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A Cartan–Hartogs version of the polydisk theorem

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Abstract

We extend the Polydisk Theorem for symmetric bounded domains to Cartan–Hartogs domains, and apply it to prove that a Cartan–Hartogs domain inherits totally geodesic submanifolds from the bounded symmetric domain which is based on, and to give a characterization of Cartan–Hartogs's geodesics with linear support.

Contents

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1 Introduction and statement of the results

For a bounded symmetric domain Ω endowed with (a multiple of) its Bergman metric g_B, the celebrated Polydisk Theorem due to J. A. Wolf [27] (see also [15]) shows that given any point $z \in \Omega$ and any direction $X \in T_z\Omega$, there exists a totally geodesic complex submanifold Π passing through z with $X \in T_z\Pi$, biholomorphically isometric to a polydisk Δ^r of dimension equals to the rank r of Ω. Moreover, the group of the (isometric) automorphisms $\mathrm{Aut}(\Omega)$ of Ω, acts transitively on the space of all such polydisks, and denoting by $\mathrm{Aut}_z(\Omega)$ the isotropy subgroup of $\mathrm{Aut}(\Omega)$ at z, one can realize Ω as union over $\gamma \in \mathrm{Aut}_z(\Omega)$ of $\gamma \cdot \Pi$.

In analogy with the symmetric case we prove a version of the Polydisk Theorem for Cartan-Hartogs domains in terms of Hartogs-Polydisk (see (4) below). For $\mu > 0$, Cartan–Hartogs domains are defined as the 1-parameter family:

$$M_\Omega(\mu) = \left\{ (z, w) \in \Omega \times \mathbb{C} \mid |w|^2 < N_\Omega^\mu(z, z) \right\}, \tag{1}$$

where Ω is a bounded symmetric domain not necessarily irreducible and $N_\Omega(z, z)$ is its generic norm. Observe that originally [28] the domain Ω the Cartan–Hartogs is based on is a Cartan domain, i.e. an irreducible bounded symmetric domain. Here Ω is allowed to be not irreducible, namely $\Omega = \Omega_1 \times \cdots \times \Omega_m$ is a product of the Cartan domains $\Omega_1, \ldots, \Omega_m$, and accordingly its generic norm N_Ω is the product of the generic norms of each factor:

$$N_\Omega(z_1, \ldots, z_m, z_1, \ldots, z_m) = N_{\Omega_1}(z_1, z_1) \cdots N_{\Omega_m}(z_m, z_m). \tag{2}$$

We consider on $M_\Omega(\mu)$ the Kobayashi metric $\omega(\mu) = \frac{i}{2}\partial\bar\partial\Phi_{\Omega,\mu}$, where:

$$\Phi_{\Omega,\mu}(z, w) = -\log\left(N_\Omega^\mu(z, z) - |w|^2\right). \tag{3}$$

We say that a Cartan–Hartogs $M_\Omega(\mu)$ domain is of *classical type* if Ω is a product of Cartan domains of classical type. When Ω is a polydisk $\Delta^n := \{z \in \mathbb{C}^n \mid |z_1|^2 < 1, \ldots, |z_n|^2 < 1\}$, the associated Cartan–Hartogs is the *Hartogs-Polydisk*:

$$M_{\Delta^n}(\mu) = \left\{ (z, w) \in \Delta^n \times \mathbb{C} \mid |w|^2 < \prod_{j=1}^n (1 - |z_j|^2)^\mu \right\}, \tag{4}$$

whose Kobayashi metric is defined by the Kähler potential:

$$\Phi_{\Delta^n,\mu}(z, w) = -\log\left(\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2\right).$$

Observe that when $\mu = 1$ and Ω is the complex hyperbolic space $\mathbb{C}H^n$, also $M_\Omega(\mu)$ reduces to be the complex hyperbolic space $\mathbb{C}H^{n+1}$. In all the other cases it is a nonhomogeneous domain that inherits symmetric peculiarities from the symmetric bounded domain it based on. For this reason Cartan–Hartogs domains represent an important class of domains in $\mathbb{C}^n$, and since their first apparence in [28] they have been studied from different points of view, see e.g. [2, 6, 7, 11, 12, 23, 25, 29, 30].

The main theorem of this paper is the following Hartogs version of the Polydisk Theorem. As his classical counterpart, which led to several applications, e.g. N. Mok and S.-C.

Ng's rigidity and extension results for holomorphic isometries [16–19, 24] (see also [9, 20, 21] where the Polydisk Theorem is used to study the diastatic exponential and the volume and diastatic entropy of symmetric bounded domains), we expect it to be a useful tool to solve geometric problems related to Cartan–Hartogs domain, improving our knowledge of nonhomogeneous domains.

Theorem 1 (Hartogs–Polydisk Theorem) *Let Ω be a bounded symmetric domain of classical type of rank r and let $M_\Omega(\mu)$ be the associated Cartan–Hartogs domain. For any point $(z, w) \in M_\Omega(\mu)$ and any $X \in T_{(z,w)} M_\Omega(\mu)$ there exists a totally geodesic complex submanifold $\tilde{\Pi}$ through (z, w) with $X \in T_{(z,w)} M_{\tilde{\Pi}}(\mu)$, such that $\tilde{\Pi}$ is biholomorphically isometric to the Hartogs-Polydisk $M_{\Delta^r}(\mu)$. Moreover,* Aut(Ω) *acts transitively on the space of all such Hartogs-polydisks, and $M_\Omega(\mu) = \cup\{\gamma \cdot \tilde{\Pi} : \gamma \in \mathrm{Aut}_z(\Omega)\}$.*

We apply the Hartogs-Polydisk Theorem to prove the following two results. The first one states that any totally geodesic Kähler submanifold of the base domain Ω is a totally geodesic submanifold of its associated Cartan–Hartogs:

Theorem 2 *Let $\Omega' \subset \Omega$ be a totally geodesic Kähler submanifold of a bounded symmetric domain of classical type. Then*

$$C_{\Omega'} = \left\{ (z, w) \in M_\Omega(\mu) \mid z \in \Omega' \right\}$$

is a totally geodesic Kähler submanifold of $M_\Omega(\mu)$ biholomorphically isometric to the Cartan-Hartogs $M_{\Omega'}(\mu)$.

The second one gives a characterization of geodesics with linear support in $M_\Omega(\mu)$:

Theorem 3 *Let $M_\Omega(\mu)$ be a Cartan–Hartogs domain not biholomorphic to $\mathbb{C}H^{n+1}$. If $M_\Omega(\mu)$ admits a geodesic with linear support passing through $(\zeta, 0)$, then up to automorphisms either the geodesic is contained in $\Omega = M_\Omega(\mu) \cap \{w = 0\}$ or in $\mathbb{C}H^1 = M_\Omega(\mu) \cap \{z = 0\}$.*

The paper is organized as follows. In the next section we recall basic facts about classical Cartan domains and we describe explicit polydisks totally geodesically embedded. In Sect. 3 we show how the totally geodesic Kähler immersions of such polydisks into the Cartan domains lift to totally geodesic Kähler immersion of Hartogs–polydisks into Cartan–Hartogs domains and prove Theorem 1. The last three sections are devoted respectively to the proofs of theorems 2 and 3.

2 Explicit polydisks in Cartan domains

In this section we are going to give an explicit totally geodesic Kähler (i.e. holomorphic and isometric) immersion of a polydisk into each one of the four irreducible classical domains. All the isometries here are intended respect to the hyperbolic metric on Ω, i.e. $\omega_{hyp}^\Omega := -\partial\bar{\partial} \log N_\Omega(z, z)$ (one has $\omega_B^\Omega = \gamma \omega_{hyp}^\Omega$, where ω_B^Ω is the Bergman metric on Ω and γ is its genus). Throughout this section we use the Jordan triple system theory, referring the reader to [4, 5, 9, 10, 14, 20–23, 26] for details and further applications.

2.1 Cartan domain of the first type

Consider the first Cartan domain of rank $r = m$ and genus $\gamma = n + m$:

$$\Omega_1[m, n] = \left\{ Z \in M_{m,n}(\mathbb{C}) \mid \det \left(I_m - ZZ^* \right) > 0 \right\}, \qquad n \geq m.$$

Its generic norm is given by:

$$N_{\Omega_1}(Z, Z) = \det\left(I_m - ZZ^*\right).\tag{5}$$

A totally geodesic polydisk $\Delta^m \xrightarrow{\varphi} \Omega_1[m, n]$ is given by

$$\varphi(z_1, \ldots, z_m) = \mathrm{diag}(z_1, \ldots, z_m) = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ & & z_m \ 0 \end{pmatrix}.\tag{6}$$

Since $\det(I_m - \varphi(z)\varphi(z)^*) = \prod_{j=1}^{m}(1 - |z_j|^2)$, φ is clearly a Kähler immersion. Moreover it is easy to check that $\varphi_*(T_0\Delta^m)$ define a sub-HJPTS of $(T_0\Omega_1[m, n], \{,,\})$, where

$$\{U, V, W\} = UV^*W + WV^*U\tag{7}$$

(see e.g. [4, (16)]), we conclude, by the one to one correspondence between sub-HJPTS e sub-HSSNT (see [4, Proposition 2.1]), that φ is totally geodesic.

2.2 Cartan domain of the second type

Consider the second Cartan domain of rank $r = [n/2]$ and genus $\gamma = 2n + 2$,

$$\Omega_2[n] = \left\{Z \in M_n(\mathbb{C}), Z = -Z^T, \det(I_n - ZZ^*) > 0\right\}.$$

A parametrization is given by:

$$u = (u_{12}, \ldots, u_{1n}, u_{23}, \ldots, u_{2n} \ldots, u_{n-1n}) \mapsto Z(u) = \begin{pmatrix} 0 & u_{12} & u_{13} & \ldots & u_{1n-1} & u_{1n} \\ -u_{12} & 0 & u_{23} & \ldots & u_{2n-1} & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -u_{1n} & -u_{2n} & -u_{3n} & \ldots & -u_{n-1n} & 0. \end{pmatrix}$$

Its generic norm is given by:

$$N_{\Omega_2}(u, u) = \det{}^{1/2}\left(I_n - Z(u)Z^*(u)\right).\tag{8}$$

A totally geodesic polydisk $\Delta^{\left[\frac{n}{2}\right]} \xrightarrow{\varphi} \Omega_2[n]$ is given by:

$$\varphi(u) = \begin{pmatrix} 0 & 0 & \ldots & 0 & u_{1\left[\frac{n}{2}\right]} \\ 0 & 0 & \ldots & u_{2\left[\frac{n}{2}\right]-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -u_{2\left[\frac{n}{2}\right]-1} & \ldots & 0 & 0 \\ -u_{1\left[\frac{n}{2}\right]} & 0 & \ldots & 0 & 0 \end{pmatrix},\tag{9}$$

where $u = \left(u_{1\left[\frac{n}{2}\right]}, u_{2\left[\frac{n}{2}\right]-1}, \ldots, u_{\left[\frac{n}{2}\right]1}\right)$. Since

$$N_{\Omega_2}(\varphi(u), \varphi(u)) = \det{}^{1/2}\left(I_n - \varphi(u)\varphi^*(u)\right) = \prod_{j=1}^{\left[\frac{n}{2}\right]}(1 - |u_{j\left[\frac{n}{2}-j+1\right]}|^2),$$

φ is a Kähler immersion, moreover it is easy to check that $\varphi_*(T_0\Delta_m)$ defines a sub-HJPTS of $(T_0\Omega_2[n], \{,,\})$, where the triple product is given by $\{U, V, W\} = UV^*W + WV^*U$, namely the restriction to $T_0\Omega_2[n]$ of the triple product of $T_0\Omega_1[n, n]$ given in (7), we conclude, by

the one to one correspondence between sub-HJPTS e sub-HSSNT (see [4, Proposition 2.1]), that φ is totally geodesic.

