diagonal type then its slope satisfies $\nu(S) = 2 + s$, where s is rational and $0 < s < 6 - 4\sqrt{2}$, and that there exist examples whose high slope violates this inequality (for instance, Catanese-Rollenske's example with $\nu(S) = 2 + 2/3$); see [28, Section 4]. For more details on the construction of diagonal double Kodaira fibrations, we refer the reader to Sect. 5.

In the light of the previous considerations, classifying diagonal double Kodaira fibrations is equivalent to describing finite groups which admit a diagonal double Kodaira structure. Our key result in this direction is the following:

Main Theorem (see Propositions 4.9, 4.11 and Theorem 4.15) *Let G be a finite group admitting a diagonal double Kodaira structure. Then* $|G| \geq 32$*, with equality if and only if G is extra-special (see Sect. 2 for the definition). Moreover, the following holds.*

(1) *Both extra-special groups G of order* 32 *admit* $2211840 = 1152 \cdot 1920$ *diagonal double Kodaira structures of type* $(b, n) = (2, 2)$*. Every such a structure $\mathfrak{S}$ is strong and satisfies* $\sigma(\mathfrak{S}) = 16$*.*
(2) *If* $G = G(32, 49) = H_5(\mathbb{Z}_2)$*, these structures form* 1920 *orbits under the action of* Aut(G)*.*
(3) *If* $G = G(32, 50) = G_5(\mathbb{Z}_2)$*, these structures form* 1152 *orbits under the action of* Aut(G)*.*

Our Main Theorem should be compared with previous results, obtained by the first author in collaboration with A. Causin, regarding the construction of diagonal double Kodaira structures on some extra-special groups of order at least $2^7 = 128$, see [9, 28]. However, even if the definition of diagonal double Kodaira structure and the construction of the corresponding diagonal double Kodaira fibration presented in Sects. 3 and 5 closely follow the ones in [28], the examples constructed here are really new, in the sense that they cannot be obtained as images of structures on extra-special groups of larger order (Remark 4.17). It is precisely the

original part of this paper, namely the subtle group theoretical analysis developed in Sects. 2 and 4 and used in the proof of the Main Theorem, which allows us to pass from $|G| = 128$ to $|G| = 32$.

The interpretation of the Main Theorem in terms of admissible epimorphisms from surface braid groups to finite groups is given in Corollary 4.18. As a consequence, we can describe all diagonal double Kodaira fibrations associated with structures of type $(2, 2)$ on extra-special groups of order 32 (Theorem 5.5), showing that they provide the sharp lower bound $\sigma(S) \geq 16$ for the signature of a diagonal double Kodaira fibration (Corollary 5.6).

These results yield, as a by-product, new "double solutions" to a problem (stated by G. Mess) from Kirby's problem list in low-dimensional topology [20, Problem 2.18 A], asking what is the smallest number b for which there exists a real surface bundle over a real surface with base genus b and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations, as shown in [9, Proposition 3.19] and [28, Theorem 4.6] by using double Kodaira structures of type $(2, 3)$ on extra-special groups of order 3^5. Those fibrations had signature 144 and fibre genera 325; the new examples presented here substantially lower both these values, in fact they have signature 16 and fibre genera 41 (Theorem 5.7).

We believe that the results described above are significant for at least two reasons.

(i) Although we know that $\mathsf{P}_2(\Sigma_b)$ is residually p-finite for all prime number $p \geq 2$, see [3, pp. 1481-1490], so far there has been no systematic work aimed to describe its admissible finite quotients. The first results in this direction were those of A. Causin and the first author, who showed that both extra-special groups of order p^{4b+1} appear as admissible quotients of $\mathsf{P}_2(\Sigma_b)$ for all $b \geq 2$ and all prime numbers $p \geq 5$; moreover, if p divides $b + 1$, then both extra-special groups of order p^{2b+1} appear as admissible quotients, too. Our work sheds some new light on this problem, by providing a sharp lower bound for the order of an admissible quotient. Moreover, for both extra-special groups of order 32 (namely, the ones for which the bound is attained) we are able to compute the number of admissible epimorphisms $\varphi \colon \mathsf{P}_2(\Sigma_2) \longrightarrow G$, and the number of their equivalence classes up to the natural action of $\mathrm{Aut}(G)$.

(ii) Constructing (double) Kodaira fibrations with small signature is a rather difficult problem, and there are few examples in the literature ([5, 24]). As far as we know, the present paper provides the first positive-dimensional families of such examples, see Remark 5.9 for more details.

Let us now describe how this paper is organized. In Sect. 2 we introduce some algebraic preliminaries, in particular we discuss the so-called CCT-groups (Definition 2.1), namely, finite non-abelian groups in which commutativity is a transitive relation on the set of non-central elements. These groups are of historical importance in the context of classification of finite simple groups, see Remark 2.3, and they play a relevant role in this paper. It turns out that there are precisely eight groups G with $|G| \leq 32$ that are not CCT-groups, namely S_4 and seven groups of order 32, see Corollary 2.6, Proposition 2.7 and Proposition 2.14.

In Sect. 3 we define diagonal double Kodaira structures and we explain the relation with their counterpart in geometric topology, namely admissible group epimorphisms from pure surface braid groups to finite groups.

Section 4 is devoted to the study of diagonal double Kodaira structures in groups of order at most 32. One crucial technical result is Proposition 4.4, stating that there are no such structures on CCT-groups. Thus, in order to prove the first part of the Main Theorem, we only need to exclude the existence of diagonal double Kodaira structures on S_4 and on the five non-abelian, non-CCT groups of order 32; this is done in Proposition 4.9 and Proposition 4.11, respectively. The second part of the Main Theorem, i.e. the computation

of number of structures in each case, is obtained by using some techniques borrowed from [38]; more precisely, we exploit the fact that $V = G/Z(G)$ is a symplectic vector space of dimension 4 over $\mathbb{Z}_2$, and that $\mathrm{Out}(G)$ embeds in $\mathrm{Sp}(4, \mathbb{Z}_2)$ as the orthogonal group associated with the quadratic form $q\colon V \longrightarrow \mathbb{Z}_2$ related to the symplectic form $(\cdot, \cdot)$ by $q(\overline{x}\,\overline{y}) = q(\overline{x}) + q(\overline{y}) + (\overline{x}, \overline{y})$.

Finally, in Sect. 5 we establish the relation between our algebraic results and the existence of diagonal double Kodaira fibrations, and we prove the consequences of the Main Theorem in this geometrical framework.

The paper ends with an Appendix, where we collect the presentations for the non-abelian groups of order 24 and 32 that we used in our calculations.

Notation and conventions. If S is a complex, non-singular projective surface, then $c_1(S)$, $c_2(S)$ denote the first and second Chern class of its tangent bundle T_S, respectively.

Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$: cyclic group of order n.
- $G = N \rtimes Q$: semi-direct product of N and Q, namely, split extension of Q by N, where N is normal in G.
- $G = N.Q$: non-split extension of Q by N.
- $\mathrm{Aut}(G)$: the automorphism group of G.
- $\mathsf{D}_{p,q,r} = \mathbb{Z}_q \rtimes \mathbb{Z}_p = \langle x, y \mid x^p = y^q = 1, \ xyx^{-1} = y^r \rangle$: split metacyclic group of order pq. The group $\mathsf{D}_{2,n,-1}$ is the dihedral group of order $2n$ and will be denoted by D_{2n}.
- If n is an integer greater or equal to 4, we denote by QD_{2^n} the quasi-dihedral group of order 2^n, having presentation

$$\mathsf{QD}_{2^n} := \langle x, y \mid x^2 = y^{2^{n-1}} = 1, \ xyx^{-1} = y^{2^{n-2}-1} \rangle.$$

- The generalized quaternion group of order $4n$ is denoted by Q_{4n} and is presented as

$$\mathsf{Q}_{4n} = \langle x, y, z \mid x^n = y^2 = z^2 = xyz \rangle.$$

For $n = 2$ we obtain the usual quaternion group Q_8, for which we adopt the classical presentation

$$\mathsf{Q}_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle,$$

denoting by -1 the unique element of order 2.

- S_n, A_n: symmetric, alternating group on n symbols. We write the composition of permutations from the right to the left; for instance, $(13)(12) = (123)$.
- $\mathsf{GL}(n, \mathbb{F}_q)$, $\mathsf{SL}(n, \mathbb{F}_q)$, $\mathsf{Sp}(n, \mathbb{F}_q)$: general linear group, special linear group and symplectic group of $n \times n$ matrices over a field with q elements.
- The order of a finite group G is denoted by $|G|$. If $x \in G$, the order of x is denoted by $o(x)$ and its centralizer in G by $C_G(x)$.
- If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$.
- The commutator subgroup of G is denoted by $[G, G]$, the center of G by $Z(G)$.
- If $S = \{s_1, \ldots, s_n\} \subset G$, the subgroup generated by S is denoted by $\langle S \rangle = \langle s_1, \ldots, s_n \rangle$.
- IdSmallGroup(G) indicates the label of the group G in the GAP4 database of small groups. For instance IdSmallGroup(D_4) = $G(8, 3)$ means that D_4 is the third in the list of groups of order 8.
- If N is a normal subgroup of G and $g \in G$, we denote by $\bar{g}$ the image of g in the quotient group G/N.

2 Group-theoretical preliminaries: CCT-groups and extra-special groups

Definition 2.1 A finite non-abelian group G is said to be a *center commutative-transitive group* (or a CCT-*group*, for short) if commutativity is a transitive relation on the set on non-central elements of G. In other words, if x, y, $z \in G - Z(G)$ and $[x, y] = [y, z] = 1$, then $[x, z] = 1$.

Other characterizations of CCT-groups are provided in the statement below, whose proof is straightforward.

Proposition 2.2 *For a finite group G, the following properties are equivalent.*

(1) *G is a* CCT-*group.*
(2) *For every pair x, y of non-central elements in G, the relation $[x, y] = 1$ implies $C_G(x) = C_G(y)$.*
(3) *For every non-central element $x \in G$, the centralizer $C_G(x)$ is abelian.*

Remark 2.3 CCT-groups are of historical importance in the context of classification of finite simple groups, see for instance [37], where they are called CA-*groups*. Further references are [29, 31, 35, 39].

Lemma 2.4 *If G is a finite group such that $G/Z(G)$ is cyclic, then G is abelian.*

Proof This is a standard exercise, cf. [16, Problem 12 p. 77]. □

Proposition 2.5 *Let G be a finite non-abelian group.*

(1) *If $|G|$ is the product of at most three prime factors (non necessarily distinct), then G is a* CCT-*group.*
(2) *If $|G| = p^4$, with p prime, then G is a* CCT-*group.*
(3) *If G contains an abelian normal subgroup of prime index, then G is a* CCT-*group.*

Proof **(1)** Assume that $|G|$ is the product of at most three prime factors, and take a non-central element y. Then the centralizer $C_G(y)$ has non-trivial center, because $1 \neq y \in C_G(y)$, and its order is the product of at most two primes. Therefore the quotient of $C_G(y)$ by its center is cyclic, hence $C_G(y)$ is abelian by Lemma 2.4.

(2) Assume $|G| = p^4$ and suppose by contradiction that there exist three elements $x, y, z \in G - Z(G)$ such that $[x, y] = [y, z] = 1$ but $[x, z] \neq 1$. They generate a non-abelian subgroup $N = \langle x, y, z \rangle$, which is not the whole of G since $y \in Z(N)$ but $y \notin Z(G)$. It follows that N has order p^3 and so, by Lemma 2.4, its center is cyclic of order p, generated by y. The group G is a finite p-group, hence a nilpotent group; being a proper subgroup of maximal order in a nilpotent group, N is normal in G (see [26, Corollary 5.2]), so we have a conjugacy homomorphism $G \longrightarrow \mathrm{Aut}(N)$, that in turn induces a conjugacy homomorphism $G \longrightarrow \mathrm{Aut}(Z(N)) \simeq \mathbb{Z}_{p-1}$. The image of such a homomorphism must have order dividing both p^4 and $p - 1$, hence it is trivial. In other words, the conjugacy action of G on $Z(N) = \langle y \rangle$ is trivial, hence y is central in G, contradiction.

(3) Let N be an abelian normal subgroup of G such that G/N has prime order p. As G/N has no non-trivial proper subgroups, it follows that N is a maximal subgroup of G. Let x be any non-central element of G, so that $C_G(x)$ is a proper subgroup of G; then there are two possibilities.

Case 1: $x \in N$. Then $N \subseteq C_G(x)$ and so, by the maximality of N, we get $C_G(x) = N$, which is abelian.

Case 2: $x \notin N$. Then the image of x generates G/N, and so every element $y \in G$ can be written in the form $y = ux^r$, where $u \in N$ and $0 \le r \le p - 1$. In particular, if $y \in C_G(x)$, the condition $[x, y] = 1$ yields $[x, u] = 1$, namely $u \in N \cap C_G(x)$. Since N is abelian, it follows that $C_G(x)$ is abelian, too.

