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The generic isogeny decomposition of the Prym Variety of a cyclic branched covering

Theodosis Alexandrou[1,2]

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Abstract

Let $f: S' \longrightarrow S$ be a cyclic branched covering of smooth projective surfaces over $\mathbb{C}$ whose branch locus $\Delta \subset S$ is a smooth ample divisor. Pick a very ample complete linear system $|\mathcal{H}|$ on S, such that the polarized surface $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. For the general member $[C] \in |\mathcal{H}|$ consider the μ_n-equivariant isogeny decomposition of the Prym variety $\mathrm{Prym}(C'/C)$ of the induced covering $f: C' := f^{-1}(C) \longrightarrow C$:

$$\mathrm{Prym}(C'/C) \sim \prod_{d|n,\ d\neq 1} \mathcal{P}_d(C'/C).$$

We show that for the very general member $[C] \in |\mathcal{H}|$ the isogeny component $\mathcal{P}_d(C'/C)$ is μ_d-simple with $\mathrm{End}_{\mu_d}(\mathcal{P}_d(C'/C)) \cong \mathbb{Z}[\zeta_d]$. In addition, for the non-ample case we reformulate the result by considering the identity component of the kernel of the map $\mathcal{P}_d(C'/C) \subset \mathrm{Jac}(C') \longrightarrow \mathrm{Alb}(S')$.

Keywords Jacobian variety · Prym variety · Isogeny decomposition · Cyclic covering

Mathematics Subject Classification 14K02 · 14K12 · 14H40 · 14H10

1 Introduction

For a cyclic cover $f: X \longrightarrow Y$ of smooth complex projective curves with $\deg(f) = n$, we fix a generator $\sigma \in \mathrm{Aut}(X/Y)$ of the automorphism group of f. The μ_n-action of X induces a $\mathbb{Q}$-algebra homomorphism

$$\rho: \mathbb{Q}[\mu_n] \cong \mathbb{Q}[T]/(T^n - 1) \to \mathrm{End}(\mathrm{Jac}(X)),\ T \mapsto \sigma^*,$$

✉ Theodosis Alexandrou
alexandrou@math.uni-hannover.de

1 Mathematisches Institut, Universität Bonn, Endenicher Allee 60, Bonn 53115, Germany

2 Present Address: Institur für Algebraische Geometrie, Leibniz Universittät Hannover, Welfengarten 1, Hannover 30167, Germany

and we define $\mathcal{P}_d(X/Y) := \ker^0(\Psi_d(\sigma^*))$ for $d|n$, where $\Psi_d \in \mathbb{Z}[T]$ is the d-th cyclotomic polynomial. In what follows we freely use the following well-known results, which can be easily checked:

(1) $\mathcal{P}_1(X/Y) = \ker^0(\sigma^* - \mathrm{id}) = f^*(\mathrm{Jac}(Y)) \sim \mathrm{Jac}(Y)$
(2) The addition map $\mathrm{Jac}(Y) \times \mathrm{Prym}(X/Y) \longrightarrow \mathrm{Jac}(X)$, $(\alpha, \beta) \mapsto f^*(\alpha) + \beta$ is an isogeny.
(3) Similarly, the addition map gives rise to the isogeny $\prod_{d|n,\ d\neq 1} \mathcal{P}_d(X/Y) \sim \mathrm{Prym}(X/Y)$.

Then, we can state the main result of this paper, which is the following:

Theorem 1.1 *Let S be a smooth projective surface over $\mathbb{C}$ with an ample line bundle $\mathcal{L}$. Assume $\Delta \in |\mathcal{L}^{\otimes n}|$ is smooth and consider the n-fold cyclic covering $f : S' \longrightarrow S$ branched along the divisor Δ. Given a very ample complete linear system $|\mathcal{H}|$ on S, such that $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. Then, for the very general member $[C] \in |\mathcal{H}|$ we have that*

$$\mathrm{Prym}(C'/C) \sim \prod_{d|n,\ d\neq 1} \mathcal{P}_d(C'/C),$$

with $\mathrm{End}_{\mu_d}(\mathcal{P}_d(C'/C)) \cong \mathbb{Z}[\zeta_d]$. Especially, each $\mathcal{P}_d(C'/C)$ is a μ_d-simple abelian variety.

If we restrict to the case of double coverings, we note that the involution σ of the covering f acts as $-\mathrm{id}$ on $\mathcal{P}_2(C'/C) = \mathrm{Prym}(C'/C)$ and thus, $\mathrm{End}_{\mu_2}(\mathrm{Prym}(C'/C)) = \mathrm{End}(\mathrm{Prym}(C'/C))$. In particular, (1.1) can be stated as follows:

Corollary 1.2 *Let S be a smooth projective surface over $\mathbb{C}$ with an ample line bundle $\mathcal{L}$. Assume $\Delta \in |\mathcal{L}^{\otimes 2}|$ is smooth and consider the double covering $f : S' \longrightarrow S$ branched along the divisor Δ. Given a very ample complete linear system $|\mathcal{H}|$ on S, such that $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. Then, for the very general member $[C] \in |\mathcal{H}|$ we have that*

$$\mathrm{End}(\mathrm{Prym}(C'/C)) \cong \mathbb{Z}.$$

The condition the line bundle $\mathcal{L}$ is ample in (1.1) implies that $\mathrm{Alb}(f) : \mathrm{Alb}(S') \longrightarrow \mathrm{Alb}(S)$ is an isomorphism cf. page 11 and therefore the map $\mathcal{P}_d(C'/C) \longrightarrow \mathrm{Alb}(S')$ is trivial. For the general situation one needs to consider the abelian subvariety

$$\mathcal{R}_d(C', C, S') := \ker^0(\mathcal{P}_d(C'/C) \longrightarrow \mathrm{Alb}(S')).$$

Then, the result can be reformulated as follows:

Theorem 1.3 *Let S be a smooth projective surface over $\mathbb{C}$ with a line bundle $\mathcal{L}$. Assume $\Delta \in |\mathcal{L}^{\otimes n}|$ is smooth and consider the n-fold cyclic covering $f : S' \longrightarrow S$ branched along the divisor Δ. Given a very ample complete linear system $|\mathcal{H}|$ on S, such that $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. Then, exactly one of the following assertions holds true:*

(i) *For the general member $[C] \in |\mathcal{H}|$ we have that $\mathcal{R}_d(C', C, S') = 0$.*
(ii) *For the very general member $[C] \in |\mathcal{H}|$ we have that $\mathrm{End}_{\mu_d}(\mathcal{R}_d(C', C, S')) \cong \mathbb{Z}[\zeta_d]$.*

In this paper we present a complete proof for the above results, inspired by Ciliberto and Van der Geer's approach in [3]. We note that this method does not capture the étale situation, cf. (3.2), (3.3) and (3.4). In addition, if we rephrase the statement for $n > 2$ by requiring simplicity instead of μ_d-simplicity to the isogeny components, we observe that this method cannot be adopted. Namely, the abelian variety B in (3.4) cannot be chosen in general to be μ_d-invariant and for this reason the last combinatorial argument in (3.4) fails. Lastly, a result due to Ortega and Lange, cf. [6] may be used to find counter-example for the case the covering f is étale of degree 7.

