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RESEARCH

A Set-Theoretic Analysis of the Black Hole Entropy Puzzle

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Abstract

Motivated by the known mathematical and physical problems arising from the current mathematical formalization of the physical spatio-temporal continuum, as a substantial technical clarification of our earlier attempt (Etesi in Found Sci 25:327–340, 2020), the aim in this paper is twofold. Firstly, by interpreting Chaitin's variant of Gödel's first incompleteness theorem as an inherent uncertainty or fuzziness present in the set of real numbers, a set-theoretic entropy is assigned to it using the Kullback–Leibler relative entropy of a pair of Riemannian manifolds. Then exploiting the non-negativity of this relative entropy an abstract Hawking-like area theorem is derived. Secondly, by analyzing Noether's theorem on symmetries and conserved quantities, we argue that whenever the four dimensional space-time continuum containing a single, stationary, asymptotically flat black hole is modeled by the set of real numbers in the mathematical formulation of general relativity, the hidden set-theoretic entropy of this latter structure reveals itself as the entropy of the black hole (proportional to the area of its "instantaneous" event horizon), indicating that this apparently physical quantity might have a pure set-theoretic origin, too.

Keywords Continuum · Chaitin incompleteness · Kullback–Leibler divergence · Black hole entropy

1 Introduction

The mathematical formalization of the continuum in its current form, what we shall call the *arithmetical continuum* or equivalently the *set of real numbers* $\mathbb{R}$ here, is widely used everywhere in mathematics, physics, engineering sciences, mathematical biology, economy and sociology, etc., etc. However, in spite of its success, the utilization of the arithmetical continuum leads to various difficulties both in pure mathematics [29] and its applications (for an excellent survey of physics see [1]);

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all of these difficulties basically originate from the fact that the arithmetical continuum has an inherent infinite transcendental structure (in many aspects). Perhaps the most relevant question related with the problematics of the arithmetical continuum on the pure mathematical side is the (in our opinion) unsettled status of *Cantor*'s continuum hypothesis; while a simple but painful example from theoretical physics is the divergence of the total electric energy of an electrically charged *point* particle in Faraday–Maxwell electrodynamics (leading eventually to the complicated *renormalization* issues in classical and quantum field theories). Fortunately, by performing simple *experiments*, we know that this and the various other occurences of divergences in classical and quantum electrodynamics are nothing but artifacts originating from the mathematical formulation (which uses the arithmetical continuum) of these theories hence we can identify and isolate these divergences quite easily. But what about the singularities or other phenomena (mathematically) predicted by general relativity for instance? Lacking unambigous experimental evidences we cannot make a commitment about their ontological status yet.

The easiest way to get rid of the various divergence problems in physical theories arising from modeling mathematically the physical spatio-temporal continuum with the arithmetical one, is to declare that the mathematical description of space-time should be simply finite; that is the mathematical structure modeling physical space-time should have non-zero finite cardinality $0 < N < +\infty$ as a set. Finiteness necessarily implies the geometric relation $N \sim V/V_{\text{Planck}}$ where V is the volume of a spatio-temporal region. In this framework it is yet reasonable to suppose that at macroscopic scales (i.e. as $V \to +\infty$) the cardinality of space-time is extremely huge, practically infinite hence the classical geometric description (provided by general relativity theory with its precise mathematics) applies as an approximation; nevertheless the macroscopic cardinality N of the observable Universe is determined by some microscopic cardinality $0 < N_{\text{Planck}} < +\infty$ hence is finite. However as one approaches microscopic scales (i.e. as $V \to V_{\text{Planck}}$) the finiteness gets more and more relevant and the supposed finite microscopic cardinality attained at the Planck scales. If one indeed wants to describe not merely a geometric continuum but the physical space-time itself then one also has to take into account that at microscopic scales the spatio-temporal continuum more-and-more resembles the vacuum state of a relativistic quantum field with its known microscopic properties (described by some yet mathematically problematic relativistic quantum field theory). Thus talking about the finiteness of space-time in fact means that one supposes that the physical vacuum has finite physical degrees of freedom. Consequently the cardinality N of a volume V containing vacuum is proportional to its energy content, i.e. expected to satisfy $N \sim E/E_{\text{Planck}}$ too, where E is the vacuum energy within V.

The quantum vacuum is subject to *Heisenberg*'s various *uncertainty principles*. But if these are also involved in the description, the cardinality of the set modeling the physical space-time, if finite, gets problematic at the microscopic (hence the macroscopic) level. This is simply because as the size of a spatial volume approaches the Planck length or its time of existence gets very short, the fluctuation of the spatio-temporal cardinality gets comparable with the cardinality itself. More precisely we assumed that $N \sim V/V_{\text{Planck}} \sim (L/\ell_{\text{Planck}})^3$. Then using $\Delta L \Delta p \geqq \hbar$ the relative fluctuation of the vacuum cardinality within a volume V is estimated from below as

$$\frac{\Delta N}{N} \sim \left(\frac{\Delta L}{L}\right)^3 \geqq \left(\frac{\hbar}{L\Delta p}\right)^3.$$

On substituting the minimal length ℓ_{Planck} and momentum uncertainty $\Delta p_{\text{Planck}} \sim p_{\text{Planck}} = m_{\text{Planck}}c$ we find

$$\frac{\Delta N_{\text{Planck}}}{N_{\text{Planck}}} \geqq \left(\frac{\hbar}{\sqrt{\hbar G/c^3}\sqrt{\hbar c/G}\, c}\right)^3 = 1$$

in a small volume comparable to the Planck length. Likewise, based on the finiteness assumption, we know that $N \sim E/E_{\text{Planck}}$ too. Putting this together with $\Delta E \Delta t \geqq \hbar$ gives

$$\frac{\Delta N}{N} \sim \frac{\Delta E}{E} \geqq \frac{\hbar}{E\Delta t}.$$

The minimal energy of the vacuum is $E_{\text{Planck}} = m_{\text{Planck}}c^2$ and the minimal observation time of it is $\Delta t_{\text{Planck}} \sim t_{\text{Planck}}$. Hence we obtain again

$$\frac{\Delta N_{\text{Planck}}}{N_{\text{Planck}}} \geqq \frac{\hbar}{c^2\sqrt{\hbar c/G}\sqrt{\hbar G/c^5}} = 1$$

shortly after the Big Bang for instance. Since the relative fluctuation is a dimensionless quantity, having unit magnitude means that it is meaningless to talk about a sharp (i.e. finite) cardinality N_{Planck} of the vacuum at short space or time scales, if quantum mechanics is also taken into account. This overall uncertainty might be the core physical reason why physical space-time is modeled by infinite mathematical structures: if one indeed wants to describe the physical space and time as the vacuum of a relativistic quantum field (and not merely as a geometric continuum) then using infinite sets is an apparently unavoidable mathematical way to grasp the overall spatio-temporal uncertainity of the relativistic quantum field comprising the true physical vacuum.

Having seen that modeling physical space-time with infinite mathematical structures is not easy to exclude, we return to our former question concerning general relativity: the purpose of this paper is to examine whether the several conceptional, technical, common sense, etc. controversies connected with *black hole entropy* [2, 11, 12, 23, 25, 27] do at least in part emanate from the fact that general relativity mathematically rests on the arithmetical continuum, an infinite structure?