□

We now want to classify non-abelian, non-CCT groups of order at most 32. First of all, as an immediate consequence of parts **(1)** and **(2)** of Proposition 2.5, we have the following

Corollary 2.6 *Let G be a finite non-abelian group such that $|G| \le 32$. If G is not a CCT-group, then either $|G| = 24$ or $|G| = 32$.*

Let us start by considering the case $G = 24$.

Proposition 2.7 *Let G be a finite non-abelian group such that $|G| = 24$ and G is not a CCT-group. Then $G = S_4$.*

Proof We start by observing that S_4 is not a CCT-group. In fact, (1234) commutes to its square (13)(24), which commutes to (12)(34), but (1234) and (12)(34) do not commute.

We are left to show that the remaining non-abelian groups of order 24 are all CCT-groups; we will do a case-by-case analysis, referring the reader to the presentations given in Table 1 of Appendix A. Apart from $G = G(24, 3) = \mathsf{SL}(2, \mathbb{F}_3)$, for which we give an ad-hoc proof, we will show that all these groups contain an abelian subgroup N of prime index, so that we can conclude by using part **(3)** of Proposition 2.5.

- $G = G(24, 1)$. Take $N = \langle x^2 y \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 3)$. The action of $\mathrm{Aut}(G)$ has five orbits, whose representative elements are $\{1, x, x^2, z, z^2\}$, see [34]. We have $\langle z^2 \rangle = Z(G)$ and so, since $C_G(x) \subseteq C_G(x^2)$, it suffices to show that the centralizers of x^2 and z are both abelian. In fact, we have

$$C_G(x^2) = \langle x \rangle \simeq \mathbb{Z}_6, \quad C_G(z) = \langle z \rangle \simeq \mathbb{Z}_4.$$

- $G = G(24, 4)$. Take $N = \langle x \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 5)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 6)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 7)$. Take $N = \langle z, x^2 y \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$.
- $G = G(24, 8)$. Take $N = \langle y, z, w \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$.
- $G = G(24, 10)$. Take $N = \langle z, y \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 11)$. Take $N = \langle z, i \rangle \simeq \mathbb{Z}_{12}$.
- $G = G(24, 13)$. Take $N = \langle z \rangle \times \mathsf{V}_4 \simeq (\mathbb{Z}_2)^3$, where $\mathsf{V}_4 = \langle (1\,2)(3\,4), \ (1\,3)(2\,4) \rangle$ is the Klein subgroup.
- $G = G(24, 14)$. Take $N = \langle z, w \rangle \times \langle (123) \rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$.

This completes the proof.   □

The next step is to classify non-abelian, non-CCT groups G with $|G| = 32$; it will turn out that there are precisely seven of them, see Proposition 2.14. Before doing this, let us introduce the following classical definition, see for instance [15, p. 183] and [18, p. 123].

Definition 2.8 Let p be a prime number. A finite p-group G is called *extra-special* if its center $Z(G)$ is cyclic of order p and the quotient $V = G/Z(G)$ is a non-trivial, elementary abelian p-group.

An elementary abelian p-group is a finite-dimensional vector space over the field $\mathbb{Z}_p$, hence it is of the form $V = (\mathbb{Z}_p)^{\dim V}$ and G fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow G \longrightarrow V \longrightarrow 1. \tag{3}$$

Note that, V being abelian, we must have $[G, G] = \mathbb{Z}_p$, namely the commutator subgroup of G coincides with its center. Furthermore, since the extension (3) is central, it cannot be split, otherwise G would be isomorphic to the direct product of the two abelian groups $\mathbb{Z}_p$ and V, which is impossible because G is non-abelian.

If G is extra-special, then we can define a map $\omega \colon V \times V \longrightarrow \mathbb{Z}_p$ as follows: for every $v_1, v_2 \in V$, we set $\omega(v_1, v_2) = [g_1, g_2]$, where g_i is any lift of v_i in G. This turns out to be a symplectic form on V, hence $\dim V$ is even and $|G| = p^{\dim V + 1}$ is an odd power of p.

For every prime number p, there are precisely two isomorphism classes $M(p), N(p)$ of non-abelian groups of order p^3, namely

$$M(p) = \langle \mathsf{r}, \mathsf{t}, \mathsf{z} \mid \mathsf{r}^p = \mathsf{t}^p = 1, \; \mathsf{z}^p = 1, [\mathsf{r}, \mathsf{z}] = [\mathsf{t}, \mathsf{z}] = 1, [\mathsf{r}, \mathsf{t}] = \mathsf{z}^{-1} \rangle$$

$$N(p) = \langle \mathsf{r}, \mathsf{t}, \mathsf{z} \mid \mathsf{r}^p = \mathsf{t}^p = \mathsf{z}, \; \mathsf{z}^p = 1, [\mathsf{r}, \mathsf{z}] = [\mathsf{t}, \mathsf{z}] = 1, [\mathsf{r}, \mathsf{t}] = \mathsf{z}^{-1} \rangle$$

and both of them are in fact extra-special, see [15, Theorem 5.1 of Chapter 5].

If p is odd, then the groups $M(p)$ and $N(p)$ are distinguished by their exponent, which equals p and p^2, respectively. If $p = 2$, the group $M(p)$ is isomorphic to the dihedral group D_8, whereas $N(p)$ is isomorphic to the quaternion group Q_8.

The classification of extra-special p-groups is now provided by the result below, see [15, Section 5 of Chapter 5] and [9, Section 2].

Proposition 2.9 *If $b \geq 2$ is a positive integer and p is a prime number, there are exactly two isomorphism classes of extra-special p-groups of order p^{2b+1}, that can be described as follows.*

- *The central product $H_{2b+1}(\mathbb{Z}_p)$ of b copies of $M(p)$, having presentation*

$$\begin{aligned} H_{2b+1}(\mathbb{Z}_p) = \langle r_1, t_1, \ldots, r_b, t_b, z \mid & r_j^p = t_j^p = z^p = 1, \\ & [r_j, z] = [t_j, z] = 1, \\ & [r_j, r_k] = [t_j, t_k] = 1, \\ & [r_j, t_k] = z^{-\delta_{jk}} \rangle. \end{aligned} \tag{4}$$

  If p is odd, this group has exponent p and is isomorphic to the matrix Heisenberg group $\mathcal{H}_{2b+1}(\mathbb{Z}_p) \subset GL(b+2, \mathbb{Z}_p)$ of dimension $2b + 1$ over the field $\mathbb{Z}_p$.

- *The central product $G_{2b+1}(\mathbb{Z}_p)$ of $b - 1$ copies of $M(p)$ and one copy of $N(p)$, having presentation*

$$\begin{aligned} G_{2b+1}(\mathbb{Z}_p) = \langle r_1, t_1, \ldots, r_b, t_b, z \mid & r_b^p = t_b^p = z, \\ & r_1^p = t_1^p = \ldots = r_{b-1}^p = t_{b-1}^p = z^p = 1, \\ & [r_j, z] = [t_j, z] = 1, \\ & [r_j, r_k] = [t_j, t_k] = 1, \\ & [r_j, t_k] = z^{-\delta_{jk}} \rangle. \end{aligned} \tag{5}$$

  If p is odd, this group has exponent p^2.

Remark 2.10 In both cases, from the relations above we deduce

$$[\mathsf{r}_j^{-1}, \mathsf{t}_k] = \mathsf{z}^{\delta_{jk}}, \quad [\mathsf{r}_j^{-1}, \mathsf{t}_k^{-1}] = \mathsf{z}^{-\delta_{jk}}$$

Remark 2.11 For both groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$, the center coincides with the derived subgroup and is equal to $\langle z \rangle \simeq \mathbb{Z}_p$. Note that, being these groups non-abelian, this condition implies that their nilpotency class is 2, see [18, p. 22].

Remark 2.12 If $p = 2$, we can distinguish the two groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$ by counting the number of elements of order 4.

Remark 2.13 The groups $H_{2b+1}(\mathbb{Z}_p)$ and $G_{2b+1}(\mathbb{Z}_p)$ are not CCT-groups. In fact, let us take two distinct indices $j, k \in \{1, \ldots, b\}$ and consider the non-central elements r_j, t_j, t_k. Then we have $[r_j, t_k] = [t_k, t_j] = 1$, but $[r_j, t_j] = z^{-1}$.

We can now analyze the case $|G| = 32$.

Proposition 2.14 *Let G be a finite non-abelian group such that $|G| = 32$ and G is not a CCT-group. Then $G = G(32, t)$, where $t \in \{6, 7, 8, 43, 44, 49, 50\}$. Here $G(32, 49) = H_5(\mathbb{Z}_2)$ and $G(32, 50) = G_5(\mathbb{Z}_2)$ are the two extra-special groups of order 32.*

Proof We first do a case-by case analysis showing that, if $t \notin \{6, 7, 8, 43, 44, 49, 50\}$, then $G = G(32, t)$ contains an abelian subgroup N of index 2, so that G is a CCT-group by part **(3)** of Proposition 2.5. In every case, we refer the reader to the presentation given in Table 2 of Appendix A.

- $G = G(32, 2)$. Take $N = \langle x, y^2, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 4)$. Take $N = \langle x, y^2 \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 5)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 9)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 10)$. Take $N = \langle ix, k \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 11)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 12)$. Take $N = \langle x^2, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 13)$. Take $N = \langle x^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 14)$. Take $N = \langle x^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 15)$. Take $N = \langle x^2, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 17)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$.
- $G = G(32, 18)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$.
- $G = G(32, 19)$. Take $N = \langle y \rangle \simeq \mathbb{Z}_{16}$.
- $G = G(32, 20)$. Take $N = \langle x \rangle \simeq \mathbb{Z}_{16}$.
- $G = G(32, 22)$. Take $N = \langle w \rangle \times \langle x, y \rangle \simeq \mathbb{Z}_8 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 23)$. Take $N = \langle z \rangle \times \langle x, y^2 \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 24)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 25)$. Take $N = \langle z \rangle \times \langle y^2 \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 26)$. Take $N = \langle z \rangle \times \langle i \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 27)$. Take $N = \langle x, y, a, b \rangle \simeq (\mathbb{Z}_2)^4$.
- $G = G(32, 28)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 29)$. Take $N = \langle x, i, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 30)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 31)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 32)$. Take $N = \langle y, z \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 33)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 34)$. Take $N = \langle x, y \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 35)$. Take $N = \langle x, k \rangle \simeq (\mathbb{Z}_4)^2$.
- $G = G(32, 37)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.

- $G = G(32, 38)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 39)$. Take $N = \langle z \rangle \times \langle y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 40)$. Take $N = \langle z \rangle \times \langle y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 41)$. Take $N = \langle w \rangle \times \langle x \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 42)$. Take $N = \langle x, y \rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$.
- $G = G(32, 46)$. Take $N = \langle z, w \rangle \times \langle y \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 47)$. Take $N = \langle z, w \rangle \times \langle i \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.
- $G = G(32, 48)$. Take $N = \langle x, y, z \rangle \simeq \mathbb{Z}_4 \times (\mathbb{Z}_2)^2$.

It remains to show that $G = G(32, t)$ is not a CCT-group for $t \in \{6, 7, 8, 43, 44, 49, 50\}$.

For $t = 49$ and $t = 50$ we have the two extra-special cases, that are not CCT-groups by Remark 2.13. Let us now deal with the remaining values of t: for each of them, we will exhibit three non-central elements for which commutativity is not a transitive relation, and this will complete the proof.

- $G = G(32, 6)$. The center of G is $Z(G) = \langle x \rangle \simeq \mathbb{Z}_2$. We have $[y, w^2] = [w^2, w] = 1$, but $[y, w] = x$.
- $G = G(32, 7)$. The center of G is $Z(G) = \langle w \rangle \simeq \mathbb{Z}_2$. We have $[y, z] = [z, u] = 1$, but $[y, u] = w$.
- $G = G(32, 8)$. The center of G is $Z(G) = \langle x^4 \rangle \simeq \mathbb{Z}_2$. We have $[x, x^2] = [x^2, y] = 1$, but $[x, y] = z^2$.
- $G = G(32, 43)$. The center of G is $Z(G) = \langle x^4 \rangle \simeq \mathbb{Z}_2$. We have $[x, x^2] = [x^2, z] = 1$, but $[x, z] = x^4$.
- $G = G(32, 44)$. The center of G is $Z(G) = \langle i^2 \rangle \simeq \mathbb{Z}_2$. We have $[x, xk] = [xk, z] = 1$, but $[x, z] = i^2$.

$\square$

3 Diagonal double Kodaira structures

For more details on the material contained in this section, we refer the reader to [9] and [28]. Let G be a finite group and let $b, n \geq 2$ be two positive integers.