Notations and Conventions. For $n \in \mathbb{N}$, μ_n is the constant group scheme over $\mathbb{C}$, which is associated to the abstract group $\mathbb{Z}/n\mathbb{Z}$. The symbol ζ_n stands for a primitive n-th root of unity. If A is an abelian variety over $\mathbb{C}$, which is endowed with a μ_n-action, then $\mathrm{End}_{\mu_n}(A)$ is the ring of μ_n-equivariant endomorphisms of A. A very general point of a given variety X is a closed point $x \in X$, that lies in the complement of a countable union of nowhere dense closed subvarieties.

2 Preliminaries

In this section, we state some well-known results, which are needed later.

Proposition 2.1 *Let $\pi : \mathcal{A} \longrightarrow S$ be a projective abelian scheme over a Noetherian base S. Then, the endomorphism functor of $\mathcal{A}$ over S is representable by an S-scheme $\mathrm{End}_{\mathcal{A}/S}$, which is a disjoint union of projective and unramified S-schemes.*

Proof This is well-known, cf. [4, pp. 133]. □

The following proposition relates the correspondences on $C \times C$ with the endomorphisms of the Jacobian $\mathrm{Jac}(C)$.

Proposition 2.2 *Let $\pi : \mathcal{X} \longrightarrow S$ be a projective smooth morphism over a Noetherian base S, whose fibres are geometrically integral curves. Furthermore, assume that the morphism π admits a section, i.e. $\mathcal{X}(S) \neq \emptyset$. Then, there is a natural and functorial isomorphism*

$$\mathrm{Corr}_S(\mathcal{X}) := \mathrm{Pic}(\mathcal{X} \times_S \mathcal{X})/(\mathrm{pr}_1)^* \mathrm{Pic}(\mathcal{X}) \otimes (\mathrm{pr}_2)^* \mathrm{Pic}(\mathcal{X}) \cong \mathrm{End}_S(\mathrm{Pic}^0_{\mathcal{X}/S}).$$

Proof Consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(\mathcal{X})/\pi^* \mathrm{Pic}(S) & \xrightarrow{(\mathrm{pr}_1)^*} & \mathrm{Pic}(\mathcal{X} \times_S \mathcal{X})/(\mathrm{pr}_2)^* \mathrm{Pic}(\mathcal{X}) & \xrightarrow{q} & \mathrm{Corr}_S(\mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow g & & \\
0 & \longrightarrow & \mathrm{Pic}_{\mathcal{X}/S}(S) & \xrightarrow{c := -\circ\pi} & \mathrm{Pic}_{\mathcal{X}/S}(\mathcal{X}) & \xrightarrow{d} & \mathrm{End}_S(\mathrm{Pic}^0_{\mathcal{X}/S}) & \longrightarrow & 0
\end{array}$$

The first row is clearly exact: Indeed, the relative Picard functor is an fppf-sheaf, cf. [13, Tag 021L], [5, Thm. 2.5] and thus, the restriction map $(\mathrm{pr}_1)^*$ is injective. Furthermore, the map q is just the cokernel of $(\mathrm{pr}_1)^*$. Next, we give the definition of the map d. Fix $x \in \mathcal{X}(S)$ and let $\phi : \mathcal{X} \longrightarrow \mathrm{Pic}_{\mathcal{X}/S}$ be any S-morphism. Then, $d\phi$ is the unique endomorphism of $\mathrm{Pic}^0_{\mathcal{X}/S}$, making the diagram below commutative.

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{can} & \mathrm{Alb}_{\mathcal{X}/S} \cong \mathrm{Pic}^0_{\mathcal{X}/S} \\
\downarrow{\scriptstyle \phi - \phi\circ x\circ\pi} & \searrow & \downarrow{\scriptstyle d\phi} \\
\mathrm{Pic}_{\mathcal{X}/S} & \hookleftarrow & \mathrm{Pic}^0_{\mathcal{X}/S}
\end{array}$$

Note that under our assumptions the Albanese map $can : \mathcal{X} \longrightarrow \mathrm{Alb}_{\mathcal{X}/S}$ exists and has the desired universal property, cf. [1, Thm. 2.17], [1, Rem. 2.19] and [[8], Thm. 10.2]. Moreover, the construction of the map d indicates that d is surjective and also that the second row in the diagram above is exact at the middle. Now, the existence of g and the fact that it is an isomorphism are clear, since the first two vertical maps are isomorphisms by [5, Thm. 4.8] and [5, Thm. 2.5]. □

The following proposition is well-known.

