Actual and virtual dimension of codimension 2 general linear subspaces in $\mathbb{P}^n$

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Abstract

In the paper we compute the virtual dimension (defined by the Hilbert polynomial) of a space of hypersurfaces of given degree containing s codimension 2 general linear subspaces in $\mathbb{P}^n$. We use Veneroni maps to find a family of unexpected hypersurfaces (in the style of B. Harbourne, J. Migliore, U. Nagel, Z. Teitler) and rigorously prove and extend examples presented in the paper by B. Harbourne, J. Migliore and H. Tutaj-Gasińska.

Keywords Unexpected varieties · Special linear systems · Line arrangements · Fat flats

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1 Introduction

While examining hypersurfaces containing general linear subspaces in $\mathbb{P}^n$ (possibly with multiplicities), the important question is how many such hypersurfaces exist. One can ask what is the dimension of the vector space of forms of given degree vanishing along those subspaces. We call it the *actual dimension*. On the other hand, we can try to find the number of conditions that are imposed on such forms. This leads to the notion of the *virtual dimension*, which is defined as the dimension of the vector spaces of all forms of degree t minus the Hilbert polynomial of a scheme defined by vanishing along given subspaces. The interesting issue is whether those two values are equal.

The reason to focus on linear subspaces of codimension 2 is that these are being widely examined in low dimensions. In $\mathbb{P}^2$ such subspaces are points. The virtual dimension of linear systems of forms of degree t vanishing at s general points with multiplicities $m_1, \dots, m_s$ is

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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equal

$$\binom{t+2}{2} - \sum_{i=1}^s \binom{m_i+1}{2}.$$

However, classification of systems for which the virtual and actual dimension are equal is the subject of SHGH (Segre-Harbourne-Gimigliano-Hirschowitz) Conjecture (see [3] for a survey on this topic).

The next step is to consider lines in $\mathbb{P}^3$. In this case it is still easy to find the virtual dimension of linear system of forms of degree t vanishing along s general lines with multiplicities $m_1, \dots, m_s$. As presented in [7] it is

$$\binom{t+3}{3} - \frac{1}{6} \sum_{i=1}^s m_i(m_i + 1)(3t + 5 - 2m_i).$$

When all multiplicities are equal 1, the actual dimension is equal to the virtual dimension (as long as the virtual dimension is non-negative). This fact is known as Hartshorne-Hirschowitz Theorem [13]. For higher multiplicities there are examples when this is no longer true (cf. [6]).

When we study planes in $\mathbb{P}^4$, it is more problematic to find the virtual dimension, since every two general planes intersect at a point. Considering s planes with multiplicities 1, we have $\binom{s}{2}$ points of intersection. The intuitive way to take it into account in calculations is to use the inclusion-exclusion principle. Since each plane imposes $\binom{t+2}{2}$ conditions on forms of degree t , the virtual dimension should be

$$\binom{t+4}{4} - s \binom{t+2}{2} + \binom{s}{2}.$$

It is also much harder to consider planes with multiplicities. For example, in [7] authors avoid this problem by taking linear subspaces in $\mathbb{P}^n$ with such codimensions that they do not intersect. Moreover, comparing the actual and virtual dimension led to the definition of unexpected hypersurfaces. As introduced in [4] and [10], a scheme *admits an unexpected hypersurface* of given degree, when the actual dimension is strictly greater than the virtual one.

One purpose of this paper is to prove the formula for the virtual dimension of the system of general linear subspaces of codimension 2 with multiplicities 1 in $\mathbb{P}^n$. As in the case of $\mathbb{P}^4$, it is given by inclusion-exclusion principle. Actually, we need to consider all the intersections of given subspaces to properly find the Hilbert polynomial. We present all the details in Sect. 3.

In Sect. 4 we introduce the Veneroni maps. They are generalizations of Cremona transformation in $\mathbb{P}^2$ and cubo-cubic transformation in $\mathbb{P}^3$. These birational maps of $\mathbb{P}^n$ allow us to transform linear systems of forms of degree $n + k$ ($k \geq 3$) vanishing along $n + 1$ codimension 2 linear subspaces with multiplicities 1 to forms of degree $kn + 1$ vanishing with multiplicities k along such subspaces. In [6] and [9] there are presented examples of using those transformations that result in the examples of unexpected hypersurfaces. Another purpose of this paper is to generalize these results to higher dimensional projective spaces. We obtain a family of examples in Theorem 7.1. What is more, the analysis of mentioned systems for $k \geq 4$ lead us to the situation, when the virtual dimension is greater than the actual dimension (as in Example 7.2). We say that a scheme *misses an expected hypersurface* in this case. Such a situation could not appear in $\mathbb{P}^2$ and $\mathbb{P}^3$, because considered subspaces are not intersecting.

In Sect. 5 we gather all the results about the systems with multiplicities 1. In Theorem 5.2 we present conditions that guarantee that the actual dimension is greater than the virtual dimension and show in Example 5.3 that in $\mathbb{P}^4$ without these conditions it may not be true. Later on, we compare dimensions in case of $n + 1$ subspaces with a view towards using the Veneroni maps. It is summarized in Corollary 5.6. Section 6 focuses on the systems resulting from the transformation by the Veneroni maps. We prove Theorem 6.4, which is needed to show that there exists a family of unexpected hypersurfaces. In the last section we present the results regarding unexpected hypersurfaces including the examples of missing expected hypersurfaces.

2 Basic definitions and auxiliary results

Let K be an algebraically closed field of characteristic 0 and $R = K[\mathbb{P}^n] = K[x_0, \dots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. We consider s distinct linear subspaces $\Lambda_1, \dots, \Lambda_s \subset \mathbb{P}^n$ of codimension 2 and denote by $X = m_1 \Lambda_1 + \dots + m_s \Lambda_s$ the scheme defined by the ideal

$$I_X = I(\Lambda_1)^{m_1} \cap \dots \cap I(\Lambda_s)^{m_s} \subseteq R$$

generated by homogeneous polynomials vanishing on each Λ_i to order at least m_i . We say that m_i are the multiplicities of X .

Definition 2.1 The *Hilbert function* of X is the function

$$\text{HF}_X : \mathbb{N} \ni t \mapsto \dim_K [R/I_X]_t \in \mathbb{N},$$

where subscript $[\]_t$ denotes the degree t part of graded homogeneous ring.

It is well-known that there exists a polynomial $\text{HP}_X \in \mathbb{Q}[t]$ (called the *Hilbert polynomial* of X) such that $\text{HP}_X(t) = \text{HF}_X(t)$ for all t sufficiently large (see e.g. [8], [11]). To examine HP_X and HF_X for X given by the vanishing along linear subspaces we need the following lemma that is proved in [7].

Lemma 2.2 Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace of codimension k ($0 < k \leq n$). Let $t \geq m$ be a positive integer. Then vanishing to order at least m along Λ imposes

$$c_{n,k,m,t} = \sum_{i=0}^{m-1} \binom{i+k-1}{k-1} \binom{t-i+n-k}{n-k}$$

linearly independent conditions on forms of degree t . Hence, for $m = 1$ that gives $c_{n,k,1,t} = \binom{t+n-k}{n-k}$ conditions.