Definition 3.1 A *diagonal double Kodaira structure* of type (b, n) on G is an ordered set of $4b + 1$ generators

$$\mathfrak{S} = (\mathsf{r}_{11}, \mathsf{t}_{11}, \ldots, \mathsf{r}_{1b}, \mathsf{t}_{1b}, \mathsf{r}_{21}, \mathsf{t}_{21}, \ldots, \mathsf{r}_{2b}, \mathsf{t}_{2b}, \mathsf{z}),$$

with $o(\mathsf{z}) = n$, such that the following relations are satisfied. We systematically use the commutator notation in order to indicate relations of conjugacy type, writing for instance $[x, y] = zy^{-1}$ instead of $xyx^{-1} = z$.

- Surface relations

$$[\mathsf{r}_{1b}^{-1}, \mathsf{t}_{1b}^{-1}]\,\mathsf{t}_{1b}^{-1}\,[\mathsf{r}_{1\,b-1}^{-1}, \mathsf{t}_{1\,b-1}^{-1}]\,\mathsf{t}_{1\,b-1}^{-1} \cdots [\mathsf{r}_{11}^{-1}, \mathsf{t}_{11}^{-1}]\,\mathsf{t}_{11}^{-1}\,(\mathsf{t}_{11}\,\mathsf{t}_{12}\cdots\mathsf{t}_{1b}) = \mathsf{z}$$
$$[\mathsf{r}_{21}^{-1}, \mathsf{t}_{21}]\,\mathsf{t}_{21}\,[\mathsf{r}_{22}^{-1}, \mathsf{t}_{22}]\,\mathsf{t}_{22} \cdots [\mathsf{r}_{2b}^{-1}, \mathsf{t}_{2b}]\,\mathsf{t}_{2b}\,(\mathsf{t}_{2b}^{-1}\,\mathsf{t}_{2\,b-1}^{-1}\cdots\mathsf{t}_{21}^{-1}) = \mathsf{z}^{-1}$$

- Conjugacy action of r_{1j}

$$[\mathsf{r}_{1j}, \mathsf{r}_{2k}] = 1 \qquad\qquad\qquad\qquad \text{if } j < k$$
$$[\mathsf{r}_{1j}, \mathsf{r}_{2j}] = 1$$
$$[\mathsf{r}_{1j}, \mathsf{r}_{2k}] = \mathsf{z}^{-1}\,\mathsf{r}_{2k}\,\mathsf{r}_{2j}^{-1}\,\mathsf{z}\,\mathsf{r}_{2j}\,\mathsf{r}_{2k}^{-1} \qquad \text{if } j > k$$

$$[r_{1j}, t_{2k}] = 1 \qquad\qquad \text{if } j < k$$
$$[r_{1j}, t_{2j}] = z^{-1}$$
$$[r_{1j}, t_{2k}] = [z^{-1}, t_{2k}] \qquad\qquad \text{if } j > k$$

$$[r_{1j}, z] = [r_{2j}^{-1}, z] \qquad\qquad\qquad (6)$$

- Conjugacy action of t_{1j}

$$[t_{1j}, r_{2k}] = 1 \qquad\qquad \text{if } j < k$$
$$[t_{1j}, r_{2j}] = t_{2j}^{-1} z\, t_{2j}$$
$$[t_{1j}, r_{2k}] = [t_{2j}^{-1}, z] \qquad\qquad \text{if } j > k$$

$$[t_{1j}, t_{2k}] = 1 \qquad\qquad \text{if } j < k$$
$$[t_{1j}, t_{2j}] = [t_{2j}^{-1}, z]$$
$$[t_{1j}, t_{2k}] = t_{2j}^{-1} z\, t_{2j}\, z^{-1} t_{2k}\, z\, t_{2j}^{-1} z^{-1} t_{2j}\, t_{2k}^{-1} \qquad \text{if } j > k$$

$$[t_{1j}, z] = [t_{2j}^{-1}, z] \qquad\qquad\qquad (7)$$

Remark 3.2 From (6) and (7) we can infer the corresponding conjugacy actions of r_{1j}^{-1} and t_{1j}^{-1}. We leave the cumbersome but standard computations to the reader.

Remark 3.3 Abelian groups admit no diagonal double Kodaira structures. Indeed, the relation $[r_{1j}, t_{2j}] = z^{-1}$ in (6) provides a non-trivial commutator in G, because $o(z) = n$.

Remark 3.4 Assume that the nilpotency class of G equals 2; since G is non-abelian, this is equivalent to $[G, G] \subseteq Z(G)$. Then the relations defining a diagonal double Kodaira structure of type (b, n) assume the following simplified form.

- Relations expressing the centrality of z

$$[r_{1j}, z] = [t_{1j}, z] = [r_{2j}, z] = [t_{2j}, z] = 1$$

- Surface relations

$$[r_{1b}^{-1}, t_{1b}^{-1}][r_{1b-1}^{-1}, t_{1b-1}^{-1}] \cdots [r_{11}^{-1}, t_{11}^{-1}] = z$$
$$[r_{21}^{-1}, t_{21}][r_{22}^{-1}, t_{22}] \cdots [r_{2b}^{-1}, t_{2b}] = z^{-1}$$

- Conjugacy action of r_{1j}

$$[r_{1j}, r_{2k}] = 1 \qquad\qquad \text{for all } j, k$$
$$[r_{1j}, t_{2k}] = z^{-\delta_{jk}}$$

- Conjugacy action of t_{1j}

$$[t_{1j}, r_{2k}] = z^{\delta_{jk}}$$
$$[t_{1j}, t_{2k}] = 1 \qquad\qquad \text{for all } j, k$$

where δ_{jk} stands for the Kronecker symbol.

The definition of diagonal double Kodaira structure can be motivated by means of some well-known concepts in geometric topology. Let Σ_b be a closed Riemann surface of genus b and let $\mathscr{P} = (p_1, \ p_2)$ be an ordered set of two distinct points on it. Let $\Delta \subset \Sigma_b \times \Sigma_b$ be the diagonal. We denote by $\mathsf{P}_2(\Sigma_b)$ the *pure braid group* of genus b on two strands, which is isomorphic to the fundamental group $\pi_1(\Sigma_b \times \Sigma_b - \Delta, \ \mathscr{P})$. By Gonçalves-Guaschi's presentation of surface pure braid groups, see [14, Theorem 7], [9, Theorem 1.7], we see that $\mathsf{P}_2(\Sigma_b)$ can be generated by $4b + 1$ elements

$$\rho_{11}, \ \tau_{11}, \ldots, \rho_{1b}, \ \tau_{1b}, \rho_{21}, \tau_{21}, \cdots, \rho_{2b}, \tau_{2b}, \ A_{12}$$

subject to the following set of relations.

- Surface relations

$$[\rho_{1b}^{-1}, \ \tau_{1b}^{-1}] \, \tau_{1b}^{-1} \, [\rho_{1b-1}^{-1}, \ \tau_{1b-1}^{-1}] \, \tau_{1b-1}^{-1} \cdots [\rho_{11}^{-1}, \ \tau_{11}^{-1}] \, \tau_{11}^{-1} \, (\tau_{11} \, \tau_{12} \cdots \tau_{1b}) = A_{12}$$

$$[\rho_{21}^{-1}, \ \tau_{21}] \, \tau_{21} \, [\rho_{22}^{-1}, \ \tau_{22}] \, \tau_{22} \cdots [\rho_{2b}^{-1}, \ \tau_{2b}] \, \tau_{2b} \, (\tau_{2b}^{-1} \, \tau_{2b-1}^{-1} \cdots \tau_{21}^{-1}) = A_{12}^{-1}$$

- Conjugacy action of ρ_{1j}

$$[\rho_{1j}, \ \rho_{2k}] = 1 \qquad\qquad\qquad\qquad \text{if } \ j < k$$

$$[\rho_{1j}, \ \rho_{2j}] = 1$$

$$[\rho_{1j}, \ \rho_{2k}] = A_{12}^{-1} \, \rho_{2k} \, \rho_{2j}^{-1} \, A_{12} \, \rho_{2j} \, \rho_{2k}^{-1} \qquad \text{if } \ j > k$$

$$[\rho_{1j}, \ \tau_{2k}] = 1 \qquad\qquad\qquad\qquad \text{if } \ j < k$$

$$[\rho_{1j}, \ \tau_{2j}] = A_{12}^{-1}$$

$$[\rho_{1j}, \ \tau_{2k}] = [A_{12}^{-1}, \ \tau_{2k}] \qquad\qquad\qquad \text{if } \ j > k$$

$$[\rho_{1j}, \ A_{12}] = [\rho_{2j}^{-1}, \ A_{12}]$$

- Conjugacy action of τ_{1j}

$$[\tau_{1j}, \ \rho_{2k}] = 1 \qquad\qquad\qquad\qquad \text{if } \ j < k$$

$$[\tau_{1j}, \ \rho_{2j}] = \tau_{2j}^{-1} \, A_{12} \, \tau_{2j}$$

$$[\tau_{1j}, \ \rho_{2k}] = [\tau_{2j}^{-1}, \ A_{12}] \qquad\qquad\qquad \text{if } \ j > k$$

$$[\tau_{1j}, \ \tau_{2k}] = 1 \qquad\qquad\qquad\qquad \text{if } \ j < k$$

$$[\tau_{1j}, \ \tau_{2j}] = [\tau_{2j}^{-1}, \ A_{12}]$$

$$[\tau_{1j}, \ \tau_{2k}] = \tau_{2j}^{-1} \, A_{12} \, \tau_{2j} \, A_{12}^{-1} \, \tau_{2k} \, A_{12} \, \tau_{2j}^{-1} \, A_{12}^{-1} \, \tau_{2j} \, \tau_{2k}^{-1} \qquad \text{if } \ j > k$$

$$[\tau_{1j}, \ A_{12}] = [\tau_{2j}^{-1}, \ A_{12}]$$

Here the elements ρ_{ij} and τ_{ij} are the braids depicted in Fig. 1, whereas A_{12} is the braid depicted in Fig. 2.
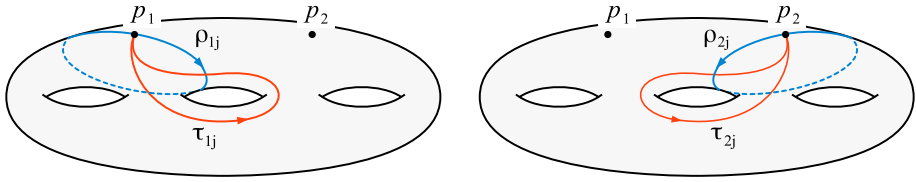
Fig. 1 The pure braids ρ_{1j} and ρ_{2j} on Σ_b. If $\ell \neq i$, the path corresponding to ρ_{ij} and τ_{ij} based at p_ℓ is the constant path

Fig. 2 The pure braid A_{12} on Σ_b

Remark 3.5 Under the identification of $P_2(\Sigma_b)$ with $\pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathscr{P})$, the generator $A_{12} \in P(\Sigma_b)$ represents the homotopy class $\gamma_\Delta \in \pi_1(\Sigma_b \times \Sigma_b - \Delta, \mathscr{P})$ of a loop in $\Sigma_b \times \Sigma_b$ that "winds once" around the diagonal Δ.

We can now state the following

Proposition 3.6 *A finite group G admits a diagonal double Kodaira structure of type (b, n) if and only if there is a surjective group homomorphism*

$$\varphi \colon P_2(\Sigma_b) \longrightarrow G \tag{8}$$

such that $\varphi(A_{12})$ has order n.

Proof If such a $\varphi \colon P_2(\Sigma_b) \longrightarrow G$ exists, we can obtain a diagonal double Kodaira structure on G by setting

$$\mathsf{r}_{ij} = \varphi(\rho_{ij}), \quad \mathsf{t}_{ij} = \varphi(\tau_{ij}), \quad \mathsf{z} = \varphi(A_{12}). \tag{9}$$

Conversely, if G admits a diagonal double Kodaira structure, then (9) defines a group homomorphism $\varphi \colon P_2(\Sigma_b) \longrightarrow G$ with the desired properties. □

The braid group $P_2(\Sigma_b)$ is the middle term of two split short exact sequences

$$1 \longrightarrow \pi_1(\Sigma_b - \{p_i\}, \; p_j) \longrightarrow P_2(\Sigma_b) \longrightarrow \pi_1(\Sigma_b, \; p_i) \longrightarrow 1, \tag{10}$$

where $\{i, \; j\} = \{1, \; 2\}$, induced by the two natural projections of pointed topological spaces

$$(\Sigma_b \times \Sigma_b - \Delta, \; \mathscr{P}) \longrightarrow (\Sigma_b, \; p_i),$$

see [14, Theorem 1]. Since we have

$$\pi_1(\Sigma_b - \{p_2\}, \; p_1) = \langle \rho_{11}, \tau_{11}, \ldots, \rho_{1b}, \tau_{1b}, A_{12} \rangle$$
$$\pi_1(\Sigma_b - \{p_1\}, \; p_2) = \langle \rho_{21}, \tau_{21}, \ldots, \rho_{2b}, \tau_{2b}, A_{12} \rangle,$$

it follows that the two subgroups

$$K_1 := \langle \mathsf{r}_{11}, \mathsf{t}_{11}, \ldots, \mathsf{r}_{1b}, \mathsf{t}_{1b}, \mathsf{z} \rangle$$
$$K_2 := \langle \mathsf{r}_{21}, \mathsf{t}_{21}, \ldots, \mathsf{r}_{2b}, \mathsf{t}_{2b}, \mathsf{z} \rangle$$

are both normal in G, and that there are two short exact sequences

$$\begin{aligned} 1 \longrightarrow K_1 \longrightarrow G \longrightarrow Q_2 \longrightarrow 1 \\ 1 \longrightarrow K_2 \longrightarrow G \longrightarrow Q_1 \longrightarrow 1, \end{aligned} \tag{11}$$

such the elements r_{21}, t_{21}, ..., r_{2b}, t_{2b} yield a complete system of coset representatives for Q_2, whereas the elements r_{11}, t_{11}, ..., r_{1b}, t_{1b} yield a complete system of coset representatives for Q_1.