Proposition 2.3 *Suppose that the polarized surface $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. Then, the following assertions hold true:*

(i) *The discriminant divisor $\mathcal{D}$ is irreducible and has codimension one in $|\mathcal{H}|$, i.e. $\mathcal{D}$ is a prime divisor of $|\mathcal{H}|$.*
(ii) *The general curve $[C] \in \mathcal{D}$ is irreducible and has a single ordinary double point as its only singularity.*

Proof Cf. [3, Lem. 3.1]. □

We close this section by introducing the μ_n-equivariant isogeny decomposition in (1.1). Let $f: C' \longrightarrow C$ be a cyclic branched covering of smooth complex projective curves with $\deg(f) = n$ and let σ stand for a generator of the Galois group of f. The μ_n-action on C' induces an action on $\mathrm{Jac}(C')$ and thus, it defines a $\mathbb{Q}$-algebra homomorphism

$$\rho: \mathbb{Q}[\mu_n] \cong \mathbb{Q}[T]/(T^n - 1) \longrightarrow \mathrm{End}^0(\mathrm{Jac}(C')),\ T \mapsto \sigma^*.$$

For any divisor $d|n$, we define $\mathcal{P}_d(C'/C) := \ker^0(\Psi_d(\sigma^*))$, where $\Psi_d(T) \in \mathbb{Z}[T]$ is the d-th cyclotomic polynomial. Then, the addition map

$$\mu: \prod_{d|n} \mathcal{P}_d(C'/C) \longrightarrow \mathrm{Jac}(C')$$

is a μ_n-equivariant isogeny. Lange and Recillas [7] have stated and proved the relation between $\mathbb{Q}$-representations and the G-equivariant isogeny decomposition of an abelian variety with G-action, in terms of the finite group G involved, cf. [7, Thm. 2.2]. The μ_n-equivariant isogeny decomposition of $\mathrm{Jac}(C')$ given above is in fact identical with the one introduced by Lange and Recillas [7]. This can be seen for example by using [2, Rem. 5.5] and [2, Cor. 5.7]. Moreover, we also note that the isogeny components $\mathcal{P}_d(C'/C)$ are non-trivial as long as the genus $g(C) \geq 1$, cf. [7, Thm. 3.1], [11, Thm. 5.12] and [11, Thm. 5.13].

3 Reduction to the generic fibre

Let S be a smooth projective surface over $\mathbb{C}$ with an ample line bundle $\mathcal{L}$. Assume $\Delta \in |\mathcal{L}^{\otimes n}|$ is smooth and consider the n-fold cyclic covering $f: S' \longrightarrow S$ branched along the divisor Δ. Furthermore, fix a very ample complete linear system $|\mathcal{H}|$ on S, such that the polarized surface $(S, |\mathcal{H}|)$ is not a scroll nor has rational hyperplane sections. In this section we reduce the proof of Theorem 1.1 to showing that $\mathcal{P}_d(C'_\eta/C_\eta)$ is a μ_d-simple abelian variety, where $[C_\eta]$ is the generic member of $|\mathcal{H}|$.

Let $x \in S$ be a closed point of S. We denote by $|\mathcal{H}|_x$ the linear system of hyperplane sections in $|\mathcal{H}|$ passing through x. In the following we impose restrictions on the point x, i.e. $x \in S$ will be taken from some appropriate non-empty open subset of S.

Let $g: \mathcal{X} \subset S \times |\mathcal{H}|_x \longrightarrow |\mathcal{H}|_x$ denote the universal family of hyperplane sections and $h: \mathcal{Y} \subset S' \times |\mathcal{H}|_x \longrightarrow |\mathcal{H}|_x$ its pullback to S', i.e. $\mathcal{Y} := \mathcal{X} \times_S S'$. Note that over the non-empty open subset $U \subset |\mathcal{H}|_x$ of smooth curves which intersect the branch locus Δ transversally both g and h are smooth families of curves having a section. The latter allows us to consider their families of Jacobians over U, which we denote by $p: \mathrm{Pic}^0_{\mathcal{X}/U} \longrightarrow U$ and $q: \mathrm{Pic}^0_{\mathcal{Y}/U} \longrightarrow U$, respectively.

A generator $\sigma : S' \longrightarrow S'$ of the Galois group of the covering f induces an automorphism of $\mathcal{Y}$ over U and thus, an automorphism $\sigma^* \colon \mathrm{Pic}^0_{\mathcal{Y}/U} \longrightarrow \mathrm{Pic}^0_{\mathcal{Y}/U}$. We define

$$\mathcal{P}_d := \ker^0(\Psi_d(\sigma^*)) \text{ for any divisor } d|n.$$

Then, $\varphi_d \colon \mathcal{P}_d \longrightarrow U$ is an abelian fibration with fibres $(\mathcal{P}_d)_{[C]} = \mathcal{P}_d(C'/C)$ for $[C] \in U$.

As a first step we use the representability of the endomorphism functor of abelian schemes cf. (2.1) to reduce the proof of Theorem 1.1 to showing that $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d]$, where $\bar{\eta}$ is a fixed geometric generic point of $|\mathcal{H}|_x$. The proof of this is standard and so we omit it.

Lemma 3.1 *Assume that* $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d]$. *Then, for the very general member* $[C] \in U$*, one has that* $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{[C]}) \cong \mathbb{Z}[\zeta_d]$.

Let $[C] \in |\mathcal{H}|_x$ be an irreducible member with a single ordinary double point as its only singularity and intersecting the branch locus Δ transversally. Then, $C' := f^{-1}(C)$ is irreducible and has n ordinary double points as its only singularities. In this case the group variety $\mathcal{P}_d(C'/C)$ is semi-abelian. In particular, the result is the following:

Lemma 3.2 *For an irreducible member* $[C] \in |\mathcal{H}|_x$ *with a single ordinary double point as its only singularity and intersecting the branch locus* Δ *transversally, there is an exact sequence:*

$$0 \longrightarrow \mathbb{G}_m^{\varphi(d)} \hookrightarrow \mathcal{P}_d(C'/C) \twoheadrightarrow \mathcal{P}_d(\tilde{C}'/\tilde{C}) \longrightarrow 0,$$

where $\nu \colon \tilde{C} \longrightarrow C$ *is the normalisation map and* $\varphi(d)$ *is the Euler's totient function.*

Proof We have a commutative diagram

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{\nu'} & C' \\ \big\downarrow \tilde{f} & & \big\downarrow f \\ \tilde{C} & \xrightarrow{\nu} & C, \end{array}$$

where $\tilde{f}$ is the cyclic covering branched along the divisor $\nu^*\Delta|_C \in |\nu^*\mathcal{L}|_C^{\otimes n}|$ and ν' is the normalisation of C'. Fix a generator σ of $\mathrm{Aut}(C'/C)$ and let $\tilde{\sigma}$ be the corresponding generator of $\mathrm{Aut}(\tilde{C}'/\tilde{C})$, i.e. the one for which the diagram below commutes

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{\nu'} & C' \\ \big\downarrow \tilde{\sigma} & & \big\downarrow \sigma \\ \tilde{C}' & \xrightarrow{\nu'} & C'. \end{array}$$