Obviously $\text{HF}_{m\Lambda}(t) = c_{n,k,m,t}$ for $t \geq m$ and $\text{HP}_{m\Lambda}(t) = c_{n,k,m,t}$ for any $t > 0$.

We also introduce the following numbers associated to $X = m_1 \Lambda_1 + \dots + m_s \Lambda_s \subseteq \mathbb{P}^n$.

Definition 2.3 The *actual dimension* of X is the dimension of the vector space of forms in I_X of degree t :

$$\text{adim}_n(X, t) = \dim_K [I_X]_t = \dim_K [R]_t - \text{HF}_X(t).$$

The *virtual dimension* is defined as:

$$\text{vdim}_n(X, t) = \dim_K [R]_t - \text{HP}_X(t).$$

We say that X admits an unexpected hypersurface of degree t , if $\text{adim}_n(X, t) > 0$ and

$$\text{adim}_n(X, t) > \text{vdim}_n(X, t)$$

and X misses an expected hypersurface of degree t , when

$$\text{vdim}_n(X, t) > \text{adim}_n(X, t) > 0.$$

Remark 2.4 Usually along with the definition of actual and virtual dimensions authors introduce the *expected dimension*, which is the maximum of the virtual dimension and 0 (as in [4,10] and [9]). Example 7.2 shows that the behaviour of the virtual dimension may be intricate and in many cases we cannot actually expect it to be equal to the actual dimension. Moreover, the situation presented in Example 7.2 allows us to introduce a new concept of missing expected hypersurfaces.

We present some facts that are helpful in studying the actual and virtual dimension. First, we need to introduce the residual and trace schemes.

Definition 2.5 Let X, Y be closed subschemes of $\mathbb{P}^n$ and $\mathcal{I}_X, \mathcal{I}_Y$ denote the corresponding ideal sheafs. The *trace* of X with respect to Y is the scheme $\text{Tr}_Y(X)$ defined by the ideal sheaf $(\mathcal{I}_X + \mathcal{I}_Y)/\mathcal{I}_Y$ (i.e. the schematic intersection of X and Y in Y). The *residual* of X with respect to Y is the scheme $\text{Res}_Y(X)$ defined by the colon ideal $(\mathcal{I}_X : \mathcal{I}_Y)$.

Let $X \subseteq \mathbb{P}^n$ be a closed subscheme and $H \subseteq \mathbb{P}^n$ be a hyperplane. We have the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(t-1) \otimes \mathcal{I}_{\text{Res}_H(X)} \rightarrow \mathcal{O}_{\mathbb{P}^n}(t) \otimes \mathcal{I}_X \rightarrow \mathcal{O}_H(t) \otimes \mathcal{I}_{\text{Tr}_H(X)} \rightarrow 0.$$

We know that $\dim_K([I]_t) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t) \otimes \mathcal{I}_X)$. So from the long exact sequence of cohomology we have the following inequality, which is called the Castelnuovo Inequality (more details can be found in [1] and [2]).

Lemma 2.6 (*Castelnuovo Inequality*) For a closed subscheme $X \subseteq \mathbb{P}^n$ and a hyperplane $H \subseteq \mathbb{P}^n$ we have

$$\dim_K([I_X]_t) \leq \dim_K([I_{\text{Res}_H(X)}]_{t-1}) + \dim_K([I_{\text{Tr}_H(X), H}]_t),$$

where $I_{\text{Tr}_H(X), H}$ denotes the ideal of forms in H vanishing on $\text{Tr}_H(X)$. Moreover, since $H \cong \mathbb{P}^{n-1}$, we have

$$\text{adim}_n(X, t) \leq \text{adim}_n(\text{Res}_H(X), t-1) + \text{adim}_{n-1}(\text{Tr}_H(X), t).$$

If we take $t \gg 0$, we have that $h^1(\mathbb{P}^n, \mathcal{I}(t)) = 0$ by Serre Vanishing Theorem (cf. [12]). Hence, we obtain the next result.

Lemma 2.7 For a closed subscheme $X \subseteq \mathbb{P}^n$ and a hyperplane $H \subseteq \mathbb{P}^n$ we have

$$\dim_K([I_{X, \mathbb{P}^n}]_t) = \dim_K([I_{\text{Res}_H(X), \mathbb{P}^n}]_{t-1}) + \dim_K([I_{\text{Tr}_H(X), H}]_t)$$

for sufficiently large t . Moreover, since $H \cong \mathbb{P}^{n-1}$, then

$$\text{vdim}_n(X, t) = \text{vdim}_n(\text{Res}_H(X), t-1) + \text{vdim}_{n-1}(\text{Tr}_H(X), t).$$

3 The virtual dimension of schemes with multiplicities 1

In this section we prove the formula for the virtual dimension of $X = \Pi_1 + \dots + \Pi_s$ which is a subscheme of $\mathbb{P}^n$ ($n \geq 2, s \geq 0$), where $\Pi_1, \dots, \Pi_s$ are general linear subspaces of codimension 2. By Lemma 2.2, one such subspace gives $c_{n,2,1,t} = \binom{t+n-2}{n-2}$ conditions. However, every $i \leq s$ subspaces intersect in a subspace of codimension $2i$ as long as $2i \leq n$. There are $\binom{s}{i}$ such intersections each giving $\binom{t+n-2i}{n-2i}$ conditions. By considering all possible intersections we can find the expected number of conditions that vanishing on X imposes on forms of degree t . Therefore, the virtual dimension should be given by the formula

$$S_{n,s,t} = \sum_{i=0}^{N(n,s)} (-1)^i \binom{s}{i} \binom{t+n-2i}{n-2i},$$

where for given n, s we take $N(n, s) = \min\{\lfloor n/2 \rfloor, s\}$. Now, we present an important property of the introduced formula that is proved in the Appendix.

Lemma 3.1 *The following formula holds*

$$S_{n,s,t} = S_{n,s-1,t-1} + S_{n-1,s-1,t-1}.$$

We move on to the main theorem of this section.

Theorem 3.2 *Let $\Pi_1, \dots, \Pi_s \subseteq \mathbb{P}^n$ be distinct general linear subspaces of codimension 2. Then for $X = \Pi_1 + \dots + \Pi_s$ we have $\text{vdim}_n(X, t) = S_{n,s,t}$.*

Proof Notice that both $\text{vdim}_n(X, t)$ and $S_{n,s,t}$ are polynomials with respect to t .

We will proceed by induction on the sum $n + s$. We assume that $n + s = 2$, so $n = 2$ and $s = 0$. Then

$$\text{vdim}_2(X, t) = \text{adim}_2(X, t) = \binom{t+2}{2} = S_{2,0,t}.$$

If $n + s = 3$, there are two possibilities. For $n = 2$ and $s = 1$, we have

$$\text{vdim}_2(X, t) = \text{adim}_2(X, t) = \binom{t+2}{2} - 1 = S_{2,1,t}$$

and for $n = 3$ and $s = 0$, we get

$$\text{vdim}_3(X, t) = \text{adim}_3(X, t) = \binom{t+3}{3} = S_{3,0,t}.$$

Now we assume that $n + s = k$ and that polynomials $\text{vdim}_n(X, t)$ and $S_{n,s,t}$ are equal for $n + s = k - 1$ and $n + s = k - 2$.

