Liftable vector fields, unfoldings and augmentations

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Abstract

We study liftable vector fields of smooth map-germs. We show how to obtain the module of liftable vector fields of any map-germ of finite singularity type from the module of liftable vector fields of a stable unfolding of it. As an application, we obtain the liftable vector fields for the family H_k in Mond's list. We then show the relation between the liftable vector fields of a stable germ and its augmentations.

Keywords Liftable vector fields · Augmentations · Stable unfoldings

Mathematics Subject Classification Primary 58K40; Secondary 58K20 · 32S05

1 Introduction

Since the beginning of the study of singularities of differentiable maps with Whitney and Thom, classification of singularities has been one of the main research subjects. Classical techniques, even with the help of computers, have, in a way, exhausted their potential and forced the appearance of new methods in order to attend the growing need of harder classifications. Examples of these new methods can be found in [3, 12]. In [1] Arnol'd defined liftable vector fields and showed how they can be used for classification purposes. In fact, the new methods developed rely in many aspects in the computation of liftable vector fields. However, it is not easy at all to compute liftable vector fields in general. Liftable vector fields of a map germ are strongly related to the Derlog of the discriminant of the map germ (the module of tangent vector fields to the discriminant), which are of great importance in algebraic geometry besides singularity theory.

Here we consider $\mathcal{A}$ -classification of map-germs $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, $S \subset \mathbb{K}$ is a finite subset and the map-germ is either smooth when $\mathbb{K} = \mathbb{R}$ or

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holomorphic when $\mathbb{K} = \mathbb{C}$. In [11], the authors gave a method to construct liftable vector fields for a certain class of map-germs f which includes stable germs. They then showed how to obtain liftable vector fields of a finite singularity type germ f from liftable vector fields of F , a stable unfolding of it. By finite singularity type, we mean a germ which admits a stable unfolding with a finite number of parameters. Furthermore, if F is a one-parameter stable unfolding, they showed that any liftable vector field of f comes from a liftable vector field of F by the previous construction. Although the class of map-germs which admit one-parameter stable unfoldings seems to be quite big amongst simple germs, it is still not fully understood when this is the case.

In this paper we generalize the result in [11], that is, we prove that all liftable vector fields of any finite singularity type map-germ can be obtained from the liftable vector fields of a stable unfolding of it, with no restriction on the number of parameters. The main result is that if F is an r -parameter stable unfolding of f , then

$$\text{Lift}(f) = \pi_1(i^*(\text{Lift}(F) \cap G)),$$

where π_1 is the projection onto the first p components, i^* is the morphism induced by $i(X) = (X, 0)$ and G is the submodule of θ_{p+r} generated by $\partial/\partial X_k$, with $1 \leq k \leq p$ and $\Lambda_i \partial/\partial \Lambda_j$, $1 \leq i, j \leq r$ (here we denote by (X, Λ) the coordinates in $\mathbb{K}^p \times \mathbb{K}^r$). As an application, we obtain the liftable vector fields for the family H_k in Mond's list of simple germs $f : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^3, 0)$ [9]. Another interesting corollary is that in the complex case and provided that either $n \leq p$ or $n > p > 2$, then

$$\text{Derlog}(\Delta(f)) = \pi_1(i^*(\text{Derlog}(\Delta(F)) \cap G)),$$

where $\Delta(f)$ and $\Delta(F)$ are the discriminants of f and F respectively. When $n > p = 1$, this equality holds if and only if f is quasihomogeneous (up to $\mathcal{A}$ -equivalence) and when $n > p = 2$ it is not known.

In the last section, we turn our attention to families of singularities which can be obtained by the method of augmentation. We show the relation between the liftable vector fields of a one-parameter stable unfolding of a map-germ and of its augmentations. We then use this result to prove the following: let F be a one-parameter stable unfolding of $f : (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ with $p \leq n + 1$ (except for $n > p \leq 2$) and let Af be the augmentation of f by F and a quasihomogeneous function ϕ , then

$$\pi_2(i^*(\text{Lift}(Af))) = \pi_2(i^*(\text{Lift}(F))),$$

where now π_2 is the projection onto the last component. What is meant here is that these two spaces are isomorphic, so the equality is understood as an equality of ideals of $\mathcal{O}_p$, the ring of smooth function germs in p -variables. This property allows us to use some results in [2, 12, 13], where this condition is required.

2 Notation

We consider map-germs $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $S \subset \mathbb{K}^n$ a finite subset. For simplicity, we will say that f is smooth if it is smooth (i.e. C^∞) when $\mathbb{K} = \mathbb{R}$ or holomorphic when $\mathbb{K} = \mathbb{C}$. We denote by $\mathcal{O}_n = \mathcal{O}_{\mathbb{K}^n, S}$ and $\mathcal{O}_p = \mathcal{O}_{\mathbb{K}^p, 0}$ the rings of smooth function germs in the source and target respectively and by $\theta_n = \theta_{\mathbb{K}^n, S}$ and $\theta_p = \theta_{\mathbb{K}^p, 0}$ the corresponding modules of vector field germs. The module of vector fields along f will be denoted by $\theta(f)$. Associated with $\theta(f)$ we have two morphisms

$tf : \theta_n \rightarrow \theta(f)$, given by $tf(\chi) = df \circ \chi$, and $wf : \theta_p \rightarrow \theta(f)$, given by $wf(\eta) = \eta \circ f$. The $\mathcal{A}_e$ -tangent space and the $\mathcal{A}_e$ -codimension of f are defined respectively as

$$T\mathcal{A}_e f = tf(\theta_n) + wf(\theta_p), \quad \mathcal{A}_e\text{-cod}(f) = \dim_{\mathbb{K}} \frac{\theta(f)}{T\mathcal{A}_e f}.$$

It follows from Mather's infinitesimal stability criterion [8] that a germ is stable if and only if its $\mathcal{A}_e$ -codimension is 0. We refer to Wall's survey paper [14] and the book [10] for general background on the theory of singularities of mappings.