The paper is organized as follows. In Sect. 2 we recall *Chaitin*'s reformulation of *Gödel*'s first incompleteness theorem (see Theorem 2.1 here) and interpret its content—with hindsight dictated by *Heisenberg*' uncertainity—as the presence of an inherent uncertainty or fuzziness within the arithmetical continuum (which might also be a consequence that its final constituents are extensionless). This strongly motivates to introduce a statistico-physical analogy and to talk in this context about the pure "set-theoretic entropy" of the arithmetical conntinuum. Quite interestingly this idea can be rigorously grasped by the aid of a key concept of current information theory, namely the *Kullback–Leibler relative entropy* or *divergence* [6, Chapter 8] adapted to a pair of Riemannian manifolds (see Theorem 2.2 here). Computing this quantity over a compact manifold-with-boundary and exploiting its non-negativity an abstract Riemannian geometric analogue of *Hawking*'s area theorem [11] is obtained (Theorem 2.3 here). Then, in Sect. 3 basically following [9, Section 3] we recall and refine *Noether*'s theorem on symmetries and conserved quantities and prove that within general relativity a conserved quantity to diffeomorphisms can be assigned which is not zero if a stationary black hole is present and can be identified with its entropy (proportional to the area of the "instantaneous" event horizon [11]). Then we argue, based on their common diffeomorphism invariance, that the "set-theoretic" and the black hole entropies are strongly related. This suggests that the latter entropy might be a consequence of the former one as a result of modeling general relativity mathematically over the arithmetical continuum therefore the long-sought physical degrees of freedom responsible for black hole entropy have at least in part simply a pure mathematical origin only, corresponding to the artificial division of the space-time continuum into points (see the discussion at the end of Sect. 3).

In the "Appendix", we make an attempt (certainly incomplete at this stage of the art) to introduce a temporal structure in a covariant way into general relativity: recall that although being our most advanced theory dealing with the structure of space and time, general relativity is in fact a "timeless theory" due to its diffeomorphism invariance (cf. e.g. [3, Chapter 3]).

2 Chaitin Incompleteness and the Entropy of the Continuum

Accepting the structure of the arithmetical continuum or equivalently the set $\mathbb{R}$ of real numbers,[1] it is the totality of, or more precisely the disjoint union of, its individual constituents called *real numbers* or—speaking geometrically—its *points*:

$$\mathbb{R} = \bigsqcup_{x\in\mathbb{R}} \{x\} \,.$$

If indeed this is the optimal mathematical structure of the continuum (e.g. it is possible to construct it without points [13]), then one would expect that using mathematical tools only, one is able to "locate" the individual constituents x within their totality $\mathbb{R}$ or equivalently, to make a mathematical distinction between them in an effective way. We know from our elementary university studies that upon fixing a notational convention every real number admits a well-defined (for instance

[1] Here by a *real number* x we mean by definition a Dedekind slice or cut of the rationals and by the *set of real numbers* $\mathbb{R}$ the collection of all of them. $\mathbb{R}$ can be equipped with the usual structures (addition, multiplication and ordering) and posssesses the completeness property rendering it a complete ordered field. It is well-known that $\mathbb{R}$ as a complete ordered field (formulable in a second order theory only!) is essentially unique.

decimal) expansion which means that this expansion exists for all real numbers and is unique in the sense that two expansions coincide if and only if the corresponding two real numbers are equal. It is however a quite surprising observation that in general the existence of this well-defined and unique expansion is the only available property of a "truely generic" real number i.e. a typical element $x \in \mathbb{R}$. Therefore our question about an effective mathematical way of "picking" a single element $x \in \mathbb{R}$ is in fact a question about the effectiveness of making distinctions between generic real numbers in terms of their (decimal) expansions.

To approach this problem first let us recall the idea of Kolmogorov complexity or algorithmic compressibility or computability of a real number (cf. e.g. [6, Chapter 14]). Let $x \in \mathbb{R}$ be given and denote by $\mathbf{T}_x$ the (probably empty) set of those Turing machines which reproduce x in the following way: if $\mathbf{T}_x \neq \emptyset$ and some $T_x \in \mathbf{T}_x$ is given with an input $n \in \mathbb{N}$ then the output $T_x(n) \in \mathbb{N}$ consists of e.g. precisely the first n digits of the expansion of x. Denoting by $|T_x| \in \mathbb{N}$ the length of the Turing machine (considered as an algorithm or program in some programming language) define

$$K(x) := \begin{cases} +\infty & \text{if} \quad \mathbf{T}_x = \emptyset \\ \inf\limits_{T_x \in \mathbf{T}_x} |T_x| < +\infty & \text{if} \quad \mathbf{T}_x \neq \emptyset \end{cases}$$

and call the resulting (extended) natural number $K(x) \in \mathbb{N} \cup \{+\infty\}$ the *Kolmogorov complexity* of $x \in \mathbb{R}$. Then $K(x) = +\infty$ corresponds to the situation when no algorithms reproducing x in the above sense exist hence x is not *algorithmically compressible* or not *computable* (by simple cardinality arguments the vast majority of real numbers belongs to this class including, as a yet definable hence mild example, Chaitin's famous Ω number [6, Section 14.8]) while $K(x) < +\infty$ corresponds to the opposite case (containing all familiar real numbers like $7, \frac{5}{8}, \sqrt{2}, \pi$, e,...). It is clear that the only important question about x in this context is whether $K(x) = +\infty$ or $K(x) < +\infty$ and in the latter case only the magnitude of $K(x)$ is relevant, for its particular value depends on the details of the sort of expansion of x, the programming language for T_x, etc. hence does not carry essential information. A remarkable observation about Kolmogorov complexity is the following result:

Theorem 2.1 (Chaitin's version of Gödel's first incompleteness theorem [4]) *For any (sufficiently rich, consistent, recursively enumerable) axiomatic system S based on a first order language L there exists a natural number* $0 < N_S < +\infty$ *such that there exists no real number x for which the proposition*

$$K(x) \geqq N_S$$

is provable within S. □

Motivated in various ways by [10, 20, 21] we interpret this quite surprising mathematical fact from our viewpoint as follows: taking into account that the only known property of a generic real number which fully characterizes it is its existing (decimal) expansion, but the Kolmogorov complexity of this expansion hence the expansion itself generally is not fully determinable (by proving theorems on it in an axiomatic system), there is in general no way, using standard mathematical tools in the broadest sense, to “sharply pick” any element from the arithmetical continuum. Consequently, from the viewpoint of an “effective mathematical activity”, *the structure of the arithmetical continuum i.e. the set* $\mathbb{R}$ *of real numbers contains an inherent uncertainty or fuzziness in the sense that its individual disjoint constituents cannot be distinguished from each other in a universal and effective mathematical way*.

The above interpretation of Theorem 2.1 serves as a motivation to introduce a *statistico-physical analogy* for the arithmetical continuum offering a fresh look into its structure. First, generalizing the above decomposition of the real line into its points, we accept as usual that every (finite dimensional, real) differentiable manifold M admits a decomposition into its disjoint constituent points:

$$M = \bigsqcup_{x \in M} \{x\}\,. \tag{1}$$

Speaking intutively, we can make three observations about this decomposition: all the points x of M (i) are homologous i.e. “look the same”, (ii) are terminal objects i.e. they do not possess any further internal structure nevertheless their collection gives back precisely M and (iii) are disjoint from each other i.e. they “do not interact”. Except its cardinality this decomposition of M therefore strongly resembles the structure of an ideal gas as usually defined in statistical physics. Take an *abstract set* X whose cardinality coincides with that of the continuum in ZFC set theory and regard it as an *abstract ideal gas* X such that its *elements* correspond to the *atoms* of the ideal gas X. Extending this analogy further, the left hand side of (1) i.e. a differentiable manifold M with its global topological, smooth, etc. “macroscopic” properties can be regarded as one possible *macrostate of* X while the right hand side of (1) i.e. the particular identification of M with its elements as a particular *microstate of* X within the macrostate M. The *equilibrium dynamics of* X in its macrostate M is generated by *diffeomorphisms*; hence another microstate of X within the same macrostate M is achieved by picking any diffeomorphism $f : M \rightarrow M$ and writing $f(M) = M$ again, and then taking the corresponding new decomposition

$$M = \bigsqcup_{x \in M} \{f(x)\}. \tag{2}$$

Another differentiable manifold N not diffeomorphic to M (in the broadest sense i.e. possibly having different dimension, number of connected components, etc.) might be interpreted as a different macrostate (with its corresponding assembly of microstates created by diffeomorphisms) of the same abstractly given ideal gas X. However this abstract ideal gas can even appear in completely different i.e. non-geometric, discontinuous macrostates as well like e.g. in the form of the *Cantor set* $C \subset \mathbb{R}$ or some other abstract *topological space* (with its *homeomorphisms* creating the corresponding assembly of microstates), or just simply in the form of some *set* (together with its *bijections*), etc., etc.