2.3 Cartan domain of the third type

Consider the Cartan domain of third type of rank $r = m$ and genus $\gamma = n + 1$:

$$\Omega_3[m] = \left\{ Z \in M_m(\mathbb{C}) \mid Z = Z^T,\ \det(I_m - ZZ^*) > 0 \right\},$$

whose generic norm is given by:

$$N_{\Omega_3}(z, z) = \det \left(I_m - ZZ^* \right). \tag{10}$$

As can be proven in a totally similar way as done for the first and second type domains, a totally geodesic polydisk $\Delta^m \xrightarrow{\varphi} \Omega_1[m]$ is given by:

$$\varphi(z) = \operatorname{diag}(z_1, \ldots, z_m) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_m \end{pmatrix}. \tag{11}$$

2.4 Cartan domain of the fourth type

Consider the fourth type domain of rank $r = 2$ and genus $\gamma = n$:

$$\Omega_4[n] = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \ \middle|\ \sum_{j=1}^{n} |z_j|^2 < 1,\ 1 + \left| \sum_{j=1}^{n} z_j^2 \right|^2 - 2 \sum_{j=1}^{n} |z_j|^2 > 0,\ n \geq 5 \right\}.$$

whose generic norm is given by:

$$N_{\Omega_4[n]}(z, z) = 1 + \left| \sum_{j=1}^{n} z_j^2 \right|^2 - 2 \sum_{j=1}^{n} |z_j|^2. \tag{12}$$

Let $\varphi : \Delta^2 \to \Omega_4[n]$ be the map:

$$\varphi(z_1, z_2) = \left(\frac{1}{2} (z_1 + z_2), \frac{i}{2} (z_1 - z_2), 0, \ldots, 0 \right). \tag{13}$$

Since:

$$N_{\Omega_4[n]}(\varphi(z_1, z_2), \varphi(z_1, z_2)) = (1 - |z_1|^2)(1 - |z_2|^2) = N_{\Delta^2}(z_1, z_2),$$

φ is Kähler. Moreover $\varphi\left(\Delta^2\right)$ is the set of points of $\Omega_4[n]$ fixed by the isometry $(z_1, \ldots, z_n) \mapsto (z_1, z_2, -z_3 \ldots, -z_n)$, thus φ is totally geodesic.

3 The polydisk theorem for Cartan-Hartogs domains

Let us begin with the following lemma.

Lemma 1 *Let Ω be a Cartan domain and let $\varphi \colon \Delta^r \to \Omega$ be a Kähler immersion fixing the origin, i.e. a holomorphic map satisfying $\varphi^* \omega_{hyp}^{\Omega} = \omega_{hyp}^{\Delta^r}$. Then:*

$$f \colon M_{\Delta^r}(\mu) \to M_{\Omega}(\mu), \quad f(z, w) = (\varphi(z), w),$$

is a Kähler immersion.

Proof Observe that $-\log(N_{\Omega}(z, z))$ and $-\log(N_{\Delta^r}(z, z))$ are the diastasis functions respectively for $(\Omega, \omega_{hyp}^{\Omega})$ and $(\Delta^r, \omega_{hyp}^{\Delta^r})$ (see [11, Prop. 7] for a proof). Since the diastasis is a Kähler potential invariant by isometries (see [3] or also [13]), one has:

$$N_{\Omega}(\varphi(z), \varphi(z)) = N_{\Delta^r}(z, z).$$

Then, it follows easily that $f(M_{\Delta^r}(\mu)) \subset M_{\Omega}(\mu)$, since $|w|^2 < N_{\Delta^r}(z, z)$ implies $|w|^2 < N_{\Omega}(\varphi(z), \varphi(z))$. Further the map is isometric since:

$$\begin{aligned}\Phi_{\Omega,\mu}(f(z, w)) &= -\log\left(N_{\Omega}((\varphi(z), \varphi(z)) - |w|^2\right) \\ &= -\log\left(N_{\Delta^r}(z, z) - |w|^2\right) = \Phi_{\Delta^r,\mu}(z, w).\end{aligned}$$

$\square$

By this lemma the totally geodesic Kähler immersions described in the previous section induce Kähler immersions of Hartogs–polydisks into Cartan–Hartogs domains. We prove now case by case that such maps are also totally geodesics.

3.1 Cartan–Hartogs domain of the first type

By (1), (3) and (5), the Cartan-Hartogs domain associated to a first type Cartan domain is:

$$M_{\Omega_1[m,n]}(\mu) = \left\{(z, w) \in \Omega_1[m, n] \times \mathbb{C} \mid |w|^2 < \det^{\mu}\left(I_m - ZZ^*\right)\right\}.$$

and its Kobayashi metric is described by the Kähler potential:

$$\Phi_{\Omega_1,\mu}(z) = -\log\left(\det^{\mu}\left(I_m - ZZ^*\right) - |w|^2\right).$$

Lemma 2 *Let $\varphi \colon \Delta^m \to \Omega_1[m, n]$ be the map in (6). Then $f \colon M_{\Delta^m}(\mu) \to M_{\Omega_1[m,n]}(\mu)$, $f(z, w) = (\varphi(z), w)$, is a totally geodesic Kähler immersion.*

Proof From Sect. 2.1 the map φ is a Kähler immersion, thus by Lemma 1 also f is.

It remains to prove that f is totally geodesic. Let $Z = (z_{jk})$. From the expression of f and (6), we see that

$$f(M_{\Delta^m}(\mu)) = \left\{(Z, w) \in M_{\Omega_1[m,n]}(\mu) \mid Z = \operatorname{diag}(z_{11}, \ldots, z_{mm})\right\} \tag{14}$$

and that $\{\partial_{z_{jj}}, \partial_w\}_{j=1,\ldots,n}$ is a basis for $Tf(M_{\Delta^m}(\mu)) \subset TM_{\Omega_1[m,n]}(\mu)$. Thus, we need to show that:

$$\nabla_{\partial_{z_{jj}}} \partial_{z_{kk}}, \nabla_{\partial_w} \partial_{z_{kk}}, \nabla_{\partial_w} \partial_w \in Tf(M_{\Delta^m}(\mu)) \quad j, k = 1, \ldots, n. \tag{15}$$

Recalling that the covariant derivative in terms of Christoffel symbols reads:

$$\nabla_{\partial_{z_{jj}}} \partial_{z_{kk}} = \Gamma_{jjkk}^0 \partial_w + \sum_{s=1}^{m} \sum_{r=1}^{n} \Gamma_{jjkk}^{sr} \partial_{z_{sr}}, \qquad \nabla_{\partial_w} \partial_{z_{kk}} = \Gamma_{0kk}^0 \partial_w + \sum_{s=1}^{m} \sum_{r=1}^{n} \Gamma_{0kk}^{sr} \partial_{z_{sr}}$$

and

$$\nabla_{\partial_w} \partial_w = \Gamma^0_{00} \partial_w + \sum_{s=1}^{m} \sum_{r=1}^{n} \Gamma^{sr}_{00} \partial_{z_{sr}},$$

where we use the index 0 for the w-entry, we see that (15) is equivalent to prove

$$\Gamma^{sr}_{jjkk} = \Gamma^{sr}_{0kk} = \Gamma^{sr}_{00} = 0, \quad s, r = 1, \ldots, n,$$

under the conditions $s \neq r$ and $Z = \mathrm{diag}(z_{11}, \ldots, z_{mm})$.

Let us start with some preliminary computations. Let $A = I - ZZ^*$, that is

$$(A)_{j\ell} = \delta_{j\ell} - (ZZ^*)_{j\ell} = \delta_{j\ell} - \sum_{k=1}^{n} z_{jk} \bar{z}_{\ell k}. \tag{16}$$

Denote by $A_{j_1 \ldots j_s, k_1 \ldots k_s}$ the matrix A after the $j_1, \ldots, j_s$-th rows and $k_1, \ldots, k_s$-th columns have been deleted. We have:

$$\begin{aligned}
\frac{\partial \det A}{\partial z_{jk}} &= \frac{\partial}{\partial z_{jk}} \left(\sum_{\ell=1}^{n} (-1)^{j+\ell} (A)_{j,\ell} \det A_{j,\ell} \right) \\
&= \sum_{\ell=1}^{n} (-1)^{j+\ell} \frac{\partial \left(\delta_{j\ell} - \sum_{i=1}^{n} z_{ji} \bar{z}_{\ell i} \right)}{\partial z_{jk}} \det A_{j,\ell} \\
&= \sum_{\ell=1}^{n} (-1)^{j+\ell+1} \bar{z}_{\ell k} \det A_{j,\ell},
\end{aligned} \tag{17}$$

Similarly we obtain:

$$\frac{\partial \det A}{\partial \bar{z}_{pq}} = \sum_{\ell=1}^{n} (-1)^{q+\ell+1} z_{p\ell} \det A_{\ell,q},$$

which evaluated at $Z = \mathrm{diag}(z_{11}, \ldots, z_{mm})$, since $\det A_{p,q}(Z) = 0$ whenever $p \neq q$, reads:

$$\frac{\partial \det A}{\partial \bar{z}_{pq}}(Z) = -\delta_{pq} z_{qq} \det A_{q,q}(Z). \tag{18}$$

By (16) it follows that $\frac{\partial}{\partial \bar{z}_{pq}} \det A_{j\ell}$ vanishes when $\ell = p$. Assume that $\ell \neq p$ and expand the determinant with respect to the p-th column. We obtain:

$$\begin{aligned}
\frac{\partial \det A_{j,\ell}}{\partial \bar{z}_{pq}} &= \sum_{h \neq j, h'=1}^{n-1} (-1)^{p'+h'} \frac{\partial (\delta_{hp} - \sum_{i=1}^{n} z_{hi} \bar{z}_{pi})_{hp}}{\partial \bar{z}_{pq}} \det A_{hj, p\ell} \\
&= \sum_{h \neq j, h'=1}^{n-1} (-1)^{p'+h'+1} z_{hq} \det A_{hj, p\ell},
\end{aligned} \tag{19}$$

where j' is the index of the j-th row in A as a row in $A_{p\ell}$, and similarly the h-th column in A is the h'-th ones in $A_{j\ell}$. An analogous computation gives:

$$\frac{\partial \det A_{j,p}}{z_{ab}} = (1 - \delta_{aj}) \sum_{h \neq p, h'=1}^{n-1} (-1)^{a'+h'+1} \bar{z}_{hb} \det A_{ja, hp}. \tag{20}$$