We will say that f has finite singularity type if

$$\dim_{\mathbb{K}} \frac{\theta(f)}{tf(\theta_n) + (f^*m_p)\theta_n} < \infty,$$

where m_p is the maximal ideal of $\mathcal{O}_p$ and $f^* : \mathcal{O}_p \rightarrow \mathcal{O}_n$ is the induced map. Another remarkable result of Mather is that f has finite singularity type if and only if it admits an r -parameter stable unfolding (see [14]). We recall that an r -parameter unfolding of f is another map-germ

$$F : (\mathbb{K}^n \times \mathbb{K}^r, S \times \{0\}) \rightarrow (\mathbb{K}^p \times \mathbb{K}^r, 0)$$

of the form $F(x, \lambda) = (f_\lambda(x), \lambda)$ and such that $f_0 = f$.

Along the paper, we use the notation of small letters $x_1, \dots, x_n, \lambda_1, \dots, \lambda_r$ for the coordinates in $\mathbb{K}^n \times \mathbb{K}^r$ and capital letters $X_1, \dots, X_p, \Lambda_1, \dots, \Lambda_r$ for the coordinates in $\mathbb{K}^p \times \mathbb{K}^r$.

Definition 1 Let $f : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be a smooth map-germ.

- (1) A vector field germ $\eta \in \theta_p$ is called *liftable over f* , if there exists $\xi \in \theta_n$ such that $df \circ \xi = \eta \circ f$ (i.e., $tf(\xi) = wf(\eta)$). The set of vector field germs liftable over f is denoted by $\text{Lift}(f)$ and is an $\mathcal{O}_p$ -submodule of θ_p .
- (2) We also define $\tilde{\tau}(f) = ev_0(\text{Lift}(f))$, the subspace of $T_0\mathbb{K}^p$ given by the evaluation at the origin of elements of $\text{Lift}(f)$.

When f has finite $\mathcal{A}_e$ -codimension, the space $\tilde{\tau}(f)$ can be interpreted geometrically as the tangent space to the isosingular locus (or analytic stratum in Mather's terminology) of f . This is the well defined manifold in the target along which the map f is trivial. See [3,6] for some basic properties of $\tilde{\tau}(f)$.

When $\mathbb{K} = \mathbb{C}$ and f has finite singularity type, we always have the inclusion $\text{Lift}(f) \subseteq \text{Derlog}(\Delta(f))$, where $\Delta(f)$ is the discriminant of f (i.e., the image of non submersive points of f) and $\text{Derlog}(\Delta(f))$ is the submodule of θ_p of vector fields which are tangent to $\Delta(f)$. Moreover, for all (n, p) except for the case $n > p \leq 2$, we have the equality $\text{Lift}(f) = \text{Derlog}(\Delta(f))$ if f has finite $\mathcal{A}_e$ -codimension. This was proven in [1] for the $n = p$ case, in [4] for when f is stable and in Proposition 8.8 and Remark 8.2 in [10] for all (n, p) except for the case $n > p \leq 2$. If $n > p = 1$ and f has finite $\mathcal{A}_e$ -codimension, $\text{Lift}(f) = \text{Derlog}(\Delta(f))$ if and only if f is quasihomogeneous. If $n > p = 2$ it is not known.

Definition 2 Let $h : (\mathbb{K}^n, S) \rightarrow (\mathbb{K}^p, 0)$ be a map-germ with a 1-parameter stable unfolding $H(x, \lambda) = (h_\lambda(x), \lambda)$. Let $g : (\mathbb{K}^q, 0) \rightarrow (\mathbb{K}, 0)$ be a function-germ. Then, the *augmentation of h by H and g* is the map $A_{H,g}(h)$ given by $(x, z) \mapsto (h_{g(z)}(x), z)$.

3 Lifiable vector fields and stable unfoldings

In this section we show that all the liftable vector fields of any map-germ of finite singularity type can be obtained from the liftable vector fields of a stable unfolding (not necessarily versal) of it.

Theorem 1 *Let $f : (\mathbb{K}^n, S) \longrightarrow (\mathbb{K}^p, 0)$ be a non-stable germ of finite singularity type and let F be an r -parameter stable unfolding. Then,*

$$\text{Lift}(f) = \pi_1(i^*(\text{Lift}(F) \cap G)),$$

where π_1 is the projection onto the first p components, i^* is the morphism induced by $i(X) = (X, 0)$ and G is the submodule of θ_{p+r} generated by $\partial/\partial X_k$, with $1 \leq k \leq p$ and $\Lambda_i \partial/\partial \Lambda_j$, $1 \leq i, j \leq r$.

Proof The proof is similar to the proof in Theorem 2 in [11]. We will prove it by a double inclusion. For simplicity, we set $\varphi = \pi_1 \circ i^* : \theta_{p+r} \rightarrow \theta_p$.

Consider $\eta \in \text{Lift}(F) \cap G$. There exists $\xi \in \theta_{n+r}$ such that $dF \circ \xi = \eta \circ F$. Evaluating this system of equations in $\lambda = 0$ we have the following:

$$\left(\frac{\partial f_\lambda}{\partial x}(x, 0) \right) \begin{pmatrix} \xi_1(x, 0) \\ \vdots \\ \xi_n(x, 0) \end{pmatrix} + \left(\frac{\partial f_\lambda}{\partial \lambda}(x, 0) \right) \begin{pmatrix} \xi_{n+1}(x, 0) \\ \vdots \\ \xi_{n+r}(x, 0) \end{pmatrix} = \begin{pmatrix} \eta_1(F(x, 0)) \\ \vdots \\ \eta_p(F(x, 0)) \end{pmatrix}$$

and

$$\xi_{n+k}(x, 0) = \eta_{p+k}(F(x, 0)) = \eta_{p+k}(f(x), 0) \quad (1 \leq k \leq r).$$

Since $\eta \in G$, $\xi_{n+k}(x, 0) = \eta_{p+k}(f(x), 0) = 0$, and so

$$\left(\frac{\partial f_\lambda}{\partial x}(x, 0) \right) \begin{pmatrix} \xi_1(x, 0) \\ \vdots \\ \xi_n(x, 0) \end{pmatrix} = \begin{pmatrix} \eta_1(F(x, 0)) \\ \vdots \\ \eta_p(F(x, 0)) \end{pmatrix},$$

which means that $\varphi(\eta(X, \Lambda)) = (\eta_1(X, 0), \dots, \eta_p(X, 0)) \in \text{Lift}(f)$.