In accord with this analogy Theorem 2.1 is interpreted as a fundamental result about the indistinguishability of the individual microstates of X realizing the same macrostate M. The next standard step in statistical mechanics is to introduce a tool, a measure, capable to capture the amount of information loss created by the passage from individual microstates to their common macrostate. This measure is known as the *entropy* of the ideal gas in a given macrostate. How could we characterize this entropy within our analogy? Proceeding completely formally along the way of Boltzmann's classical approach to entropy we can argue as follows. Certainly all possible microstates of the abstract ideal gas X are parameterized by the elements of the group of its all set-theoretic bijections $\mathrm{Bij}(X)$ while its possible microstates within the macrostate M are parameterized by its subgroup of diffeomorphisms $\mathrm{Diff}(M)$. Therefore restricting the dynamics of X from $\mathrm{Bij}(X)$ to $\mathrm{Diff}(M)$ *by construction* means that M is an equilibrium state of X. If we further assume that all microstates appear with equal probability i.e. a sort of ergodicity holds for X then as a first trial we *formally* put

$$\text{Entropy of the set } X \text{ in its manifold-macrostate } M \sim \log \Gamma(\mathrm{Diff}(M))$$

with Γ being an, at this state of the art admittedly hypothetical, volume measure on $\mathrm{Bij}(X)$ depending on a particular choice of the axiomatic system S in Theorem 2.1. Taking into account that if M has positive dimension then $1 \subsetneqq \mathrm{Diff}(M) \subsetneqq \mathrm{Bij}(X)$, we expect $1 \lneqq \Gamma(\mathrm{Diff}(M)) \lneqq \Gamma(\mathrm{Bij}(X))$ to hold such that the resulting entropy expression is a finite positive number and at least in its magnitude being independent of any choice for S as dictated by the universality of Theorem 2.1. Note that despite being formally ill-defined, *by construction* this entropy formula is invariant under diffeomorphisms of M since a diffeomorphism does not change the given macrostate.

Let us try to grasp this set-theoretic entropy more precisely from a mathematical viewpoint. To achieve this we follow [6, Chapter 8] and introduce the Kullback–Leibler relative entropy adapted to a pair of Riemannian manifolds. Let (M, g) be an m dimensional Riemannian manifold what we assume to be oriented and compact for technical reasons. Consider the associated volume measure $\mu_g \in \Omega^m(M)$ defined by the aid of the Hodge operator associated with the orientation and the metric as $\mu_g := *_g 1$. Suppose that it is normalized i.e. $\int_M \mu_g = 1$. If $\mathscr{A}_g$ denotes the σ-algebra of μ_g-measurable subsets of M then g improves M to a *Kolmogorov probability measure space* $(M, \mathscr{A}_g, p_g)$. It is remarkable that these probability spaces in fact do not depend on which particular normalized-volume metrics g or h they come from. This is because by a theorem of Moser [17] the only diffeomorphism invariant of a smooth positive density over M is its volume. Therefore taking two different $(M, \mathscr{A}_g, \mu_g)$ and $(M, \mathscr{A}_h, \mu_h)$ there exists an orientation-preserving diffeomorphism $f : M \to M$ such that for every $A \in \mathscr{A}_g$ one finds $f(A) \in \mathscr{A}_h$ and $\int_A \mu_g = \int_{f(A)} \mu_h$. Thus switching to another probability space simply corresponds to identify the manifold M not with its particular microstate as in (1) but with its different one (2) still belonging to the same manifold macrostate M of the abstract ideal gas X.

Keeping in mind this universality of the Riemannian probablity spaces and fixing from now on a particular one $(M, \mathscr{A}_g, \mu_g)$ we can assign a meaning to the emerging probabilities $p_g(A) := \int_A \mu_g$ for every $A \in \mathscr{A}_g$ as follows. Motivated in a straightforward way by interpreting Theorem 2.1 above as the inherent indistinguishability of the points of the continuum we accept that the apparently simple task of identifying or localizing a point $x_0 \in M$ within M cannot be carried out. The best we can do is to introduce the following

Assumption–mathematical form. *The number* $0 \leqq p_g(A) \leqq 1$*is the probability that a distinguished point satisfies that* $x_0 \in A \subsetneqq M$.

Accepting this interpretation let $\{A_i\}_{i=1,\dots,n}$ be a *finite covering of M by* μ_g*-measurable almost-disjoint subsets* i.e. $M = \cup_i A_i$ such that $A_i \in \mathscr{A}_g$ thus $0 \leqq p_g(A_i) \leqq 1$ exists for all $i = 1, \dots, n$ and $p_g(A_i \cap A_j) = 0$ for all $i \neq j$. This implies that $\sum_i p_g(A_i) = 1$. Associated with the metric g and the covering $\{A_i\}_{i=1,\dots,n}$ one introduces in a natural way the *approximate Shannon entropy* of M with respect to the covering:

$$S\big(M, g, \{A_i\}_{i=1,\dots,n}\big) := -\sum_{i=1}^{n} p_g(A_i) \log p_g(A_i). \tag{3}$$

If the information that the point $x_0 \in M$ actually satisfies $x_0 \in A_i$ is interpreted as saying that "M is in its i^{th} state" then taking into account the interpretation of the probabilities involved we can say that (3) describes the entropy of a "state" of M which is the mixture of the "pure states" $i = 1, \dots, n$ with corresponding probabilities $p_g(A_i)$. Observe that this formally agrees with the general definition of the entropy of a system in statistical physics. It is clear that the "knowledge" about the "point distribution" of M is improved if the covering is refined. Therefore it is challenging to define the entropy of M by taking the limit of (3) over all coverings, if exists. However we cannot expect to come up with any reasonable number in this way since taking for example an equipartition i.e. for which $p_g(A_i) = \frac{1}{n}$ for all $i = 1, \dots, n$ with corresponding entropy (3) then

$$\lim_{n\to+\infty} S\big(M, g, n\big) = \lim_{n\to+\infty} \log n = +\infty$$

demonstrating that the naive Shannon entropy of the continuum diverges at least logarithmically. Nevertheless using the physical language this equipartition corresponds to ergodicity of the equilbrium dynamics of the continuum in its manifold-macrostate M provided by its orientation-preserving diffeomorphisms; hence comparing this formula with the formal entropy expression above which also expresses entropy in an equilibrium state under ergodic dynamics we find that $\Gamma(\mathrm{Diff}^+(M)) \sim n$ as $n \to +\infty$ hence indeed regularization needed.