Let us now give a couple of definitions, whose geometrical meaning will become clear in Sect. 5, see in particular Proposition 5.2 and Remark 5.3.

Definition 3.7 Let $\mathfrak{S}$ be a diagonal double Kodaira structure of type (b, n) on a finite group G. Its *signature* is defined as

$$\sigma(\mathfrak{S}) = \frac{1}{3}|G|(2b-2)\left(1 - \frac{1}{n^2}\right).$$

Definition 3.8 A diagonal double Kodaira structure on G is called *strong* if $K_1 = K_2 = G$.

For later use, let us write down the special case consisting of a diagonal double Kodaira structure of type $(2, n)$. It is an ordered set of nine generators of G

$$(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z),$$

with $o(z) = n$, subject to the following relations.

(S1) $[r_{12}^{-1}, t_{12}^{-1}]t_{12}^{-1}[r_{11}^{-1}, t_{11}^{-1}]t_{11}^{-1}(t_{11}t_{12}) = z$

(S2) $[r_{21}^{-1}, t_{21}]t_{21}[r_{22}^{-1}, t_{22}]t_{22}(t_{22}^{-1}t_{21}^{-1}) = z^{-1}$

(R1) $[r_{11}, r_{22}] = 1$ **(R6)** $[r_{12}, r_{22}] = 1$

(R2) $[r_{11}, r_{21}] = 1$ **(R7)** $[r_{12}, r_{21}] = z^{-1}\,r_{21}\,r_{22}^{-1}\,z\,r_{22}\,r_{21}^{-1}$

(R3) $[r_{11}, t_{22}] = 1$ **(R8)** $[r_{12}, t_{22}] = z^{-1}$

(R4) $[r_{11}, t_{21}] = z^{-1}$ **(R9)** $[r_{12}, t_{21}] = [z^{-1}, t_{21}]$

(R5) $[r_{11}, z] = [r_{21}^{-1}, z]$ **(R10)** $[r_{12}, z] = [r_{22}^{-1}, z]$

(T1) $[t_{11}, r_{22}] = 1$ **(T6)** $[t_{12}, r_{22}] = t_{22}^{-1}\,z\,t_{22}$

(T2) $[t_{11}, r_{21}] = t_{21}^{-1}\,z\,t_{21}$ **(T7)** $[t_{12}, r_{21}] = [t_{22}^{-1}, z]$

(T3) $[t_{11}, t_{22}] = 1$ **(T8)** $[t_{12}, t_{22}] = [t_{22}^{-1}, z]$

(T4) $[t_{11}, t_{21}] = [t_{21}^{-1}, z]$ **(T9)** $[t_{12}, t_{21}] = t_{22}^{-1}\,z\,t_{22}\,z^{-1}\,t_{21}\,z\,t_{22}^{-1}\,z^{-1}\,t_{22}\,t_{21}^{-1}$

(T5) $[t_{11}, z] = [t_{21}^{-1}, z]$ **(T10)** $[t_{12}, z] = [t_{22}^{-1}, z]$

$$(12)$$

Remark 3.9 When $[G, G] \subseteq Z(G)$, we have

$$[r_{11}, z] = [t_{11}, z] = [r_{12}, z] = [t_{12}, z] = 1$$
$$[r_{21}, z] = [t_{21}, z] = [r_{22}, z] = [t_{22}, z] = 1$$

and the previous relations become

$$(\textbf{S1}') \ [r_{12}^{-1}, t_{12}^{-1}] [r_{11}^{-1}, t_{11}^{-1}] = z$$
$$(\textbf{S2}') \ [r_{21}^{-1}, t_{21}] [r_{22}^{-1}, t_{22}] = z^{-1}$$

$$
\begin{array}{ll}
(\textbf{R1}') \ [r_{11}, r_{22}] = 1 \qquad & (\textbf{R6}') \ [r_{12}, r_{22}] = 1 \\
(\textbf{R2}') \ [r_{11}, r_{21}] = 1 & (\textbf{R7}') \ [r_{12}, r_{21}] = 1 \\
(\textbf{R3}') \ [r_{11}, t_{22}] = 1 & (\textbf{R8}') \ [r_{12}, t_{22}] = z^{-1} \\
(\textbf{R4}') \ [r_{11}, t_{21}] = z^{-1} & (\textbf{R9}') \ [r_{12}, t_{21}] = 1 \\
\\
(\textbf{T1}') \ [t_{11}, r_{22}] = 1 & (\textbf{T6}') \ [t_{12}, r_{22}] = z \\
(\textbf{T2}') \ [t_{11}, r_{21}] = z & (\textbf{T7}') \ [t_{12}, r_{21}] = 1 \\
(\textbf{T3}') \ [t_{11}, t_{22}] = 1 & (\textbf{T8}') \ [t_{12}, t_{22}] = 1 \\
(\textbf{T4}') \ [t_{11}, t_{21}] = 1 & (\textbf{T9}') \ [t_{12}, t_{21}] = 1
\end{array}
\tag{13}
$$

4 Structures on groups of order at most 32

4.1 Prestructures

Definition 4.1 Let G be a finite group. A *prestructure* on G is an ordered set of nine elements

$$(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z),$$

with $o(z) = n \geq 2$, subject to the relations (R1), ..., (R10), (T1), ..., (T10) in (12).

In other words, the nine elements must satisfy all the relations defining a diagonal double Kodaira structure of type $(2, n)$, except the surface relations. In particular, no abelian group admits prestructures. Note that we are *not* requiring that the elements of the prestructure generate G.

Proposition 4.2 *If a finite group G admits a diagonal double Kodaira structure of type (b, n), then it admits a prestructure with $o(z) = n$.*

Proof Consider the ordered set of nine elements $(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)$ in Definition 3.1 and the relations satisfied by them, with the exception of the surface relations. □

Remark 4.3 Let G be a finite group that admits a prestructure. Then z and all its conjugates are non-trivial elements of G and so, from relations (R4), (R8), (T2), (T6), it follows that $r_{11}, r_{12}, r_{21}, r_{22}$ and $t_{12}, t_{12}, t_{21}, t_{22}$ are non-central elements of G.

Proposition 4.4 *If G is a CCT-group, then G admits no prestructures and, subsequently, no diagonal double Kodaira structures.*

Proof The second statement is a direct consequence of the first one (see Proposition 4.2), hence it suffices then to check that G admits no prestructures. Otherwise, keeping in mind Remark 4.3, we see that (R6) and (T1) imply $[r_{12}, t_{11}] = 1$. From this and (T3) we get $[r_{12}, t_{22}] = 1$, that contradicts (R8). □

Given a finite group G, we define the *socle* of G, denoted by $\mathrm{soc}(G)$, as the intersection of all non-trivial, normal subgroups of G. For instance, G is simple if and only if $\mathrm{soc}(G) = G$.

Definition 4.5 A finite group G is called *monolithic* if $\mathrm{soc}(G) \neq \{1\}$. Equivalently, G is monolithic if it contains precisely one minimal non-trivial, normal subgroup.

Example 4.6 If G is an extra-special p-group, then G is monolithic and $\mathrm{soc}(G) = Z(G)$. Indeed, since $Z(G) \simeq \mathbb{Z}_p$ is normal in G, by definition of socle we always have $\mathrm{soc}(G) \subseteq Z(G)$. On the other hand, every non-trivial, normal subgroup of an extra-special group contains the center (see [30, Exercise 9 p. 146]), hence $Z(G) \subseteq \mathrm{soc}(G)$.

Proposition 4.7 *The following holds.*

(1) *Assume that G admits a prestructure, whereas no proper quotient of G does. Then G is monolithic and $z \in \mathrm{soc}(G)$.*
(2) *Assume that G admits a prestructure, whereas no proper subgroup of G does. Then the elements of the prestructure generate G.*

Proof **(1)** Let $\mathfrak{S} = (r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)$ be a prestructure in G. Assume that there is a non-trivial normal subgroup N of G such that $z \notin N$. Then $\bar{z} \in G/N$ is non-trivial, and so $\tilde{\mathfrak{S}} = (\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}, \bar{r}_{22}, \bar{t}_{22}, \bar{z})$ is a prestructure in the quotient group G/N, contradiction. Therefore we must have $z \in \mathrm{soc}(G)$, in particular, G is monolithic.

(2) Clear, because every prestructure $\mathfrak{S}$ in G is also a prestructure in the subgroup $\langle \mathfrak{S} \rangle$. □

Corollary 4.8 *Given a prestructure on an extra-special p-group G, the element z is a generator of $Z(G) \simeq \mathbb{Z}_p$.*

Proof If G is extra-special, every proper quotient of G is abelian, hence it admits no prestructures. The result now follows from Example 4.6 and Proposition 4.7 (1). □

Note that, by Corollary 4.8, in the case of extra-special p-groups the choice of calling z the element in the prestructure is coherent with presentations (4) and (5). The case of diagonal double Kodaira structures on extra-special groups of order 32 will be studied in Subsection 4.4.

4.2 The case $|G| < 32$

Proposition 4.9 *If $|G| < 32$, then G admits no diagonal double Kodaira structures.*

Proof By Corollary 2.6, Proposition 2.7 and Proposition 4.4, it remains only to check that the symmetric group S_4 admits no prestructures. We start by observing that

$$\mathrm{soc}(S_4) = V_4 = \langle (1\,2)(3\,4), \ (1\,3)(2\,4) \rangle$$

and so, by part (1) of Proposition 4.7, if $\mathfrak{S}$ is a prestructure on S_4 then $z \in V_4$. Let $x, y \in S_4$ be such that $[x, y] = z$. Examining the tables of subgroups of S_4 given in [33], by straightforward computations we deduce that either $x, y \in C_{S_4}(z) \simeq D_8$ or $x, y \in A_4$. Every pair in A_4 includes at least a 3-cycle and so, if $[x, y] = z$ and both x and y have even order, then x and y centralize z.

If $x \in S_4$ is a 3-cycle, then $C_{S_4}(x) = \langle x \rangle \simeq \mathbb{Z}_3$. So, from relations (R1), (R2), (R3), (R6), it follows that, if one of the elements $r_{11}, r_{12}, r_{21}, r_{22}, t_{22}$ is a 3-cycle, then all these elements

generate the same cyclic subgroup. This contradicts (R8), hence r_{11}, r_{12}, r_{21}, r_{22}, t_{22} all have even order.

Let us look now at relation (R8). Since r_{12}, t_{22} have even order, from the previous remark we infer r_{12}, $t_{22} \in C_{S_4}(z)$. Let us consider r_{11}. If r_{11} belongs to A_4, being an element of even order it must be conjugate to z, and so it commutes with z; otherwise, by (R4), both r_{11} and t_{21} commute with z. Summing up, in any case we have $r_{11} \in C_{S_4}(z)$.

Relation (R5) can be rewritten as $r_{11}r_{21} \in C_{S_4}(z)$, hence $r_{21} \in C_{S_4}(z)$. Analogously, relation (R10) can be rewritten as $r_{12}r_{22} \in C_{S_4}(z)$, hence $r_{22} \in C_{S_4}(z)$.

Using relation (R9), we get $r_{12}z \in C_{S_4}(t_{21})$. Since r_{12} and z commute and their orders are powers of 2, it follows that $o(r_{12}z)$ is also a power of 2. Therefore t_{21} cannot be a 3-cycle, otherwise $C_{S_4}(t_{21}) \simeq \mathbb{Z}_3$ and so $r_{12}z = 1$ that, in turn, would imply $[r_{12}, t_{22}] = 1$, contradicting (R8). It follows that t_{21} has even order and so, since r_{11} has even order as well, by (R4) we infer $t_{21} \in C_{S_4}(z)$.