Now let $\bar{\eta} \in \text{Lift}(f)$. By definition, there exists a vector field $\bar{\xi} \in \theta_n$ such that $d\bar{f}(\bar{\xi}) = \bar{\eta} \circ f$. Define $\eta(X, \Lambda) = (\bar{\eta}(X), 0)$ and $\xi(x, \lambda) = (\bar{\xi}(x), 0)$. Consider $\tilde{\eta} = \eta \circ F - dF(\xi) \in \theta(F)$. We have $\tilde{\eta}(x, 0) = (0, 0)$ so by Hadamard's Lemma there exist vector fields $\tilde{\eta}_i \in \theta(F)$ with $1 \leq i \leq r$ such that $\tilde{\eta} = \sum_{i=1}^r \Lambda_i \tilde{\eta}_i$.

Since F is stable, there exist $\hat{\xi}_i \in \theta_{n+r}$ and $\hat{\eta}_i \in \theta_{p+r}$ for $1 \leq i \leq r$ such that $\tilde{\eta}_i = dF(\hat{\xi}_i) + \hat{\eta}_i \circ F$ for all i . Therefore,

$$\begin{aligned} \eta \circ F - dF(\xi) &= \tilde{\eta} = \sum_{i=1}^r \Lambda_i \tilde{\eta}_i = \sum_{i=1}^r \Lambda_i (dF(\hat{\xi}_i) + \hat{\eta}_i \circ F) \\ &= dF \left(\sum_{i=1}^r \Lambda_i \hat{\xi}_i \right) + \left(\sum_{i=1}^r \Lambda_i \hat{\eta}_i \right) \circ F. \end{aligned}$$

Since η is 0 in the last r components, we have that $\eta - \sum_{i=1}^r \Lambda_i \hat{\eta}_i \in \text{Lift}(F) \cap G$ and $\varphi(\eta - \sum_{i=1}^r \Lambda_i \hat{\eta}_i) = \bar{\eta}$. $\square$

Remark 1 If $r = 1$, then $G = \text{Lift}(g)$, where $g : (\mathbb{K}^p \times \mathbb{K}, 0) \rightarrow (\mathbb{K}^p \times \mathbb{K}, 0)$ is the map germ $g(X, \Lambda) = (X, \Lambda^2)$ as was stated in Theorem 2 in [11]. However, if $r > 1$ and $g : (\mathbb{K}^p \times \mathbb{K}^r, S) \rightarrow (\mathbb{K}^p \times \mathbb{K}^r, 0)$, where $S = \{z_1, \dots, z_r\}$ and the branch at z_i is given by $g_i(X, \Lambda) = (X, \Lambda_1, \dots, \Lambda_i^2, \dots, \Lambda_r)$, then

$$\text{Lift}(g) = \mathcal{O}_{p+r} \left\{ \frac{\partial}{\partial X_k}, 1 \leq k \leq p, \Lambda_i \frac{\partial}{\partial \Lambda_i}, 1 \leq i \leq r \right\} \subsetneq G.$$

When $\mathbb{K} = \mathbb{C}$ and f has finite $\mathcal{A}_e$ -codimension $\text{Lift}(f) = \text{Derlog}(\Delta(f))$ (unless $n > p \leq 2$), so we have the following:

Corollary 1 Let $f : (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ be a non-stable germ of finite $\mathcal{A}_e$ -codimension such that either $n \leq p$ or $n > p > 2$ and let F be an r -parameter stable unfolding. Then,

$$\text{Derlog}(\Delta(f)) = \pi_1(i^*(\text{Derlog}(\Delta(F)) \cap G)).$$

Moreover, when $n > p = 1$, the above equality holds if and only if f is quasihomogeneous.

3.1 Lifiable vector fields of H_k

As an application of Theorem 1 we compute the liftable vector fields of a family of germs which do not admit a 1-parameter stable unfolding. Consider $H_k : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^3, 0)$, from Mond's list [9] given by

$$H_k(x, y) = (x, y^3, y^{3k-1} + xy).$$

For any $k \geq 2$, H_k admits a 2-parameter stable unfolding $\mathcal{A}$ -equivalent to $F : (\mathbb{K}^4, 0) \rightarrow (\mathbb{K}^5, 0)$,

$$F(u_1, v_1, v_2, y) = (u_1, v_1, v_2, y^3 + u_1y, v_1y + v_2y^2),$$

which is versal for $k = 2$. In fact, the change of variables in the source $v'_2 = v_2 - y^{3k-3}$, the change $G_k(U_1, V_1, V_2, W_1, W_2) = (U_1, V_1, V_2 - W_1^{k-1}, W_1, W_2)$ in the target, and a further change in the source, show that F is $\mathcal{A}$ -equivalent to

$$F_k(u_1, v_1, v_2, y) = (u_1, v_1, v_2, y^3 + u_1y, v_1y + v_2y^2 + \psi(u_1, y)y^2 + y^{3k-1}),$$

for some function ψ such that $\psi(0, y) = 0$. So F_k is a 2-parameter stable unfolding of H_k .

In this case,

$$G = \mathcal{O}_{3+2} \left\{ U_1 \frac{\partial}{\partial U_1}, V_2 \frac{\partial}{\partial U_1}, U_1 \frac{\partial}{\partial V_2}, V_2 \frac{\partial}{\partial V_2}, \frac{\partial}{\partial V_1}, \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2} \right\},$$

and by Theorem 1, $\text{Lift}(H_k) = \varphi(\text{Lift}(F_k) \cap G)$, where $\varphi = \pi_1 \circ i^*$.