However it turns out that this is the only sort of divergence and one can renormalize the entropy of a compact Riemannian manifold essentially by removing a single logarithmically divergent universal term from (3). To this end we will follow [6, Section 8.3]. Assume that in $\{A_i\}_{i=1,\dots,n}$ every A_i has the form of a closed

m-ball hence we can choose a local coordinate system $(A_i, x^1, \dots, x^m)$ in each of them such that there exists a uniform number

$$\Delta := \int_{A_i} \mathrm{d}x^1 \dots \mathrm{d}x^m$$

for all $i = 1, \dots, n$ satisfying $0 < \Delta < +\infty$ taking into account the orientability of M. Moreover $p_g(A_i) = \int_{A_i} \mu_g = \int_{A_i} \sqrt{\det g(x^1, \dots, x^m)}\mathrm{d}x^1 \dots \mathrm{d}x^m$ therefore introducing the smooth strictly positive local function $\rho_i := \sqrt{\det(g|_{A_i})}$ by the mean value theorem there exists a point $y_i \in A_i$ such that $p_g(A_i) = \rho_i(y_i)\Delta$. Thus (3) takes the shape

$$S\big(M, g, \Delta\big) = -\sum_{i=1}^{n} \rho_i(y_i)\Delta \log\left(\rho_i(y_i)\Delta\right)$$

which is however a highly coordinate-depending expression. To overcome this difficulty introduce another Riemannian structure (M, h) having normalized volume too i.e. $\int_M \mu_h = 1$ with corresponding strictly positive local density functions $\sigma_j := \sqrt{\det(h|_{A_j})}$ hence $p_h(A_j) = \sigma_j(z_j)\Delta$ with some point $z_j \in A_j$. Then the Shannon entropy can be expanded like

$$\begin{aligned} S\big(M, g, \Delta\big) &= -\sum_{i=1}^{n} \rho_i(y_i)\Delta \log\left(\frac{\rho_i(y_i)}{\sigma_i(z_i)}\sigma_i(z_i)\Delta\right) \\ &= -\sum_{i=1}^{n} \rho_i(y_i)\Delta\left(\log\frac{\rho_i(y_i)}{\sigma_i(z_i)} + \log\sigma_i(z_i) + \log\Delta\right) \\ &= -\sum_{i=1}^{n} \log\left(\frac{\rho_i(y_i)}{\sigma_i(z_i)}\right)\rho_i(y_i)\Delta - \sum_{i=1}^{n} \log(\sigma_i(z_i))\rho_i(y_i)\Delta - \log\Delta\sum_{i=1}^{n} \rho_i(y_i)\Delta. \end{aligned}$$

Let us say that the countable sequence $\{A_1\}, \{A_1, A_2\}, ..., \{A_i\}_{i=1,\dots,n}, \dots$ is a *refinement* (of a finite covering of M by μ_g-measurable almost-disjoint subsets as above) if for every points $x, y \in M$ with $x \neq y$ there exists an $n_{x,y}$ such that for every $n > n_{x,y}$ the corresponding $\{A_i\}_{i=1,\dots,n}$ in this sequence contains no single element A_j satisfying $x, y \in A_j$. Applying this to our covering with uniform balls, refinement implies $\Delta \to 0$ but not the other way round. We make now the following four observations. The first and most important is that the ratios of the local functions already extend globally: using the globally existing volume-forms $\mu_g, \mu_h \in \Omega^m(M)$ there exists a positive smooth function $f : M \to \mathbb{R}$ satisfying $\mu_g = f\mu_h$ and obviously $f|_{A_i} = \frac{\rho_i}{\sigma_i}$. We can write this fact as $\frac{\rho_i}{\sigma_i} = \frac{\mathrm{d}\mu_g}{\mathrm{d}\mu_h}\big|_{A_i}$ in terms of the globally well-defined Radon–Nikodym derivative of the involved measures. The second observation is that $\sum_i \rho_i(y_i)\Delta = \int_M \mu_g = 1$. These two observations make sure that the first term on the right hand side converges to a coordinate-free i.e. globaly well-defined integral $-\int_M \log\left(\frac{\mathrm{d}\mu_g}{\mathrm{d}\mu_h}\right)\mu_g$ during a refinement. Thirdly, the second term on the right hand side is a coordinate-dependent hence not well-defined number $I(M,g,h,\Delta)$ nevertheless satisfying $|I(M,g,h,\Delta)| \leqq c(M,h)\int_M \mu_g = c(M,h)$ hence remains bounded (possibly vanishes) during a refinement. Finally the third term is equal to $\log\Delta$ representing the already recognized logarithmic divergence in the Shannon entropy.

Putting all of these findings together we arrive at the following result which can be understood as the appropriate renormalization of the Shannon entropy of a compact Riemannian space; that is upon removing two ill-defined terms from it we come up with a well-defined i.e. diffeomorphism-invariant quantity:

Theorem 2.2 (cf. [6, Theorem 8.3.1]) *Let $(M,\ g)$ be an m dimensional compact oriented Riemannian manifold having unit volume and $\{A_i\}_{i=1,\dots,n}$ as above which is uniform in the sense that there exists a local coordinate system $(A_i, x^1,\dots,x^m)$ such that $\Delta = \int_{A_i} \mathrm{d}x^1 \dots \mathrm{d}x^m$ is a positive number independent of $i = 1,\dots,n$. Let $(M,\ h)$ be another Riemannian structure having unit volume too.*

Then the approximate Shannon entropy $S(M,g,\Delta)$ under the refinement of the corresponding covering behaves like

$$\lim_{\Delta\to 0}\big(S(M,g,\Delta) + I(M,g,h,\Delta) + \log\Delta\big) = -\int\limits_M \log\left(\frac{\mathrm{d}\mu_g}{\mathrm{d}\mu_h}\right)\mu_g$$

which means that the sum of three expressions including the approximate Shannon entropy and which are ill-defined separately in different ways, already converge under refinements to a well-defined expression.

The quantity $-\int_M \log\left(\frac{\mathrm{d}\mu_g}{\mathrm{d}\mu_h}\right)\mu_g$ is called the Kullback–Leibler relative entropy of $(M,\ g)$ with respect to $(M,\ h)$ regarded as an ambient fixed Riemannian structure.

□

Remark The Kullback–Leibler relative entropy is invariant under orientation-preserving diffeomorphisms of M however is *not* symmetric under $g \leftrightarrow h$; in particular it follows from the Jensen inequality that it is always non-negative and is equal to zero if and only if μ_g-almost everywhere $\mu_g = \mu_h$ holds [6, Theorem 8.6.1]. This is the case for instance if $(M,\ g)$ and $(M,\ h)$ are isometric. Moreover recalling again Moser's theorem [17] without loss of generality we can assume that for instance μ_h is equal to a once and for all fixed density $\mu_0 \in \Omega^m(M)$ having unit volume. Therefore the Kullback–Leibler relative entropy does not really depend on $(M,\ h)$ hence it can be understood as a quantity which measures the "knowledge" on various changing geometries like $(M,\ g)$ from the "viewpoint" of a once and for all fixed but otherwise arbitrary geometry like $(M,\ h)$.

The Kullback–Leibler relative entropy admits a hypersurface formulation too:

Theorem 2.3 (An analogue of Hawking's area theorem [11]) *Let M be an $m > 1$ dimensional compact oriented manifold with non-empty connected boundary ∂M and let (M, g_i) with $i = 0, 1$ be two smooth Riemannian structures on it having (non-normalized) volume-forms $\mu_i \in \Omega^m(M)$ and corresponding volumes $0 < V_i < +\infty$ respectively.*

Then, upon modifying the metric g_0 with an inessential homothety if necessary, an equality

$$-\frac{1}{V_1}\int\limits_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\right)\mu_1 + \log\frac{V_1}{V_0} = \mathrm{Area}_1(\partial M) - \mathrm{Area}_0(\partial M)$$

holds where $\mathrm{Area}_i(\partial M) = \int_{\partial M}\sigma_i$ is the area of the boundary with respect to its induced orientation and the surface-form σ_i provided by the volume-form μ_i.

Therefore taking into account the non-negativity of the left hand side as well, an inequality

$$\mathrm{Area}_1(\partial M) \geqq \mathrm{Area}_0(\partial M)$$

exists. Referring to our interpretation of the Kullback–Leibler relative entropy above, the surface area of the boundary with respect to the "unknown" geometry (M, g_1) is not smaller than the surface area of the boundary with respect to the "known" (fixed but modified with a homothety if necessary) geometry (M, g_0).

Remark

1. Before embarking upon the proof let us recall that a *homothety* is the scaling of a Riemannian metric with a positive constant i.e. $g_0 \mapsto c^2 g_0$ with an arbitrary $0 \neq c \in \mathbb{R}$. Such a constant scaling is generally considered (by both mathematicians and physicists) as irrelevant. At this level of generality an application of a homothety on (M, g_0) might be necessary in order our statements to be valid, see the proof below. However it is possible that in more restricted situations (like being (M, g_1) a spatial section "preceded by" (M, g_0) in a common ambient space-time satisfying the Einstein equation, etc.) performing homotheties turns out to be unnecessary.
2. Moreover with some technical effort the theorem in an appropriate form could be stated over non-compact manifolds as well however we skip that formulation here.