Choose $H \cong \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ in such a way that $\Pi_s \subseteq H$. Then

$$\text{Res}_H(X) = \Pi_1 + \dots + \Pi_{s-1}$$

and

$$\text{Tr}_H(X) = \Pi_1|_H + \dots + \Pi_s|_H.$$

By Lemma 2.7 we have

$$\text{vdim}_n(X, t) = \text{vdim}_n(\Pi_1 + \dots + \Pi_{s-1}, t - 1) + \text{vdim}_{n-1}(\Pi_1|_H + \dots + \Pi_s|_H, t).$$

Since $\Pi_s \subseteq H$ is a fixed component for the locus of forms in H , we have

$$\text{vdim}_{n-1}(\Pi_1|_H + \dots + \Pi_s|_H, t) = \text{vdim}_{n-1}(\Pi_1|_H + \dots + \Pi_{s-1}|_H, t - 1).$$

By induction and Lemma 3.1 we obtain our result:

$$\begin{aligned} \text{vdim}_n(X, t) &= \text{vdim}_n(\Pi_1 + \dots + \Pi_{s-1}, t-1) \\ &\quad + \text{vdim}_{n-1}(\Pi_1|_H + \dots + \Pi_{s-1}|_H, t-1) \\ &= S_{n,s-1,t-1} + S_{n-1,s-1,t-1} = S_{n,s,t}. \end{aligned}$$

□

4 Veneroni maps

Now we recall the construction of Veneroni maps. One can find more details in [5], where authors present basic facts about Veneroni maps with a modern approach. In [6] there is a thorough analysis of the case of a cubo-cubic transformation in $\mathbb{P}^3$, which is a special case of the Veneroni map.

Let $\Pi_1, \dots, \Pi_{n+1} \subseteq \mathbb{P}^n$ be codimension 2 general linear subspaces. The linear system of all forms of degree n vanishing on those subspaces has dimension $n+1$. Hence, it defines a rational map $v_n: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. This map is called the *Veneroni map*. As it is proved in [5], its inverse is again the Veneroni map defined by the degree n forms vanishing on some general linear subspaces $\Pi'_1, \dots, \Pi'_{n+1}$ of codimension 2 and v_n is in fact a birational map.

Following the notation from [9] we write $dH - m_1\Pi_1 - \dots - m_{n+1}\Pi_{n+1}$ to denote the linear system of forms of degree d that vanish on each Π_i to order at least m_i . Let H and H' denote the linear system of all hyperplanes in $\mathbb{P}^n$, where H is considered in the source of the map v_n and H' in its target. Then v_n pulls back H' to $nH - \Pi_1 - \dots - \Pi_{n+1}$. By the construction of Π'_j from [5], v_n pulls back Π'_j to a unique hypersurface of degree $n-1$ containing $\Pi_1, \dots, \Pi_{j-1}, \Pi_{j+1}, \dots, \Pi_{n+1}$, which is denoted by $(n-1)H - \Pi_1 - \dots - \Pi_{n+1} + \Pi_j$.

To generalize the results from [9] we examine the linear systems of the form $S = (n+k)H - \Pi_1 - \dots - \Pi_{n+1}$. Then

$$\begin{aligned} v_n^*(S) &= S' = (n+k)H' - \Pi'_1 - \dots - \Pi'_{n+1} \\ &= (n+k)(nH - \Pi_1 - \dots - \Pi_{n+1}) - ((n-1)H - \Pi_2 - \dots - \Pi_{n+1}) \\ &\quad - \dots - ((n-1)H - \Pi_1 - \dots - \Pi_n) = (nk+1)H - k\Pi_1 - \dots - k\Pi_{n+1}. \end{aligned}$$

Since v_n is a birational map, we have

$$\text{adim}_n(\Pi_1 + \dots + \Pi_{n+1}, n+k) = \text{adim}_n(k\Pi_1 + \dots + k\Pi_{n+1}, kn+1).$$

In the following sections we focus on those two types of linear systems. First, we will examine the actual dimension of S (and therefore of S'). Then, we will analyse the virtual dimension of S' for $k=3$. As a consequence, we will show that the systems S' for $k=3$ form a family of examples of unexpected hypersurfaces.

5 Linear systems with base loci in subspaces of codimension 2

This section presents the analysis of the relation between the virtual and actual dimension of the linear systems of hypersurfaces containing general linear subspaces of codimension 2 with multiplicities 1. We prove Theorem 5.2 that provides sufficient conditions under which the actual dimension is greater than the virtual dimension. Afterwards, we show that that systems of hypersurfaces of degree $n+k$ vanishing along $n+1$ such subspaces in $\mathbb{P}^n$ satisfy

those conditions and that the virtual dimension is positive in such case. Then, we check that the opposite equality between dimensions also holds in this situation.

Notation It is convenient to introduce, for numbers a, b, c the following notation:

$$\begin{array}{ccc} a & \dashrightarrow & b \\ \downarrow & & \\ c & & \end{array}$$

to denote that $a \leq b + c$ and

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array}$$

for $a = b + c$.

The following example was presented in [9, Ex. 4.2]. Hereby, we show how to calculate the actual dimension of a given system without using computer methods.

Example 5.1 Let $X = \Pi_1 + \dots + \Pi_5 \subseteq \mathbb{P}^4$. We want to show that

$$\text{adim}_4(X, 7) = \text{vdim}_4(X, 7) = 160.$$

By Theorem 3.2 we have $\text{vdim}_4(X, 7) = S_{4,5,7} = \binom{7+4}{4} - 5\binom{7+2}{2} + \binom{5}{2} = 160$.

First, we prove that $\text{adim}_4(X, 7) \geq 160$. Given a form f of degree 7 we want to construct the sets W_i of linear equations on the coefficients of f such that the equations in W_i guarantee the vanishing of f along Π_i and $|W_1 \cup \dots \cup W_5| = 170$. The equations in $W_1 \cup \dots \cup W_5$ may be as well linearly dependent. In any case,

$$\text{adim}_4(X, 7) \geq \binom{7+4}{4} - |W_1 \cup \dots \cup W_5| = 330 - 170 = 160.$$

We start by constructing the sets W_{ij} of equations giving the vanishing along the intersections $\Pi_i \cap \Pi_j$. There are $\binom{5}{2} = 10$ such intersections. In the considered case they are points, thus $|W_{ij}| = 1$ since W_{ij} necessarily consists of one equation imposing vanishing at a point. Now, having established W_{ij} 's, we require each set W_i to contain the equations that are already in the sets W_{ij} for $j \neq i$ (4 equations). Then W_i can be completed with 32 linearly independent equations so that $|W_i| = \binom{7+2}{2} = 36$. Hence, by the inclusion-exclusion principle

$$|W_1 \cup \dots \cup W_5| = \sum_{i=1}^5 |W_i| - \sum_{1 \leq i < j \leq 5} |W_{ij}| = 5 \cdot 36 - 10 = 170.$$

On the other hand, we take a hyperplane $H_5 \cong \mathbb{P}^3$ in such a way that $\Pi_5 \subseteq H_5$. Then

$$\text{Res}_{H_5}(X) = \Pi_1 + \dots + \Pi_4$$

and

$$\text{Tr}_{H_5}(X) = \Pi_1|_{H_5} + \dots + \Pi_5|_{H_5}.$$

By Lemma 2.6 we get

$$\text{adim}_4(X, t) \leq \text{adim}_4(\text{Res}_{H_5}(X), t-1) + \text{adim}_3(\text{Tr}_{H_5}(X), t).$$