In Example 3.4 in [7], the generators for $\text{Lift}(F)$ are calculated. We show these vector fields as matrices for simplicity:

$$\eta_e = \begin{pmatrix} 2U_1 \\ 2V_1 \\ V_2 \\ 3W_1 \\ 3W_2 \end{pmatrix}, \eta_1^{1,2,3} = \begin{pmatrix} 4U_1^2 \\ -3U_1V_1 + 3V_2W_1 \\ -5U_1V_2 - 3W_2 \\ 6U_1W_1 \\ -3V_1W_1 + 2U_1W_2 \end{pmatrix}, \begin{pmatrix} 6U_1 \\ -3V_1 \\ -6V_2 \\ 9W_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9V_1 \\ -6V_2^2 \\ 0 \\ 9W_2 + 3U_1V_2 \\ 3V_1V_2 \end{pmatrix},$$

$$\eta_2^{1,2,3} = \begin{pmatrix} 0 \\ -3U_1V_2 - 3W_2 \\ 3V_1 \\ 0 \\ -3V_2W_1 \end{pmatrix}, \begin{pmatrix} -9W_1 \\ 2U_1V_2 \\ -3V_1 \\ 2U_1^2 \\ 6V_2W_1 + 2U_1V_1 \end{pmatrix}, \begin{pmatrix} -9W_2 - 3U_1V_2 \\ -3V_1V_2 \\ 0 \\ 3U_1V_1 \\ 6V_2W_2 + 3V_1^2 \end{pmatrix}.$$

In [11, Lemma 6.1], it is shown that if two maps f and g are $\mathcal{A}$ -equivalent, i.e. there exist h and H diffeomorphisms in source and target such that $f = H \circ g \circ h$, then the map $L_{f,g} : \text{Lift}(g) \rightarrow \text{Lift}(f)$ such that $L_{f,g}(\eta) = dH \circ \eta \circ H^{-1}$ is a bijection. We therefore can obtain $\text{Lift}(F_k)$ from $\text{Lift}(F)$ by composing its generators by G_k^{-1} on the right and multiplying by dG_k on the left.

$$\begin{aligned} \eta_{ke} &= \begin{pmatrix} 2U_1 \\ 2V_1 \\ V_2 - (3k-4)W_1^{k-1} \\ 3W_1 \\ 3W_2 \end{pmatrix}, \eta_{k1}^1 = \begin{pmatrix} 4U_1^2 \\ -3U_1V_1 + 3V_2W_1 + 3W_1^k \\ -5U_1V_2 - (6k-1)U_1W_1^{k-1} - 3W_2 \\ 6U_1W_1 \\ -3V_1W_1 + 2U_1W_2 \end{pmatrix}, \\ \eta_{k1}^2 &= \begin{pmatrix} 6U_1 \\ -3V_1 \\ -6V_2 - (9k-3)W_1^{k-1} \\ 9W_1 \\ 0 \end{pmatrix}, \eta_{k2}^1 = \begin{pmatrix} 0 \\ -3U_1V_2 - 3U_1W_1^{k-1} - 3W_2 \\ 3V_1 \\ 0 \\ -3V_2W_1 - 3W_1^k \end{pmatrix}, \\ \eta_{k1}^3 &= \begin{pmatrix} 9V_1 \\ -6V_2^2 - 12V_2W_1^{k-1} - 6W_1^{2k-2} \\ -9(k-1)W_1^{k-2}W_2 - 3(k-1)U_1V_2W_1^{k-2} - 3(k-1)U_1W_1^{2k-3} \\ 9W_2 + 3U_1V_2 + 3U_1W_1^{k-1} \\ 3V_1V_2 + 3V_1W_1^{k-1} \end{pmatrix}, \\ \eta_{k2}^2 &= \begin{pmatrix} -9W_1 \\ 2U_1V_2 + 2U_1W_1^{k-1} \\ -3V_1 - 2(k-1)U_1^2W_1^{k-2} \\ 2U_1^2 \\ 6V_2W_1 + 6W_1^k + 2U_1V_1 \end{pmatrix}, \eta_{k2}^3 = \begin{pmatrix} -9W_2 - 3U_1V_2 - 3U_1W_1^{k-1} \\ -3V_1V_2 - 3V_1W_1^{k-1} \\ -3(k-1)U_1V_1W_1^{k-2} \\ 3U_1V_1 \\ 6V_2W_2 + 6W_1^{k-1}W_2 + 3V_1^2 \end{pmatrix}. \end{aligned}$$

The module $\text{Lift}(F_k) \cap G$ is formed by vector fields of type

$$\eta = \tilde{\alpha}_1\eta_{ke} + \tilde{\alpha}_2\eta_{k1}^1 + \tilde{\alpha}_3\eta_{k1}^2 + \tilde{\alpha}_4\eta_{k2}^1 + \tilde{\alpha}_5\eta_{k1}^3 + \tilde{\alpha}_6\eta_{k2}^2 + \tilde{\alpha}_7\eta_{k2}^3,$$

such that the first and third components are divisible by U_1 or V_2 , where the $\tilde{\alpha}_i$ are functions in U_1, V_1, V_2, W_1 and W_2 . This means that the first and third components evaluated at $U_1 = V_2 = 0$ are equal to 0, which yields two conditions on $\alpha_i(V_1, W_1, W_2) = \tilde{\alpha}_i(0, V_1, 0, W_1, W_2)$:

$$\begin{aligned} 9\alpha_5V_1 - 9\alpha_6W_1 - 9\alpha_7W_2 &= 0, -(3k-4)\alpha_1W_1^{k-1} - 3\alpha_2W_2 \\ -(9k-3)\alpha_3W_1^{k-1} + 3\alpha_4V_1 - 9(k-1)\alpha_5W_1^{k-2}W_2 - 3\alpha_6V_1 &= 0. \end{aligned}$$

Now, set $\beta = -(3k-4)\alpha_1 - (9k-3)\alpha_3$, so that $\alpha_3 = -\frac{1}{9k-3}\beta - \frac{3k-4}{9k-3}\alpha_1$. The conditions can be rewritten as

$$\alpha_5V_1 = \alpha_6W_1 + \alpha_7W_2, \quad (1)$$

$$\alpha_4 V_1 = -\frac{1}{3} \beta W_1^{k-1} + \alpha_2 W_2 + 3(k-1) \alpha_5 W_1^{k-2} W_2 + \alpha_6 V_1. \quad (2)$$

By Theorem 1, the liftable vector fields of H_k are obtained from the second, fourth and fifth components vector fields of the type of η evaluated at $U_1 = V_2 = 0$. That is,

$$\begin{aligned} \text{Lift}(H_k) = \{ & \alpha_1 (2V_1 \frac{\partial}{\partial V_1} + 3W_1 \frac{\partial}{\partial W_1} + 3W_2 \frac{\partial}{\partial W_2}) + \alpha_2 (3W_1^k \frac{\partial}{\partial V_1} - 3V_1 W_1 \frac{\partial}{\partial W_2}) \\ & + \alpha_3 (-3V_1 \frac{\partial}{\partial V_1} + 9W_1 \frac{\partial}{\partial W_1}) + \alpha_4 (-3W_2 \frac{\partial}{\partial V_1} - 3W_1^k \frac{\partial}{\partial W_2}) \\ & + \alpha_5 (-6W_1^{2k-2} \frac{\partial}{\partial V_1} + 9W_2 \frac{\partial}{\partial W_1} + 3V_1 W_1^{k-1} \frac{\partial}{\partial W_2}) + \alpha_6 (6W_1^k \frac{\partial}{\partial W_2}) \\ & + \alpha_7 (-3V_1 W_1^{k-1} \frac{\partial}{\partial V_1} + (6W_1^{k-1} W_2 + 3V_1^2) \frac{\partial}{\partial W_2}) \}, \end{aligned}$$

where the α_i are germs of functions.