Thus using (17) and (19):

$$\frac{\partial^2 \det A}{\partial \bar{z}_{pq} \partial z_{jk}} = \frac{\partial}{\partial \bar{z}_{pq}} \sum_{\ell=1}^{n} (-1)^{j+\ell+1} \bar{z}_{\ell k} \det A_{j,\ell}$$

$$= (-1)^{j+p+1} \delta_{qk} \det A_{j,p} + \sum_{p \neq \ell=1}^{n} (-1)^{j+\ell+1} \bar{z}_{\ell k} \frac{\partial \det A_{j,\ell}}{\partial \bar{z}_{pq}} \tag{21}$$

$$= (-1)^{j+p+1} \delta_{qk} \det A_{j,p} + \sum_{p \neq \ell=1}^{n} \sum_{h \neq j, h'=1}^{n-1} (-1)^{j+\ell+p'+h'} \bar{z}_{\ell k} z_{hq} \det A_{hj,p\ell},$$

which evaluated at $Z = \mathrm{diag}(z_{11}, \ldots, z_{mm})$ reads:

$$\frac{\partial^2 \det A}{\partial \bar{z}_{pq} \partial z_{jk}}(Z) = -\delta_{jp} \delta_{qk} \det A_{j,j}(Z)$$

$$+ (1 - \delta_{pk})(1 - \delta_{qj}) \bar{z}_{kk} z_{qq} (-1)^{j+k+p'+q'} \det A_{qj,pk}(Z), \tag{22}$$

$$= -\delta_{jp} \delta_{qk} \det A_{j,j}(Z) + \delta_{qp} \delta_{kj} (1 - \delta_{qj}) \bar{z}_{jj} z_{qq} \det A_{qj,qj}(Z)$$

$$+ \delta_{qk} \delta_{pj} (1 - \delta_{jk}) \bar{z}_{kk} z_{jj} \det A_{jk,kj}(Z),$$

where we used that $\det A_{j_1 \ldots j_s, k_1 \ldots k_s}(Z) = 0$ whenever $\{j_1, \ldots, j_s\} \neq \{k_1, \ldots, k_s\}$. Finally consider that by (20) and (21) it follows:

$$\frac{\partial^3 \det A}{\partial z_{ab} \partial \bar{z}_{pq} \partial z_{jk}} = \frac{\partial}{z_{ab}} \left((-1)^{j+p+1} \delta_{qk} \det A_{j,p} + \sum_{p \neq \ell=1}^{n} \sum_{h \neq j, h'=1}^{n-1} (-1)^{j+\ell+p'+h'} \bar{z}_{\ell k} z_{hq} \det A_{hj,p\ell}, \right)$$

$$= (-1)^{j+p+1} \delta_{qk} (1 - \delta_{aj}) \sum_{h \neq p, h'=1}^{n-1} (-1)^{a'+h'+1} \bar{z}_{hb} \det A_{ja,hp} +$$

$$+ \sum_{p \neq \ell=1}^{n} (-1)^{j+\ell+p'+a'} \bar{z}_{\ell k} (1 - \delta_{aj}) \delta_{qb} \det A_{aj,p\ell} + \tag{23}$$

$$+ \sum_{p \neq \ell=1}^{n} \sum_{h \neq j, h'=1}^{n-1} (-1)^{j+\ell+p'+h'} \bar{z}_{\ell k} z_{hq} \frac{\partial}{\partial z_{ab}} \det A_{hj,p\ell}.$$

In order to deal with the third and last term in the above formula, observe that by (16), for $\frac{\partial}{\partial z_{ab}} \det A_{hj,p\ell}$ to not vanish, we need $a \notin \{h, j\}$. When $a \notin \{h, j\}$ we can expand the determinant with respect to the a-th row and we obtain:

$$\frac{\partial}{\partial z_{ab}} \det A_{hj,p\ell} = \sum_{r \neq \{p,\ell\}, r'=1}^{n-2} (-1)^{r'+a'} \frac{\partial (\delta_{ar} - \sum_{i=1}^{n} z_{ai} \bar{z}_{ri})}{\partial z_{ab}} \det A_{hja,p\ell r}$$

$$= \sum_{r \neq \{p,\ell\}, r'=1}^{n-2} (-1)^{r'+a'+1} \bar{z}_{rb} \det A_{hja,p\ell r}. \tag{24}$$

In particular (23) and (24) imply:

$$\frac{\partial^3 \det A}{\partial z_{aa} \partial \bar{z}_{pq} \partial z_{jj}}(Z) = \delta_{pq} \delta_{jp} (1 - \delta_{aj}) \bar{z}_{aa} \det A_{aj,aj}(Z) + \delta_{pq} \delta_{qa} (1 - \delta_{aj}) \bar{z}_{jj} \det A_{aj,aj}(Z) +$$

$$- \delta_{qp} (1 - \delta_{jq})(1 - \delta_{aj}) \bar{z}_{jj} z_{qq} \bar{z}_{aa} \det A_{qja,qja}(Z) +$$

$$- \delta_{pq} \delta_{ap} (1 - \delta_{aj}) \bar{z}_{jj} |z_{qq}|^2 \det A_{qja,ajq}(Z),$$

and thus:

$$\frac{\partial^3 \det A}{\partial z_{aa} \partial \overline{z}_{pq} \partial z_{jj}}(Z) = 0 \quad \text{for } p \neq q. \tag{25}$$

We will now use (18), (22) and (25) to compute the Christoffel symbols at $Z = \mathrm{diag}(z_{11}, \ldots, z_{mm})$. Let us start with $\Gamma^{rs}_{jjkk} = \sum_\ell g^{rs,\overline{\ell}} \frac{\partial g_{kk,\overline{\ell}}}{\partial z_{jj}}$. We are going to show that $\frac{\partial g_{kk,\overline{pq}}}{\partial z_{jj}}$ is equal to zero when $p \neq q$, and that when $p = q, r \neq s$ implies $g^{rs,\overline{pq}} = 0$. We have:

$$\frac{\partial \Phi_{\Omega_1,\mu}}{\partial \overline{z}_{pq}} = -\frac{\partial \log((\det A)^\mu - |w|^2)}{\partial \overline{z}_{pq}} = \frac{\mu(\det A)^{\mu-1}}{(\det A)^\mu - |w|^2} \frac{\partial \det A}{\partial \overline{z}_{pq}},$$

$$\frac{\partial^2 \Phi_{\Omega_1,\mu}}{\partial z_{jj} \partial \overline{z}_{pq}} = \frac{\mu(\det A)^{\mu-2}}{(\det A)^\mu - |w|^2} \left(\det A \frac{\partial^2 \det A}{\partial z_{jj} \partial \overline{z}_{pq}} - \left(\frac{(\det A)^\mu + (\mu-1)|w|^2}{(\det A)^\mu - |w|^2} \right) \frac{\partial \det A}{\partial \overline{z}_{pq}} \frac{\partial \det A}{\partial z_{jj}} \right)$$

$$= B_1(B_2 - B_3), \tag{26}$$

where we set:

$$B_1 := \frac{\mu(\det A)^{\mu-2}}{(\det A)^\mu - |w|^2}, \quad B_2 := \det A \frac{\partial^2 \det A}{\partial z_{jj} \partial \overline{z}_{pq}};$$

$$B_3 := \frac{(\det A)^\mu + (\mu-1)|w|^2}{(\det A)^\mu - |w|^2} \frac{\partial \det A}{\partial \overline{z}_{pq}} \frac{\partial \det A}{\partial z_{jj}}.$$

Then:

$$\frac{\partial^3 \Phi_{\Omega_1,\mu}}{\partial z_{aa} \partial z_{jj} \partial \overline{z}_{pq}} = \left(\frac{\partial}{\partial z_{aa}} B_1 \right)(B_2 - B_3) + B_1 \left(\frac{\partial}{\partial z_{aa}} B_1 - \frac{\partial}{\partial z_{aa}} B_2 \right).$$

By (18) and since by (22) at $Z = \mathrm{diag}(z_{11}, \ldots, z_{mm})$:

$$\frac{\partial^2 \det A}{\partial z_{jj} \partial \overline{z}_{pq}} = -\delta_{pq}\delta_{qj} \det A_{j,j} + \delta_{pq}(1 - \delta_{qj})\overline{z}_{jj} z_{qq} \det A_{qj,qj}, \tag{27}$$

$B_2 - B_3$ vanishes when $p \neq q$. By (18), (25) and (27), the same holds also for $\frac{\partial}{\partial z_{aa}} B_1$ and $\frac{\partial}{\partial z_{aa}} B_2$. At this point it is easy to see that, by (27), $g_{rs,\overline{pp}} = \frac{\partial^2 \Phi_{\Omega_1,\mu}}{\partial z_{rs} \partial \overline{z}_{pp}}$ vanishes for $r \neq s$, thus also its inverse has the same property, and we are done.

The same conclusions can readily be reached also for Γ^{rs}_{0kk} and Γ^{rs}_{00}. In particular, with the same notation as above we have:

$$\frac{\partial^3 \Phi_{\Omega_1,\mu}}{\partial w \partial z_{jj} \partial \overline{z}_{pq}} = \left(\frac{\partial}{\partial w} B_1 \right)(B_2 - B_3) + B_1 \left(-\frac{\partial}{\partial w} B_3 \right).$$

By the discussion above the term $B_2 - B_3$ vanishes when $p \neq q$. Further the derivative $\frac{\partial}{\partial w} B_3$ contains the factor $\frac{\partial \det A}{\partial \overline{z}_{pq}}$ which vanishes when $p \neq q$ by (18). Thus

$$\Gamma^{rs}_{0kk} = \sum_\ell g^{rs,\overline{\ell}} \frac{\partial g_{0,\overline{\ell}}}{\partial z_{kk}} = 0, \quad \text{for } r \neq s, \ Z = \mathrm{diag}(z_{11}, \ldots, z_{mm}).$$

Finally:

$$\frac{\partial^2}{\partial w^2} \frac{\partial \Phi_{\Omega_1,\mu}}{\partial \overline{z}_{pq}} = 2 \frac{\mu \overline{w}^2 (\det A)^{\mu-1}}{((\det A)^\mu - |w|^2)^3} \frac{\partial \det A}{\partial \overline{z}_{pq}},$$

thus again by (18),

$$\Gamma_{00}^{rs} = \sum_\ell g^{rs,\overline{\ell}} \frac{\partial g_{0,\overline{\ell}}}{\partial w} = 0, \quad \text{for } r \neq s, \ Z = \text{diag}(z_{11}, \dots, z_{mm}),$$

completing the proof. □

3.2 Cartan–Hartogs domain of the second type

By (1), (3) and (8), the Cartan–Hartogs associated to the second type Cartan domain is:

$$M_{\Omega_2[n]}(\mu) = \{(u, w) \in \Omega_2[n] \times \mathbb{C} \mid |w|^2 < \det^{\mu/2}(I_m - Z(u)Z(u)^*)\},$$

and a Kähler potential for its Kobayashi metric is:

$$\Phi_{\Omega_2,\mu}(u, w) = -\log\left(\det^{\mu/2}(I_n - Z(u)Z(u)^*) - |w|^2\right).$$

Lemma 3 *Let $\varphi : \Delta^{\frac{n}{2}} \to \Omega_2[n]$ be the map in (9). Then $f : M_{\Delta^{n/2}}(\mu) \to M_{\Omega_2[n]}(\mu)$, $f(u, w) = (\varphi(u), w)$, is a totally geodesic Kähler immersion.*