And as before

$$\operatorname{adim}_3(\operatorname{Tr}_{H_5}(X), t) = \operatorname{adim}_3(\operatorname{Tr}_{H_5}(X) - \Pi_5|_{H_5}, t - 1)$$

because Π_5 is a component for the locus of forms in H_5 . Moreover, by Hartshorne-Hirschowitz Theorem we have

$$\operatorname{adim}_3(\operatorname{Tr}_{H_5}(X) - \Pi_5|_{H_5}, 6) = \operatorname{vdim}_3(\operatorname{Tr}_{H_5}(X) - \Pi_5|_{H_5}, 6) = S_{3,4,6} = 56.$$

We illustrate the situation as a diagram.

$$\begin{array}{ccc} \operatorname{adim}_4(X, 7) & \dashrightarrow & \operatorname{adim}_3(\Pi_1|_{H_5} + \dots + \Pi_4|_{H_5}, 6) = S_{3,4,6} = 56 \\ \downarrow & & \\ \operatorname{adim}_4(\Pi_1 + \dots + \Pi_4, 6) & & \end{array}$$

We continue that procedure by taking $X' = \Pi_1 + \dots + \Pi_4$ and H_4 containing Π_4 .

$$\begin{array}{ccc} \operatorname{adim}_4(\Pi_1 + \dots + \Pi_4, 6) & \dashrightarrow & \operatorname{adim}_3(\Pi_1|_{H_4} + \Pi_2|_{H_4} + \Pi_3|_{H_4}, 5) = S_{3,3,5} = 38 \\ \downarrow & & \\ \operatorname{adim}_4(\Pi_1 + \Pi_2 + \Pi_3, 5) & & \end{array}$$

Then we take consecutively $H_3 \supseteq \Pi_3$ and $H_2 \supseteq \Pi_2$.

$$\begin{array}{ccc} \operatorname{adim}_4(\Pi_1 + \Pi_2 + \Pi_3, 5) & \dashrightarrow & \operatorname{adim}_3(\Pi_1|_{H_3} + \Pi_2|_{H_3}, 4) = 25 \\ \downarrow & & \\ \operatorname{adim}_4(\Pi_1 + \Pi_2, 4) & \dashrightarrow & \operatorname{adim}_3(\Pi_1|_{H_2}, 3) = 16 \\ \downarrow & & \\ \operatorname{adim}_4(\Pi_1, 3) & & \end{array}$$

That gives us the inequality:

$$\operatorname{adim}_4(X, 7) \leq 135 + \operatorname{adim}_4(\Pi_1, 3).$$

Since $\operatorname{adim}_4(\Pi_1, 3) = \binom{7}{4} - \binom{5}{2} = 25$, we have $\operatorname{adim}_4(X, 7) \leq 160$.

Theorem 5.2 Let $\Pi_1, \dots, \Pi_s$ be general linear subspaces of codimension 2 in $\mathbb{P}^n$. Consider the scheme $X = \Pi_1 + \dots + \Pi_s$. If

$$S_{n-2p, s-p, t} > 0$$

for $p = 1, \dots, N(n, s) - 1$, then

$$\operatorname{adim}_n(X, t) \geq \operatorname{vdim}_n(X, t).$$

Proof Recall that

$$S_{n-2p, s-p, t} = \sum_{i=0}^{N(n, s)-p} (-1)^i \binom{s-p}{i} \binom{t+n-2p-2i}{n-2p-2i} = \sum_{i=p}^{N(n, s)} (-1)^{i-p} \binom{s-p}{i-p} \binom{t+n-2i}{n-2i},$$

where $N(n, s) = \min\{\lfloor n/2 \rfloor, s\}$. Also notice that vanishing along the subspace of codimension c imposes $\binom{t+n-c}{n-c}$ conditions on forms of degree t .

We proceed as in Example 5.1. We want to construct the sets $W_{i_1 \dots i_l}$ ($1 \leq l \leq N(n, s)$) of linear equations that guarantee the vanishing of forms of degree t along the intersections $\Pi_{i_1} \cap \dots \cap \Pi_{i_l}$ and $\binom{t+n}{n} - |W_1 \cup \dots \cup W_s| = S_{n, s, t}$.

Take $l = N(n, s)$, then any intersection of $l + 1$ subspaces is empty. First, we choose the sets $W_{i_1 \dots i_l}$ consisting of $\binom{t+n-2l}{n-2l}$ linearly independent equations. For $p = l - 1$ each intersection of p subspaces contains $s - p$ intersections of $p + 1 = l$ subspaces. Hence, we require each set $W_{i_1 \dots i_p}$ to contain $(s - p)\binom{t+n-2l}{n-2l}$ equations that were previously chosen. If

$$\binom{t+n-2p}{n-2p} - (s - p)\binom{t+n-2l}{n-2l} = S_{n-2p, s-p, t} > 0,$$

then $W_{i_1 \dots i_p}$ can be completed (with linearly independent equations) so that $|W_{i_1 \dots i_p}| = \binom{t+n-2p}{n-2p}$.

Similarly, for any $p = N(n, s) - 2, \dots, 1$ we use the equations that were previously chosen for smaller subspaces and complete them to the set $W_{i_1 \dots i_p}$. By the inclusion-exclusion principle, if the inequalities of the form

$$\begin{aligned} \binom{t+n-2p}{n-2p} - (s - p)|W_{i_1 \dots i_{p+1}}| + \binom{s-p}{2}|W_{i_1 \dots i_{p+2}}| - \dots + (-1)^{l-p} \binom{s-p}{l-p} |W_{i_1 \dots i_l}| \\ = S_{n-2p, s-p, t} > 0 \end{aligned}$$

are satisfied, then we can do such construction.

Hence, if the inequalities $S_{n-2p, s-p, t} > 0$ are satisfied for $p = 1, \dots, N(n, s) - 1$, then by the inclusion-exclusion principle we have that

$$\text{adim}_n(X, t) \geq \binom{t+n}{n} - |W_1 \cup \dots \cup W_s| = S_{n, s, t} = \text{vdim}_n(X, t).$$

□

In the following example we show that the conditions of the form $S_{n-2p, s-p, t} > 0$ are needed in the previous theorem.

Example 5.3 Consider $X = \Pi_1 + \dots + \Pi_{12}$ in $\mathbb{P}^4$. There are no forms of degree 2 vanishing on 12 general planes. Indeed, taking a hyperplane $H \cong \mathbb{P}^3$ such that $\Pi_{12} \subseteq H$ and using Lemma 2.6 we get

$$\text{adim}_4(X, 2) \leq \text{adim}_4(\Pi_1 + \dots + \Pi_{11}, 1) + \text{adim}_3(\Pi_1|_H + \dots + \Pi_{11}|_H, 1) = 0.$$

On the other hand, by Theorem 3.2

$$\text{vdim}_4(X, 2) = \binom{2+4}{4} - 12\binom{2+2}{2} + \binom{12}{2} = 9.$$

The assumptions of Theorem 5.5 are not satisfied, since $S_{2, 11, 2} = -5$.