Now we can rewrite (T2) as $[t_{11}, r_{21}] = z$. If t_{11} were a 3-cycle, from (T1) we would get $r_{22} \in C_{S_4}(t_{11}) \simeq \mathbb{Z}_3$, a contradiction since r_{22} has even order. Thus t_{11} has even order and so it belongs to $C_{S_4}(z)$, because r_{21} has even order, too. Analogously, by using (T6) and (T7), we infer $t_{12} \in C_{S_4}(z)$.

Summarizing, if $\mathfrak{S}$ were a prestructure on S_4 we should have

$$\langle \mathfrak{S} \rangle = C_{S_4}(z) \simeq D_8,$$

contradicting part (2) of Proposition 4.7.   □

4.3 The case $|G| = 32$ and G non-extra-special

We start by proving the following partial strengthening of Proposition 4.4.

Proposition 4.10 *Let G be a finite non-abelian group, and let H be the subgroup of G generated by those elements whose centralizer is non-abelian. If H is abelian and $[H : Z(G)] \leq 4$, then G admits no prestructures with $z \in Z(G)$.*

Proof First of all, remark that $Z(G)$ is a (normal) subgroup of H because G is non-abelian. Assume now, by contradiction, that the elements $(r_{11}, t_{11}, r_{12}, t_{12}, r_{21}, t_{21}, r_{22}, t_{22}, z)$ form a prestructure on G, with $z \in Z(G)$. Then these elements satisfy relations $(R1'), \ldots, (R9'), (T1'), \ldots, (T9')$ in (13). As H is abelian, $(R4')$ implies that at least one between r_{11}, t_{21} does not belong to H.

Let us assume $r_{11} \notin H$. Thus $C_G(r_{11})$ is abelian, and so $(R2')$ and $(R3')$ yield $[r_{21}, t_{22}] = 1$. From this, using $(T2')$ and $(T3')$, we infer that $C_G(t_{22})$ is non-abelian. Similar considerations show that $C_G(r_{21})$ and $C_G(r_{22})$ are non-abelian, and so we have r_{21}, r_{22}, $t_{22} \in H$. Using $(T2')$, $(T6')$, $(R8')$, together with the fact that H is abelian, we deduce t_{11}, t_{12}, $r_{12} \notin H$. In particular, $C_G(r_{12})$ is abelian, so $(R7')$ and $(R9')$ yield $[r_{21}, t_{21}] = 1$; therefore $(T2')$ and $(T4')$ imply that $C_G(t_{21})$ is non-abelian, and so $t_{21} \in H$. Summing up, we have proved that the four elements r_{21}, t_{21}, r_{22}, t_{22} belong to H; since they are all non-central, we infer that they yield four non-trivial elements in the quotient group $H/Z(G)$. On the other hand, we have $[H : Z(G)] \leq 4$, and so $H/Z(G)$ contains at most three non-trivial elements; it follows that (at least) two among the elements r_{21}, t_{21}, r_{22}, t_{22} have the same image in $H/Z(G)$. This means that these two elements are of the form g, gz, with $z \in Z(G)$, and so they have the same centralizer. But this is impossible: in fact, relations (13) show that each element in the set $\{r_{21}, t_{21}, r_{22}, t_{22}\}$ fails to commute with exactly

one element in the set $\{r_{11}, t_{11}, r_{12}, t_{12}\}$, and no two elements in $\{r_{21}, t_{21}, r_{22}, t_{22}\}$ fail to commute with the same element in $\{r_{11}, t_{11}, r_{12}, t_{12}\}$.

The remaining case, namely $t_{21} \notin H$, can be dealt with in an analogous way. Indeed, in this situation we obtain $\{r_{11}, t_{11}, r_{12}, t_{12}\} \subseteq H$, that leads to a contradiction as before.   □

We can now rule out the non-extra-special groups of order 32.

Proposition 4.11 *Let G be a finite group of order* 32 *which is not extra-special. Then G admits no diagonal double Kodaira structures.*

Proof If G is a CCT-group, then the result follows from Proposition 4.4. Thus, by Proposition 2.14, we must only consider the cases $G = G(32, t)$, where $t \in \{6, 7, 8, 43, 44\}$. Standard computations using the presentations in Table 2 of Appendix A show that all these groups are monolithic, and that for all of them $\mathrm{soc}(G) = Z(G) \simeq \mathbb{Z}_2$. Since no proper quotients of G admit diagonal double Kodaira structures (Proposition 4.9), it follows from Proposition 4.7 that every diagonal double Kodaira structure on G is such that z is the generator of $Z(G)$. Let H be the subgroup of G generated by those elements whose centralizer is non-abelian; by Proposition 4.10 we are now done, provided that in every case H is abelian and $[H : Z(G)] \leq 4$. Let us now show that this is indeed true, leaving the straightforward computations to the reader.

- $G = G(32, 6)$. In this case $\mathrm{soc}(G) = Z(G) = \langle x \rangle$ and $H = \langle x, y, w^2 \rangle$. Then $H \simeq (\mathbb{Z}_2)^3$ and $[H : Z(G)] = 4$.
- $G = G(32, 7)$. In this case $\mathrm{soc}(G) = Z(G) = \langle w \rangle$ and $H = \langle z, u, w \rangle$. Then $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $[H : Z(G)] = 4$.
- $G = G(32, 8)$. In this case $\mathrm{soc}(G) = Z(G) = \langle x^4 \rangle$ and $H = \langle x^2, y, z^2 \rangle$. Then $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $[H : Z(G)] = 4$.
- $G = G(32, 43)$. In this case $\mathrm{soc}(G) = Z(G) = \langle x^4 \rangle$ and $H = \langle x^2, z \rangle$. Then $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $[H : Z(G)] = 4$.
- $G = G(32, 44)$. In this case $\mathrm{soc}(G) = Z(G) = \langle i^2 \rangle$ and $H = \langle x, k \rangle$. Then $H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$ and $[H : Z(G)] = 4$.

This completes the proof.   □

4.4 The case $|G| = 32$ and G extra-special

We are now ready to address the case where $|G| = 32$ and G is extra-special. Let us first recall some additional results on extra-special p-groups, referring the reader to [38] for more details.

Let G be an extra-special p-group of order p^{2b+1} and $x, y \in G$. Setting $(\bar{x}, \bar{y}) = \bar{a}$ where $[x, y] = z^a$, the quotient group $V = G/Z(G) \simeq (\mathbb{Z}_p)^{2b}$ becomes a non-degenerate symplectic vector space over $\mathbb{Z}_p$. Looking at (4) and (5), we see that in both cases $G = \mathsf{H}_{2b+1}(\mathbb{Z}_p)$ and $G = \mathsf{G}_{2b+1}(\mathbb{Z}_p)$ we have

$$(\bar{r}_j, \bar{r}_k) = 0, \quad (\bar{t}_j, \bar{t}_k) = 0, \quad (\bar{r}_j, \bar{t}_k) = -\delta_{jk}$$

for all $j, k \in \{1, \ldots, b\}$, so that

$$\bar{r}_1, \bar{t}_1, \ldots, \bar{r}_b, \bar{t}_b \tag{14}$$

is an ordered symplectic basis for $V \simeq (\mathbb{Z}_p)^{2b}$. If $p = 2$, we can also set $q(\bar{x}) = \bar{c}$, where $x^2 = z^c$ and $c \in \{0, 1\}$; this is a quadratic form on V. If $\bar{x} \in G/Z(G)$ is expressed in

coordinates, with respect to the symplectic basis (14), by the vector $(\xi_1, \psi_1, \ldots, \xi_b, \psi_b) \in (\mathbb{Z}_2)^{2b}$, then a straightforward computation yields

$$q(\bar{x}) = \begin{cases} \xi_1\psi_1 + \cdots + \xi_b\psi_b, & \text{if } G = \mathsf{H}_{2b+1}(\mathbb{Z}_2) \\ \xi_1\psi_1 + \cdots + \xi_b\psi_b + \xi_b^2 + \psi_b^2 & \text{if } G = \mathsf{G}_{2b+1}(\mathbb{Z}_2). \end{cases} \tag{15}$$

These are the two possible normal forms for a non-degenerate quadratic form of dimension $2b$ over $\mathbb{Z}_2$; they have Arf invariant equal to 0 and 1, respectively, see for instance [11] or [25, Chapter 10]. In both cases, the symplectic and the quadratic form are related by

$$q(\bar{x}\bar{y}) = q(\bar{x}) + q(\bar{y}) + (\bar{x}, \bar{y}) \quad \text{for all } \bar{x}, \bar{y} \in V.$$

If $\phi \in \mathrm{Aut}(G)$, then ϕ induces a linear map $\bar{\phi} \in \mathrm{End}(V)$; moreover, if $p = 2$, then ϕ acts trivially on $Z(G) = [G, G] \simeq \mathbb{Z}_2$, and this in turn implies that ϕ preserves the symplectic form on V. In other words, if we identify V with $(\mathbb{Z}_2)^{2b}$ via the symplectic basis (14), we have $\bar{\phi} \in \mathsf{Sp}(2b, \mathbb{Z}_2)$.

We are now in a position to describe the structure of $\mathrm{Aut}(G)$, see [38, Theorem 1].

Proposition 4.12 *Let G be an extra-special group of order 2^{2b+1}. Then the kernel of the group homomorphism $\mathrm{Aut}(G) \longrightarrow \mathsf{Sp}(2b, \mathbb{Z}_2)$ given by $\phi \mapsto \bar{\phi}$ is the subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G. Therefore $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ embeds in $\mathsf{Sp}(2b, \mathbb{Z}_2)$. More precisely, $\mathrm{Out}(G)$ coincides with the orthogonal group $\mathsf{O}_\epsilon(2b, \mathbb{Z}_2)$, of order*

$$|\mathsf{O}_\epsilon(2b, \mathbb{Z}_2)| = 2^{b(b-1)+1}(2^b - \epsilon) \prod_{i=1}^{b-1} (2^{2i} - 1), \tag{16}$$

associated with the quadratic form (15). Here $\epsilon = 1$ if $G = \mathsf{H}_{2b+1}(\mathbb{Z}_2)$ and $\epsilon = -1$ if $G = \mathsf{G}_{2b+1}(\mathbb{Z}_2)$.

Corollary 4.13 *Let G be an extra-special group of order 2^{2b+1}. We have*

$$|\mathrm{Aut}(G)| = 2^{b(b+1)+1}(2^b - \epsilon) \prod_{i=1}^{b-1} (2^{2i} - 1). \tag{17}$$

Proof By Proposition 4.12 we get $|\mathrm{Aut}(G)| = |\mathrm{Inn}(G)| \cdot |\mathsf{O}_\epsilon(2b, \mathbb{Z}_2)|$. Since $\mathrm{Inn}(G) \simeq G/Z(G)$ has order 2^{2b}, the claim follows from (16). □

In particular, plugging $b = 2$ in (17), we can compute the orders of automorphism groups of extra-special groups of order 32, namely

$$|\mathrm{Aut}(\mathsf{H}_5(\mathbb{Z}_2))| = 1152, \quad |\mathrm{Aut}(\mathsf{G}_5(\mathbb{Z}_2))| = 1920. \tag{18}$$

Assume now that $\mathfrak{S} = (\mathsf{r}_{11}, \mathsf{t}_{11}, \mathsf{r}_{12}, \mathsf{t}_{12}, \mathsf{r}_{21}, \mathsf{t}_{21}, \mathsf{r}_{22}, \mathsf{t}_{22}, \mathsf{z})$ is a diagonal double Kodaira structure of type $(2, n)$ on an extra-special group G of order 32; by Corollary 4.8, the element z is the generator of $Z(G) \simeq \mathbb{Z}_2$, hence $n = 2$. Then

$$\bar{\mathfrak{S}} = (\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}, \bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}, \bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}, \bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) \tag{19}$$

is an ordered set of generators for the symplectic $\mathbb{Z}_2$-vector space $V = G/Z(G) \simeq (\mathbb{Z}_2)^4$, and (13) yields the relations

$$\begin{aligned} &(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}) + (\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}) = 1, \\ &(\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}) + (\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) = 1, \\ &(\bar{\mathsf{r}}_{1j}, \bar{\mathsf{t}}_{2k}) = \delta_{jk}, \quad (\bar{\mathsf{r}}_{1j}, \bar{\mathsf{r}}_{2k}) = 0 \\ &(\bar{\mathsf{t}}_{1j}, \bar{\mathsf{r}}_{2k}) = \delta_{jk}, \quad (\bar{\mathsf{t}}_{1j}, \bar{\mathsf{t}}_{2k}) = 0. \end{aligned} \tag{20}$$

Conversely, given any set of generators $\bar{\mathfrak{S}}$ of V as in (19), whose elements satisfy (20), a diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on G inducing $\bar{\mathfrak{S}}$ is necessarily of the form

$$\mathfrak{S} = (\mathsf{r}_{11}\mathsf{z}^{a_{11}}, \, \mathsf{t}_{11}\mathsf{z}^{b_{11}}, \, \mathsf{r}_{12}\mathsf{z}^{a_{12}}, \, \mathsf{t}_{12}\mathsf{z}^{b_{12}}, \, \mathsf{r}_{21}\mathsf{z}^{a_{21}}, \, \mathsf{t}_{21}\mathsf{z}^{b_{21}}, \, \mathsf{r}_{22}\mathsf{z}^{a_{22}}, \, \mathsf{t}_{22}\mathsf{z}^{b_{22}}, \, \mathsf{z}),$$

where $a_{ij}, \, b_{ij} \in \{0, 1\}$. This proves the following

Lemma 4.14 *The total number of diagonal double Kodaira structures of type $(b, n) = (2, 2)$ on an extra-special group G of order* 32 *is obtained multiplying by 2^8 the number of ordered sets of generators $\bar{\mathfrak{S}}$ of V as in (19), whose elements satisfy (20). In particular, such a number does not depend on G.*

We are now ready to state the main result of this section.