We use conditions (1) and (2) to eliminate the functions α_4, α_5 . Then, after some easy simplifications in the obtained generators, we arrive to the following:

Theorem 2 *The module of liftable vector fields over the map germs $H_k : (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^3, 0)$, given by $H_k(x, y) = (x, y^3, y^{3k-1} + xy)$ is generated by the following vector fields*

$$\begin{aligned} & \left(\begin{pmatrix} (3k-2)V_1 \\ 3W_1 \\ (3k-1)W_2 \end{pmatrix}, \begin{pmatrix} V_1^2 + (3k-1)W_1^{k-1}W_2 \\ -3V_1W_1 \\ (3k-1)W_1^{2k-1} \end{pmatrix}, \begin{pmatrix} W_2^2 - V_1W_1^k \\ 0 \\ V_1^2W_1 + W_1^kW_2 \end{pmatrix} \right), \\ & \left(\begin{pmatrix} (3k-1)W_1^{2k-1} + V_1W_2 \\ -3W_1W_2 \\ -(3k-1)V_1W_1^k \end{pmatrix}, \begin{pmatrix} -(3k-1)W_1^{2k-2}W_2 - V_1^2W_1^{k-1} \\ 3W_2^2 \\ 3kV_1W_1^{k-1}W_2 + V_1^3 \end{pmatrix} \right). \end{aligned}$$

Remark 2 In fact, these vector fields can be obtained by applying φ to the following vector fields in $\text{Lift}(F_k) \cap G$:

$$\begin{aligned} & (9k-3)\eta_{ke} - (3k-4)\eta_{k1}^2, \\ & 3V_1\eta_{k1}^2 + (9k-3)W_1^{k-1}\eta_{k2}^1, \\ & 2V_1\eta_{k1}^1 + W_2(\eta_{k2}^1 - \eta_{k2}^2) + W_1\eta_{k2}^3, \\ & W_1\eta_{k1}^3 - 3(k-1)W_1^{k-1}\eta_{k1}^1 + V_1(\eta_{k2}^1 + \eta_{k2}^2), \\ & W_2\eta_{k1}^3 + V_1\eta_{k2}^3 - 3(k-1)W_1^{k-2}W_2\eta_{k2}^1. \end{aligned}$$

4 Liftable vector fields and augmentations

In this section, we restrict ourselves to the complex case $\mathbb{K} = \mathbb{C}$. Suppose $f : (\mathbb{C}^n, S) \rightarrow (\mathbb{C}^p, 0)$ admits a one-parameter stable unfolding $F(x, \lambda) = (f_\lambda(x), \lambda)$. Let $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$, $\phi \neq 0$, and let $Af(x, z) = (f_{\phi(z)}(x), z) = (X, Z)$ be the augmentation of f by F and ϕ . We shall establish a relation between the liftable vector fields of an augmentation Af and those of the stable unfolding F .

As we pointed out in Sect. 2, if g is a complex analytic germ of finite $\mathcal{A}_e$ -codimension and $n \leq p$ or $n > p > 2$, then $\text{Lift}(g) = \text{Derlog}(\Delta(g))$. When $p \leq n+1$, $\Delta(g)$ is always a hypersurface in $(\mathbb{C}^p, 0)$, hence $\text{Derlog}(h) \subset \text{Derlog}(\Delta(g))$, where h is the reduced defining equation for $\Delta(g)$ and

$$\text{Derlog}(h) = \{\eta \in \theta_p : \eta(h) = 0\}.$$

Furthermore, if g is quasihomogeneous $\text{Derlog}(\Delta(g)) = \langle e \rangle \oplus \text{Derlog}(h)$, where e is the Euler vector field (see for instance [5]).

Lemma 1 Let $H(X, \Lambda)$ be a defining equation of $\Delta(F)$, then we can take $h(X, Z) = H(X, \phi(Z))$ as a defining equation of $\Delta(Af)$.

Proof Let $F(x, \lambda) = (f_\lambda(x), \lambda)$. The critical set of F is given by $C_F = \{(x, \lambda) \in \mathbb{C}^n \times \mathbb{C} : F \text{ is not a submersion in } (x, \lambda)\}$ and this is equal to $\{(x, \lambda) \in \mathbb{C}^n \times \mathbb{C} : f_\lambda \text{ is not a submersion in } x\}$. Notice that when $n < p$ the critical set is the whole space and instead of discriminant we are considering the image. Now, $\Delta(F) = F(C_F)$ is given by $H(X, \Lambda) = 0$, so for all (x, λ) such that f_λ is not a submersion in x we have $H(f_\lambda(x), \lambda) = 0$. Changing λ for $\phi(z)$ we get that for all (x, z) such that $f_{\phi(z)}$ is not a submersion in x we have $H(f_{\phi(z)}(x), \phi(z)) = 0$.