For the rest of this section we assume that $X = \Pi_1 + \dots + \Pi_{n+1} \subseteq \mathbb{P}^n$, where $\Pi_1, \dots, \Pi_{n+1}$ are general linear subspaces of codimension 2 and $n \geq 2$. We want to compare the virtual and actual dimension of the system of forms of degree $n + k$ vanishing along X . For this purpose, we prove the next two theorems.

Theorem 5.4 *If $k \geq 3$, then we have*

$$S_{n-2p, n+1-p, n+k} > 0$$

for $p = 0, \dots, \lfloor n/2 \rfloor - 1$.

Proof We will prove by induction on n that

$$S_{n-2p, n+1-p-j, n+k-j} > 0$$

for $j = 0, \dots, n - p$.

For $n = 2$ we need to consider $p = 0$ and $j = 0, 1, 2$ and we get

$$S_{2,3,k+2} = \binom{k+4}{2} - 3 > 0, \quad S_{2,2,k+1} = \binom{k+3}{2} - 2 > 0, \quad S_{2,1,k} = \binom{k+2}{2} - 1 > 0.$$

Fix n and p . First, we consider the case $j = n - p$ and show that $S_{n-2p,1,k+p} > 0$. Indeed, notice that for $\tilde{n} = n - 2p \geq 2$ and $\tilde{k} = k + p \geq 3$, we have

$$S_{n-2p,1,k+p} = S_{\tilde{n},1,\tilde{k}} = \binom{\tilde{k}+\tilde{n}}{\tilde{n}} - \binom{\tilde{k}+\tilde{n}-2}{\tilde{n}-2} = \binom{\tilde{k}+\tilde{n}-2}{\tilde{n}-2} \cdot \left(\frac{(\tilde{k}+\tilde{n}-1)(\tilde{k}+\tilde{n})}{(\tilde{n}-1)\tilde{n}} - 1 \right) > 0.$$

For $j \leq n - p - 1$ by Lemma 3.1

$$\begin{aligned} S_{n-2p,n+1-p-j,n+k-j} &= S_{n-2p,n+1-p-(j+1),n+k-(j+1)} \\ &\quad + S_{(n-1)-2p,(n-1)+1-p-j,(n-1)+k-j}. \end{aligned}$$

By reverse induction on j we have

$$S_{n-2p,n+1-p-(j+1),n+k-(j+1)} > 0.$$

If $p \leq \lfloor \frac{n-1}{2} \rfloor - 1$, we have

$$S_{(n-1)-2p,(n-1)+1-p-j,(n-1)+k-j} > 0$$

by induction on n . For odd n we consider $p = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-1}{2} \rfloor - 1$ and for even n we take $p = 0, \dots, \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-1}{2} \rfloor$. However, in the case of $p = \lfloor \frac{n-1}{2} \rfloor$ for even n , we have

$$S_{(n-1)-2p,(n-1)+1-p-j,(n-1)+k-j} = S_{0,(n-1)+1-p-j,(n-1)+k-j} = 1.$$

□

Theorem 5.5 For $k \geq 3$, we have that

$$\text{adim}_n(X, n+k) \leq \text{vdim}_n(X, n+k).$$

Proof First, notice that

$$\text{adim}_n(\Pi, k) = \text{vdim}_n(\Pi, k) = S_{n,1,k},$$

where Π is a codimension 2 linear subspace in $\mathbb{P}^n$. We will prove by induction on n that

$$\text{adim}_n(\Pi_1 + \dots + \Pi_{n+1-j}, n+k-j) \leq S_{n,n+1-j,n+k-j}$$

for $j = 0, \dots, n-1$. We proceed similarly to the proof of the previous theorem.

For $n = 2$ the linear subspaces of codimension 2 are points, so

$$\begin{aligned} \text{adim}_2(\Pi_1 + \Pi_2 + \Pi_3, k+2) &= \text{vdim}_2(\Pi_1 + \Pi_2 + \Pi_3, k+2) = S_{2,3,k+2}, \\ \text{adim}_2(\Pi_1 + \Pi_2, k+1) &= \text{vdim}_2(\Pi_1 + \Pi_2, k+1) = S_{2,2,k+1}. \end{aligned}$$

We fix n and use Lemma 2.6 by taking a hyperplane H such that $\Pi_{n+1-j} \subseteq H \cong \mathbb{P}^{n-1}$. We get

$$\begin{aligned} \text{adim}_n(\Pi_1 + \dots + \Pi_{n+1-j}, n+k-j) &\leq \text{adim}_n(\Pi_1 + \dots + \Pi_{n+1-(j+1)}, n+k-(j+1)) \\ &\quad + \text{adim}_{n-1}(\Pi_1 + \dots + \Pi_{(n-1)+1-j}, (n-1)+k-j). \end{aligned}$$

By reverse induction on j and induction on n the right-hand side is not greater than

$$S_{n,n-j,n+k-j-1} + S_{n-1,n-j,n+k-j-1}.$$

By Lemma 3.1 it is equal to $S_{n,n+1-j,n+k-j}$.

□

Corollary 5.6 *By Theorems 5.2, 5.4 and 5.5 we have*

$$\operatorname{adim}_n(X, n+k) = \operatorname{vdim}_n(X, n+k) > 0.$$

for $k \geq 3$.

6 Images of linear systems under Veneroni transformations

Now we focus on the linear systems that are transformations of

$$S = (n+k)H - \Pi_1 - \dots - \Pi_{n+1}$$

by the Veneroni maps, where $\Pi_1, \dots, \Pi_{n+1} \subseteq \mathbb{P}^n$ are codimension 2 general linear subspaces and $k \geq 3$. As we explained in Sect. 4, such systems are of the form

$$S' = (nk+1)H - k\Pi_1 - \dots - k\Pi_{n+1}.$$

We will prove that for $k=3$ and $n \geq 6$ the virtual dimension of S' is always non-positive.

First, we continue the analysis of the example presented in [9, Ex. 4.2.], which shows in particular that the assumption $n \geq 6$ is essential.

Example 6.1 Let $X' = 3\Pi_1 + \dots + 3\Pi_5 \subseteq \mathbb{P}^4$. We want to find $\operatorname{vdim}_4(X', 13)$. As before, we assume that $H \cong \mathbb{P}^3$ is a hyperplane such that $\Pi_5 \subseteq H$. Then

$$\operatorname{Res}_H(X') = 3\Pi_1 + \dots + 3\Pi_4 + 2\Pi_5$$

and

$$\operatorname{Tr}_H(X') = 3\Pi_1|_H + \dots + 3\Pi_5|_H.$$

By Lemma 2.7 and the fact that $3\Pi_5$ is a component for the zero locus of forms in H we have

$$\operatorname{vdim}_4(X', t) = \operatorname{vdim}_4(\operatorname{Res}_H(X'), t-1) + \operatorname{vdim}_3(\operatorname{Tr}_H(X') - 3\Pi_5|_H, t-3).$$

We repeat that procedure as presented in the diagram. To alleviate notation, at each level we write $\Pi_i|_H$, although at each step we take different hyperplanes H_i .