Let $\{y, \sigma(y), \sigma^2(y), \dots, \sigma^{n-1}(y)\}$ be the set of ordinary double points of C'. Then, we find a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\alpha) & \hookrightarrow & \Psi_d(\sigma^*)\,\mathrm{Pic}^0(C') & \xrightarrow{\alpha} \!\!\!\!\rightarrow & \Psi_d(\tilde{\sigma}^*)\,\mathrm{Pic}^0(\tilde{C}') & \longrightarrow & 0 \\ & & \gamma \big\uparrow & & \big\uparrow\!\!\uparrow & & \big\uparrow\!\!\uparrow & & \\ 0 & \longrightarrow & \mathbb{C}^*_y \times \cdots \times \mathbb{C}^*_{\sigma^{n-1}(y)} & \hookrightarrow & \mathrm{Pic}^0(C') & \xrightarrow{\nu'^*} \!\!\!\!\rightarrow & \mathrm{Pic}^0(\tilde{C}') & \longrightarrow & 0 \\ & & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\ 0 & \longrightarrow & \mathbb{C}^*_y \times \cdots \times \mathbb{C}^*_{\sigma^{\varphi(d)-1}(y)} & \hookrightarrow & \ker(\Psi_d(\sigma^*)) & \xrightarrow{\beta} & \ker(\Psi_d(\tilde{\sigma}^*)). & & \end{array}$$

We show that β induces a surjection $\mathcal{P}_d(C'/C) = \ker^0(\Psi_d(\sigma^*)) \to \mathcal{P}_d(\tilde{C}'/\tilde{C}) = \ker^0(\Psi_d(\tilde{\sigma}^*))$. Indeed, by Snake lemma we have the exact sequence

$$\ker(\Psi_d(\sigma^*)) \longrightarrow \ker(\Psi_d(\tilde{\sigma}^*)) \longrightarrow \operatorname{coker}(\gamma) \longrightarrow 0.$$

Note that $\operatorname{coker}(\gamma)$ is an affine algebraic group, as it is the quotient of a commutative affine algebraic group by an algebraic subgroup. Since $\ker(\Psi_d(\tilde{\sigma}^*))$ is a projective variety and the last arrow in the above sequence is surjective, [14, Cor. 12.67] shows that $\operatorname{coker}(\gamma)$ is finite. The latter provides the surjectivity of the map $\ker^0(\Psi_d(\sigma^*)) \longrightarrow \mathcal{P}_d(\tilde{C}'/\tilde{C}) = \ker^0(\Psi_d(\tilde{\sigma}^*))$, as claimed. □

We are now in the position to prove the following:

Proposition 3.3 *The abelian variety $(\mathcal{P}_d)_{\bar{\eta}}$ is μ_d-simple if and only if* $\operatorname{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d]$.

Proof The one direction is clear: Indeed, if $\operatorname{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \cong \mathbb{Z}[\zeta_d]$, then every non-zero μ_d-equivariant endomorphism of $(\mathcal{P}_d)_{\bar{\eta}}$ is an isogeny and thus, $(\mathcal{P}_d)_{\bar{\eta}}$ is a μ_d-simple abelian variety. Conversely, assume that $(\mathcal{P}_d)_{\bar{\eta}}$ is μ_d-simple. We divide the proof into steps.
Step 1. There is a closed subscheme $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}(0) \subset \operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}$ whose points parametrise the μ_d-equivariant endomorphisms of $\mathcal{P}_d$, which are not isogenies, i.e. the ones, which are of degree 0. □

Proof of Step 1 Observe that the functor of μ_d-equivariant endomorphisms of $\mathcal{P}_d$ denoted by $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}$ is representable by a closed subscheme of $\operatorname{End}_{\mathcal{P}_d/U}$, since the equivariant condition is closed. It follows that we have a universal endomorphism α, such that every other μ_d-equivariant endomorphism of $\mathcal{P}_d$ over some scheme T is obtained by pulling-back α along a morphism $T \longrightarrow \operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}$. By [14, Prop. 12.93] the set

$$\mathcal{V}:=\{x \in \operatorname{End}^{\mu_d}_{\mathcal{P}_d/U} \mid \alpha_x:=\alpha \times \mathrm{id}_{\kappa(x)} \text{ is an isogeny}\}$$

is open. Therefore, $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}(0):=\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U} \setminus \mathcal{V}$ with the reduced induced closed subscheme structure has the desired property. □

Step 2. The fibre $(\mathcal{P}_d)_{[C]}$ for the very general member $[C] \in |\mathcal{H}|_x$ is a μ_d-absolutely simple abelian variety.

Proof of Step 2 Recall that the U-scheme $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}(0)$ is unramified cf. (2.1). It follows that a geometric fibre of this U-scheme is a disjoint union of points, corresponding to the μ_d-equivariant endomorphisms of $\mathcal{P}_d$, which are not isogenies cf. Step 1. Since $(\mathcal{P}_d)_{\bar{\eta}}$ is a μ_d-simple abelian variety, the only μ_d-equivariant endomorphism of $(\mathcal{P}_d)_{\bar{\eta}}$, that is not an isogeny is the zero-morphism. In particular, this means that the geometric generic fibre of the U-scheme $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}(0)$ is connected and therefore, we can determine countably many non-empty open subsets $U_i \subset U$, such that the U-scheme $\operatorname{End}^{\mu_d}_{\mathcal{P}_d/U}(0)$ has (geometrically) connected fibres for all points lying in the intersection of the U_i's, cf. [13, Tag 055C]. Thus, for the very general member $[C] \in |\mathcal{H}|_x$, the only μ_d-equivariant endomorphism of $(\mathcal{P}_d)_{[C]}$, which is not an isogeny is the zero-morphism. The latter is equivalent to the μ_d-simplicity of $(\mathcal{P}_d)_{[C]}$, proving the claim. □

Pick a Lefschetz pencil $(C_t)_{t\in\mathbb{P}^1} \subset |\mathcal{H}|_x$. We may assume that all its singular members are irreducible and intersect the branch locus Δ transversally, cf. (2.3).
Step 3. Given a Lefschetz pencil $(C_t)_{t\in\mathbb{P}^1}$ as above, we construct a homomorphism:

$$\rho\colon \operatorname{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \longrightarrow \operatorname{End}(\mathbb{G}_m^{\varphi(d)}),$$

where $\bar{\mu}$ is a fixed geometric generic point of $\mathbb{P}^1$.