Theorem 4.15 *A finite group G of order* 32 *admits a diagonal double Kodaira structure if and only if G is extra-special. In this case, the following holds.*

(1) *Both extra-special groups of order* 32 *admit* $2211840 = 1152 \cdot 1920$ *distinct diagonal double Kodaira structures of type $(b, n) = (2, 2)$. Every such a structure $\mathfrak{S}$ is strong and satisfies $\sigma(\mathfrak{S}) = 16$.*

(2) *If $G = G(32, 49) = H_5(\mathbb{Z}_2)$, these structures form* 1920 *orbits under the action of* $\mathrm{Aut}(G)$.

(3) *If $G = G(32, 50) = G_5(\mathbb{Z}_2)$, these structures form* 1152 *orbits under the action of* $\mathrm{Aut}(G)$.

Proof We already know that non-extra-special groups of order 32 admit no diagonal double Kodaira structures (Proposition 4.11) and so, in the sequel, we can assume that G is extra-special.

Looking at the first two relations in (20), we see that we must consider four cases:

(a) $(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}) = 0,$ $(\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}) = 1,$ $(\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}) = 0,$ $(\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) = 1,$
(b) $(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}) = 1,$ $(\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}) = 0,$ $(\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}) = 1,$ $(\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) = 0,$
(c) $(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}) = 0,$ $(\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}) = 1,$ $(\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}) = 1,$ $(\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) = 0,$
(d) $(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}) = 1,$ $(\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}) = 0,$ $(\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21}) = 0,$ $(\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}) = 1.$

Case (a). In this case the vectors $\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{11}, \bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{22}$ are a symplectic basis of V, whereas the subspace $W = \langle \bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}, \bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{21} \rangle$ is isotropic, namely the symplectic form is identically zero on it. Since V is a symplectic vector space of dimension 4, the Witt index of V, i.e. the dimension of a maximal isotropic subspace of V, is $\frac{1}{2} \dim(V) = 2$, see [1, Théorèmes 3.10, 3.11]. On the other hand, we have $(\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{22}) = 1$ and $(\bar{\mathsf{t}}_{12}, \bar{\mathsf{t}}_{22}) = 0$, hence $\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12}$ are linearly independent and so they must generate a maximal isotropic subspace; it follows that $W = \langle \bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{12} \rangle$. Let us set now

$$(\bar{\mathsf{r}}_{11}, \bar{\mathsf{r}}_{12}) = a, \quad (\bar{\mathsf{r}}_{11}, \bar{\mathsf{t}}_{12}) = b, \quad (\bar{\mathsf{r}}_{12}, \bar{\mathsf{t}}_{11}) = c, \quad (\bar{\mathsf{t}}_{11}, \bar{\mathsf{t}}_{12}) = d,$$
$$(\bar{\mathsf{r}}_{21}, \bar{\mathsf{r}}_{22}) = e, \quad (\bar{\mathsf{r}}_{21}, \bar{\mathsf{t}}_{22}) = f, \quad (\bar{\mathsf{r}}_{22}, \bar{\mathsf{t}}_{21}) = g, \quad (\bar{\mathsf{t}}_{21}, \bar{\mathsf{t}}_{22}) = h,$$

where $a, \, b, \, c, \, d, \, e, \, f, \, g, \, h \in \mathbb{Z}_2$, and let us express the remaining vectors of $\bar{\mathfrak{S}}$ in terms of the symplectic basis. Standard computations yield

$$\begin{aligned}
\bar{\mathsf{r}}_{12} &= c\bar{\mathsf{r}}_{11} + a\bar{\mathsf{t}}_{11} + \bar{\mathsf{r}}_{22}, & \bar{\mathsf{t}}_{12} &= d\bar{\mathsf{r}}_{11} + b\bar{\mathsf{t}}_{11} + \bar{\mathsf{t}}_{22}, \\
\bar{\mathsf{r}}_{21} &= \bar{\mathsf{r}}_{11} + f\bar{\mathsf{r}}_{22} + e\bar{\mathsf{t}}_{22}, & \bar{\mathsf{t}}_{21} &= \bar{\mathsf{t}}_{11} + h\bar{\mathsf{r}}_{22} + g\bar{\mathsf{t}}_{22}.
\end{aligned} \qquad (21)$$

Now recall that W is isotropic; then, using the expressions in (21) and imposing the relations

$$(\bar{r}_{12}, \bar{t}_{12}) = 0, \quad (\bar{r}_{12}, \bar{r}_{21}) = 0, \quad (\bar{r}_{12}, \bar{t}_{21}) = 0,$$
$$(\bar{r}_{21}, \bar{t}_{12}) = 0, \quad (\bar{t}_{12}, \bar{t}_{21}) = 0, \quad (\bar{r}_{21}, \bar{t}_{21}) = 0,$$

we get

$$ad + bc = 1, \quad a + e = 0, \quad c + g = 0,$$
$$b + f = 0, \quad d + h = 0, \quad eh + fg = 1.$$

Summing up, the elements $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}$ can be determined from the symplectic basis via the relations

$$\bar{r}_{12} = c\bar{r}_{11} + a\bar{t}_{11} + \bar{r}_{22}, \qquad\qquad \bar{t}_{12} = d\bar{r}_{11} + b\bar{t}_{11} + \bar{t}_{22},$$
$$\bar{r}_{21} = \bar{r}_{11} + b\bar{r}_{22} + a\bar{t}_{22}, \qquad\qquad \bar{t}_{21} = \bar{t}_{11} + d\bar{r}_{22} + c\bar{t}_{22}, \qquad (22)$$

where $a, b, c, d \in \mathbb{Z}_2$ and $ad + bc = 1$. Conversely, given any symplectic basis $\bar{r}_{11}, \bar{t}_{11}, \bar{r}_{22}, \bar{t}_{22}$ of V and elements $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}$ as in (22), with $ad + bc = 1$, we get a set of generators $\tilde{\mathfrak{S}}$ of V having the form (19), and whose elements satisfy (20). Thus, the total number of such $\tilde{\mathfrak{S}}$ in this case is given by

$$|\mathsf{Sp}(4, \mathbb{Z}_2)| \cdot |\mathsf{GL}(2, \mathbb{Z}_2)| = 720 \cdot 6 = 4320$$

and so, by Lemma 4.14, the corresponding number of diagonal double Kodaira structures is $2^8 \cdot 4320 = 1105920$. All these structures are strong: in fact, we have

$$K_1 = \langle \mathsf{r}_{11}, \mathsf{t}_{11}, \mathsf{r}_{12}, \mathsf{t}_{12} \rangle = \langle \mathsf{r}_{11}, \mathsf{t}_{11}, \mathsf{r}_{11}^c \mathsf{t}_{11}^a \mathsf{r}_{22}, \mathsf{r}_{11}^d \mathsf{t}_{11}^b \mathsf{t}_{22} \rangle$$
$$= \langle \mathsf{r}_{11}, \mathsf{t}_{11}, \mathsf{r}_{22}, \mathsf{t}_{22} \rangle = G$$
$$K_2 = \langle \mathsf{r}_{21}, \mathsf{t}_{21}, \mathsf{r}_{22}, \mathsf{t}_{22} \rangle = \langle \mathsf{r}_{11} \mathsf{r}_{22}^b \mathsf{t}_{22}^a, \mathsf{t}_{11} \mathsf{r}_{22}^d \mathsf{t}_{22}^c, \mathsf{r}_{22}, \mathsf{t}_{22} \rangle$$
$$= \langle \mathsf{r}_{11}, \mathsf{t}_{11}, \mathsf{r}_{22}, \mathsf{t}_{22} \rangle = G,$$

the last equality following in both cases because $\langle \bar{r}_{11}, \bar{t}_{11}, \bar{r}_{22}, \bar{t}_{22} \rangle = V$ and $[\mathsf{r}_{11}, \mathsf{t}_{11}] = \mathsf{z}$.

Case (b). In this situation, the elements $\{\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}, \bar{t}_{21}\}$ form a symplectic basis for V, whereas $W = \langle \bar{r}_{11}, \bar{t}_{11}, \bar{r}_{22}, \bar{t}_{22} \rangle$ is an isotropic subspace. The same calculations as in case (a) show that there are again 1105920 diagonal double Kodaira structures.

Case (c). This case do not occur. In fact, in this situation the subspace $W = \langle \bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21} \rangle$ is isotropic. Take a linear combination of its generators giving the zero vector, namely

$$a\bar{r}_{12} + b\bar{t}_{12} + c\bar{r}_{21} = 0.$$

Pairing with $\bar{t}_{21}, \bar{t}_{22}, \bar{r}_{22}$, we get $c = a = b = 0$. Thus, $\bar{r}_{12}, \bar{t}_{12}, \bar{r}_{21}$ are linearly independent, and W is an isotropic subspace of dimension 3 inside the 4-dimensional symplectic space V, contradiction.

Case (d). This case is obtained from (c) by exchanging the indices 1 and 2, so it does not occur, either.

Summarizing, we have found 1105920 diagonal double Kodaira structures in cases (a) and (b), and no structure at all in cases (c) and (d). So the total number of diagonal double Kodaira structures on G is 2211840, and this concludes the proof of part **(1)**.

Now observe that, since every diagonal double Kodaira structure $\mathfrak{S}$ generates G, the only automorphism ϕ of G fixing $\mathfrak{S}$ elementwise is the identity. This means that $\mathrm{Aut}(G)$ acts freely on the set of diagonal double Kodaira structures, hence the number of orbits is obtained dividing 2211840 by $|\mathrm{Aut}(G)|$. Parts **(2)** and **(3)** now follow from (18), and we are done. $\square$

Example 4.16 Let us give an explicit example of diagonal double Kodaira structure on an extra-special group G of order 32, by using the construction described in the proof of part **(1)** of Theorem 4.15. Referring to the presentations for $\mathsf{H}_5(\mathbb{Z}_2)$ and $\mathsf{G}_5(\mathbb{Z}_2)$ given in Proposition 2.9, we start by choosing in both cases the following elements, whose images give a symplectic basis for V:

$$\mathsf{r}_{11} = \mathsf{r}_1, \quad \mathsf{t}_{11} = \mathsf{t}_1, \quad \mathsf{r}_{22} = \mathsf{r}_2, \quad \mathsf{t}_{22} = \mathsf{t}_2.$$

Choosing $a = d = 1$ and $b = c = 0$ in (22), we find the remaining elements, obtaining the diagonal double Kodaira structure

$$\begin{aligned}
&\mathsf{r}_{11} = \mathsf{r}_1, && \mathsf{t}_{11} = \mathsf{t}_1, && \mathsf{r}_{12} = \mathsf{r}_2\,\mathsf{t}_1, && \mathsf{t}_{12} = \mathsf{r}_1\,\mathsf{t}_2 \\
&\mathsf{r}_{21} = \mathsf{r}_1\,\mathsf{t}_2, && \mathsf{t}_{21} = \mathsf{r}_2\,\mathsf{t}_1, && \mathsf{r}_{22} = \mathsf{r}_2, && \mathsf{t}_{22} = \mathsf{t}_2.
\end{aligned}$$

Remark 4.17 Theorem 4.15 should be compared with previous results of [9] and [28], regarding the construction of diagonal double Kodaira structures on some extra-special groups of order at least $2^7 = 128$. We emphasize that the examples on extra-special groups of order 32 presented here are really new, in the sense that they cannot be obtained by taking the image of structures on extra-special groups of bigger order: in fact, an extra-special group admits no non-abelian proper quotients, cf. Example 4.6.

Let us end this section with the restatement of Theorem 4.15 in terms of admissible epimorphisms from surface braid groups to finite groups.