Proof From Sect. 2.2 the map φ is Kähler, thus by Lemma 1 f also is. Let us use the parametrization described in Sect. 2.2. In terms of the Christoffel symbols, since:

$$\nabla_{\partial_{u_{j\,n+1-j}}} \partial_{u_{k\,n+1-k}} = \sum_{s,r=1}^n \Gamma_{j\,n+1-j,\,k\,n+1-k}^{sr} \partial_{u_{sr}} + \Gamma_{j\,n+1-j,\,k\,n+1-k}^0 \partial_w,$$

and

$$\nabla_{\partial_w} \partial_{u_{k\,n+1-k}} = \sum_{s,r=1}^n \Gamma_{0,\,k\,n+1-k}^{sr} \partial_{u_{sr}} + \Gamma_{0,\,k\,n+1-k}^0 \partial_w,$$

the map f to be totally geodesic is equivalent to:

$$\Gamma_{j\,n+1-j,\,k\,n+1-k}^{sr} = \Gamma_{0,\,k\,n+1-k}^{sr} = 0, \tag{28}$$

for $s \neq n+1-r$ and $1 \leq j, k \leq n/2$. Observing that $\Phi_{\Omega_2,\mu}(u, w) = \Phi_{\Omega_1,\mu}(Z(u), w)$ once substituted μ with $\mu/2$ in the second term, we have:

$$\begin{aligned}
\frac{\partial \Phi_{\Omega_2,\mu}}{\partial \overline{u}_{rs}} &= \frac{\partial \Phi_{\Omega_1,\mu}}{\partial \overline{z}_{rs}} - \frac{\partial \Phi_{\Omega_1,\mu}}{\partial \overline{z}_{sr}} = -\frac{\partial \log(\det^{\frac{\mu}{2}}A - |w|^2)}{\partial \overline{z}_{rs}} + \frac{\partial \log(\det^{\frac{\mu}{2}}A - |w|^2)}{\partial \overline{z}_{sr}} \\
&= \frac{\frac{\mu}{2}\det^{\frac{\mu}{2}-1}A\left(\sum_{\ell=1}^n \epsilon_{\ell r} \det \widetilde{A}_{\ell r} z_{\ell s}(u) - \sum_{\ell=1}^n \epsilon_{\ell s} \det \widetilde{A}_{\ell s} z_{\ell r}(u)\right)}{\det^{\frac{\mu}{2}}A - |w|^2},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\frac{\partial^2 \Phi_{\Omega_2,\mu}}{\partial w \partial \overline{u}_{rs}} &= \frac{\partial^2 \Phi_{\Omega_1,\mu}}{\partial w \partial \overline{z}_{rs}} - \frac{\partial^2 \Phi_{\Omega_1,\mu}}{\partial w \partial \overline{z}_{sr}} \\
&= \frac{\overline{w}\frac{\mu}{2}\det^{\frac{\mu}{2}-1}A\left(\sum_{\ell=1}^n \epsilon_{\ell r} \det \widetilde{A}_{\ell r} z_{\ell s}(u) - \sum_{\ell=1}^n \epsilon_{\ell s} \det \widetilde{A}_{\ell s} z_{\ell r}(u)\right)}{\left(\det^{\frac{\mu}{2}}A - |w|^2\right)^2}.
\end{aligned} \tag{30}$$

For $(u, w) \in f(\Delta_{[n/2]})$ and $r \neq n + 1 - s$, we get

$$\frac{\partial^2 \phi_2}{\partial w \partial \overline{u}_{rs}}$$

$$= \frac{\frac{\mu}{2}\det^{\frac{\mu}{2}-1}A\left(\epsilon_{n+1-r\,r}\det\widetilde{A}_{n+1-s\,r}\,z_{n+1-s\,s}(u) - \epsilon_{n+1-s\,s}\det\widetilde{A}_{n+1-r\,s}\,z_{n+1-r\,r}(u)\right)}{\det^{\frac{\mu}{2}}A - |w|^2} = 0.$$

Notice that

$$\frac{\partial^2\Phi_{\Omega_2,\mu}}{\partial u_{jk}\,\partial\overline{u}_{rs}} = \left(\frac{\partial\Phi_{\Omega_1,\mu}}{\partial z_{jk}} - \frac{\partial\Phi_{\Omega_1,\mu}}{\partial z_{kj}}\right)\left(\frac{\partial\Phi_{\Omega_1,\mu}}{\partial\overline{z}_{rs}} - \frac{\partial\Phi_{\Omega_1,\mu}}{\partial\overline{z}_{sr}}\right) \tag{31}$$
$$= \frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{rs}} + \frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{sr}} - \frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{rs}} - \frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{sr}}.$$

If we take $\overline{u}_{rs}$ with $r \neq n+1-s$ and u_{jk} with $(j,k) \neq (r,s)$, then the indexes of $\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{rs}}$, $\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{sr}}$, $\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{rs}}$, $\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{sr}}$ in (31) must satisfies $r \neq n+1-s$, $(j,k) \neq (r,s)$ and $(k,j) \neq (r,s)$. Under this conditions on the indexes, it is just a straightforward computation to prove that $\left(\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{rs}}\right)_{|(u,w)\in f(\Delta_{[n/2]})} = \left(\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{sr}}\right)_{|(u,w)\in f(\Delta_{[n/2]})} = \left(\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{kj}\,\partial\overline{z}_{rs}}\right)_{|(u,w)\in f(\Delta_{[n/2]})} = \left(\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{jk}\,\partial\overline{z}_{sr}}\right)_{|(u,w)\in f(\Delta_{[n/2]})} = 0$, and in particular that $\left(\frac{\partial^2\Phi_{\Omega_2,\mu}}{\partial u_{jk}\,\partial\overline{u}_{rs}}\right)_{|Z(u)\in\Delta} = 0$. We conclude that for $(u,w)\in f(\Delta_{[n/2]})$ and $r\neq n+1-s$ we have

$$\Gamma^{sr}_{0,\,k\,n+1-k} = \sum_{\ell} g^{rs,\overline{\ell}}\frac{\partial g_{k\,n+1-k,\overline{\ell}}}{\partial w} = g^{rs,\overline{rs}}\frac{\partial g_{k\,n+1-k,\overline{rs}}}{\partial w},$$

and

$$\Gamma^{sr}_{n+1-j\,j,\,n+1-k\,k} = \sum_{\ell} g^{rs,\overline{\ell}}\frac{\partial g_{n+1-k\,k,\overline{\ell}}}{\partial u_{n+1-j\,j}} = g^{rs,\overline{rs}}\frac{\partial g_{n+1-k\,k,\overline{rs}}}{\partial u_{n+1-j\,j}}.$$

Deriving (30), we can see that $\frac{\partial g_{k\,[n+1-k],\overline{rs}}}{\partial w} = 0$, which readily implies that $\Gamma^{sr}_{0,\,k\,n+1-k} = 0$. It remains to prove that, under the above conditions $\frac{\partial g_{n+1-k\,k,\overline{rs}}}{\partial u_{n+1-j\,j}} = 0$ (or equivalently that $\frac{\partial^3\phi_2}{\partial u_{n+1-j\,j}\,\partial u_{n+1-k\,k}\,\partial\overline{u}_{rs}} = 0$). We have

$$\frac{\partial^3\Phi_{\Omega_1,\mu}}{\partial z_{j\,[n+1-j]}\,\partial z_{k\,[n+1-k]}\,\partial\overline{z}_{rs}}$$
$$= \frac{\partial}{\partial z_{j\,[n+1-j]}}\left(\frac{\partial^2\Phi_{\Omega_1,\mu}}{\partial z_{k\,[n+1-k]}\,\partial\overline{z}_{rs}}\right)$$
$$= \frac{\partial}{\partial z_{j\,[n+1-j]}}\left(\frac{\frac{\mu}{2}(\frac{\mu}{2}-1)\det^{\frac{\mu}{2}-2}A\,\epsilon_{kk}\det\widetilde{A}_{kk}\,\overline{z}_{k\,[n+1-k]}\,\epsilon_{n+1-s\,r}\det\widetilde{A}_{[n+1-s]r}\,z_{[n+1-s]s}}{\det^{\frac{\mu}{2}}A - |w|^2} + \right.$$
$$+ \frac{\frac{\mu}{2}\det^{\frac{\mu}{2}-1}A\left(\epsilon_{[n+1-s]r}\epsilon'_{kk}\det\widetilde{A}_{[n+1-s]r,kk}\,\overline{z}_{k\,[n+1-k]}\,z_{[n+1-s]s} + \epsilon_{kr}\det\widetilde{A}_{kr}\,\delta_{[n+1-k]s}\right)}{\det^{\frac{\mu}{2}}A - |w|^2} + \tag{32}$$
$$\left. - \frac{\left(\frac{\mu}{2}\det^{\frac{\mu}{2}-1}A\right)^2\epsilon_{[n+1-s]r}\det\widetilde{A}_{[n+1-s]r}\,z_{[n+1-s]s}\,\epsilon_{kk}\det\widetilde{A}_{kk}\,\overline{z}_{k\,[n+1-k]}}{\left(\det^{\frac{\mu}{2}}A - |w|^2\right)^2}\right).$$

If we assume that $(u, w) \in f(\Delta_{[n/2]})$, we obtain

$$
\begin{aligned}
&\frac{\partial^3 \Phi_{\Omega_1,\mu}}{\partial z_{j[n+1-j]} \partial z_{k[n+1-k]} \partial \overline{z}_{rs}} \\
&= \frac{\frac{\mu}{2}(\frac{\mu}{2} - 1) \det^{\frac{\mu}{2}-2} A \epsilon_{kk} \det \widetilde{A}_{kk} \overline{z}_{k[n+1-k]} \epsilon_{[n+1-s]r} \frac{\partial \det \widetilde{A}_{[n+1-s]r}}{\partial z_{j[n+1-j]}} z_{[n+1-s]s}}{\det^{\frac{\mu}{2}} A - |w|^2} + \\
&+ \frac{\frac{\mu}{2} \det^{\frac{\mu}{2}-1} A \left(\epsilon_{[n+1-s]r} \epsilon'_{kk} \frac{\partial \det \widetilde{A}_{[n+1-s]r,kk}}{\partial z_{j[n+1-j]}} \overline{z}_{k[n+1-k]} z_{[n+1-s]s} + \epsilon_{kr} \det \widetilde{A}_{kr} \delta_{[n+1-k]s} \right)}{\det^{\frac{\mu}{2}} A - |w|^2} + \\
&- \frac{\left(\frac{\mu}{2} \det^{\frac{\mu}{2}-1} A \right)^2 \epsilon_{[n+1-s]r} \frac{\partial \det \widetilde{A}_{[n+1-s]r}}{\partial z_{j[n+1-j]}} z_{[n+1-s]s} \epsilon_{kk} \det \widetilde{A}_{kk} \overline{z}_{k[n+1-k]}}{\left(\det^{\frac{\mu}{2}} A - |w|^2 \right)^2},
\end{aligned} \tag{33}
$$

hence if we also assume $s \neq n + 1 - r$ and $1 \leq j, k \leq [n/2]$ we see that $\frac{\partial^3 \Phi_{\Omega_1,\mu}}{\partial z_{j[n+1-j]} \partial z_{k[n+1-k]} \partial \overline{z}_{rs}} = 0$. Thus (28) holds true, concluding the proof. $\square$

3.3 Cartan–Hartogs domain of the third type

By (1), (3) and (10), the Cartan–Hartogs associated to a third type domain is given by:

$$
M_{\Omega_3[m]}(\mu) = \{(Z, w) \in \Omega_2[n] \times \mathbb{C} \mid |w|^2 < \det^\mu (I_m - ZZ^*)\},
$$

and its Kobayashi metric is described by the Kähler potential:

$$
\Phi_{\Omega_3,\mu}(Z, w) = -\log \left(\det^\mu (I_m - ZZ^*) - |w|^2 \right).
$$

Lemma 4 *Let* $\varphi : \Delta^m \to \Omega_3[m]$ *be the map in* (11). *Then* $f : M_{\Delta^m}(\mu) \to M_{\Omega_3[m]}(\mu)$, $f(z, w) = (\varphi(z), w)$, *is a totally geodesic Kähler immersion.*