Proof of Step 3 Since the endomorphism ring of any abelian variety is finitely generated, cf. [[9], Thm. 12.5], we find a finite field extension $L \supset \kappa(\mu)$, such that every endomorphism of $\mathcal{P}_d$ over $\kappa(\bar{\mu})$ is defined over L, i.e. $\mathrm{End}((\mathcal{P}_d)_{\bar{\mu}}) = \mathrm{End}((\mathcal{P}_d)_L)$. Consider the smooth projective model E of L together with the morphism $E \longrightarrow \mathbb{P}^1$ induced by this field extension and fix a closed point $y \in E$ lying over a point of the pencil that corresponds to a nodal curve. The map $\rho\colon \mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \longrightarrow \mathrm{End}(\mathbb{G}_m^{\varphi(d)})$ is constructed as follows: Let $f \in \mathrm{End}_{\mu_d}((\mathcal{P}_d)_L)$. Then, f extends to an endomorphism over the local ring R of E at the point y, cf. [12, Prop. 7.4.3]. The restriction of the first projection of $\mathcal{P}_d \times_R \mathcal{P}_d$ to the graph of f is an isomorphism. We set $\alpha := \mathrm{pr}_1 |_{(\Gamma_f)_y}$. By pulling back α along $\mathbb{G}_m^{\varphi(d)} \hookrightarrow (\mathcal{P}_d)_y$, we get an isomorphism $\alpha\colon \alpha^{-1}(\mathbb{G}_m^{\varphi(d)}) \longrightarrow \mathbb{G}_m^{\varphi(d)}$. We claim that α^{-1} is the graph of a homomorphism $\mathbb{G}_m^{\varphi(d)} \longrightarrow \mathbb{G}_m^{\varphi(d)}$. Indeed, it suffices to show that $\mathrm{pr}_2(\alpha^{-1}(\mathbb{G}_m^{\varphi(d)})) \subset \mathbb{G}_m^{\varphi(d)}$. To see this, observe that the composite

$$\mathbb{G}_m^{\varphi(d)} \xrightarrow{\cong} \alpha^{-1}(\mathbb{G}_m^{\varphi(d)}) \subset (\Gamma_f)_y \xrightarrow{\mathrm{pr}_2} (\mathcal{P}_d)_y \longrightarrow \mathcal{P}_d(\tilde{C}_y'/\tilde{C}_y)$$

is the zero map by [[9], Cor. 3.9] and hence, $\mathrm{pr}_2 |_{\mathbb{G}_m^{\varphi(d)}}$ factors through the kernel of $(\mathcal{P}_d)_y \longrightarrow \mathcal{P}_d(\tilde{C}_y'/\tilde{C}_y)$ which is $\mathbb{G}_m^{\varphi(d)}$. Finally, we define $\rho(f)$ to be this endomorphism of $\mathbb{G}_m^{\varphi(d)}$. One checks that ρ is a homomorphism of rings. □

Conclusion Eventually, we are in the position to complete the proof. Suppose $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\eta}}) \neq \mathbb{Z}[\zeta_d]$ and choose a μ_d-equivariant endomorphism f not in $\mathbb{Z}[\zeta_d]$. The endomorphism f can be described as a $\kappa(\bar{\eta})$-point of $\mathrm{End}^{\mu_d}_{\mathcal{P}_d/U}$ and we let $Z \subset \mathrm{End}^{\mu_d}_{\mathcal{P}_d/U}$ be the irreducible component containing this point. Then, the generic point $\theta \in Z$ corresponds to a μ_d-equivariant endomorphism not in $\mathbb{Z}[\zeta_d]$. Consider the finite set

$$\Gamma := \{n := (n_0, n_1, \ldots, n_{\varphi(d)-1}) \in \mathbb{Z}^{\varphi(d)} \mid \mathrm{im}([n]^1) \cap Z \neq \emptyset\}.$$

Each $\mathrm{im}([n]) \cap Z$ is a proper closed subset of Z. Setting[1]

$$Z_n := \pi(\mathrm{im}([n]) \cap Z),$$

for $n \in \Gamma$, we get finitely many nowhere dense closed subsets of U, such that for every point $u \in U \setminus \bigcup_{n \in \Gamma} Z_n$ the fibre $\pi^{-1}(u)$ contains a point, which is not in $\mathbb{Z}[\zeta_d]$. We can choose a Lefschetz pencil as above, such that $(\mathcal{P}_d)_{\bar{\mu}}$ is μ_d-simple, cf. Step 2 and $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \neq \mathbb{Z}[\zeta_d]$. By Step 3 this leads to a contradiction. Indeed, using that every non-zero element of $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}})$ is invertible in $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \otimes \mathbb{Q}$, it is readily checked that the composition of the map ρ constructed in Step 3 with $\psi := \mathrm{pr}_1 \circ -\colon \mathrm{End}(\mathbb{G}_m^{\varphi(\delta)}) \longrightarrow \mathrm{Hom}(\mathbb{G}_m^{\varphi(\delta)}, \mathbb{G}_m) \cong \mathbb{Z}^{\varphi(\delta)}$ is injective. It follows that $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \otimes \mathbb{Q} \cong \mathbb{Q}(\zeta_d)$. Since $\mathbb{Z}[\zeta_d]$ is a maximal order in $\mathbb{Q}(\zeta_d)$, we also obtain $\mathrm{End}_{\mu_d}((\mathcal{P}_d)_{\bar{\mu}}) \cong \mathbb{Z}[\zeta_d]$. The proof is complete. □

The next lemma consists of the final reduction step.