Proof Expand the relative entropy expression like

$$
\begin{aligned}
-\int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\right)\frac{\mu_1}{V_1} =& \mathrm{Area}_1(\partial M) - \int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\right)\frac{\mu_1}{V_1} - \mathrm{Area}_1(\partial M)\\
=& \mathrm{Area}_1(\partial M) - \left(\int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\right)\frac{\mu_1}{V_1} + \int_M \mathrm{Area}_1(\partial M)\frac{\mu_1}{V_1}\right)\\
=& \mathrm{Area}_1(\partial M) - \int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\mathrm{e}^{\mathrm{Area}_1(\partial M)}\right)\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\frac{\mu_0}{V_0}
\end{aligned}
$$

and consider the following Dirichlet problem:

$$
\begin{cases}
\Delta_0 u &= \frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\,\mathrm{e}^{\mathrm{Area}_1(\partial M)}\right)\\
u|_{\partial M} &= 0
\end{cases}
$$

where Δ_0 is the scalar Laplacian on (M, g_0). It is known (cf. e.g. [24, Section 5.1]) that this problem has a unique solution $u \in C^\infty(M;\mathbb{R})$ whose smoothness follows from that of g_0 and the inhomogeneous term on the right hand side. We proceed further by applying the divergence expression $\mathrm{div}X = L_X\mu_0$ where L_X is the Lie derivative along a vector field $X \in C^\infty(M;TM)$ and then Cartan's magic formula $L_X\omega = \mathrm{d}\omega(X,\cdot) + \mathrm{d}(\omega(X,\cdot))$ and finally Stokes' theorem to get

$$
\begin{aligned}
\int_M \frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\mathrm{e}^{\mathrm{Area}_1(\partial M)}\right)\frac{\mu_0}{V_0} =& \frac{1}{V_0}\int_M (\Delta_0 u)\mu_0 = \frac{1}{V_0}\int_M \mathrm{div}(\mathrm{grad}\,u)\mu_0 = \frac{1}{V_0}\int_M L_{\mathrm{grad}\,u}\,\mu_0\\
=& \frac{1}{V_0}\int_M \big(\mathrm{d}\mu_0(\mathrm{grad}\,u,\cdot) + \mathrm{d}(\mu_0(\mathrm{grad}\,u,\cdot))\big) = \frac{1}{V_0}\int_{\partial M}\mu_0(\mathrm{grad}\,u,\cdot)\\
=& \frac{1}{V_0}\int_{\partial M} g_0(N_0,\ \mathrm{grad}\,u)\sigma_0
\end{aligned}
$$

where N_0 denotes the unit normal to ∂M with respect to the orientation of M and the metric g_0.

Consider the function $g_0(N_0, \mathrm{grad}\,u) : \partial M \to \mathbb{R}$. Because u is surely not constant over M but is surely constant along ∂M we know that $\mathrm{grad}\,u \neq 0$ and is parallel with N_0 hence $g_0(N_0, \mathrm{grad}\,u)$ is a not-identically-zero function. Let $Y \in C^\infty(\partial M;T(\partial M))$ be a tangent field. If ∇ denotes the Levi–Civita connection of g_0 over M then $Yg_0(N_0, \mathrm{grad}\,u) = g_0(\nabla_Y N_0, \mathrm{grad}\,u) + g_0(N_0, \nabla_Y \mathrm{grad}\,u)$. However by the definitions of Y and N_0 they are not only orthogonal but even $\nabla_Y N_0 = 0$ holds; moreover we also find that $\nabla_Y \mathrm{grad}\,u = \mathrm{grad}(Yu) = 0$ because $u = 0$ along the boundary. Therefore we conclude that $Yg_0(N_0, \mathrm{grad}\,u) = 0$ for every tangent field hence by the connectivity of ∂M in fact $g_0(N_0, \mathrm{grad}\,u) = a$ is a non-zero constant (depending on g_1 through u as well).

We want to eliminate this constant upon applying a homothety with $0 \neq c \in \mathbb{R}$ on the metric g_0. Beyond the scaling $g_0 \mapsto c^2 g_0$ of the metric itself let us collect the induced scalings of the other things involved in the last integral too. These are $N_0 \mapsto c^{-1}N_0$ and $\mu_0 \mapsto c^m\mu_0$ hence $V_0 \mapsto c^m V_0$ however $\sigma_0 \mapsto c^{\frac{m-1}{m}}\sigma_0$. Next, the function comprising the inhomogeneous term in the Dirichlet problem above is not sensitive for the homothety on g_0 meanwhile $\Delta_0 \mapsto c^{-2}\Delta_0$; therefore $u \mapsto c^2 u$. In addition to this by definition $\mathrm{grad} f = g_0(\mathrm{d}f, \cdot)$ with g_0 here being the *inverse* metric hence scaling as $g_0 \mapsto c^{-2}g_0$; thus $\mathrm{grad} \mapsto c^{-2}\mathrm{grad}$ yielding eventually $\mathrm{grad}\, u \mapsto \mathrm{grad}\, u$. Putting all of these together the last integral on the right hand side above scales as

$$\frac{1}{V_0}\int\limits_{\partial M} g_0(N_0, \mathrm{grad}\, u)\sigma_0 \longmapsto \frac{1}{c^m V_0}\int\limits_{\partial M} c^2 g_0\big(c^{-1}N_0, \mathrm{grad}\, u\big)c^{\frac{m-1}{m}}\sigma_0$$

$$= \frac{c^{1-m}a}{V_0}\mathrm{Area}_{c^2 g_0}(\partial M)$$

demonstrating that if $\dim_{\mathbb{R}} M = m > 1$ then we can adjust the homothety so that $\frac{c^{1-m}a}{V_0} = 1$ rendering the integral in question equal to $\mathrm{Area}_{c^2 g_0}(\partial M)$. However the original integral on the left hand side is invariant under homotheties. Therefore, upon modifying the metric g_0 with a homothety $g_0 \mapsto c^2 g_0$ if necessary, we come up with the equality of the theorem.

The inequality then also follows taking into account the non-negativity of the left hand side provided by the aforementioned non-negativity of the Kullback–Leibler relative entropy (observe again the asymmetry of the relative entropy formula in its metric content!). □

After these preparations we are in a position to offer a mathematically meaningful definition of the entropy of the continuum at least in a relative way by comparing its two particular microstates within a common compact-orientable-manifold-macrostate. Let M be a compact orientable m-manifold carrying two strictly positive measures $\mu_0, \mu_1 \in \Omega^m(M)$ satisfying $\int_M \mu_i = V_i$ for $i = 0, 1$. Using $\frac{\mu_0}{V_0}$ as a fixed reference measure as so far we can make $\Omega^0(M)$, the space of smooth functions over the compact M, a Banach space $L^\infty(M, \frac{\mu_0}{V_0})$ by completing it with respect to the norm $\|f\|_{L^\infty} := \frac{\mu_0}{V_0} - \mathrm{ess\,sup}_{x\in M}|f(x)|$ for every $f \in \Omega^0(M)$. The dual space of $L^\infty(M, \frac{\mu_0}{V_0})$ is $L^1(M, \frac{\mu_0}{V_0})$ and contains $\Omega^m(M)$ because the formula $F_\omega(f) := \int_M f\omega$ by extension gives rise to a continuous linear functional on $L^\infty(M, \frac{\mu_0}{V_0})$ for every $\omega \in \Omega^m(M)$. The norm on $\Omega^m(M) \subset L^1(M, \frac{\mu_0}{V_0})$ is $\|\omega\|_{L^1} = \int_M |\omega| = \int_M |\frac{\mathrm{d}\omega}{\mathrm{d}\mu_0}|\mu_0$ and it follows that $\frac{\mu_1}{V_1}$ belongs to the unit ball in the dual space $L^1(M, \frac{\mu_0}{V_0})$. Introducing the weak*-topology on $L^1(M, \frac{\mu_0}{V_0})$ generated by the seminorms Φ_f of the form $\Phi_f(\omega) := |\int_M f\omega|$ it is easy to check that $\frac{\mu_1}{V_1} \mapsto -\int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\right)\frac{\mu_1}{V_1}$ is continuous. But by Alaoglu's theorem (e.g. [24, p. 484]) the unit ball in $L^1(M, \frac{\mu_0}{V_0})$ is compact in the weak*-topology consequently this map attains its maximum somewhere hence