$$\begin{array}{ccc}
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_5, 13) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + \dots + 3\Pi_4|_H, 10) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_4 + 2\Pi_5, 12) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + \dots + 3\Pi_4|_H, 10) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_4 + \Pi_5, 11) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + \dots + 3\Pi_4|_H, 10) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_4, 10) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H + 3\Pi_3|_H, 7) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + 3\Pi_2 + 3\Pi_3 + 2\Pi_4, 9) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H + 3\Pi_3|_H, 7) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + 3\Pi_2 + 3\Pi_3 + \Pi_4, 8) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H + 3\Pi_3|_H, 7) \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + 3\Pi_2 + 3\Pi_3, 7) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H, 4) \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1 + 3\Pi_2, 4) & \longrightarrow & \text{vdim}_3(3\Pi_1|_H, 1) \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\text{vdim}_4(3\Pi_1, 1) & &
\end{array}$$

It follows that

$$\begin{aligned}
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_5, 13) &= 3[\text{vdim}_3(3\Pi_1|_H + \dots + 3\Pi_4|_H, 10)] \\
&\quad + \text{vdim}_4(3\Pi_1 + \dots + 3\Pi_4, 10)
\end{aligned}$$

and so forth. That gives us

$$\begin{aligned}
\text{vdim}_4(3\Pi_1 + \dots + 3\Pi_5, 13) &= 3\left[\text{vdim}_3(3\Pi_1|_H + \dots + 3\Pi_4|_H, 10) \right. \\
&\quad + \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H + 3\Pi_3|_H, 7) + \text{vdim}_3(3\Pi_1|_H + 3\Pi_2|_H, 4) \\
&\quad \left. + \text{vdim}_3(3\Pi_1|_H, 1)\right] + \text{vdim}_4(3\Pi_1, 1). \tag{1}
\end{aligned}$$

One can check that $\text{vdim}_3(3\Pi_1|_H, 1) = \text{vdim}_4(3\Pi_1, 1) = 0$ (we will prove it in Lemma 6.3). To find remaining values we repeat the procedure in $\mathbb{P}^3$, we denote $S_i := \Pi_i|_H$ and hyperplanes in $\mathbb{P}^3$ as H' .

$$\begin{array}{ccc}
 \text{vdim}_3(3S_1 + \dots + 3S_4, 10) & \longrightarrow & \text{vdim}_2(3S_1|_{H'} + 3S_2|_{H'} + 3S_3|_{H'}, 7) = 18 \\
 \downarrow & & \\
 \vdots & & \\
 \downarrow & & \\
 \text{vdim}_3(3S_1 + 3S_2 + 3S_3, 7) & \longrightarrow & \text{vdim}_2(3S_1|_{H'} + 3S_2|_{H'}, 4) = 3 \\
 \downarrow & & \\
 \vdots & & \\
 \downarrow & & \\
 \text{vdim}_3(3S_1 + 3S_2, 4) & \longrightarrow & \text{vdim}_2(3S_1|_{H'}, 1) = -3 \\
 \downarrow & & \\
 \vdots & & \\
 \downarrow & & \\
 \text{vdim}_3(3S_1, 1) = 0 & &
 \end{array}$$

The numbers on the right are easily calculated, since we consider vanishing at points in $\mathbb{P}^2$. Again by Lemma 2.7

$$\text{vdim}_3(3S_1 + 3S_2, 4) = 3 \cdot \text{vdim}_2(3S_1|_{H'}, 1) + \text{vdim}_3(3S_1, 1) = -9,$$

$$\text{vdim}_3(3S_1 + 3S_2 + 3S_3, 7) = 3 \cdot \text{vdim}_2(3S_1|_{H'} + 3S_2|_{H'}, 4) + \text{vdim}_3(3S_1 + 3S_2, 4) = 0$$

and

$$\begin{aligned}
 \text{vdim}_3(3S_1 + \dots + 3S_4, 10) &= 3 \cdot \text{vdim}_2(3S_1|_{H'} + 3S_2|_{H'} + 3S_3|_{H'}, 7) \\
 &\quad + \text{vdim}_3(3S_1 + 3S_2 + 3S_3, 7) = 54.
 \end{aligned}$$

By the formula (1) we get $\text{vdim}_4(X', 13) = 135$.

Remark 6.2 In the case of multiplicities 1 we proved that there is a formula for the virtual dimension of a given system. When we consider higher multiplicities the situation becomes much more complicated. We can use Lemma 2.7 to find the virtual dimension as in the previous example. The procedure that returns the value

$$\text{vdim}_n(m_1\Pi_1 + \dots + m_s\Pi_s, t)$$

for given $n \geq 2$, $m_i > 0$ and $s \geq 0$ can be easily implemented in Singular or Maple. Notice that if $\Pi_s \subseteq H \cong \mathbb{P}^{n-1}$, then

$$\text{Res}_H(m_1\Pi_1 + \dots + m_s\Pi_s) = m_1\Pi_1 + \dots + (m_s - 1)\Pi_s$$

and

$$\text{Tr}_H(m_1\Pi_1 + \dots + m_s\Pi_s) - m_s\Pi_s = m_1\Pi_1 + \dots + m_{s-1}\Pi_{s-1}.$$

By Lemma 2.7 $\text{vdim}_n(m_1\Pi_1 + \dots + m_s\Pi_s, t)$ is equal to

$$\text{vdim}_n(m_1\Pi_1 + \dots + (m_s - 1)\Pi_s, t - 1) + \text{vdim}_{n-1}(m_1\Pi_1 + \dots + m_{s-1}\Pi_{s-1}, t - m_s).$$

We define the procedure recursively starting from the conditions

$$\text{vdim}_n(X, t) = \binom{t+n}{n}$$

for $X = \emptyset$ and

$$\text{vdim}_2(m_1\Pi_1 + \cdots + m_s\Pi_s, t) = \binom{t+2}{2} - \sum_{i=1}^s \binom{m_i+1}{2}.$$

Lemma 6.3 For $n \geq k \geq 3$ we have

$$\text{vdim}_n(k\Pi, 1) = 0.$$

Proof By Lemma 2.2 we have that

$$\text{adim}_n(k\Pi, t) = \binom{t+n}{n} - c_{n,2,k,t} = \binom{t+n}{n} - \sum_{i=0}^{k-1} (i+1) \binom{t-i+n-2}{n-2}$$

for $t \geq k$. Denote the summands in $c_{n,2,k,t}$ as $c_i(t) = (i+1) \binom{t-i+n-2}{n-2}$ treated as polynomials with respect to t . Since $\text{vdim}_n(k\Pi, t)$ is a polynomial that agrees with $\text{adim}_n(k\Pi, t)$ for all sufficiently large t , we have

$$\text{vdim}_n(k\Pi, 1) = n + 1 - \sum_{i=0}^{k-1} c_i(1).$$

If $n > 2$, then $c_0(1) = \binom{1+n-2}{n-2} = n-1$, $c_1(1) = 2 \cdot \binom{n-2}{n-2} = 2$ and $c_i(1) = 0$ for $i = 2, \dots, k-1$. Indeed,

$$c_i(t) = (i+1) \cdot \frac{(t-i+1)(t-i+2) \cdots (t-i+n-2)}{(n-2)!}$$

and $1 \leq i-1 \leq k-1 \leq n-2$, so one of the terms in the numerator is equal 0. Hence, $\text{vdim}_n(k\Pi, 1) = n+1 - (n-1) - 2 = 0$. $\square$

Theorem 6.4 For $n \geq 6$ we have that

$$\text{vdim}_n(3\Pi_1 + \cdots + 3\Pi_{n+1}, 3n+1) \leq 0.$$

Proof First, we see that $\text{vdim}_n(3\Pi, 1) = 0$ for $n \geq 3$ by Lemma 6.3. We will prove by induction on n that

$$\text{vdim}_n(3\Pi_1 + \cdots + 3\Pi_{n+1-j}, 3(n-j)+1) \leq 0$$

for $j = 0, \dots, n-1$.