Corollary 4.18 *Let G be a finite group admitting an admissible epimorphism $\varphi \colon \mathsf{P}_2(\Sigma_b) \longrightarrow G$. Then $|G| \geq 32$, with equality if and only if G is extra-special. Moreover, the following holds.*

(1) *For both extra-special groups G of order 32, there are $2211840 = 1152 \cdot 1920$ admissible epimorphisms $\varphi \colon \mathsf{P}_2(\Sigma_2) \longrightarrow G$. For all of them, $\varphi(A_{12})$ is the generator of $Z(G)$, so $n = 2$.*

(2) *If $G = G(32, 49) = \mathsf{H}_5(\mathbb{Z}_2)$, these epimorhisms form 1920 orbits under the natural action of $\mathrm{Aut}(G)$.*

(3) *If $G = G(32, 50) = \mathsf{G}_5(\mathbb{Z}_2)$, these epimorhisms form 1152 orbits under the natural action of $\mathrm{Aut}(G)$.*

5 Geometrical application: diagonal double Kodaira fibrations

The aim of this section is to show how the existence of diagonal double Kodaira structures is equivalent to the existence of some special double Kodaira fibrations (see the Introduction for the definition), that we call *diagonal double Kodaira fibrations*. We closely follow the treatment given in [28, Section 4].

In the sequel we use the symbol Σ_b to indicate both a smooth complex curve of genus b and its underlying real surface, and we assume that the finite group G admits a diagonal double Kodaira structure $\mathfrak{S}$ of type (b, n). By Grauert-Remmert's extension theorem and Serre's GAGA, the group epimorphism $\varphi \colon \mathsf{P}_2(\Sigma_b) \longrightarrow G$ described in Proposition 3.6 yields the existence of a smooth, complex, projective surface S endowed with a Galois cover

$$\mathbf{f} \colon S \longrightarrow \Sigma_b \times \Sigma_b$$

with Galois group G and branched precisely over the diagonal Δ with branching order n, see [9, Proposition 3.4]. Composing the left homomorphisms in (10) with $\varphi\colon \mathsf{P}_2(\Sigma_b) \longrightarrow G$, we get two homomorphisms

$$\varphi_1\colon \pi_1(\Sigma_b - \{p_2\},\, p_1) \longrightarrow G, \quad \varphi_2\colon \pi_1(\Sigma_b - \{p_1\},\, p_2) \longrightarrow G,$$

whose respective images coincide with the subgroups K_1 and K_2 defined in (11). By construction, these are the homomorphisms induced by the restrictions $\mathbf{f}_i\colon \Gamma_i \longrightarrow \Sigma_b$ of the Galois cover $\mathbf{f}\colon S \longrightarrow \Sigma_b \times \Sigma_b$ to the fibres of the two natural projections $\pi_i\colon \Sigma_b \times \Sigma_b \longrightarrow \Sigma_b$. Since Δ intersects transversally at a single point all the fibres of the natural projections, it follows that both such restrictions are branched at precisely one point, and the number of connected components of the smooth curve $\Gamma_i \subset S$ equals the index $m_i := [G : K_i]$ of K_i in G.

So, taking the Stein factorizations of the compositions $\pi_i \circ \mathbf{f}\colon S \longrightarrow \Sigma_b$ as in the diagram below

$$
\begin{array}{ccc}
S & \xrightarrow{\ \pi_i \circ \mathbf{f}\ } & \Sigma_b \\
& \searrow{\scriptstyle f_i} \quad \nearrow{\scriptstyle \theta_i} & \\
& \Sigma_{b_i} &
\end{array}
\tag{23}
$$

we obtain two distinct Kodaira fibrations $f_i\colon S \longrightarrow \Sigma_{b_i}$, hence a double Kodaira fibration by considering the product morphism

$$f = f_1 \times f_2\colon S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}.$$

Definition 5.1 We call $f\colon S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ the *diagonal double Kodaira fibration* associated with the diagonal double Kodaira structure $\mathfrak{S}$ on the finite group G. Conversely, we will say that a double Kodaira fibration $f\colon S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is *of diagonal type* (b, n) if there exists a finite group G and a diagonal double Kodaira structure $\mathfrak{S}$ of type (b, n) on it such that f is associated with $\mathfrak{S}$.

Since the morphism $\theta_i\colon \Sigma_{b_i} \longrightarrow \Sigma_b$ is étale of degree m_i, by using the Hurwitz formula we obtain

$$b_1 - 1 = m_1(b - 1), \quad b_2 - 1 = m_2(b - 1).$$

Moreover, the fibre genera g_1, g_2 of the Kodaira fibrations $f_1\colon S \longrightarrow \Sigma_{b_1}$, $f_2\colon S \longrightarrow \Sigma_{b_2}$ are computed by the formulae

$$2g_1 - 2 = \frac{|G|}{m_1}(2b - 2 + \mathfrak{n}), \quad 2g_2 - 2 = \frac{|G|}{m_2}(2b - 2 + \mathfrak{n}),
\tag{24}$$

where $\mathfrak{n} := 1 - 1/n$. Finally, the surface S fits into a diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad \mathbf{f} \quad} & \Sigma_b \times \Sigma_b \\
& \searrow{\scriptstyle f} \quad \nearrow{\scriptstyle \theta_1 \times \theta_2} & \\
& \Sigma_{b_1} \times \Sigma_{b_2} &
\end{array}
$$

so that the diagonal double Kodaira fibration $f\colon S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is a finite cover of degree $\frac{|G|}{m_1 m_2}$, branched precisely over the curve

$$(\theta_1 \times \theta_2)^{-1}(\Delta) = \Sigma_{b_1} \times_{\Sigma_b} \Sigma_{b_2}.$$

Such a curve is always smooth, being the preimage of a smooth divisor via an étale morphism. However, it is reducible in general, see [9, Proposition 3.11]. The invariants of S can be now computed as follows, see [9, Proposition 3.8].

Proposition 5.2 *Let $f \colon S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure $\mathfrak{S}$ of type (b, n) on a finite group G. Then we have*

$$c_1^2(S) = |G|\,(2b-2)(4b-4+4\mathfrak{n}-\mathfrak{n}^2)$$
$$c_2(S) = |G|\,(2b-2)(2b-2+\mathfrak{n})$$

where $\mathfrak{n} = 1 - 1/n$. As a consequence, the slope and the signature of S can be expressed as

$$
\begin{aligned}
\nu(S) &= \frac{c_1^2(S)}{c_2(S)} = 2 + \frac{2\mathfrak{n} - \mathfrak{n}^2}{2b - 2 + \mathfrak{n}} \\
\sigma(S) &= \frac{1}{3}\left(c_1^2(S) - 2c_2(S)\right) = \frac{1}{3}\,|G|\,(2b-2)\left(1 - \frac{1}{n^2}\right) = \sigma(\mathfrak{S})
\end{aligned}
\tag{25}
$$

Remark 5.3 By definition, the diagonal double Kodaira structure $\mathfrak{S}$ is strong if and only if $m_1 = m_2 = 1$, that in turn implies $b_1 = b_2 = b$, i.e., $f = \mathbf{f}$. In other words, $\mathfrak{S}$ is strong if and only if no Stein factorization as in (23) is needed or, equivalently, if and only if the Galois cover $\mathbf{f} \colon S \longrightarrow \Sigma_b \times \Sigma_b$ induced by (8) is already a double Kodaira fibration, branched on the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$.

Remark 5.4 Every Kodaira fibred surface S satisfies $\sigma(S) > 0$, see the introduction to [24]; moreover, since S is a differentiable 4-manifold that is a real surface bundle, its signature is divisible by 4, see [27]. In addition, if S is associated with a diagonal double Kodaira structure of type (b, n), with n odd, then K_S is 2-divisible in $\mathrm{Pic}(S)$ and so $\sigma(S)$ is a positive multiple of 16 by Rokhlin's theorem, see [9, Remark 3.9].

We are now ready to give a geometric restatement of the algebraic results of Sect. 4 in terms of double Kodaira fibrations.

Theorem 5.5 *Let G be a finite group and*

$$\mathbf{f} \colon S \longrightarrow \Sigma_b \times \Sigma_b \tag{26}$$

be a Galois cover with Galois group G, branched over the diagonal Δ with branching order $n \geq 2$. Then the following hold.

(1) *We have $|G| \geq 32$, with equality precisely when G is extra-special.*
(2) *If $G = G(32, 49) = H_5(\mathbb{Z}_2)$ and $b = 2$, there are 1920 G-covers of type (26), up to cover isomorphisms.*
(3) *If $G = G(32, 50) = G_5(\mathbb{Z}_2)$ and $b = 2$, there are 1152 G-covers of type (26), up to cover isomorphisms.*

Finally, in both cases (2) and (3), we have $n = 2$ and each cover $\mathbf{f}$ is a double Kodaira fibration with

$$b_1 = b_2 = 2, \quad g_1 = g_2 = 41, \quad \sigma(S) = 16.$$

Proof A cover as in (26), branched over Δ with order n, exists if and only if G admits a double Kodaira structure of type (b, n); additionally, the number of such covers, up to cover isomorphisms, equals the number of structures up the natural action of $\mathrm{Aut}(G)$. Then, **(1)**,

(2) and **(3)** can be deduced from the corresponding statements in Theorem 4.15. The same theorem tells us that all double Kodaira structures on an extra-special group of order 32 are strong, hence the cover **f** is already a double Kodaira fibration and no Stein factorization is needed (Remark 5.3). The fibre genera, the slope and the signature of S can be now computed by using (24) and (25).   □

As a consequence, we obtain a sharp lower bound for the signature of a diagonal double Kodaira fibration or, equivalently, of a diagonal double Kodaira structure.

Corollary 5.6 *Let* $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ *be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure of type* (b, n) *on a finite group G. Then* $\sigma(S) \geq 16$, *and equality holds precisely when* $(b, n) = (2, 2)$ *and G is an extra-special group of order* 32.

Proof Theorem 4.15 implies $|G| \geq 32$. Since $b \geq 2$ and $n \geq 2$, from (25) we get

$$\sigma(S) = \frac{1}{3} |G| (2b - 2) \left(1 - \frac{1}{n^2}\right) \geq \frac{1}{3} \cdot 32 \cdot (2 \cdot 2 - 2) \left(1 - \frac{1}{2^2}\right) = 16,$$

and equality holds if and only if we are in the situation described in Theorem 5.5, namely, $b = n = 2$ and G an extra-special group of order 32.   □

These results provide, in particular, new "double solutions" to a problem, posed by G. Mess, from Kirby's problem list in low-dimensional topology [20, Problem 2.18 A], asking what is the smallest number b for which there exists a real surface bundle over a real surface with base genus b and non-zero signature. We actually have $b = 2$, also for double Kodaira fibrations, as shown in [9, Proposition 3.19] and [28] by using double Kodaira structures of type $(2, 3)$ on extra-special groups of order 3^5. Those fibrations had signature 144 and fibre genera 325; by using our new examples, we can now substantially lower both these values.

Theorem 5.7 *Let S be the diagonal double Kodaira surface associated with a strong diagonal double Kodaira structure of type* $(b, n) = (2, 2)$ *on an extra-special group G of order* 32. *Then the real manifold M underlying S is a closed, orientable* 4*-manifold of signature* 16 *that can be realized as a real surface bundle over a real surface of genus* 2*, with fibre genus* 41*, in two different ways.*

Theorem 5.5 also implies the following partial answer to [9, Question 3.20].

Corollary 5.8 *Let* $g_{\min}$ *and* $\sigma_{\min}$ *be the minimal possible fibre genus and signature for a double Kodaira fibration* $f : S \longrightarrow \Sigma_2 \times \Sigma_2$. *Then we have*

$$g_{\min} \leq 41, \quad \sigma_{\min} \leq 16.$$

In fact, it is an interesting question whether 16 and 41 are the minimum possible values for the signature and the fibre genus of a (non necessarily diagonal) double Kodaira fibration $f : S \longrightarrow \Sigma_2 \times \Sigma_2$, but we will not address this problem here.

Remark 5.9 Constructing (double) Kodaira fibrations with small signature is a rather difficult problem. As far as we know, before our work the only examples with signature 16 were the one described in [5, Theorem 1.1] and the ones listed in [24, Table 3, Cases 6.2, 6.6, 6.7 (Type 1), 6.9]. The examples provided by Theorem 5.5 are new, since both the base genera and the fibre genera are different. Note that our results also show that *every* curve of genus 2 (and not only some special curve with extra automorphisms) is the base of a double Kodaira fibration with signature 16. Thus, we obtain two families of dimension 3 of such fibrations that, to the best of our knowledge, provide the first examples of a positive-dimensional families of double Kodaira fibrations with small signature.

Research Data Policy and Data Availability Statements Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix: Non abelian groups of order 24 and 32

See Tables 1 and 2.