$$S(M) := \sup_{\mu_1 \in \Omega^m(M)} - \int_M \log\left(\frac{\mathrm{d}\mu_1}{\mathrm{d}\mu_0}\frac{V_0}{V_1}\right)\frac{\mu_1}{V_1}$$

is a finite number. It is independent of the choice for the reference measure $\frac{\mu_0}{V_0}$ too by the aid of its diffeomorphism invariance and Moser's theorem [17]. Moreover it satisfies the (sub)additivity $S(M\#N) \leqq S(M) + S(N)$ and $S(M \times N) = S(M) + S(N)$ under taking connected sum or Descartes product, respectively. Thus collecting all of our (including interpretational, with special attention to **Assumption–mathematical form**) efforts so far we put

Definition 2.1 Let X be an abstract set whose cardinality coincides with that of the continuum in ZFC set theory. Then

$$\text{Entropy of the set } X \text{ in its compact-orientable-manifold-macrostate } M := S(M) \tag{4}$$

where $0 < S(M) < +\infty$ is the quantity introduced above.

In the case of a manifold-with-boundary we can compare (4) with Theorem 2.3 to obtain a two-sided inequality

$$0 \leqq \mathrm{Area}_1(\partial M) - \mathrm{Area}_0(\partial M) \leqq S(M) \tag{5}$$

for an arbitrary pair (M, g_0) and (M, g_1) (upon applying a homothery on the former member).

Before closing this section let us also return to the problem of the volume of the diffeomorphism group here for a moment; comparing our entropy definitions so far we come up with

$$\Gamma(\mathrm{Diff}^+(M)) := \mathrm{e}^{S(M)} < +\infty$$

as a reasonable choice at least in the compact orientable setting.

To summarize, we have sketched a framework in which the *inherent uncertainty* or *fuzziness* of the arithmetical continuum i.e. the set $\mathbb{R}$ of real numbers or more generally any differentiable manifold, can be interpreted as a *non-zero entropy* of the arithmetical continuum (cf. [20, 21]), quantitatively captured by (4) at least in the compact case.

3 Secondary Noether Theory and the Entropy of Black Holes

Keeping in mind the results of Sect. 2 and recalling now [9, Section 3] we would like to approach the formula (4) again but by passing from mathematics to physics. Namely, we shall consider classical physical theories over physical space-time such that in the mathematical description of these theories the physical space-time is modeled on a differentiable manifold possessing the property (1) or more generally (2). Then, we shall ask ourselves: does the inherent uncertainty or fuzziness of the arithmetical continuum, just recognized in the mathematical model of the physical theory, "pop up" somehow among the physical propositions of the physical theory? Putting differently: does this fuzziness somehow "lift" from the mathematical level to the physical level of the physical theory? Since we have found some similarities between this purely mathematical uncertainty or fuzziness of the arithmetical continuum and the physical concept of *entropy*, we are going to seek entropylike phenomena in those physical theories which are particularly sensitive for the physical structure of space-time. If these sought entropylike phenomena happen to have a pure set-theoretical origin introduced by the mathematical description of the physical theory, then we expect them to have something to do with *diffeomorphisms* of the underlying differentiable manifold modeling physical space-time; for the expression (4) is diffeomorphism-invariant hence the entropy it describes is invariant under diffeomorphisms. Apart from this, if diffeomorphisms are in addition *symmetries* of the physical theory we are dealing with then we may as well try to identify these entropylike things with Noether charges associated with diffeomorphism symmetry.

By *Noether's theorem* in a broad physical sense one means that "to every continuous symmetry of a physical theory a quantity can be assigned which is conserved". It may happen however that this conserved quantity, the *Noether charge*, vanishes. Our goal is to demonstrate that even in this trivial case certain non-trivial de Rham cohomology classes can still be interpreted as "secondary Noether charges" associated with this symmetry of the theory. There is an analogouos situation in algebraic topology. Consider a complex vector bundle E over a topological space X. Recall that for all $i = 0, \dots, \mathrm{rk}\, E$ the i^{th} Chern class of E takes value in $H^{2i}(X;\mathbb{Z})$. Therefore, if it happens that X has vanishing *even* dimensional singular cohomology then *characteristic classes* cannot be used to distinguish complex vector bundles over it.[2] However if X is a manifold M then one can still introduce the so-called *secondary* or *Chern–Simons characteristic classes* taking values, as a cohomological shift, in *odd* dimensional cohomology [5]. Motivated by this consideration as well as those in [7, 27] we proceed as follows.

For completeness and convenience let us recall how standard Noether theory works in case of a classical relativistic field theory. We are going to skip all the technical details here but emphasize that in case of a closed, orientable Riemannian 4-manifold all of our considerations below are rigorous mathematical statements; therefore we have a reason to expect that with appropriate technical modifications all the stuff remains valid in physically more realistic situations.

So let (N, h) be a four dimensional (non-)closed oriented (pseudo-)Riemannian manifold representing space-time and let Φ denote the full field content of a classical field theory over (N, h) defined by a Lagrangian density $L(\Phi, h) \in \Omega^4(N)$. Note that by definition the Lagrangian is not a function but a 4-form over N allowing one to talk about the corresponding action $S(\Phi, h) = \int_N L(\Phi, h)$ defined by integration over N. Let $\mathscr{N}$ be the configuration space of *all* (but belonging to a nice

[2] A simple example for this failure is provided by complex rank two vector bundles with structure group SU(2) over the 5-sphere S^5. Then, on the one hand, isomorphism classes of these type of vector bundles are classified by the group $\pi_4(\mathrm{SU}(2)) \cong \pi_4(S^3) \cong \mathbb{Z}_2$ hence there are precisely two different such bundles up to isomorphism over S^5; meanwhile, on the other hand, $H^k(S^5;\mathbb{Z}) \cong \{0\}$ if $k \neq 0, 5$ hence all Chern classes of these bundles are trivial.

function class) (Φ, h)-field configurations over N i.e. its elements are *not* identified by diffeomorphisms, gauge, etc. transformations. Consider a differentiable curve $C : \mathbb{R} \to \mathscr{N}$. We say that C is a *symmetry* of the theory $L(\Phi, h)$ if its action $S(\Phi, h)$ is constant along C that is, $S(C(t)) = S(\Phi(t), h(t))$ =const. for all $t \in \mathbb{R}$. Writing $(\Phi, h) := (\Phi(0), h(0))$ and using physicists' usual notation define "the infinitesimal variation of the action at (Φ, h) along C" by

$$\delta_C S(\Phi, h) := \lim_{t\to 0} \frac{1}{t}(S(C(t)) - S(C(0)))$$
$$= \int_N \lim_{t\to 0} \frac{1}{t}(L(C(t)) - L(C(0))) =: \int_N \delta_C L(\Phi, h)$$