Explicit calculation along the lines of Example 6.1 or appealing to the procedure explained in the previous remark shows that

$$\begin{aligned} \text{vdim}_6(3\Pi_1 + \cdots + 3\Pi_7, 19) &= 0, & \text{vdim}_6(3\Pi_1 + \cdots + 3\Pi_6, 16) &= -729, \\ \text{vdim}_6(3\Pi_1 + \cdots + 3\Pi_5, 13) &= -243, & \text{vdim}_6(3\Pi_1 + \cdots + 3\Pi_4, 10) &= 0, \\ \text{vdim}_6(3\Pi_1 + 3\Pi_2 + 3\Pi_3, 7) &= 0, & \text{vdim}_6(3\Pi_1 + 3\Pi_2, 4) &= 0. \end{aligned}$$

Now we fix n and use Lemma 2.7 three times to obtain

$$\begin{aligned} &\text{vdim}_n(3\Pi_1 + \cdots + 3\Pi_{n+1-j}, 3(n-j)+1) \\ &= \text{vdim}_n(3\Pi_1 + \cdots + 3\Pi_{n+1-(j+1)}, 3(n-(j+1))+1) \\ &\quad + 3(\text{vdim}_{n-1}(3\Pi_1 + \cdots + 3\Pi_{(n-1)+1-j}, 3((n-1)-j)+1)). \end{aligned}$$

The right-hand side is smaller than 0 by reverse induction on j and induction on n . $\square$

7 Application to unexpected hypersurfaces

In this section we present our results concerning unexpected hypersurfaces. Again we assume that $\Pi_1, \dots, \Pi_{n+1}$ are codimension 2 general linear subspaces in $\mathbb{P}^n$ and $X = \Pi_1 + \dots + \Pi_{n+1}$. We consider the linear system of forms of degree $n + k$ vanishing along X and its transformation by the Veneroni map to the system of forms of degree $kn + 1$ vanishing along $X' = k\Pi_1 + \dots + k\Pi_{n+1}$. First, we examine the case $k = 3$.

Theorem 7.1 *Let $X' = 3\Pi_1 + \dots + 3\Pi_{n+1}$, where $n \geq 3$. Then X' admits an unexpected hypersurface of degree $3n + 1$.*

Proof The linear system $S' = (3n + 1)H - 3\Pi_1 - \dots - 3\Pi_{n+1}$ is the pullback of the system $S = (n + 3)H - \Pi_1 - \dots - \Pi_{n+1}$ by the Veneroni map v_n . So $\text{adim}_n(X', 3n + 1) = \text{adim}_n(X, n + 3)$ and by Corollary 5.6 this value is strictly greater than 0.

For $n = 4$ we have $\text{adim}_n(X, n + 3) = 160$ by Example 5.1 and $\text{vdim}_n(X', 3n + 1) = 135$ by Example 6.1. Hence,

$$\text{adim}_n(X', 3n + 1) = \text{adim}_n(X, n + 3) > \text{vdim}_n(X', 3n + 1).$$

One can check that the same inequality holds for $n = 3$ and $n = 5$ using similar techniques or implemented procedure described in Remark 6.2. The cases $n = 3, 4, 5$ are also analysed in [9] (using computer methods). For $n \geq 6$ by Theorem 6.4 $\text{vdim}_n(X', 3n + 1) \leq 0$. Hence, X' admits an unexpected surface of degree $3n + 1$ for all $n \geq 3$. $\square$

When considering higher values of k , it turns out that not only unexpected hypersurfaces appear. Surprisingly, for $k = 4$ the virtual dimension of a given scheme in $\mathbb{P}^{21}$ exceeds the actual dimension.

Example 7.2 Consider $Y = \Pi_1 + \dots + \Pi_{22}$ and $Y' = 4\Pi_1 + \dots + 4\Pi_{22}$ in $\mathbb{P}^{21}$. The system of forms of degree 25 on Y is transformed to the system of forms of degree 85 on Y' by the Veneroni map. The results obtained using procedure from Remark 6.2 are as follows. By Corollary 5.6 we have

$$\text{adim}_{21}(Y', 85) = \text{adim}_{21}(Y, 25) = \text{vdim}_{21}(Y, 25) = 1\,337\,982\,976,$$

whereas

$$\text{vdim}_{21}(Y', 85) = 12\,094\,627\,905\,536.$$

Hence, Y' misses an expected hypersurface of degree 85.

Moreover, the next values of $\text{adim}_{21}(Y', t)$ are

$$\text{vdim}_{21}(Y', 86) = -157\,230\,162\,771\,968,$$

$$\text{vdim}_{21}(Y', 87) = 96\,757\,023\,244\,288,$$

$$\text{vdim}_{21}(Y', 88) = 2\,366\,593\,604\,971\,209.$$

Hence, it seems that the virtual dimension does not provide a good prediction of the actual dimension in this case.

Example 7.3 We also present the analysis of the cases $k = 4, 5, 6$ for $n = 3, \dots, 50$. Using suitable procedure (in Maple or Singular) we check what is the sign of the difference

$$\text{adim}_n(X', kn + 1) - \text{vdim}_n(X', kn + 1).$$

In the following cases we obtain:

for $k = 4$

- unexpected hypersurfaces for $n = 3, \dots, 20$;
- missing expected hypersurfaces for $n = 21, \dots, 50$;

for $k = 5$

- unexpected hypersurfaces for $n = 3, \dots, 17, n = 42, \dots, 50$;
- missing expected hypersurfaces for $n = 18, \dots, 41$;

for $k = 6$

- unexpected hypersurfaces for $n = 3, \dots, 15, n = 37, \dots, 50$;
- missing expected hypersurfaces for $n = 16, \dots, 36$.

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Appendix

Lemma 3.1 *The following formula holds*

$$S_{n,s,t} = S_{n,s-1,t-1} + S_{n-1,s-1,t-1}.$$

Proof Recall that

$$S_{n,s,t} = \sum_{i=0}^{N(n,s)} (-1)^i \binom{s}{i} \binom{t+n-2i}{n-2i},$$

where $N(n, s) = \min\{\lfloor n/2 \rfloor, s\}$. Denote the summands in the formula as

$$a_i = (-1)^i \binom{s}{i} \binom{t+n-2i}{n-2i}, \quad b_i = (-1)^i \binom{s-1}{i} \binom{t+n-2i-1}{n-2i-1}, \quad c_i = (-1)^i \binom{s-1}{i} \binom{t+n-2i-2}{n-2i-1}.$$

Then we want to prove that

$$S := S_{n,s,t} - S_{n,s-1,t-1} - S_{n-1,s-1,t-1} = \sum_{i=0}^{N(n,s)} a_i - \sum_{i=0}^{N(n,s-1)} b_i - \sum_{i=0}^{N(n-1,s-1)} c_i = 0.$$

Notice that for $i = 0, \dots, N(n-2, s-1)$ we have that

$$a_i - b_i - c_i + r_{i-1} = r_i, \quad (*)$$

where $r_{-1} = 0$ and $r_i = (-1)^i \binom{s-1}{i} \binom{t+n-2i-2}{n-2i-2}$ for $i = 0, \dots, N(n-2, s-1)$.