Similarly $C_{Af} = \{(x, z) \in \mathbb{C}^n \times \mathbb{C} : f_{\phi(z)} \text{ is not a submersion in } x\}$. Setting $h(X, Z) = H(X, \phi(Z))$ we get that

$$h(Af(x, z)) = h(f_{\phi(z)}(x), z) = H(f_{\phi(z)}(x), \phi(z)) = 0,$$

so $h(X, Z)$ is a defining equation of $\Delta(Af)$. $\square$

In particular,

$$\frac{\partial h}{\partial X_i}(X, Z) = \frac{\partial H}{\partial X_i}(X, \phi(Z)), \quad i = 1, \dots, p, \quad \frac{\partial h}{\partial Z}(X, Z) = \frac{\partial H}{\partial \Lambda}(X, \phi(Z))\phi'(Z).$$

Given a vector field $\eta(X, \Lambda) = \sum_{i=1}^p \eta_i(X, \Lambda) \frac{\partial}{\partial X_i} + \eta_{p+1}(X, \Lambda) \frac{\partial}{\partial \Lambda} \in \theta_{p+1}$, we set:

$$\tilde{\eta}(X, Z) = \sum_{i=1}^p \eta_i(X, \phi(Z))\phi'(Z) \frac{\partial}{\partial X_i} + \eta_{p+1}(X, \phi(Z)) \frac{\partial}{\partial Z} \in \theta_{p+1}.$$

We have the following two lemmas:

Lemma 2 If $\eta \in \text{Derlog}(\Delta(F))$, then $\tilde{\eta} \in \text{Derlog}(\Delta(Af))$. Furthermore, suppose that $\eta_{p+1}(X, 0) = 0$, then $\frac{\tilde{\eta}}{\phi'} \in \text{Derlog}(\Delta(Af))$.

Proof If $\eta \in \text{Derlog}(\Delta(F))$, $\eta(H) = \alpha H$ for some $\alpha \in \mathcal{O}_{p+1}$. That is,

$$\sum_{i=1}^p \eta_i(X, \Lambda) \frac{\partial H}{\partial X_i}(X, \Lambda) + \eta_{p+1}(X, \Lambda) \frac{\partial H}{\partial \Lambda}(X, \Lambda) = \alpha(X, \Lambda)H(X, \Lambda).$$

Substituting $\Lambda = \phi(Z)$ and multiplying by $\phi'(Z)$ yields

$$\begin{aligned} \sum_{i=1}^p \eta_i(X, \phi(Z))\phi'(Z) \frac{\partial H}{\partial X_i}(X, \phi(Z)) + \eta_{p+1}(X, \phi(Z))\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)) \\ = \alpha(X, \phi(Z))\phi'(Z)H(X, \phi(Z)). \end{aligned}$$

Hence, $\tilde{\eta}(h) = \phi'\tilde{\alpha}h$, where $\tilde{\alpha}(X, Z) = \alpha(X, \phi(Z))$ and thus $\tilde{\eta} \in \text{Derlog}(\Delta(Af))$.

On the other hand, if $\eta_{p+1}(X, 0) = 0$ we can write $\eta_{p+1} = \Lambda a$, for some $a \in \mathcal{O}_{p+1}$. We also put $\phi = b\phi'$, for some $b \in \mathcal{O}_1$. We have:

$$\eta_{p+1}(X, \phi(Z)) = \phi(Z)a(X, \phi(Z)) = b(Z)\phi'(Z)a(X, \phi(Z)).$$

Thus $\tilde{\eta}$ is divisible by ϕ' and $\frac{\tilde{\eta}(h)}{\phi'} = \tilde{\alpha}h$, so $\frac{\tilde{\eta}}{\phi'} \in \text{Derlog}(\Delta(Af))$.

Lemma 3 Suppose $\phi \neq 0$ is quasihomogeneous and let $\bar{\eta} \in \text{Derlog}(\Delta(Af))$. Then there exists $\eta \in \text{Derlog}(\Delta(F))$ such that $\tilde{\eta} = \bar{\eta} + \varphi$ with $\varphi \in \text{Derlog}(\Delta(Af))$. Furthermore, if $\bar{\eta}_{p+1}(X, 0) = 0$ then there exists $\eta \in \text{Derlog}(\Delta(F))$ such that $\frac{\tilde{\eta}}{\phi'} = \bar{\eta} + \varphi$, with $\varphi \in \text{Derlog}(\Delta(Af))$.

Proof We put $\bar{\eta}_{p+1}(X, Z) = Z\psi(X, Z) + \beta(X)$, for some functions ψ, β , hence

$$\bar{\eta}(X, Z) = \sum_{i=1}^p \bar{\eta}_i(X, Z) \frac{\partial}{\partial X_i} + (Z\psi(X, Z) + \beta(X)) \frac{\partial}{\partial Z}.$$

For simplicity we suppose $\bar{\eta} \in \text{Derlog}(h)$ and comment on the general case later. So, we have

$$\sum_{i=1}^p \bar{\eta}_i(X, Z) \frac{\partial h}{\partial X_i}(X, Z) = -(Z\psi(X, Z) + \beta(X)) \frac{\partial h}{\partial Z}(X, Z),$$

which means

$$\begin{aligned} & \sum_{i=1}^p \bar{\eta}_i(X, Z) \frac{\partial H}{\partial X_i}(X, \phi(Z)) + Z\psi(X, Z)\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)) \\ &= -\beta(X) \frac{\partial H}{\partial \Lambda}(X, \phi(Z))\phi'(Z). \end{aligned}$$

Since $\phi(Z) = Z^k$ and $\phi'(Z) = kZ^{k-1}$, the right hand side of this equality can be seen as a series in Z of type

$$c_{k-1}(X)Z^{k-1} + c_{2k-1}(X)Z^{2k-1} + c_{3k-1}(X)Z^{3k-1} + \dots$$

On the other hand, for $i = 1, \dots, p$,

$$\begin{aligned} \bar{\eta}_i(X, Z) &= a_{i0}(X) + a_{i1}(X)Z + a_{i2}(X)Z^2 + \dots, \\ \frac{\partial H}{\partial X_i}(X, \phi(Z)) &= b_{i0}(X) + b_{ik}(X)Z^k + b_{i,2k}(X)Z^{2k} + \dots \end{aligned}$$

Analogously for $Z\psi(X, Z)$ and $\frac{\partial H}{\partial \Lambda}(X, \phi(Z))$. By identifying the coefficients at both sides of the equality, the left hand side has a part equal to 0 (the sum of the coefficients whose terms do not appear in the right hand side) and the rest is used to rewrite the equality

$$\begin{aligned} & \sum_{i=1}^p A_i(X, \phi(Z))\phi'(Z) \frac{\partial H}{\partial X_i}(X, \phi(Z)) + A_{p+1}(X, \phi(Z))\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)) \\ &= -\beta(X)\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)), \end{aligned}$$

for some functions A_i , $i = 1, \dots, p+1$. The part which was equal to 0 corresponds to a certain $\varphi \in \text{Derlog}(h)$.