where $\delta_C L(\Phi, h) \in \Omega^4(N)$ is the "infinitesimal variation of the Lagrangian at (Φ, h) along C".[3] Assume that (N, h) is smooth and let $\Delta_h := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$ denote its Hodge Laplacian; recall [24, Chapter 5] that if $\omega \in \Omega^4(N)$ the partial differential equation $\Delta_h \varphi = \omega$ has a smooth solution φ if and only if $\int_N \omega = 0$. By definition of a symmetry $\int_N \delta_C L(\Phi, h) = 0$ hence there exists an element $\varphi_C \in \Omega^4(N)$ satisfying $\Delta_h \varphi_C = \delta_C L(\Phi, h)$. However $\Delta_h \varphi_C = \mathrm{dd}^* \varphi_C + \mathrm{d}^*\mathrm{d}\varphi_C = \mathrm{dd}^* \varphi_C$ consequently picking any $\eta_C \in \Omega^2(N)$ and putting

$$\theta_C := \mathrm{d}^* \varphi_C + \mathrm{d}\eta_C \tag{6}$$

we succeeded to find an element $\theta_C \in \Omega^3(N)$ such that $\mathrm{d}\theta_C = \delta_C L(\Phi, h)$. Note first that, although φ_C is well-defined only up to a harmonic 4-form i.e. an element $\varphi \in \ker \Delta_h$, the 3-form θ_C is not sensitive for this ambiguity because taking into account its harmonicity, the general solution φ above is both closed ($\mathrm{d}\varphi = 0$) and co-closed ($\mathrm{d}^*\varphi = 0$) hence $\theta_C = \mathrm{d}^*\varphi_C + \mathrm{d}\eta_C = \mathrm{d}^*(\varphi_C + \varphi) + \mathrm{d}\eta_C$. Secondly, the "gauge freedom" i.e. the η_C-ambiguity can be fixed as well by imposing the "Coulomb gauge condition" $\mathrm{d}^*\theta_C = 0$. Indeed, $\mathrm{d}^*\theta_C = \mathrm{d}^{*2}\varphi_C + \mathrm{d}^*\mathrm{d}\eta_C = \mathrm{d}^*\mathrm{d}\eta_C = 0$ (together with the Hodge decomposition theorem) implies $\mathrm{d}\eta_C = 0$. Therefore, given a symmetry C of the theory we come up with a $\theta_C \in \Omega^3(N)$ which satisfies

$$\begin{cases} \mathrm{d}\theta_C & = \delta_C L(\Phi, h) \\ \mathrm{d}^*\theta_C & = 0 \\ \int_N \mathrm{d}\theta_C & = 0 \end{cases}$$

along N and this 3-form is well-defined in the sense that it is unique and depends only on the symmetry represented by the curve C as expected.[4]

[3] Using standard notations of differential geometry if $\dot{C}$ is the derivative of C at $t = 0$ then $\delta_C L = \dot{C}(L) : \mathscr{N} \to \Omega^4(N)$. By specializing the variation further we can demand $\delta^2 = 0$ hence we can formally treat δ as an exterior derivative on the infinite dimensional manifold $\mathscr{N}$ [7, and references therein] and can introduce the $\Omega^4(N)$-valued 1-form δL on $\mathscr{N}$. Then $\delta_C L = \delta L(\dot{C}) : \mathscr{N} \to \Omega^4(N)$ as well.

[4] If $(\Phi_0, h_0) \in \mathscr{N}$ is a *critical point* of the action then $\delta_C S(\Phi_0, h_0) = \int_N \delta_C L(\Phi_0, h_0) \delta_C S(\Phi_0, h_0) = \int_N \delta_C L(\Phi_0, h_0) = \int_N \delta L(\Phi_0, h_0)(\dot{C}) = 0$ for *all* curves passing through the critical point. Hence $\int_N \delta L(\Phi_0, h_0)(\dot{C}) = 0$ for all $\dot{C} \neq 0$ thus in fact $\delta L(\Phi_0, h_0) = 0$ which is the resulting *field equation* of the theory.

Proceeding further, we call the Hodge dual 1-form $j_C := *_h \theta_C \in \Omega^1(N)$ the *Noether current* associated with the symmery C moreover for a (spacelike) hypersurface-without-boundary $S \subset N$ put $q_{C,S} := \int_S *_h j_C$ and call it the *Noether charge* associated with the symmetry C. The Noether charge satisfies

$$q_{C,S_1} - q_{C,S_2} = \pm \int\limits_{W(S_1,S_2)} \mathrm{d}\theta_C = 0$$

by applying Stokes' theorem on a domain $W(S_1, S_2) \subsetneqq N$ with induced orientation and oriented boundary $\partial W(S_1, S_2) = S_1 \sqcup (-S_2)$. (Here we strictly speaking assume that the variation vanishes on the complementum $N \setminus W(S_1, S_2)$ hence $\int_{W(S_1,S_2)} \mathrm{d}\theta_C = \int_N \mathrm{d}\theta_C = 0$ indeed.) Consequently the real number $q_C := q_{C,S}$ is a well-defined conserved (i.e. independent of the spacelike surface S) quantity associated with the symmetry of the theory in this sense.

Thanks to the gauge fixing condition $\mathrm{d}^*\theta_C = 0$ the Noether current looks like $j_C = *_h \theta_C = \pm \mathrm{d} *_h \varphi_C$ consequently $j_C \in [0] \in H^1(N)$ i.e. the current represents the trivial cohomology class in the first de Rham cohomology. We may then ask ourselves what about θ_C from the de Rham theoretic viewpoint? Does it represent a cohomology class in $H^3(N)$? Still working in the gauge $\mathrm{d}^*\theta_C = 0$, assume $\mathrm{d}\theta_C = 0$ holds; then via (6) we get $\Delta_h \varphi_C = 0$ implying φ_C is harmonic hence $\theta_C = \mathrm{d}^*\varphi_C = 0$. Therefore we find that $\mathrm{d}\theta_C = 0$ if and only if $\theta_C = 0$. Consequently $\theta_C \in \Omega^3(N)$ represents a cohomology class $[\theta_C] \in H^3(N)$ if and only if $\theta_C = 0$ and the associated Noether charge $q_C = \int_S *_h j_C = \pm \int_S \theta_C = 0$ is trivial rendering the classical Noether theory useless in this situation.

Let us focus attention to this trivial case i.e. when for a symmetry C of the theory $L(\Phi, h)$ the associated total derivative satisfies $0 = [\theta_C] \in H^3(N)$ (by exploiting the gauge fixing condition $\mathrm{d}^*\theta_C = 0$ too). The gauge fixing also implies $\mathrm{d}^*\varphi_C = 0$ as we have seen hence the general expression (6) reduces to

$$0 = \mathrm{d}\eta_C$$

saying that η_C itself represents a cohomology class in $H^2(N)$. Consequently in this situation—which is trivial from the variational viewpoint in the sense that it yields vanishing primary Noether theory, but not trivial from the topological viewpoint in the sense that $H^2(N) \ncong \{0\}$ may hold—we can still introduce a *secondary* or *topological Noether current* $J_C \in \Omega^2(N)$ by putting $J_C := *_h \eta_C$. Then taking any two dimensional submanifold-without-boundary $\Sigma \subset N$ the corresponding *secondary* or *topological Noether charge* $Q_{C,\Sigma} := \int_\Sigma *_h J_C$ is well-defined in the sense that it depends only on the chosen de Rham cohomology class $[*_h J_C] \in H^2(N)$. Moreover

$$Q_{C,\Sigma_1} - Q_{C,\Sigma_2} = \pm \int\limits_{W(\Sigma_1,\Sigma_2)} \mathrm{d}\eta_C = 0$$

by Stokes' theorem as before. (This time $W(\Sigma_1, \Sigma_2) \subset N$ is a sub-3-manifold with induced orientation and oriented boundary $\partial W(\Sigma_1, \Sigma_2) = \Sigma_1 \sqcup (-\Sigma_2)$.) Consequently we have a conserved quantity in the sense that the number $Q_{C,\Sigma} =: Q_{C,[\Sigma]}$

depends only on $[*_h J_C] \in H^2(N)$ and the singular homology class $[\Sigma] \in H_2(N;\mathbb{Z})$. Although it is not necessary, just for aesthetical reasons we can suppose without loss of generality that η_C is the unique harmonic represnentative of $[\eta_C]$ hence both η_C and $J_C =*_h \eta_C$ are closed that is, represent cohomology classes within $H^2(N)$.