Lemma 3.4 *The abelian variety $(\mathcal{P}_d)_\eta$ is μ_d-simple if and only if it is μ_d-absolutely simple.*

Proof Clearly, if $(\mathcal{P}_d)_\eta$ is μ_d-absolutely simple, then it is μ_d-simple. Conversely, assume that $(\mathcal{P}_d)_\eta$ is μ_d-simple but not μ_d-absolutely simple. Then, there is a finite field extension $L \supset \kappa(\eta)$ and a non-zero and proper μ_d-simple abelian subvariety B of $(\mathcal{P}_d)_L$, such that $(\mathcal{P}_d)_L$ can be written up to isogeny as a product $\prod B^\tau$, where B^τ stands for a Galois conjugate of B and τ runs through a finite subset $J \subset \mathrm{Gal}(L/\kappa(\eta))$ of cardinality greater equal to 2. The field extension $L \supset \kappa(\eta)$ gives rise to a morphism $g\colon U' \longrightarrow U$, which we may assume

[1] $[n] := n_0 \,\mathrm{id} + n_1\sigma^* + n_2(\sigma^*)^2 + \cdots + n_{\varphi(d)-1}(\sigma^*)^{\varphi(d)-1}$.

is étale. For $\tau \in J$, we let φ_τ be the endomorphism of $(\mathcal{P}_d)_L$ whose image is B^τ. More explicitly, φ_τ is given by

$$(\mathcal{P}_d)_L \xrightarrow{\sim} \prod B^\tau \xrightarrow{proj} B^\tau \subset (\mathcal{P}_d)_L.$$

Pick a Lefschetz pencil $(C_t)_{t\in\mathbb{P}^1}$, such that its singular members are irreducible and intersect the branch locus Δ transversally. Let X be any irreducible component of $g^{-1}(\mathbb{P}^1 \cap U)$. Then, X dominates $\mathbb{P}^1 \cap U$ and if $\theta \in X$ is its generic point, then each φ_τ determines an endomorphism of $\mathcal{P}_d$ over θ, e.g. using the Néron mapping property, such that if $B^\tau := \mathrm{im}(\varphi_\tau)$, then $\prod B^\tau \sim (\mathcal{P}_d)_\theta$. Let $\bar{X}$ be a smooth compactification of X and $\bar{X} \longrightarrow \mathbb{P}^1$ the extension of $g\colon X \longrightarrow \mathbb{P}^1 \cap U$. Fix a point $y \in \bar{X}$ lying over a point of the pencil which corresponds to a nodal curve and consider the local ring R of $\bar{X}$ at y. Since $\mathcal{P}_d$ admits a semi-abelian reduction over R , cf. (3.2) the same is true for all B^τ, cf. [12, Cor. 7.1.6]. We denote by $\tilde{B}^\tau$ the identity component of the Néron model of B^τ. Then, the isogeny of the generic fibre extends to an isogeny $\prod \tilde{B}^\tau \sim \mathcal{P}_d$ over R, cf. [12, Prop. 7.3.6]. Since $(\mathcal{P}_d)_y$ is an extension of an abelian variety by a torus of rank $\varphi(d)$, cf. (3.2), it follows that the toric part of $\tilde{B}^\tau_y$ has rank δ, $1 \le \delta \le \varphi(d)$, such that $\delta|J| = \varphi(d)$. As in Step 3, one constructs a homomorphism $\rho_\tau\colon \mathrm{End}_{\mu_d}(B^\tau) \longrightarrow \mathrm{End}(\mathbb{G}_m^\delta)$. Since the restriction of $\psi \circ \rho_\tau$ to $\mathbb{Z}[\zeta_d] \subset \mathrm{End}_{\mu_d}(B^\tau)$ is injective, where $\psi := \mathrm{pr}_1 \circ -\colon \mathrm{End}(\mathbb{G}_m^\delta) \longrightarrow \mathrm{Hom}(\mathbb{G}_m^\delta, \mathbb{G}_m) \cong \mathbb{Z}^\delta$ and $\mathbb{Z}[\zeta_d]$ has rank $\varphi(d)$ as a free abelian group, we conclude that $\delta = \varphi(d)$. But then $|J| = 1$, which is absurd. □

4 The Proof of Theorem 1.1

According to the results of Sect. 3, our task to prove Theorem 1.1 is reduced to showing $(\mathcal{P}_d)_\eta$ is a μ_d-simple abelian variety. Recall, that we have an isogeny

$$\mathrm{Jac}(C'_\eta) \sim \mathrm{Jac}(C_\eta) \times \prod_{d|n,\ d\neq 1} (\mathcal{P}_d)_\eta.$$

Given a non-zero endomorphism $\varepsilon \in \mathrm{End}_{\mu_d}((\mathcal{P}_d)_\eta)$. Then, by considering the composite

$$\varepsilon'\colon \mathrm{Jac}(C'_\eta) \xrightarrow{\sim} \mathrm{Jac}(C_\eta) \times \prod_{d|n,\ d\neq 1} (\mathcal{P}_d)_\eta \xrightarrow{\mathrm{pr}_d} (\mathcal{P}_d)_\eta \xrightarrow{\varepsilon} (\mathcal{P}_d)_\eta \hookrightarrow \mathrm{Jac}(C'_\eta),$$

we get an endomorphism of $\mathrm{Jac}(C'_\eta)$ whose restriction to $(\mathcal{P}_d)_\eta$ is simply $\varepsilon \circ [n]$. Hence, it suffices to show that that the restriction of ε' to $(\mathcal{P}_d)_\eta$ lies in $\mathbb{Z}[\zeta_d]$. Recall, that abelian schemes satisfy a stronger Néron mapping property, cf. [10, Sec. 3.1.5]. Thus, the endomorphism ε' extends to an endomorphism

$$\varepsilon'\colon \mathrm{Pic}^0_{\mathcal{Y}/U} \longrightarrow \mathcal{P}_d \subset \mathrm{Pic}^0_{\mathcal{Y}/U}.$$

Let $[T] \in \mathrm{Corr}_U(\mathcal{Y})$ be the class of a correspondence T on $\mathcal{Y} \times_U \mathcal{Y}$ associated to the endomorphism ε', cf. (2.2). We write $T = \sum n_i T_i$, where T_i are prime divisors. Let Σ be a general two dimensional linear system in $|\mathcal{H}|_x$, i.e. the general member of Σ is smooth and intersects the branch locus Δ transversally. Then, the correspondences T_i are all defined over a non-empty open subset of Σ and we can construct a rational map $\phi_{\Sigma,T_i}\colon S' \dashrightarrow \mathrm{Div}^+(S'),\ y \mapsto \Gamma^i_y$, cf. [3, pp. 38]. Especially, we get a rational map

$$\phi_{\Sigma,T}\colon S' \dashrightarrow \mathrm{Pic}(S'),\ y \mapsto [\Gamma_y] := \sum n_i [\Gamma^i_y].$$

Let $[C] \in |\mathcal{H}|_x$ be a general member and choose a general two-dimensional linear system Σ containing $[C]$. Consider the rational map $\phi_{\Sigma,T}$. Then, for a general point $y \in C'$ we get a divisor $\Gamma_y = \phi_{\Sigma,T}(y)$ on S'. Set $w = f(y) \in C$, $f^{-1}(w) = \{y, \sigma(y), \ldots, \sigma^{n-1}(y)\}$ and $f^{-1}(x) = \{z, \sigma(z), \ldots, \sigma^{n-1}(z)\}$, where σ is a generator of the Galois group of the covering f. The following lemma computes the divisor E_y in C' corresponding to the intersection of C' with Γ_y.