Dividing the previous equation by $\phi'(Z)$ and substituting $\Lambda = \phi(Z)$ (since $\phi(Z) = Z^k$, ϕ is surjective), we obtain

$$\sum_{i=1}^p A_i(X, \Lambda) \frac{\partial H}{\partial X_i}(X, \Lambda) + (A_{p+1}(X, \Lambda) + \beta(X)) \frac{\partial H}{\partial \Lambda}(X, \Lambda) = 0,$$

which means that

$$\eta = \sum_{i=1}^p A_i \frac{\partial}{\partial X_i} + (A_{p+1} + \beta) \frac{\partial}{\partial Z} \in \text{Derlog}(H)$$

and $\tilde{\eta} = \bar{\eta} - \varphi$.

If $\bar{\eta} \notin \text{Derlog}(h)$, that is $\bar{\eta}(h) = \alpha(X, Z)h(X, Z) = \alpha(X, Z)H(X, \phi(Z))$, in the same way as above we would take only the part of $\alpha(X, Z)H(X, \phi(Z))$ in the powers Z^{nk-1} , that is $\gamma(X, \phi(Z))\phi'(Z)H(X, \phi(Z))$ and proceed in the same way.

Second, suppose that $\bar{\eta}_{p+1}(X, 0) = 0$, so $\beta = 0$. Then $\bar{\eta}(h) = \alpha h$ yields

$$\sum_{i=1}^p \bar{\eta}_i(X, Z) \frac{\partial H}{\partial X_i}(X, \phi(Z)) + Z\psi(X, Z)\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)) = \alpha(X, Z)H(X, \phi(Z)).$$

If we take only the terms in Z^{nk} with $n \in \{0, 1, \dots\}$ on both sides of the equation we are left with

$$\begin{aligned} \sum_{i=1}^p A_i(X, \phi(Z)) \frac{\partial H}{\partial X_i}(X, \phi(Z)) + ZA_{p+1}(X, \phi(Z))\phi'(Z) \frac{\partial H}{\partial \Lambda}(X, \phi(Z)) \\ = \gamma(X, \phi(Z))H(X, \phi(Z)). \end{aligned}$$

Notice that $Z\phi'(Z) = kZ^k = k\phi(Z)$, so if we substitute $\Lambda = \phi(Z)$, we get that

$$\eta = \sum_{i=1}^p A_i \frac{\partial}{\partial X_i} + k\Lambda A_{p+1} \frac{\partial}{\partial \Lambda} \in \text{Derlog}(\Delta(F)).$$

□

Remark 3 Lemmas 2 and 3 imply that the number of generators of $\text{Derlog}(\Delta(Af))$ (and therefore of $\text{Lift}(Af)$) is greater than or equal to the number of generators of $\text{Derlog}(\Delta(F))$ ($\text{Lift}(F)$ resp.). Notice too that if $\eta \in \text{Derlog}(H)$, then $\tilde{\eta} \in \text{Derlog}(h)$.

Example 1 Consider $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ given by $f(x, y) = (x^4 + yx, y)$, its 1-parameter stable unfolding $F(x, y, z) = (x^4 + yx + zx^2, y, z)$ and the augmentations by $\phi(z) = z^k$, $A^k f(x, y, z) = (x^4 + yx + z^k x^2, y, z)$. The defining equation of the discriminant is

$$h(X, Y, Z) = 256X^3 + 27Y^4 + 144XY^2Z^k + 128X^2Z^{2k} + 4Y^2Z^{3k} + 16XZ^{4k}.$$

When $k = 1$ we have H . It can be seen (see [1] for example) that $\text{Lift}(F) = \langle \eta^1, \eta^2, \eta^3 \rangle$, where

$$\begin{aligned} \eta^1 &= 4X \frac{\partial}{\partial X} + 3Y \frac{\partial}{\partial Y} + 2Z \frac{\partial}{\partial Z}, \\ \eta^2 &= -(9Y^2 + 16XZ) \frac{\partial}{\partial X} + 12YZ \frac{\partial}{\partial Y} + (48X + 4Z^2) \frac{\partial}{\partial Z}, \\ \eta^3 &= YZ \frac{\partial}{\partial X} - (8X + 2Z^2) \frac{\partial}{\partial Y} + 6Y \frac{\partial}{\partial Z}. \end{aligned}$$

On the other hand, $\text{Lift}(A^k f) = \langle \tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3, \tilde{\eta}^4 \rangle$, where

$$\tilde{\eta}^1 = 4kX \frac{\partial}{\partial X} + 3kY \frac{\partial}{\partial Y} + 2Z \frac{\partial}{\partial Z},$$

$$\begin{aligned}\tilde{\eta}^2 &= -(9kY^2Z^{k-1} + 16kXZ^{2k-1})\frac{\partial}{\partial X} + 12kYZ^{2k-1}\frac{\partial}{\partial Y} + (48X + 4Z^{2k})\frac{\partial}{\partial Z}, \\ \tilde{\eta}^3 &= kYZ^{2k-1}\frac{\partial}{\partial X} - (8kXZ^{k-1} + 2kZ^{3k-1})\frac{\partial}{\partial Y} + 6Y\frac{\partial}{\partial Z}, \\ \tilde{\eta}^4 &= -(9kY^3 + 24kXYZ^k)\frac{\partial}{\partial X} + (64kX^2 + 12kY^2Z^k + 16kXZ^{2k})\frac{\partial}{\partial Y} \\ &\quad + 4YZ^{k+1}\frac{\partial}{\partial Z}.\end{aligned}$$

Following Lemma 2, if we substitute Z by Z^k in η^2 and η^3 and multiply their first two components by kZ^{k-1} we obtain $\tilde{\eta}^2$ and $\tilde{\eta}^3$. Now consider $\xi = kY\eta^2(X, Y, Z) - 8kX\eta^3(X, Y, Z)$. Since $\xi_3(X, Y, 0) = 0$, again by Lemma 2, substituting Z by Z^k and then dividing the last component by $\phi'(Z) = kZ^{k-1}$ gives $\tilde{\eta}^4$. Finally, if we take $k\eta^1(X, Y, Z)$, substitute Z by Z^k and then divide the last component by $\phi'(Z) = kZ^{k-1}$ we get $\tilde{\eta}^1$.