Proof The proof is similar to those of Lemma 2 and Lemma 3 and therefore is omitted. $\square$

3.4 Cartan–Hartogs domain of the fourth type

By (1), (3) and (12), the Cartan–Hartogs associated to a fourth type domain is given by:

$$
M_{\Omega_4[n]}(\mu) = \left\{ (u, w) \in \Omega_4[n] \times \mathbb{C} \mid |w|^2 < \left(1 + \left| \sum_{j=1}^n z_j^2 \right|^2 - 2 \sum_{j=1}^n |z_j|^2 \right)^\mu \right\},
$$

and a Kähler potential for the Kobayashi metric is:

$$
\Phi_{\Omega_4,\mu}(z, w) = -\log \left(\left(1 + \left| \sum_{j=1}^n z_j^2 \right|^2 - 2 \sum_{j=1}^n |z_j|^2 \right)^\mu - |w|^2 \right).
$$

Lemma 5 *Let* $\varphi : \Delta^2 \to \Omega_4[n]$ *be the map in* (13). *Then* $f : M_{\Delta^2}(\mu) \to M_{\Omega_4[n]}(\mu)$, $f(z_1, z_2, w) = (\varphi(z_1, z_2), w)$, *is a totally geodesic Kähler immersion.*

Proof From Sect. 2.4 the map φ is a Kähler immersion, thus by Lemma 1 f also is. It remains to prove that f is totally geodesic, which is equivalent to $\Gamma^\ell_{jk} = 0$ for $\ell > 2$ and $0 \le j, k \le 2$, where:

$$\nabla_{\partial_{z_j}} \partial_{z_k} = \sum_{\ell=1}^n \Gamma^\ell_{jk} \, \partial_{z_\ell} + \Gamma^0_{jk} \, \partial_w, \qquad \nabla_{\partial_w} \partial_{z_k} = \sum_{\ell=1}^n \Gamma^\ell_{0k} \, \partial_{z_\ell} + \Gamma^0_{0k} \, \partial_w.$$

We have

$$
\begin{aligned}
\frac{\partial \Phi_{\Omega_4,\mu}}{\partial \overline{z}_k} =& \frac{-\partial \log\left(\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2\right)}{\partial \overline{z}_k} \\
=& \frac{-\mu\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^{\mu-1}\left(2\overline{z}_k \sum_{\ell=1}^n z_\ell^2 - 2z_k\right)}{\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2}, \\
\frac{\partial^2 \Phi_{\Omega_4,\mu}}{\partial w \partial \overline{z}_k} =& \frac{-\mu\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^{\mu-1}\left(2\overline{z}_k \sum_{\ell=1}^n z_\ell^2 - 2z_k\right)\overline{w}}{\left(\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2\right)^2},
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \Phi_{\Omega_4,\mu}}{\partial z_h \partial \overline{z}_k} =& \frac{-\mu(\mu-1)\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^{\mu-2}\left(2\overline{z}_k \sum_{\ell=1}^n z_\ell^2 - 2z_k\right)\left(2z_h \sum_{\ell=1}^n \overline{z}_\ell^2 - 2\overline{z}_h\right)}{\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2} \\
&+ \frac{-\mu\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^{\mu-1}\left(4\overline{z}_k z_h - 2\delta_{hk}\right)}{\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2} \\
&+ \frac{\mu^2\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^{2\mu-2}\left(2\overline{z}_k \sum_{\ell=1}^n z_\ell^2 - 2z_k\right)\left(2z_h \sum_{\ell=1}^n \overline{z}_\ell^2 - 2\overline{z}_h\right)}{\left(\left(1 + \left|\sum_{\ell=1}^n z_\ell^2\right|^2 - 2\sum_{\ell=1}^n |z_\ell|^2\right)^\mu - |w|^2\right)^2}.
\end{aligned}
$$

Hence for $(z, w) \in f(M_{\Delta_2})$, $2 \ge j, h \ge 1$ and $k > 2$, we have

$$\Gamma^k_{0h} = \sum_\ell g^{k,\overline{\ell}} \frac{\partial g_{h\overline{\ell}}}{\partial w} = g^{k,\overline{k}} \frac{\partial g_{h\overline{k}}}{\partial w}$$

and

$$\Gamma^k_{jh} = \sum_\ell g^{k,\overline{\ell}} \frac{\partial g_{h\overline{\ell}}}{\partial z_j} = g^{k,\overline{k}} \frac{\partial g_{k\overline{h}}}{\partial z_j},$$

where we used that for $(z, w) \in f(M_{\Delta_2}) = \left\{(z, w) \in M_{\Omega_4[n]} \mid z_3 = \cdots = z_n = 0\right\}$, $k > 2$ and $k \ne \ell$ we have $g^{k,\overline{\ell}}(z, w) = 0$. It is straightforward to check that under this conditions $\frac{\partial^3 \Phi_{\Omega_4,\mu}}{\partial w \partial z_h \partial \overline{z}_k}(z, w) = 0$ and $\frac{\partial^3 \Phi_{\Omega_4,\mu}}{\partial z_j \partial z_h \partial \overline{z}_k}(z, w) = 0$, namely that $\Gamma^\ell_{jk}(z, w) = 0$. Therefore $f(M_{\Delta_2})$ is totally geodesic in $M_{\Omega_4[n]}$. The proof is complete. $\qquad\square$

3.5 Cartan–Hartogs domains with reducible base

Let us consider a Cartan–Hartogs domain $M_\Omega(\mu)$ where $\Omega = \Omega_1 \times \cdots \times \Omega_m$ is a product of the irreducible Cartan domains $\Omega_1, \ldots, \Omega_m$. The generic norm and the Kobayashi metric are given by (2) and (3).

Let r_j be the rank of Ω_j and let $\varphi_j : \Delta^{r_j} \to \Omega_j$ the map given in (6), (9), (11) or (13), accordingly with the type of Ω_j, and let us denote by Δ^r the polydisk of Ω of rank $r = r_1 + \cdots + r_m$.

Lemma 6 *Let $\varphi : \Delta^r \to \Omega$ be the product $\varphi = \varphi_1 \times \cdots \times \varphi_m$. Then $f : M_{\Delta^r}(\mu) \to M_\Omega(\mu)$, $f(z, w) = (\varphi(z), w)$, is a totally geodesic Kähler immersion.*

Proof By applying Lemma 1 we see that f is a Kähler immersion. Let us prove that f is totally geodesic for $m = 2$, i.e. $\Omega = \Omega_1 \times \Omega_2$.

Let $Z = (z_1, \ldots, z_n, u_1, \ldots, u_h) \in \Omega_1 \times \Omega_2$, for $n = \dim(\Omega_1)$ and $h = \dim(\Omega_2)$, and denote by r and s the ranks of Ω_1 and Ω_2 respectively. We will use j, k, i, ℓ, for the indices of z and α, β, γ, for those of u. By construction of f and up to reordering the coordinates we can write

$$f(M_{\Delta^r}(\mu)) = \{(Z, w) \in M_\Omega(\mu) \mid z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}.$$

In order to prove that f is totally geodesic it is enough to show that

$$\nabla_{\partial_{z_j}} \partial_{z_k}, \nabla_{\partial_{z_j}} \partial_{u_\alpha}, \nabla_{\partial_w} \partial_{z_k}, \nabla_{\partial_w} \partial_w \in Tf(M_{\Delta^r}(\mu)), \tag{35}$$

for $j, k = 1, \ldots, r, \alpha = 1, \ldots, s$.

Let us define:

$$\phi := \Phi_{\Omega, \mu} = -\log\left(N_1^\mu N_2^\mu - |w|^2\right) \quad \text{and}$$
$$\phi_i := \Phi_{\Omega_i, \mu} = -\log\left(N_i^\mu - |w|^2\right), \quad i = 1, 2.$$

In order to compute the Christoffel symbols at $(Z, w) \in f(M_{\Delta^r}(\mu))$, that is at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$, we will first write the derivatives of ϕ in terms of those of ϕ_i. Then, we will apply the computations in the proofs of Lemma 2, 3, 4 and 5, and in particular that at $(z, w) \in f_1(M_{\Delta^r}(\mu))$, where f_1 denotes the map in Lemma 2, 3, 4 or 5 accordingly with the type of Ω_1, for $i, j = 1, \ldots, r$, and $k = r + 1, \ldots, n$, we have:

$$\frac{\partial\phi_1}{\partial z_k} = \frac{\partial^2\phi_1}{\partial\overline{z}_j \partial z_k} = \frac{\partial^2\phi_1}{\partial\overline{z}_k \partial w} = \frac{\partial^3\phi_1}{\partial z_i \partial\overline{z}_k \partial z_j} = \frac{\partial^3\phi_1}{\partial^2 w \partial\overline{z}_k} = \frac{\partial^3\phi_1}{\partial z_j \partial\overline{z}_k \partial w} = 0. \tag{36}$$

Observing that $\frac{\partial N_1^\mu}{\partial z_k} = -(N_1^\mu - |w|^2)\frac{\partial\phi_1}{\partial z_k}$, we have:

$$\frac{\partial\phi}{\partial z_k} = -\frac{N_2^\mu}{N_1^\mu N_2^\mu - |w|^2} \frac{\partial N_1^\mu}{\partial z_k} = \frac{\partial\phi_1}{\partial z_k} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}, \tag{37}$$

thus, at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$, (36) applied to (37) gives:

$$\frac{\partial\phi}{\partial z_k} = 0, \quad \text{for } k = r + 1, \ldots, n. \tag{38}$$

Further we have:

$$\frac{\partial^2\phi}{\partial\overline{z}_k \partial z_j} = \frac{\partial^2\phi_1}{\partial\overline{z}_k \partial z_j} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} + \frac{\partial\phi_1}{\partial z_j} \frac{\partial}{\partial\overline{z}_k}\left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right), \tag{39}$$

$$\frac{\partial^2\phi}{\partial\overline{u}_\alpha \partial z_k} = \frac{\partial\phi_1}{\partial z_k} \frac{\partial}{\partial\overline{u}_\alpha}\left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right), \tag{40}$$

$$\frac{\partial^2\phi}{\partial\overline{w} \partial z_k} = \frac{\partial^2\phi_1}{\partial\overline{w} \partial z_k} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} + \frac{\partial\phi_1}{\partial z_k} \frac{\partial}{\partial\overline{w}}\left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right). \tag{41}$$

By (36) evaluating (39), (40) and (41) at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ one has:

$$\frac{\partial^2 \phi}{\partial \overline{z}_k \partial z_j} = \frac{\partial^2 \phi}{\partial \overline{u}_\alpha \partial z_k} = \frac{\partial^2 \phi}{\partial \overline{w} \partial z_k} = 0 \quad \text{for } j = 1, \ldots, r, \ k = r+1, \ldots, n, \tag{42}$$

Finally, let us deal with the derivatives of the third order. Observe first that from $\frac{\partial N_1^\mu}{\partial z_j} = -(N_1^\mu - |w|^2) \frac{\partial \phi_1}{\partial z_j}$ (and similarly $\frac{\partial N_2^\mu}{\partial u_\alpha} = -(N_2^\mu - |w|^2) \frac{\partial \phi_2}{\partial u_\alpha}$) we get:

$$\frac{\partial}{\partial z_j} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} = -\frac{N_2^\mu (N_1^\mu - |w|^2)}{N_1^\mu N_2^\mu - |w|^2} \left[1 - \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right] \frac{\partial \phi_1}{\partial z_j}, \tag{43}$$

$$\frac{\partial}{\partial u_\alpha} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} = -\frac{\left(N_1^\mu - |w|^2\right)\left(N_2^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} \left[1 - \frac{N_2^\mu N_1^\mu}{N_1^\mu N_2^\mu - |w|^2}\right] \frac{\partial \phi_2}{\partial u_\alpha}, \tag{44}$$

$$\frac{\partial^2}{\partial z_j \partial u_\alpha} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} = -\frac{\partial}{\partial z_j} \left[\frac{\left(N_1^\mu - |w|^2\right)\left(N_2^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} \left(1 - \frac{N_2^\mu N_1^\mu}{N_1^\mu N_2^\mu - |w|^2}\right)\right] \frac{\partial \phi_2}{\partial u_\alpha}. \tag{45}$$