Lemma 4.1 *We have that* $E_y = \alpha_0 z + \alpha_1\sigma(z) + \ldots + \alpha_{n-1}\sigma^{n-1}(z) + \beta_0 y + \beta_1\sigma(y) + \ldots + \beta_{n-1}\sigma^{n-1}(y) + \gamma\mathcal{B}'_{x,w} + T_{C'}(y)$, *where* $\alpha_i, \beta_i, \gamma \in \mathbb{Z}$ *and* $\mathcal{B}'_{x,w}$ *is the pull-back of the divisor of base points different from* x *and* w *of* Σ_w *under the covering* f.

Proof Cf. [3, Lem. 3.6]. □

4.1 Regular case

The branched locus Δ of the covering f is a smooth ample divisor and thus, the canonical map $\mathrm{Alb}(f)\colon \mathrm{Alb}(S') \longrightarrow \mathrm{Alb}(S)$ induced by f is an isomorphism. Indeed, since $f_*\mathcal{O}_{S'} \cong \bigoplus_{i=0}^{n-1}\mathcal{L}^{-i}$, the Kodaira Vanishing theorem gives $H^1(\mathcal{O}_{S'}) = H^1(\mathcal{O}_S)$ and hence, $\mathrm{Alb}(f)$ is an isogeny. From this one immediately sees that the induced action on $\mathrm{Alb}(S')$ is trivial, i.e. $\mathrm{Alb}(\sigma) = \mathrm{id}$. Consider the Albanese map $\mathrm{Alb}_{\xi_o}\colon S' \longrightarrow \mathrm{Alb}(S')$, where the point $\xi_o \in S'$ lies over a point of the branch locus $\Delta \subset S$ and observe that the map is invariant under the μ_n-action. Therefore, we find a homomorphism $\mathrm{Alb}(S) \longrightarrow \mathrm{Alb}(S')$ that is inverse to $\mathrm{Alb}(f)$, proving the claim. In particular, we deduce that $q(S) = q(S')$. Here, we give the proof for the case S is regular, i.e. $q(S) = 0$.

Proof of Theorem 1.1 for the regular case If S is regular, then $\mathrm{Pic}(S')$ is discrete and thus, the rational map $\phi_{\Sigma,T}$ is constant. Hence, for a general point $y \in C'$, the curves Γ_y and $\Gamma_{\sigma(y)}$ are linearly equivalent. It follows that E_y and $E_{\sigma(y)}$ are also linearly equivalent and so, $E_y - E_{\sigma(y)} = \beta_0(y - \sigma(y)) + \beta_1\sigma(y - \sigma(y)) + \ldots + \beta_{n-1}\sigma^{n-1}(y - \sigma(y)) + T_{C'}(y - \sigma(y)) \sim 0$. Since $\mathrm{Prym}(C'/C) = \mathrm{im}(\mathrm{id} - \sigma^*)$, the latter forces $T_{C'}(y) = (-\beta_0)y + \ldots + (-\beta_{n-1})\sigma^{n-1}(y)$ for all $y \in \mathrm{Prym}(C'/C)$. Eventually, we see that the restriction of $T_{C'}$ to $\mathcal{P}_d(C'/C)$ takes the desired form. This yields that the restriction of ε' to $(\mathcal{P}_d)_\eta$ lies in $\mathbb{Z}[\zeta_d]$, as claimed. □

4.2 Irregular case

The closed embedding $i\colon C' \hookrightarrow S'$ defines the natural map $i^*\colon \mathrm{Pic}^0(S') \longrightarrow \mathrm{Pic}^0(C')$ whose kernel is finite, since $H^1(S', \mathcal{O}_{S'}(-C')) = 0$. In what follows we view $\mathrm{Pic}^0(S')$ as an abelian subvariety of $\mathrm{Jac}(C')$ by identifying it with $\mathrm{im}(i^*)$. We shall use the following lemma.

Lemma 4.2 *Let* $a\colon \mathrm{Jac}(C') \longrightarrow \mathcal{P}_d(C'/C) \subset \mathrm{Jac}(C')$ *be a homomorphism and let* T_a *be a correspondence associated to it, cf.* (2.2)*. Assume that there exist* $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}$*, such that for general* $y \in C'$ *the divisor class* $T_a(y-\sigma(y)) + \alpha_0(y-\sigma(y)) + \ldots + \alpha_{n-1}\sigma^{n-1}(y-\sigma(y))$ *lies in* $\mathrm{Pic}^0(S')$*. Then, the restriction of* a *to* $\mathcal{P}_d(C'/C)$ *lies in* $\mathbb{Z}[\zeta_d] \subset \mathrm{End}(\mathcal{P}_d(C'/C))$.

Proof Recall that $\mathrm{Prym}(C'/C) = \mathrm{im}(\mathrm{id} - \sigma^*)$ and for this reason the closed points of $\mathrm{Prym}(C'/C)$ are generated by elements of the form $y - \sigma(y)$, where $y \in C'$. Hence, the assumption clearly implies that $\eta(y) := a(y) + \alpha_0 y + \ldots + \alpha_{n-1}\sigma^{n-1}(y) \in \mathrm{im}(i^*) \cap$

$\mathrm{Prym}(C'/C)$ (note that $\mathcal{P}_d(C'/C) \subset \mathrm{Prym}(C'/C)$) for all $y \in \mathrm{Prym}(C'/C)$, where $i^*\colon \mathrm{Pic}^0(S') \longrightarrow \mathrm{Pic}^0(C') = \mathrm{Jac}(C')$ is the natural pull-back induced by $C' \hookrightarrow S'$. We show that the intersection $\mathrm{im}(i^*) \cap \mathrm{Prym}(C'/C)$ is finite. Indeed, consider the commutative square:

$$\begin{array}{ccc} \mathrm{Pic}^0(S) & \xrightarrow{i^*} & \mathrm{Pic}^0(C) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{Pic}^0(S') & \xrightarrow{i^*} & \mathrm{Pic}^0(C'). \end{array}$$