In [12] the operation of simultaneous augmentation and concatenation was defined as the multigerm defined by $\{Af, g\}$ where g is a fold map or an immersion. The authors proved a formula relating the $\mathcal{A}_e$ -codimensions of $\{Af, g\}$ and of f under a certain condition, namely that $\pi_2(i^*(\text{Lift}(Af))) = \pi_2(i^*(\text{Lift}(F)))$ where π_2 is the projection onto the last component and i is the immersion $i(X) = (X, 0)$. Here the equality stands for equality of ideals in $\mathcal{O}_p$. This condition appears again in [2] and in [13] where the authors say that all examples they have studied satisfy this condition but they were not able to prove it. We will use Theorem 1 and Lemmas 2, 3 in order to give a proof of this fact when the augmenting function ϕ is quasihomogeneous.

Theorem 3 *Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, 0)$ be a map germ of finite codimension with $p \leq n + 1$ (except for $n > p \leq 2$) which admits a one-parameter stable unfolding F . Let $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ be a quasihomogeneous function and let Af be the augmentation of f by F and ϕ . Then,*

$$\pi_2(i^*(\text{Lift}(Af))) = \pi_2(i^*(\text{Lift}(F))),$$

as ideals of $\mathcal{O}_p$.

Proof Let us see that $\pi_2(i^*(\text{Lift}(F))) \subseteq \pi_2(i^*(\text{Lift}(Af)))$. Suppose $F(x, \lambda) = (f_\lambda(x), \lambda)$ and consider

$$AF(x, z, \mu) = (f_{\phi(z)+\mu}(x), z, \mu) = (X, Z, M),$$

the one-parameter stable unfolding of Af and the immersion $i_1(X, Z) = (X, Z, 0)$. Then, by Theorem 1,

$$\text{Lift}(Af) = \varphi(\text{Lift}(AF) \cap G) = \pi_1(i_1^*(\text{Lift}(AF) \cap G)),$$

where $G = \mathcal{O}_{p+1+1}\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_p}, \frac{\partial}{\partial Z}, M\frac{\partial}{\partial M}\}$ and π_1 is the projection onto the first $p + 1$ components. On the other hand, AF is $\mathcal{A}$ -equivalent to a trivial one-parameter unfolding of F , and if we consider the immersion $i'_1(X, M) = (X, 0, M)$, then, by Theorem 1,

$$\text{Lift}(F) = \varphi'(\text{Lift}(AF) \cap G') = \pi'_1(i'^1_1(\text{Lift}(AF) \cap G')),$$

where $G' = \mathcal{O}_{p+1+1}\{\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_p}, Z\frac{\partial}{\partial Z}, \frac{\partial}{\partial M}\}$ and π'_1 is the projection onto the components in $\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_p}$ and $\frac{\partial}{\partial M}$.

Now let $\chi = \beta(X) \frac{\partial}{\partial M} \in \pi_2(i^*(\text{Lift}(F))) = \pi_2(i^*(\pi_1'(i_1'^*(\text{Lift}(AF) \cap G'))))$. Notice that the previous equality is amongst ideals in $\mathcal{O}_p$, but, alternatively, there is an isomorphism between the two spaces that takes $\beta(X) \frac{\partial}{\partial M}$ to $\beta(X) \frac{\partial}{\partial \Lambda}$. There exists $\xi \in \text{Lift}(AF) \cap G'$ such that

$$\xi = \sum_{i=1}^p \xi_i \frac{\partial}{\partial X_i} + \xi_{p+1} \frac{\partial}{\partial Z} + (\beta + \xi_{p+2}) \frac{\partial}{\partial M},$$

where $\xi_{p+1}(X, 0, M) = 0$ and $\xi_{p+2}(X, 0, 0) = 0$. Notice that $\dim \tilde{\tau}(AF) > 0$, in fact $\frac{\partial}{\partial Z}|_0 \in \tilde{\tau}(AF)$ so there exists $\eta \in \text{Lift}(AF)$ such that

$$\eta = \sum_{i=1}^p \eta_i \frac{\partial}{\partial X_i} + \eta_{p+1} \frac{\partial}{\partial Z} + \eta_{p+2} \frac{\partial}{\partial M},$$

where $\eta_{p+1}(0) \neq 0$. Without loss of generality we can assume $\eta_{p+1}(0) = 1$.

We define $\zeta = \frac{1}{\eta_{p+1}}(\eta_{p+2}\xi - (\beta + \xi_{p+2})\eta)$ and we get

$$\zeta = \frac{1}{\eta_{p+1}} \left(\sum_{i=1}^p (\eta_{p+2}\xi_i - (\beta + \xi_{p+2})\eta_i) \frac{\partial}{\partial X_i} + (\xi_{p+1}\eta_{p+2} - (\beta + \xi_{p+2})\eta_{p+1}) \frac{\partial}{\partial Z} \right).$$

We have that $\zeta \in \text{Lift}(AF) \cap G$, so $\chi \in \pi_2(i^*(\pi_1(i_1^*(\text{Lift}(AF) \cap G)))) = \pi_2(i^*(\text{Lift}(Af)))$, where this equality is amongst ideals of $\mathcal{O}_p$.

Now we will prove $\pi_2(i^*(\text{Lift}(Af))) \subseteq \pi_2(i^*(\text{Lift}(F)))$. Since f has finite codimension and ϕ is quasihomogeneous, by Theorem 4.4 in [6], Af has finite codimension. Let $\chi = \beta(X) \frac{\partial}{\partial Z} \in \pi_2(i^*(\text{Lift}(Af)))$, i.e. there exists $\bar{\eta} \in \text{Lift}(Af) = \text{Derlog}(\Delta(Af))$ such that

$$\bar{\eta}(X, Z) = \sum_{i=1}^p \bar{\eta}_i(X, Z) \frac{\partial}{\partial X_i} + (\psi_{p+1}(X, Z) + \beta(X)) \frac{\partial}{\partial Z},$$

where $\psi_{p+1}(X, 0) = 0$. By Lemma 3, there exists $\eta \in \text{Derlog}(\Delta(F))$ of type

$$\eta = \sum_{i=1}^p A_i \frac{\partial}{\partial X_i} + (\eta_{p+1} + \beta) \frac{\partial}{\partial Z},$$

where $\eta_{p+1}(X, 0) = \psi_{p+1}(X, 0) = 0$, and so $\pi_2(i^*(\eta)) = \chi$. The result is proved. $\square$

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