Then from (39), (40) and (41) we have:

$$\frac{\partial^3 \phi}{\partial z_i \partial \overline{z}_k \partial z_j} = \frac{\partial^3 \phi_1}{\partial z_i \partial \overline{z}_k \partial z_j} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} + \frac{\partial^2 \phi_1}{\partial \overline{z}_k \partial z_j} \frac{\partial}{\partial z_i} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}$$
$$+ \frac{\partial^2 \phi_1}{\partial z_i \partial z_j} \frac{\partial}{\partial \overline{z}_k} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right) + \frac{\partial \phi_1}{\partial z_j} \frac{\partial^2}{\partial z_i \partial \overline{z}_k} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right),$$

$$\frac{\partial^3 \phi}{\partial z_i \partial \overline{u}_\alpha \partial z_j} = \frac{\partial^2 \phi_1}{\partial z_i \partial z_j} \frac{\partial}{\partial \overline{u}_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right) + \frac{\partial \phi_1}{\partial z_j} \frac{\partial^2}{\partial z_i \partial \overline{u}_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right),$$

$$\frac{\partial^3 \phi}{\partial z_i \partial u_\alpha \partial \overline{z}_j} = \frac{\partial^2 \phi_1}{\partial z_i \partial \overline{z}_j} \frac{\partial}{\partial u_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right) + \frac{\partial \phi_1}{\partial \overline{z}_j} \frac{\partial^2}{\partial z_i \partial u_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right),$$

$$\frac{\partial^3 \phi}{\partial w \partial \overline{u}_\alpha \partial z_j} = \frac{\partial^2 \phi_1}{\partial w \partial z_j} \frac{\partial}{\partial \overline{u}_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right) + \frac{\partial \phi_1}{\partial z_j} \frac{\partial^2}{\partial w \partial \overline{u}_\alpha} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right),$$

$$\frac{\partial^3 \phi}{\partial w \partial \overline{z}_k \partial z_j} = \frac{\partial^3 \phi_1}{\partial w \partial \overline{z}_k \partial z_j} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} + \frac{\partial^2 \phi_1}{\partial \overline{z}_k \partial z_j} \frac{\partial}{\partial w} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} +$$
$$+ \frac{\partial^2 \phi_1}{\partial w \partial z_j} \frac{\partial}{\partial \overline{z}_k} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right) + \frac{\partial \phi_1}{\partial z_j} \frac{\partial^2}{\partial w \partial \overline{z}_k} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right),$$

$$\frac{\partial^3 \phi}{\partial w^2 \partial \overline{z}_j} = \frac{\partial^2 \phi_1}{\partial w^2 \partial \overline{z}_j} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} + 2 \frac{\partial^2 \phi_1}{\partial w \partial \overline{z}_j} \frac{\partial}{\partial w} \frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2} +$$
$$+ \frac{\partial \phi_1}{\partial \overline{z}_j} \frac{\partial^2}{\partial w^2} \left(\frac{N_2^\mu \left(N_1^\mu - |w|^2\right)}{N_1^\mu N_2^\mu - |w|^2}\right).$$

which, evaluating at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ and applying (36), (43), (44) and (45), gives:

$$\frac{\partial^3 \phi}{\partial z_i \partial \overline{z}_k \partial z_j} = \frac{\partial^3 \phi}{\partial z_i \partial u_\alpha \partial \overline{z}_k} = \frac{\partial^3 \phi}{\partial w \partial \overline{z}_k \partial z_j}$$
$$= \frac{\partial^3 \phi}{\partial w^2 \partial \overline{z}_k} = 0, \text{ for } i, j = 1, \ldots, r; \ k = r+1, \ldots, n, \tag{46}$$

$$\frac{\partial^3 \phi}{\partial z_i \partial \overline{u}_\alpha \partial z_j} = \frac{\partial^3 \phi}{\partial w \partial \overline{u}_\alpha \partial z_j} = 0, \quad \text{for } \alpha = s+1, \ldots, h. \tag{47}$$

We can now proceed with the proof of (35). Let us first show that $\nabla_{\partial_{z_j}} \partial_{z_k} \in Tf\left(M_{\Delta^r}(\mu)\right)$, for $j, k = 1, \ldots, r$. Recalling that

$$\nabla_{\partial_{z_j}} \partial_{z_k} = \Gamma^0_{jk} \partial_w + \sum_{i=1}^n \Gamma^i_{jk} \partial_{z_i} + \sum_{\alpha=1}^h \Gamma^\alpha_{jk} \partial_{u_\alpha},$$

this is equivalent to show that for $j, k = 1, \ldots, r$, $\Gamma^i_{jk} = \Gamma^\alpha_{jk} = 0$ for any $i = r+1, \ldots, n$ and any $\alpha = s+1, \ldots, h$. By (42) $g_{j\bar{\ell}} = 0$ when $j = 1, \ldots, r$ and $\ell = r+1, \ldots, n$, and the same of course holds for $g_{j\bar{\alpha}}$ since g is a product metric. Thus, also the inverse $g^{j\bar{\ell}}$ enjoys the same property and at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ we have:

$$\Gamma^i_{jk} = \sum_{\ell=1}^n g^{i\bar{\ell}} \frac{\partial g_{k\bar{\ell}}}{\partial z_j} + \sum_{\beta=1}^h g^{i\bar{\beta}} \frac{\partial g_{k\bar{\beta}}}{\partial z_j} = \sum_{\ell=r+1}^n g^{i\bar{\ell}} \frac{\partial g_{k\bar{\ell}}}{\partial z_j} = 0, \quad j, k = 1, \ldots, r,$$

where to obtain the last equality we apply (46). Similarly, applying (47) instead of (46), at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ we have:

$$\Gamma^\alpha_{jk} = \sum_{\ell=1}^n g^{\alpha\bar{\ell}} \frac{\partial g_{k\bar{\ell}}}{\partial z_j} + \sum_{\beta=1}^h g^{\alpha\bar{\beta}} \frac{\partial g_{k\bar{\beta}}}{\partial z_j} = \sum_{\beta=s+1}^h g^{\alpha\bar{\beta}} \frac{\partial g_{k\bar{\beta}}}{\partial z_j} = 0, \quad j, k = 1, \ldots r.$$

We move now to show that $\nabla_{\partial_{z_j}} \partial_{u_\alpha} \in Tf\left(M_{\Delta^r}(\mu)\right)$, for $j = 1, \ldots, r, \alpha = 1, \ldots, s$. By definition:

$$\nabla_{\partial_{z_j}} \partial_{u_\alpha} = \Gamma^0_{j\alpha} \partial_w + \sum_{i=1}^n \Gamma^i_{j\alpha} \partial_{z_i} + \sum_{\beta=1}^h \Gamma^\beta_{j\alpha} \partial_{u_\beta},$$

thus we need to show that $\Gamma^i_{j\alpha} = \Gamma^\beta_{j\alpha} = 0$ for all $j = 1, \ldots, r, \alpha = 1, \ldots, s, i = r+1, \ldots, n,$ $\beta = s+1, \ldots, m$. At $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ we have:

$$\Gamma^i_{j\alpha} = \sum_{\ell=1}^n g^{i\bar{\ell}} \frac{\partial g_{j\bar{\ell}}}{\partial z_\alpha} + \sum_{\beta=1}^h g^{i\bar{\beta}} \frac{\partial g_{j\bar{\beta}}}{\partial z_\alpha} = \sum_{\ell=r+1}^n g^{i\bar{\ell}} \frac{\partial g_{j\bar{\ell}}}{\partial z_\alpha} = 0,$$

where last equality follows by (46). The case $\Gamma^\beta_{j\alpha}$ is obtained by this one exchanging the role of the first and second domain.

Let us now deal with $\nabla_{\partial_w} \partial_{z_k} \in Tf\left(M_{\Delta^r}(\mu)\right)$. We have:

$$\nabla_{\partial_w} \partial_{z_k} = \Gamma^0_{0k} \partial_w + \sum_{i=1}^n \Gamma^i_{0k} \partial_{z_i} + \sum_{\alpha=1}^h \Gamma^\alpha_{0k} \partial_{u_\alpha},$$

and we need $\Gamma^i_{0k} = \Gamma^\alpha_{0k} = 0$ at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$ for $k = 1, \ldots, r, i = r+1, \ldots, n, \alpha = s+1, \ldots, h$. Similarly as before and using (46) and (47) we get at $Z \in \{z_{r+1} = \cdots = z_n = u_{s+1} = \cdots = u_h = 0\}$:

$$\Gamma^i_{0k} = \sum_{\ell=1}^n g^{i\bar{\ell}} \frac{\partial g_{0\bar{\ell}}}{\partial z_k} + \sum_{\beta=1}^h g^{i\bar{\beta}} \frac{\partial g_{0\bar{\beta}}}{\partial z_k} = \sum_{\ell=r+1}^n g^{i\bar{\ell}} \frac{\partial g_{0\bar{\ell}}}{\partial z_k} = 0,$$

$$\Gamma_{0k}^{\alpha} = \sum_{\ell=1}^{n} g^{\alpha\bar{\ell}} \frac{\partial g_{0\bar{\ell}}}{\partial z_k} + \sum_{\beta=1}^{h} g^{\alpha\bar{\beta}} \frac{\partial g_{0\bar{\beta}}}{\partial z_k} = \sum_{\beta=s+1}^{n} g^{\alpha\bar{\beta}} \frac{\partial g_{0\bar{\beta}}}{\partial z_k} = 0.$$

Finally, to show that $\nabla_{\partial_w} \partial_w \in Tf\,(M_{\Delta^r}(\mu))$ recall that:

$$\nabla_{\partial_w} \partial_w = \Gamma_{00}^0 \partial_w + \sum_{i=1}^{n} \Gamma_{00}^i \partial_{z_i} + \sum_{\alpha=1}^{h} \Gamma_{00}^{\alpha} \partial_{u_\alpha},$$

thus we need to show that $\Gamma_{00}^i = \Gamma_{00}^{\alpha} = 0$ for $i = r+1, \ldots, n, \alpha = s+1, \ldots, h$. Conclusion follows by (46):

$$\Gamma_{00}^i = \sum_{\ell=1}^{n} g^{i\bar{\ell}} \frac{\partial g_{0\bar{\ell}}}{\partial w} + \sum_{\beta=1}^{h} g^{i\bar{\beta}} \frac{\partial g_{0\bar{\beta}}}{\partial w} = \sum_{\ell=r+1}^{n} g^{i\bar{\ell}} \frac{\partial g_{0\bar{\ell}}}{\partial w} = 0,$$

and the same holds for Γ_{00}^{α}.

The proof for $m = 2$ is complete. The general case can be recursively obtained replacing (36) with (46) and (47). □

3.6 The proof of the Hartogs–Polydisk theorem

We need two further preliminary results.