The canonical map $\mathrm{Alb}(S') \longrightarrow \mathrm{Alb}(S)$ induced by f is an isomorphism and so, is its dual, which is f^*. Hence, the latter yields that $\mathrm{im}(i^*\colon \mathrm{Pic}^0(S') \longrightarrow \mathrm{Pic}^0(C')) \subset f^*(\mathrm{Pic}^0(C))$. By the definition of $\mathrm{Prym}(C'/C)$, we know that $f^*(\mathrm{Pic}^0(C)) \cap \mathrm{Prym}(C'/C)$ is finite and so, is the intersection $\mathrm{im}(i^*) \cap \mathrm{Prym}(C'/C)$, as claimed. From the latter one deduces that the endomorphism η of $\mathrm{Prym}(C'/C)$ defined above is the zero-map, simply because $\eta(\mathrm{Prym}(C'/C))$ is irreducible subvariety of $\mathrm{im}(i^*) \cap \mathrm{Prym}(C'/C)$, which is a finite union of points. Finally, by restricting to $\mathcal{P}_d(C'/C) \subset \mathrm{Prym}(C'/C)$, we conclude that a lies in the image of the map $\mathbb{Z}[\zeta_d] \subset \mathrm{End}(\mathcal{P}_d(C'/C))$, $\zeta_d \mapsto \sigma$. The proof is complete. □

Proof of Theorem 1.1 for the irregular case Using the curves Γ_y we find that $E_y - E_{\sigma(y)}$ lies in the image of $\mathrm{Pic}(S') \longrightarrow \mathrm{Pic}(C')$. Therefore, we have that $T_{C'}(y-\sigma(y)) + \beta_0(y-\sigma(y)) + \beta_1\sigma(y-\sigma(y)) + \ldots + \beta_{n-1}\sigma^{n-1}(y-\sigma(y)) \in \mathrm{im}(i^*\colon \mathrm{Pic}(S') \longrightarrow \mathrm{Pic}(C'))$ for general $y \in C'$. It follows that $\varepsilon' \in \mathbb{Z}[\zeta_d] \subset \mathrm{End}((\mathcal{P}_d)_\eta)$, cf. (4.2). □

5 The proof of Theorem 1.3

The proof is similar to the case of (1.1). First, we need to replace our earlier family $\varphi_d\colon \mathcal{P}_d \longrightarrow U$. In particular, we consider the abelian fibration

$$\mathcal{R}_d := \ker^0(\mathcal{P}_d \longrightarrow \mathrm{Alb}(S') \times U).$$

Assume that the abelian fibration $\varphi_d\colon \mathcal{R}_d \longrightarrow U$ is non-zero, i.e. $\mathcal{R}_{[C]} \neq 0$ for $[C] \in U$. Then, we show that for the very general member $[C] \in U$, we have that $\mathrm{End}_{\mu_d}((\mathcal{R}_d)_{[C]}) \cong \mathbb{Z}[\zeta_d]$. One checks that the results (3.3) and (3.4) still hold true for the family $\varphi_d\colon \mathcal{R}_d \longrightarrow U$.

We proceed as in the proof of Theorem 1.1. A non-zero endomorphism $\varepsilon \in \mathrm{End}_{\mu_d}((\mathcal{R}_d)_\eta)$ gives rise to an endomorphism $\varepsilon' \in \mathrm{End}(\mathrm{Jac}(C'_\eta))$ and it is enough to check that the restriction of ε' to $(\mathcal{R}_d)_\eta$ lies in $\mathbb{Z}[\zeta_d]$. The following lemma is needed.

Lemma 5.1 *Let $a\colon \mathrm{Jac}(C') \longrightarrow \mathcal{R}_d(C', C, S') \subset \mathrm{Jac}(C')$ be a homomorphism and let T_a be a correspondence associated to it, cf. (2.2). Assume that there exist $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{Z}$, such that for general $y \in C'$ the divisor class $T_a(y-\sigma(y)) + \alpha_0(y-\sigma(y)) + \ldots + \alpha_{n-1}\sigma^{n-1}(y-\sigma(y))$ lies in $\mathrm{Pic}^0(S')$. Then, the restriction of a to $\mathcal{R}_d(C', C, S')$ lies in $\mathbb{Z}[\zeta_d] \subset \mathrm{End}(\mathcal{R}_d(C', C, S'))$.*

Proof Clearly, we have that $a(y) + \alpha_0 y + \ldots + \alpha_{n-1}\sigma^{n-1}(y) \in \mathrm{im}(i^*)$ for all $y \in \mathrm{Prym}(C'/C)$, where $i^*\colon \mathrm{Pic}^0(S') \longrightarrow \mathrm{Pic}^0(C') = \mathrm{Jac}(C')$ is the pull-back induced by $C' \hookrightarrow S'$. Let $\mathcal{K}(C', S') := \ker(\mathrm{Jac}(C') \longrightarrow \mathrm{Alb}(S'))$ and observe that the intersection $\mathrm{im}(i^*) \cap \mathcal{K}(C', S')$ is finite. Since $\mathcal{R}_d(C', C, S') \subset \mathcal{K}(C', S')$, we find that $a(y) + \alpha_0 y + \ldots + \alpha_{n-1}\sigma^{n-1}(y) = 0$ for all $y \in \mathcal{R}_d(C', C, S')$. Therefore, the restriction of a to $\mathcal{R}_d(C', C, S')$ belongs to $\mathbb{Z}[\zeta_d]$, as claimed. □

Proof of Theorem 1.3 Using the curves Γ_y one sees that $E_y - E_{\sigma(y)}$ lies in the image of $\mathrm{Pic}(S') \longrightarrow \mathrm{Pic}(C')$. It follows that $T_{C'}(y - \sigma(y)) + \beta_0(y - \sigma(y)) + \beta_1\sigma(y - \sigma(y)) + \ldots + \beta_{n-1}\sigma^{n-1}(y - \sigma(y)) \in \mathrm{im}(i^*\colon\ \mathrm{Pic}(S') \longrightarrow \mathrm{Pic}(C'))$. Now, the result is an immediate consequence of (5.1). □

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