Table 1 Nonabelian groups of order 24 . Source: https://groupprops.subwiki.org/wiki/Groups_of_order_24

IdSmallGroup(G)	G	Presentation
$G(24, 1)$	$D_{8, 3, -1}$	$\langle x, y \mid x^8 = y^3 = 1, xyx^{-1} = y^{-1} \rangle$
$G(24, 3)$	$SL(2, \mathbb{F}_3)$	$\langle x, y, z \mid x^3 = y^3 = z^2 = xyz \rangle$
$G(24, 4)$	Q_{24}	$\langle x, y, z \mid x^6 = y^2 = z^2 = xyz \rangle$
$G(24, 5)$	$D_{2, 12, 5}$	$\langle x, y \mid x^2 = y^{12} = 1, xyx^{-1} = y^5 \rangle$
$G(24, 6)$	D_{24}	$\langle x, y \mid x^2 = y^{12} = 1, xyx^{-1} = y^{-1} \rangle$
$G(24, 7)$	$\mathbb{Z}_2 \times D_{4, 3, -1}$	$\langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle$
$G(24, 8)$	$((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	$\langle x, y, z, w \mid x^2 = y^2 = z^2 = w^3 = 1,$
		$[y, z] = [y, w] = [z, w] = 1,$
		$xyx^{-1} = y, xzx^{-1} = zy, xwx^{-1} = w^{-1} \rangle$
$G(24, 10)$	$\mathbb{Z}_3 \times D_8$	$\langle z \mid z^3 = 1 \rangle \times \langle x, y \mid x^2 = y^4 = 1, xyx^{-1} = y^{-1} \rangle$
$G(24, 11)$	$\mathbb{Z}_3 \times Q_8$	$\langle z \mid z^3 = 1 \rangle \times \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$
$G(24, 12)$	S_4	$\langle x, y \mid x = (12), y = (1234) \rangle$
$G(24, 13)$	$\mathbb{Z}_2 \times A_4$	$\langle z \mid z^2 = 1 \rangle \times \langle x, y \mid x = (12)(34), y = (123) \rangle$
$G(24, 14)$	$(\mathbb{Z}_2)^2 \times S_3$	$\langle z, w \mid z^2 = w^2 = [z, w] = 1 \rangle$
		$\times \langle x, y \mid x = (12), y = (123) \rangle$

Table 2 Nonabelian groups of order 32. Source: https://groupprops.subwiki.org/wiki/Groups_of_order_32

IdSmallGroup(G)	G	Presentation
$G(32, 2)$	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	$\langle x, y, z \mid x^4 = y^4 = z^2 = 1,$ $[x, y] = z, [x, z] = [y, z] = 1\rangle$
$G(32, 4)$	$D_{4,8,5}$	$\langle x, y \mid x^4 = y^8 = 1, xyx^{-1} = y^5 \rangle$
$G(32, 5)$	$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^8 = y^2 = z^2 = 1, [x, y] = 1,$ $zxz^{-1} = x^5 y, \, zyz^{-1} = y\rangle$
$G(32, 6)$	$(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_4$	$\langle x, y, z, w \mid x^2 = y^2 = z^2 = w^4 = 1,$ $[x, y] = 1, [x, z] = 1, [y, z] = 1,$ $wxw^{-1} = x, \, wyw^{-1} = xy, \, wzw^{-1} = yz\rangle$
$G(32, 7)$	$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z, u, w \mid y^2 = z^2 = w^2 = 1,$ $u^2 = w^{-1}, x^2 = u, (yz)^2 = 1, (yu^{-1})^2 = 1,$ $uzu^{-1} = z^{-1}, xyzx^{-1} = y^{-1}\rangle$
$G(32, 8)$	$(\mathbb{Z}_2)^2 \,.\, (\mathbb{Z}_4 \times \mathbb{Z}_2)$	$\langle x, y, z \mid x^8 = y^2 = 1, z^2 = x^4,$ $xy = yx^5, [y, z] = 1, xz = zxy^{-1}\rangle$
$G(32, 9)$	$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^8 = y^2 = z^2 = 1, [x, y] = 1,$ $zxz^{-1} = x^3 y, \, zyz^{-1} = y\rangle$
$G(32, 10)$	$Q_8 \rtimes \mathbb{Z}_4$	$\langle i, j, k, x \mid i^2 = j^2 = k^2 = ijk, x^4 = 1,$ $xix^{-1} = j, \, xjx^{-1} = i, \, xkx^{-1} = k^{-1}\rangle$
$G(32, 11)$	$(\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^4 = y^4 = [x, y] = 1, z^2 = 1,$ $zxz^{-1} = y, \, zyz^{-1} = x\rangle$
$G(32, 12)$	$D_{8,4,3}$	$\langle x, y \mid x^8 = y^4 = 1, xyx^{-1} = y^3\rangle$
$G(32, 13)$	$D_{4,8,3}$	$\langle x, y \mid x^4 = y^8 = 1, xyx^{-1} = y^3\rangle$
$G(32, 14)$	$D_{4,8,-1}$	$\langle x, y \mid x^4 = y^8 = 1, xyx^{-1} = y^{-1}\rangle$
$G(32, 15)$	$\mathbb{Z}_4 \,.\, D_8$	$\langle x, y, z, u, w \mid w^2 = 1, z^2 = u^2 = w^{-1},$ $x^2 = u, y^2 = z, xzx^{-1} = z^{-1},$ $[y, u] = 1, xyxu = y^{-1}\rangle$
$G(32, 17)$	$D_{2,16,9}$	$\langle x, y \mid x^2 = y^{16} = 1, xyx^{-1} = y^9\rangle$
$G(32, 18)$	D_{32}	$\langle x, y \mid x^2 = y^{16} = 1, xyx^{-1} = y^{-1}\rangle$
$G(32, 19)$	QD_{32}	$\langle x, y \mid x^2 = y^{16} = 1, xyx^{-1} = y^7\rangle$
$G(32, 20)$	Q_{32}	$\langle x, y, z \mid x^8 = y^2 = z^2 = xyz\rangle$
$G(32, 22)$	$\mathbb{Z}_2 \times ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$	$\langle w \mid w^2 = 1\rangle \times$ $\langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x, y] = 1,$ $zxz^{-1} = xy, \, zyz^{-1} = y\rangle$
$G(32, 23)$	$\mathbb{Z}_2 \times D_{4,4,3}$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^4 = y^4 = 1, xyx^{-1} = y^3\rangle$
$G(32, 24)$	$(\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^4 = y^4 = z^2 = 1,$ $[x, y] = 1, zxz^{-1} = x, \, zyz^{-1} = x^2 y\rangle$
$G(32, 25)$	$\mathbb{Z}_4 \times D_8$	$\langle z \mid z^4 = 1\rangle \times \langle x, y \mid x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
$G(32, 26)$	$\mathbb{Z}_4 \times Q_8$	$\langle z \mid z^4 = 1\rangle \times \langle i, j, k \mid i^2 = j^2 = k^2 = ijk\rangle$

Table 2 continued

IdSmallGroup(G)	G	Presentation
$G(32, 27)$	$(\mathbb{Z}_2)^3 \rtimes (\mathbb{Z}_2)^2$	$\langle x, y, z, a, b \mid$ $x^2 = y^2 = z^2 = a^2 = b^2 = 1,$ $[x, y] = [y, z] = [x, z] = [a, b] = 1,$ $axa^{-1} = x,\ aya^{-1} = y,\ aza^{-1} = xz,$ $bxb^{-1} = x,\ byb^{-1} = y,\ bzb^{-1} = yz\rangle$
$G(32, 28)$	$(\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \rtimes \mathbb{Z}_2$	$\langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1,$ $[x, y] = [x, z] = [y, z] = 1,$ $wxw^{-1} = x^{-1},\ wyw^{-1} = z,\ wzw^{-1} = y\rangle$
$G(32, 29)$	$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$	$\langle x, i, j, k, z \mid x^2 = z^2 = 1,\ i^2 = j^2 = k^2 = ijk,$ $[x, i] = [x, j] = [x, k] = 1,$ $zxz^{-1} = x,\ ziz^{-1} = i,\ zjz^{-1} = xj^{-1}\rangle$
$G(32, 30)$	$(\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \rtimes \mathbb{Z}_2$	$\langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1,$ $[x, y] = [x, z] = [y, z] = 1,$ $wxw^{-1} = xy,\ wyw^{-1} = y,\ wzw^{-1} = x^2z\rangle$
$G(32, 31)$	$(\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^4 = y^4 = [x, y] = 1,\ z^2 = 1,$ $zxz^{-1} = xy^2,\ zyz^{-1} = x^2y\rangle$
$G(32, 32)$	$(\mathbb{Z}_2)^2 . (\mathbb{Z}_2)^3$	$\langle x, y, z, u, w \mid u^2 = w^2 = 1,$ $u = z^2,\ u = x^{-2},\ w = y^{-2},$ $yxy^{-1} = x^{-1},\ [y, z] = 1,\ xzxwz = 1\rangle$
$G(32, 33)$	$(\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^4 = y^4 = [x, y] = 1,\ z^2 = 1,$ $zxz^{-1} = xy^2,\ zyz^{-1} = x^2y^{-1}\rangle$
$G(32, 34)$	$(\mathbb{Z}_4)^2 \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^4 = y^4 = [x, y] = 1,\ z^2 = 1,$ $zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1}\rangle$
$G(32, 35)$	$\mathbb{Z}_4 \rtimes Q_8$	$\langle x, i, j, k \mid x^4 = 1,\ i^2 = j^2 = k^2 = ijk,$ $ixi^{-1} = x^{-1},\ jxj^{-1} = x^{-1},\ kxk^{-1} = x\rangle$
$G(32, 37)$	$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^8 = y^2 = z^2 = 1,\ [x, y] = 1,$ $zxz^{-1} = x^5,\ zyz^{-1} = y\rangle$
$G(32, 38)$	$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^8 = y^2 = z^2 = 1,\ [x, y] = 1,$ $zxz^{-1} = x,\ zyz^{-1} = x^4y\rangle$
$G(32, 39)$	$\mathbb{Z}_2 \times D_{16}$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^2 = y^8 = 1,\ xyx^{-1} = y^{-1}\rangle$
$G(32, 40)$	$\mathbb{Z}_2 \times QD_{16}$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^2 = y^8 = 1,\ xyx^{-1} = y^3\rangle$
$G(32, 41)$	$\mathbb{Z}_2 \times Q_{16}$	$\langle w \mid w^2 = 1\rangle \times \langle x, y, z \mid x^4 = y^2 = z^2 = xyz\rangle$
$G(32, 42)$	$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\langle x, y, z \mid x^8 = y^2 = z^2 = 1,\ [x, y] = 1,$ $zxz^{-1} = x^3,\ zyz^{-1} = x^4y\rangle$
$G(32, 43)$	$\mathbb{Z}_8 \rtimes (\mathbb{Z}_2)^2$	$\langle x, y, z \mid x^8 = 1,\ y^2 = z^2 = [y, z] = 1,$ $yxy^{-1} = x^{-1},\ zxz^{-1} = x^5\rangle$

Table 2 continued

IdSmallGroup(G)	G	Presentation
$G(32, 44)$	$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$	$\langle x, i, j, k, z \mid x^2 = z^2 = 1, \, i^2 = j^2 = k^2 = ijk,$ $[x, i] = [x, j] = [x, k] = 1,$ $zxz^{-1} = xi^2, \, ziz^{-1} = j, \, zjz^{-1} = i \rangle$
$G(32, 46)$	$(\mathbb{Z}_2)^2 \times D_8$	$\langle z, w \mid z^2 = w^2 = [z, w] = 1 \rangle$ $\times \langle x, y \mid x^2 = y^4 = 1, \, xyx^{-1} = y^{-1} \rangle$
$G(32, 47)$	$(\mathbb{Z}_2)^2 \times Q_8$	$\langle z, w \mid z^2 = w^2 = [z, w] = 1 \rangle$ $\times \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$
$G(32, 48)$	$(\mathbb{Z}_4 \times (\mathbb{Z}_2)^2) \rtimes \mathbb{Z}_2$	$\langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1,$ $[x, y] = [x, z] = [y, z] = 1,$ $wxw^{-1} = x, \, wyw^{-1} = y, \, wzw^{-1} = x^2z \rangle$
$G(32, 49)$	$H_5(\mathbb{Z}_2)$	$\langle r_1, t_1, r_2, t_2, z \mid r_j^2 = t_j^2 = z^2 = 1,$ $[r_j, z] = [t_j, z] = 1,$ $[r_j, r_k] = [t_j, t_k] = 1,$ $[r_j, t_k] = z^{-\delta_{jk}} \rangle, \quad \text{see (4)}$
$G(32, 50)$	$G_5(\mathbb{Z}_2)$	$\langle r_1, t_1, r_2, t_2, z \mid, r_1^2 = t_1^2 = z^2 = 1,$ $r_2^2 = t_2^2 = z,$ $[r_j, z] = [t_j, z] = 1,$ $[r_j, r_k] = [t_j, t_k] = 1,$ $[r_j, t_k] = z^{-\delta_{jk}} \rangle, \quad \text{see (5)}$

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