Lemma 7 *Let Ω be a bounded symmetric domain.*

(1) If $\phi \colon \Omega \to \Omega$ is an isometric automorphism of Ω then ϕ lifts to an isometric automorphism $\tilde{\phi} \colon M_{\Omega}(\mu) \to M_{\Omega}(\mu)$ defined by

$$\tilde{\phi}(z, w) = \left(\phi(z), e^{\mu h_\phi(z)} w \right),$$

for an appropriate holomorphic function $h_\phi \colon \Omega \to \mathbb{C}$.

(2) If $\phi \colon \Omega \to \Omega$ is an automorphism of Ω which fix the origin, then ϕ lifts to an isometric automorphism $\tilde{\phi} \colon M_{\Omega}(\mu) \to M_{\Omega}(\mu)$ defined by

$$\tilde{\phi}(z, w) = (\phi(z), w).$$

Proof Let $\phi \colon \Omega \to \Omega$ be an isometric automorphism of Ω. Then ϕ satisfies:

$$\partial\bar{\partial} \log \left(N\left(\phi(z), \overline{\phi(z)} \right) \right) = \partial\bar{\partial} \log \left(N\,(z, \bar{z}) \right),$$

and hence $N\left(\phi(z), \overline{\phi(z)} \right) = N\,(z, \bar{z})\,e^{h_\phi(z)+\bar{h}_\phi(z)}$ for an opportune holomorphic function $h_\phi \colon \Omega \to \mathbb{C}$. The holomorphic map $f \colon M_{\Omega}(\mu) \to M_{\Omega}(\mu)$ defined by:

$$f(z, w) = \left(\phi(z), e^{\mu h_\phi(z)} w \right),$$

is well defined, as $\left| e^{\mu h_\phi(z)} w \right|^2 < \left| e^{\mu h_\phi(z)} \right|^2 N\,(z, \bar{z}) = N\left(\phi(z), \overline{\phi(z)} \right)$, and it is an isometry of $M_{\Omega}(\mu)$, since:

$$\partial\bar{\partial} \log \left(N^{\mu}\left(\phi(z), \overline{\phi(z)} \right) - \left| e^{\mu h_\phi(z)} w \right|^2 \right) = \partial\bar{\partial} \log \left(N^{\mu}\,(z, \bar{z}) - |w|^2 \right).$$

For the second part, it is enough to recall that automorphisms of Ω that fix the origin preserves the minimal polynomial N_Ω (see e.g [1, Prop. III.2.7] or [23, Section 2.2]), thus in this case $h_\phi = 0$. □

Proposition 1 *Let $\Delta^r \subset \Omega$ be an r-dimensional totally geodesic polydisk of a bounded symmetric domain of classical type of rank r. Then*

$$C_{\Delta^r} = \left\{ (z, w) \in M_\Omega(\mu) \mid z \in \Delta^r \right\}$$

is a totally geodesic Kähler submanifold of $M_\Omega(\mu)$ biholomorphically isometric to $M_{\Delta^r}(\mu)$.

Proof By (1) of Lemma 7 we can assume without loss of generality that Δ^r passes through the origin. Observe that $N_{\Delta^r} = N_{\Omega|_{\Delta^r}}$ (see [26, Proposition VI.2.4 and VI.3.6]). Now the proof is an immediate consequence of lemmata 2, 3, 4, 5, 7 and the Polydisk Theorem that assure us that $\mathrm{Aut}_0(\Omega)$ acts transitively on the set of the r-dimensional totally geodesic polydisk through the origin of Ω (see also [26, Theorem VI.3.5]). □

Proof of Theorem 1 Let $X \in T_{(z,w)} M_\Omega(\mu)$ be a fixed tangent vector. Consider the decomposition $X = X_1 + X_2$, where $X_1 \in T_z\Omega$ and $X_2 \in \mathbb{C}$. From the Polydisk Theorem we know that there exists a totally geodesic polydisk $\Delta^r \subset \Omega$, through z, such that $X_1 \in T_z\Delta^r$. By Proposition 1 we know that $\{(z, w) \in M_\Omega(\mu) \mid z \in \Delta^r\}$ is the Cartan-Hartogs $M_{\Delta^r}(\mu)$ realized as a totally geodesic Kähler submanifold of $M_\Omega(\mu)$. The proof is complete by observing that by construction $X \in T_0\Delta^r \times \mathbb{C} \cong T_{(z,w)} M_{\Delta^r}(\mu)$. □

4 Proof of theorem 2

In order to proof Theorem 2 we need the following lemma, which generalize Proposition 1 to polydisks of dimension less than the rank of Ω.

Lemma 8 *Let $\Delta^r \subset \Delta^n$ be a totally geodesic r-dimensional polydisk of an n-dimensional polydisk. Then*

$$\left\{ (z, w) \in M_{\Delta^n}(\mu) \mid z \in \Delta^r \right\} \tag{48}$$

is a totally geodesic Kähler submanifold of $M_{\Delta^n}(\mu)$ biholomorphic isometric to $M_{\Delta^r}(\mu)$.

Proof Let us first show that the inclusion $i_j : \mathbb{C}H^1 \to \Delta^n$ of $\mathbb{C}H^1$ in the j-th factor of Δ^r, is a holomorphic and totally geodesic immersion of $\mathbb{C}H^1$ in Δ^n. Let us denote by $K^{\mathbb{C}H^1}$ and K^{Δ^n} the holomorphic sectional curvatures of $\mathbb{C}H^1$ and Δ^n respectively. We have (see [8, Propostion IX.9.2]),

$$K^{\mathbb{C}H^1}(X) = K^{\Delta^n}(i_{j*}X) = \sum_{\ell=1}^n K^{\Delta^n}\left(a_\ell \frac{\partial}{\partial z_\ell}\right), \qquad \forall X \in T_z\mathbb{C}H^1,$$

where $i_{j*}(X) = \sum_{\ell=1}^n a_\ell \frac{\partial}{\partial z_\ell}$. We conclude that all but one of the $a_1, \dots, a_n$ are forced to be zero. We can therefore assume, without loss of generality, that $\mathbb{C}H^1 \times \cdots \times \mathbb{C}H^1 = \Delta^r = \{z \in \Delta^n \mid z_j = 0, \ j > r\}$. Clearly $(z_1, \dots, z_r, w) \overset{f}{\mapsto} (z_1, \dots, z_r, 0, \dots, 0, w)$ defines an holomorphic isometric immersion of $M_{\Delta^r}(\mu)$ in $M_{\Delta^n}(\mu)$, in order to complete the proof of the lemma we are going to prove that it is also totally geodesic.

Let ∇ be the Levi-Civita connection of $M_{\Delta^n}(\mu)$, let us denote $\frac{\partial}{\partial z_0} = \frac{\partial}{\partial w}$ and let Γ_{ij}^k be the associated Christoffel symbols defined by $\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \sum_{k=0}^n \Gamma_{ij}^k \frac{\partial}{\partial z_k}$. In order to prove that f

is totally geodesic we need to show that $\Gamma_{ij}^k = 0$ for $0 \leq i, j \leq r$ and $k > r$. For $k, \ell > 0$ and $k \neq \ell$, we have

$$
\begin{aligned}
g_{k\bar{\ell}} &= -\frac{i}{2} \partial_{z_k} \partial_{\bar{z}_\ell} \log \left(\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2 \right) \\
&= \frac{i}{2} \partial_{z_k} \frac{\mu z_\ell (1 - |z_\ell|^2)^{\mu-1} \prod_{j=1, j\neq\ell}^n (1 - |z_j|^2)^\mu}{\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2} \\
&= \frac{i}{2} \frac{\mu^2 z_\ell \bar{z}_k (1 - |z_\ell|^2)^{\mu-1} (1 - |z_k|^2)^{\mu-1} \prod_{j=1, j\neq\ell}^n (1 - |z_j|^2)^\mu \prod_{j=1, j\neq k}^n (1 - |z_j|^2)^\mu}{\left(\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2 \right)^2} \\
&\quad - \frac{i}{2} \frac{\mu^2 z_\ell \bar{z}_k (1 - |z_\ell|^2)^{\mu-1} (1 - |z_k|^2)^{\mu-1} \prod_{j=1, j\neq\ell,k}^n (1 - |z_j|^2)^\mu}{\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2}
\end{aligned}
$$

and

$$
\begin{aligned}
g_{0\bar{\ell}} &= -\frac{i}{2} \partial_w \partial_{\bar{z}_\ell} \log \left(\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2 \right) \\
&= \frac{i}{2} \partial_w \frac{\mu z_\ell (1 - |z_\ell|^2)^{\mu-1} \prod_{j=1, j\neq\ell}^n (1 - |z_j|^2)^\mu}{\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2} \\
&= \frac{i}{2} \frac{\overline{w} \mu z_\ell (1 - |z_\ell|^2)^{\mu-1} \prod_{j=1, j\neq\ell}^n (1 - |z_j|^2)^\mu}{\prod_{j=1}^n (1 - |z_j|^2)^\mu - |w|^2}.
\end{aligned}
$$

Therefore, for $k > r$ and $z_k = 0$, we get

$$
\Gamma_{ij}^k = \sum_\ell g^{k\bar{\ell}} \frac{\partial g_{j\bar{\ell}}}{\partial z_i} = g^{k\bar{k}} \frac{\partial g_{j\bar{k}}}{\partial z_i} = 0,
$$

for any $i, j \neq k$. The proof is complete.   □

Proof of Theorem 2 As Ω' is a totally geodesic Kähler submanifold of the bounded symmetric domain Ω, it is an HSSNCT and therefore can be realized as a bounded symmetric domain $\Omega' \subset \mathbb{C}^m$, where $m = \dim(\Omega')$. With a slight abuse of notation, let us denote by $f : \Omega' \subset \mathbb{C}^m \to \Omega' \subset \mathbb{C}^n$ the totally geodesic Kähler immersion of Ω' in Ω. Without loss of generality (up to automorphisms of Ω and Ω') we can assume $f(0) = 0$. Once observed that $N_{\Omega'} = N_{\Omega|_{\Omega'}}$ it is easy to verify that $\tilde{f} : M_{\Omega'}(\mu) \to M_\Omega(\mu)$ given by $\tilde{f}(z, w) = (f(z), w)$ defines a Kähler embedding, with $C_{\Omega'} = f(M_{\Omega'}(\mu)) \simeq M_{\Omega'}(\mu)$.

It remains to prove that $C_{\Omega'}$ is totally geodesic in $M_\Omega(\mu)$. Let $p \in C_{\Omega'} \subset M_\Omega(\mu)$ and let $X \in T_p C_{\Omega'} \subset T_p M_\Omega(\mu)$. We want to prove that the geodesic γ of $M_\Omega(\mu)$ with $\gamma(0) = p$ and $\gamma'(0) = X$ is also a geodesic of $C_{\Omega'}$. By Theorem 1 and Proposition 1, we know that there exist t.g. Kähler immersed polydisks $\Delta^{r'} \subset \Omega'$ and $\Delta^r \subset \Omega$ such that the associated Hartogs-Polydisk $C_{\Delta^{r'}} \subset M_{\Omega'}(\mu)$ and $C_{\Delta^r} \subset M_\Omega(\mu)$ are totally geodesics (here r' and r are the ranks of Ω and Ω respectively). Using similar argument to that used in first part of the proof of Lemma 8 we can see that $\Delta' \cap \Delta$ is a t.g. polydisk of Ω' (and therefore of Ω). By Lemma 8 we conclude that $C_{\Delta' \cap \Delta} = \{(z, w) \in M_\Omega(\mu)\}$ is a totally geodesic Kähler submanifold of $C_{\Omega'}$ and $M_\Omega(\mu)$ at the same time. It is a simple observation that $p \in C_{\Delta' \cap \Delta}$ and $X \in T_p C_{\Delta' \cap \Delta}$, hence γ is a geodesic of $M_{\Omega'}(\mu)$ as wished.   □

5 Proof of theorem 3

We start this section giving the explicit expression of a holomorphic and isometric immersion f of $(M_{\Delta^r}(\mu), \omega_{\Delta^r}(\mu))$ in $(l^2(\mathbb{C}), \omega_0)$.