Indeed,

$$\begin{aligned}
 a_0 - b_0 - c_0 &= \binom{t+n}{n} - \binom{t+n-1}{n} - \binom{t+n-2}{n-1} = \binom{t+n-2}{n-2} = r_0, \\
 a_i - b_i - c_i + r_{i-1} &= (-1)^i \left[\binom{s}{i} \binom{t+n-2i}{n-2i} - \binom{s-1}{i} \binom{t+n-2i-1}{n-2i} - \binom{s-1}{i} \binom{t+n-2i-2}{n-2i-1} - \binom{s-1}{i-1} \binom{t+n-2i}{n-2i} \right] \\
 &= (-1)^i \left[\left(\binom{s-1}{i} + \binom{s-1}{i-1} \right) \binom{t+n-2i}{n-2i} - \binom{s-1}{i} \binom{t+n-2i-1}{n-2i} \right. \\
 &\quad \left. - \binom{s-1}{i} \binom{t+n-2i-2}{n-2i-1} - \binom{s-1}{i-1} \binom{t+n-2i}{n-2i} \right] \\
 &= (-1)^i \binom{s-1}{i} \left[\binom{t+n-2i}{n-2i} - \binom{t+n-2i-1}{n-2i} - \binom{t+n-2i-2}{n-2i-1} \right] \\
 &= (-1)^i \binom{s-1}{i} \binom{t+n-2i-2}{n-2i-2} = r_i
 \end{aligned}$$

for $i = 1, \dots, N(n-2, s-1)$.

The equalities (*) allow us to simplify the formula for S since we have that

$$\sum_{i=0}^m a_i - \sum_{i=0}^m b_i - \sum_{i=0}^m c_i = r_m \quad (**)$$

for $m = 0, \dots, N(n-2, s-1)$.

Now we consider the following cases.

I. First, we assume that $s < \lfloor \frac{n-1}{2} \rfloor$. Then $N(n, s) = s$ and $N(n, s-1) = N(n-1, s-1) = N(n-2, s-1) = s-1$. Using (**) we have that $S = 0$, since

$$S = \sum_{i=0}^s a_i - \sum_{i=0}^{s-1} b_i - \sum_{i=0}^{s-1} c_i = a_s + r_{s-1} = (-1)^s \binom{t+n-2s}{n-2s} + (-1)^{s-1} \binom{t+n-2s}{n-2s} = 0.$$

II. Now we assume that $s \geq \lfloor \frac{n-1}{2} \rfloor$ and $n = 2l + 1$ for some $l \in \mathbb{N}$. Then $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor = l$ and $\lfloor \frac{n-2}{2} \rfloor = l-1$.

If $s = l$, then $N(n, s) = s$ and $N(n, s-1) = N(n-1, s-1) = N(n-2, s-1) = s-1$. We get that $S = 0$ by (**) and the fact that $a_s + r_{s-1} = 0$.

If $s > l$, then $N(n, s) = N(n, s-1) = N(n-1, s-1) = l$ and $N(n-2, s-1) = l-1$. We have by (**) that

$$\begin{aligned}
 S = a_l - b_l - c_l + r_{l-1} &= (-1)^l \left[\binom{s}{l} \binom{t+1}{1} - \binom{s-1}{l} \binom{t}{1} - \binom{s-1}{l} \binom{t-1}{0} - \binom{s-1}{l-1} \binom{t+1}{1} \right] \\
 &= (-1)^l \left[\left(\binom{s-1}{l} + \binom{s-1}{l-1} \right) (t+1) - \binom{s-1}{l} t - \binom{s-1}{l} - \binom{s-1}{l-1} (t+1) \right] = 0.
 \end{aligned}$$

III. Finally, we take $s \geq \lfloor \frac{n-1}{2} \rfloor$ and $n = 2l$ for some $l \in \mathbb{N}$. Then $\lfloor \frac{n}{2} \rfloor = l$ and $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor = l-1$.

If $s = l-1$ or $s = l$, then $N(n, s) = s$ and $N(n, s-1) = N(n-1, s-1) = N(n-2, s-1) = s-1$ and again, we have that $S = 0$ using the equality $a_s + r_{s-1} = 0$.

If $s > l$, then $N(n, s) = N(n, s-1) = l$ and $N(n-1, s-1) = N(n-2, s-1) = l-1$. We have that

$$\begin{aligned}
 S = a_l - b_l + r_{l-1} &= (-1)^l \left[\binom{s}{l} \binom{t}{0} - \binom{s-1}{l} \binom{t-1}{0} - \binom{s-1}{l-1} \binom{t}{0} \right] \\
 &= (-1)^l \left[\binom{s}{l} - \binom{s-1}{l} - \binom{s-1}{l-1} \right] = 0.
 \end{aligned}$$

That gives all possible cases and proves the formula. $\square$

References

1. Alexander, J., Hirschowitz, A.: Polynomial interpolation in several variables. *J. Alg. Geom.* 4(2), 201–222 (1995)
2. Alexander, J., Hirschowitz, A.: An asymptotic vanishing theorem for generic unions of multiple points. *Invent. Math.* 140(2), 303–325 (2000)
3. C. Ciliberto (2001) *Geometric aspects of polynomial interpolation in more variables and of Waring's problem*, European Congress of Mathematics. Springer, Birkhuser
4. Cook, D., II., Harbourne, B., Migliore, J., Nagel, U.: Line arrangements and configurations of points with an unusual geometric property. *Compositio Math.* 154, 2150–2194 (2018)
5. M. Dumnicki, Ł. Farnik, B. Harbourne, T. Szemberg, H. Tutaj-Gasińska: *Veneroni maps*, Combinatorial Structures in Algebra and Geometry, NSA 26, Constantin 2018
6. Dumnicki, M., Harbourne, B., Roé, J., Szemberg, T., Tutaj-Gasińska, H.: Unexpected surfaces singular on lines in P^3 . *Eur. J. Math.* 7, 570 (2021)
7. Dumnicki, M., Harbourne, B., Szemberg, T., Tutaj-Gasińska, H.: Linear subspaces, symbolic powers and Nagata type conjectures. *Adv. Math.* 252, 471–491 (2014)
8. Eisenbud, D.: *Commutative Algebra with a View Toward Algebraic Geometry*. Springer-Verlag, New York (1995)
9. Harbourne, B., Migliore, J., Tutaj-Gasińska, H.: New Constructions of Unexpected Hypersurfaces in $\mathbb{P}^n$. *Revista Matemática Complutense* (2020)
10. B. Harbourne, J. Migliore, U. Nagel, Z. Teitler (2021) *Unexpected hypersurfaces and where to find them*. *Mich J Math* 70: 301-339
11. J. Harris: *Algebraic Geometry, A First Course*, Graduate Texts in Mathematics. Springer, New York (1992)
12. Hartshorne, R.: *Algebraic Geometry*. Springer-Verlag, New York (1977)
13. Hartshorne, R., Hirschowitz, A.: *Droites en position generale dans l'espace projectif* Algebraic Geometry, La Rabida, (1981)

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