Lemma 9 *The holomorphic map* $f: M_{\Delta^r}(\mu) \to l^2(\mathbb{C})$ *given by:*

$$f(z, w) = (\psi_1, \ldots, \psi_r, \Psi),$$

where for $j = 1, \ldots, r$:

$$\psi_j := \sqrt{\mu}\left(z_j, \ldots, \frac{z_j^k}{\sqrt{k}}, \ldots\right), \tag{49}$$

$$\Psi := \left(\ldots, \frac{1}{\sqrt{a}}\sqrt{\binom{\mu a + k_1 - 1}{k_1} \cdots \binom{\mu a + k_r - 1}{k_1}} z_1^{k_1} \ldots z_r^{k_r} w^a, \ldots\right), \tag{50}$$

for $k = (k_1, \ldots, k_r)$, $|k| = 0, 1, 2, \ldots,$ *and* $a = 1, 2, \ldots,$ *satisfies* $f^*\omega_0 = \omega_{\Delta^r}(\mu)$.

Proof We have:

$$\sum_{j=0}^{\infty} |f_j|^2 = \mu \sum_j \sum_{k_j} \frac{|z_j|^{2k_j}}{k_j}$$
$$+ \sum_{k,a} \binom{\mu a + k_1 - 1}{k_1} \cdots \binom{\mu a + k_r - 1}{k_r} |z_1|^{2k_1} \ldots |z_r|^{2k_r} \frac{|w|^{2a}}{a},$$

(to avoid confusion the sums are always taken in the parameters' range) and:

$$\sum_j \sum_{k_j} \frac{|z_j|^{2k_j}}{k_j} = -\sum_{j=1}^{r} \log(1 - |z_j|^2) = -\log\left(\prod_{j=1}^{r}(1 - |z_j|^2)\right),$$

$$\sum_{k,a} \binom{\mu a + k_1 - 1}{k_1} \cdots \binom{\mu a + k_r - 1}{k_r} |z_1|^{2k_1} \ldots |z_r|^{2k_r} \frac{|w|^{2a}}{a} = \sum_{a=1}^{\infty} \frac{|w|^{2a}}{a \prod_{j=1}^{r}(1 - |z_j|^2)^{\mu a}}$$

$$= -\log\left(1 - \frac{|w|^2}{\prod_{j=1}^{r}(1 - |z_j|^2)^{\mu}}\right),$$

which imply:

$$\sum_{j=0}^{\infty} |f_j|^2 = -\log\left(\prod_{j=1}^{r}(1 - |z_j|^2)^{\mu} - |w|^2\right),$$

as requested. □

We use Lemma 9 to obtain geodesic equations for $M_{\Delta^r}(\mu)$. From (49), deriving twice $\psi_j(\gamma(t))$ w.r.t. t gives:

$$\psi_j(\gamma)'' = \sqrt{\mu}\left(\ddot{u}_j, \ldots, \frac{k(u_j^{k-1}\dot{u}_j)'}{\sqrt{k}}, \ldots\right),$$

and, denoting by $A(\mu, a, k) := \frac{1}{\sqrt{a}}\sqrt{\binom{\mu a + k_1 - 1}{k_1} \cdots \binom{\mu a + k_r - 1}{k_1}}$, from (50) we get:

$$\Psi(\gamma)'' := \left(\dots, A(\mu, a, k)(u_1^{k_1} \dots u_r^{k_r} u_w^a)'', \dots\right).$$

The tangent space $T_{f(\gamma)} f(M_{\Delta^r}(\mu))$ is spanned by

$$\nabla f(\gamma) = (\partial_1 f, \dots, \partial_r f, \partial_w f)(\gamma),$$

and the condition for γ to be a geodesic is equivalent to the system:

$$\langle f(\gamma)'', \partial_1 f \rangle = \cdots = \langle f(\gamma)'', \partial_r f \rangle = \langle f(\gamma)'', \partial_w f \rangle = 0, \tag{51}$$

namely:

$$\langle f(\gamma)'', \partial_w f \rangle = \sum_{k,a} a A^2(\mu, a, k)(\bar{u}_1^{k_1} \dots \bar{u}_r^{k_r} \bar{u}_w^a)'' u_1^{k_1} \dots u_r^{k_r} u_w^{a-1} = 0, \tag{52}$$

and for $s = 1, \dots, r$:

$$\begin{aligned}
\langle f(\gamma)'', \partial_s f \rangle = &\mu \sum_{k=1}^{\infty} u_s^{k-1} (\bar{u}_s^k)'' + \\
&+ \sum_{k,a} k_s A^2(\mu, a, k)(\bar{u}_1^{k_1} \dots \bar{u}_r^{k_r} \bar{u}_w^a)'' u_1^{k_1} \dots u_s^{k_s-1} \dots u_r^{k_r} v^a = 0.
\end{aligned} \tag{53}$$

Let us now prove Theorem 3.

Proof of Theorem 3 Let γ be a geodesic with linear support in $M_\Omega(\mu)$, passing through $(\zeta, 0)$ with direction ξ. By Lemma 7 up to automorphisms we can assume $\zeta = 0$ and by the Hartogs polydisk Theorem γ is contained in an Hartogs polydisk $M_{\Delta^r}(\mu)$ passing through 0 with direction ξ. Then γ is a geodesic with linear support passing through the origin in $M_{\Delta^r}(\mu)$ and conclusion follows by Lemma 10 below. □

Lemma 10 *If $\gamma(t) = (\xi_1 v(t), \dots, \xi_r v(t), \xi_0 v(t))$ is a geodesic in $M_{\Delta^r}(\mu)$, then either $\gamma \subset \Delta^r = M_{\Delta^r}(\mu) \cap \{w = 0\}$ or $\gamma \subset \mathbb{CH}^1 = M_{\Delta^r}(\mu) \cap \{z = 0\}$ or $r = 1 = \mu$, i.e. $M_{\Delta^r}(\mu) \simeq \mathbb{CH}^2$.*

Proof A geodesic in $M_{\Delta^r}(\mu)$ must satisfy (52) and (53). Plugging γ respectively into (52) and (53) gives:

$$\bar{\xi}_0 \sum_{k,a} a A^2(\mu, a, k)|\xi_1|^{2k_1} \cdots |\xi_r|^{2k_r} |\xi_0|^{2a-2} (\bar{v}(t)^{|k|+a})'' v(t)^{|k|+a-1} = 0, \tag{54}$$

$$\begin{aligned}
&\mu \bar{\xi}_s \sum_{k=1}^{\infty} |\xi_s|^{2(k-1)} v(t)^{k-1} (\bar{v}(t)^k)'' \\
&+ \bar{\xi}_s \sum_{k,a} k_s A^2(\mu, a, k)|\xi_1|^{2k_1} \cdots |\xi_s|^{2(k_s-1)} \cdots |\xi_r|^{2k_r} |\xi_0|^{2a} (\bar{v}(t)^{|k|+a})'' v(t)^{|k|+a-1} = 0.
\end{aligned} \tag{55}$$

for $s = 1, \dots, r$, $|k| = 0, 1, 2, \dots$, and $a = 1, 2, \dots$. Evaluating at $t = 0$ we get:

$$\xi_0 \ddot{v}(0) = \mu \xi_1 \ddot{v}(0) = \cdots = \mu \xi_r \ddot{v}(0) = 0,$$

and since $\xi_0, \xi_j, j = 1, \dots, r$ cannot be all vanishing, it implies $\ddot{v}(0) = 0$.

Taking into account that $v(0) = \ddot{v}(0) = 0$ and $\dot{v}(0) = 1$, deriving (54) and (55) once with respect to t and evaluating at $t = 0$ gives:

$$\bar{\xi}_0 \left(v'''(0) + 2|\xi_0|^2 + 2\mu \sum_{j=1}^{r} |\xi_j|^2 \right) = 0 = \mu \bar{\xi}_s \left(v'''(0) + 2|\xi_s|^2 + 2|\xi_0|^2 \right). \quad (56)$$

If $\xi_0 = 0$ or $\xi_s = 0$ for all $s = 1, \ldots, r$ then, since by Theorem 2 $M_{\Delta^r}(\mu) \cap \{z = 0\}$ and $M_{\Delta^r}(\mu) \cap \{w = 0\}$ are totally geodesic in $M_{\Delta^r}(\mu)$, $\gamma \subset \Delta^r = M_{\Delta^r}(\mu) \cap \{w = 0\}$ or $\gamma \subset \mathbb{CH}^1 = M_{\Delta^r}(\mu) \cap \{z = 0\}$. Thus, assume that $\xi_0 \neq 0$ and at least one between the ξ_j's is different from 0. From (56) we get:

$$\mu \sum_{j=1}^{r} |\xi_j|^2 = |\xi_s|^2, \quad \text{for any } s = 1, \ldots, r, \quad (57)$$

which implies that all the ξ_s's are equal in module, and thus $r\mu = 1$.

To conclude that $r = \mu = 1$, we need to consider the third order derivative of (54) and (55) evaluated at $t = 0$. Observe first that:

$$\left[(v(t)^2)'' v(t) \right]'''(0) = 26v'''(0), \quad \left[(v(t)^3)'' v(t)^2 \right]'''(0) = 36,$$

and recall that from (56) we get $v'''(0) = -2(|\xi_0|^2 + |\xi_s|^2)$. Deriving three times (54) with respect to t and evaluating at $t = 0$ we get:

$$\bar{\xi}_0 \left[v^{(v)}(0) + \left(|\xi_0|^2 + \mu \sum_{j=1}^{r} |\xi_j|^2 \right) 26v'''(0) + 36 \left(|\xi_0|^4 + 2\mu|\xi_0|^2 \sum_{j=1}^{r} |\xi_j|^2 + \right. \right.$$

$$\left. \left. + \mu(\mu - 1) \sum_{j=1}^{r} |\xi_j|^4 + \mu^2 \sum_{j,k=1}^{r} |\xi_j|^2 |\xi_k|^2 \right) \right] = 0,$$

which by (57) reads:

$$\bar{\xi}_0 \left[v^{(v)}(0) - 16 \left(|\xi_0|^2 + |\xi_s|^2 \right)^2 + 36(\mu - 1)|\xi_s|^4 \right] = 0. \quad (58)$$

On the other hand, (55) gives:

$$\mu \bar{\xi}_s \left[v^{(v)}(0) + 26 \left(|\xi_s|^2 + |\xi_0|^2 \right) v'''(0) + 36 \left(|\xi_s|^2 + |\xi_0|^2 \right)^2 + 36|\xi_0|^2 \left(\mu \sum_{j=1}^{r} |\xi_j|^2 - |\xi_s|^2 \right) \right] = 0,$$

i.e.:

$$\mu \bar{\xi}_s \left[v^{(v)}(0) - 16 \left(|\xi_s|^2 + |\xi_0|^2 \right)^2 \right] = 0. \quad (59)$$

Comparing (58) and (59) we get $\mu = r = 1$ and we are done. □

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