Note that, regardless what the symmetry C actually is, in order $Q_{C,[\Sigma]}$ not to be trivial i.e., $Q_{C,[\Sigma]} \neq 0$, we need $[0] \neq [\Sigma] \in H_2(N;\mathbb{Z})$ as well as $[0] \neq [*_h J_C] \in H^2(N)$. Both conditions are met if we demand N to satisfy the topological condition that the free part of its second singular homology group $H_2(N;\mathbb{Z})_{\text{free}} \cong \mathbb{Z}^{\operatorname{rk} H_2(N;\mathbb{Z})}$ be non-zero (i.e., the rank of $H_2(N;\mathbb{Z})$ be non-zero). Moreover, at this level of generality all cohomology classes in $H^2(N)$ are permitted to play the role of $[*_h J_C]$ consequently the number of linearly independent secondary Noether currents is equal to $b^2(N)$. Therefore if $\operatorname{rk} H_2(N;\mathbb{Z}) > 1$ that is, $b^2(N) > 1$ then we have "too many" options to introduce non-trivial secondary Noether charges for a given symmetry. Consequently, the optimal situation for this secondary theory is when $\operatorname{rk} H_2(N;\mathbb{Z}) = 1$ (at this level of generality).

To summarize, we have proved:

Lemma 3.1 (cf. [9, Lemma 3.1]) *Let* (N, h) *be a (non-)closed oriented (pseudo-) Riemannian* 4*-manifold satisfying the topological condition* $H_2(N;\mathbb{Z})_{\text{free}} \ncong \{0\}$. *Let moreover a classical relativistic field theory be given over* (N, h) *defined by its Lagrangian density* $L(\Phi, h) \in \Omega^4(N)$. *Assume that* $C : \mathbb{R} \to \mathcal{N}$ *is a symmetry of the theory such that the corresponding total derivative* $\theta_C \in \Omega^3(N)$ *satisfying the gauge fixing condition* $\mathrm{d}^*\theta_C = 0$ *is closed i.e.* $\mathrm{d}\theta_C = 0$ *(hence* $\theta_C = 0$ *therefore in fact* $[\theta_C] = 0 \in H^3(N)$*).*

Then there exist a 2*-form* $0 \neq J_C \in \Omega^2(N)$ *representing a non-trivial de Rham cohomology class* $0 \neq [*_h J_C] \in H^2(N)$ *as well as a closed oriented surface* $\Sigma \subset N$ *representing a non-trivial singular homology class* $[0] \neq [\Sigma] \in H_2(N;\mathbb{Z})_{\text{free}}$ *such that the associated quantity* $Q_{C,\Sigma} := \int_\Sigma *_h J_C \in \mathbb{R}$ *is not zero and depends only on* $[*_h J_C] \in H^2(N)$ *and* $[\Sigma] \in H_2(N;\mathbb{Z})_{\text{free}}$. *We denote this quantity by* $Q_{C,[\Sigma]}$ *and call the secondary or topological Noether charge associated with the symmetry C and the classes* $[*_h J_C]$ *and* $[\Sigma]$. *For a given symmetry C the number of linearly independent cohomology classes* $[*_h J_C]$ *is equal to* $\operatorname{rk} H_2(N;\mathbb{Z})$. □

Note that what we have done is in fact simple: We interpret the *a priori* existing cohomology classes of N as certain physical quantities whenever a theory $L(\Phi, h)$, possessing certain type of symmetries, has been formulated over N.

Let us apply this theory for diffeomorphisms in pure gravity in four dimensions. Let $\nu_h \in \Omega^4(N)$ defined by $\nu_h :=*_h 1$ be the volume-form of (N, h) and take the usual Einstein–Hilbert Lagrangian $L_{EH}(h) := (\mathrm{Scal}_h - 2\Lambda)\nu_h$ with cosmological constant $\Lambda \in \mathbb{R}$ and consider its variation with respect to a 1-parameter subgroup $\{f_t\}_{t\in\mathbb{R}}$ of the orientation-preserving diffeomorphism group $\mathrm{Diff}^+(N)$ of the underlying space-time manifold N while the metric h is kept fixed. That is we define our curve by $C(t) := f_t^* h \in \mathcal{N}$ for all $t \in \mathbb{R}$. The infinitesimal generator of this 1-parameter subgroup is a compactly supported vector field $X \in C_c^\infty(N;TN)$.

Since a diffeomorphism acts on k-forms via pullback and the *scalar* curvature is invariant under diffeomorphisms, $L_{EH}(C(t)) = f_t^* L_{EH}(f_t^* h) = f_t^* L_{EH}(h)$ hence the corresponding infinitesimal variation takes the shape

$$\delta_C L_{EH}(h) = \lim_{t\to 0} \frac{1}{t}(1 - f_t^*)L_{EH}(h) = L_X(L_{EH}(h))$$

where L_X denotes the Lie derivative with respect to X. Substituting the Lagrangian and applying Cartan's formula we thus get

$$\begin{aligned}\delta_C L_{EH}(h) &= \mathrm{d}\big((\mathrm{Scal}_h - 2\Lambda)\nu_h\big)(X,\ \cdot) + \mathrm{d}\big((\mathrm{Scal}_h - 2\Lambda)\nu_h(X,\cdot)\big)\\ &= \mathrm{d}\big((\mathrm{Scal}_h - 2\Lambda)\nu_h(X,\cdot)\big).\end{aligned}$$

While $\delta_C L_{EH}(h) \neq 0$ in general, nevertheless we find that $\delta_C S_{EH}(h) = \int_M \mathrm{d}\big((\mathrm{Scal}_h - 2\Lambda)\nu_h(X,\cdot)\big) = 0$ by Stokes' theorem hence the Einstein–Hilbert action itself is invariant consequently diffeomorphisms are both off or on shell symmetries of general relativity with possibly non-vanishing cosmological constant. The associated total derivative up to an exact term looks like $\theta_C = (\mathrm{Scal}_h - 2\Lambda)\nu_h(X,\cdot)$ in some gauge probably *not* satisfying the condition $\mathrm{d}^*\theta_C = 0$.

In order not to get lost in the gauge fixing problem assume instead that (i) we are *on shell* i.e., Einstein's equation $\mathrm{Ric}_h = \Lambda h$ is valid hence $\theta_C = (4\Lambda - 2\Lambda)\nu_h(X,\cdot) = 2\Lambda\nu_h(X,\cdot)$ and (ii) the cosmological constant vanishes yielding $\theta_C = 0$. Consequently *the diffeomorphism symmetry in on shell pure gravity with vanishing cosmological constant has vanishing associated (primary) Noether charge*. However substituting $\theta_C = 0$ into (6) (and referring to the Hodge decomposition theorem) we get $\mathrm{d}^*\varphi_C = 0$ and $\mathrm{d}\eta_C = 0$ consequently in this physically important situation we can interpret the cohomology classes $[\eta_C] \in H^2(N)$ as Hodge duals of currents J_C in secondary Noether theory.

As we stressed even in the formulation of Lemma 3.1, interesting secondary Noether theory emerges only if the underlying manifold is topologically non-trivial in the sense formulated there. At this point, by taking e.g. a survey on known solutions [22], we make an observation which is completely independent of our considerations taken so far—hence in our opinion is very interesting!—namely: *apparently all explicitly known 4 dimensional black hole solutions in vacuum general relativity with vanishing cosmological constant satisfy the topological condition formulated in Lemma* 3.1. This intuitively means that because of some general reason a black hole is even topologically recognizable as a two dimensional "hole" in space-time. In fact with an appropriate restriction this observation can be proved [8] and can be considered as a global topological counterpart of well-known black hole uniqueness theorems [14]. In accordance with this provable version [8] we suppose from now on that: (N, h) is a 4 dimensional solution of the Einstein's equation $\mathrm{Ric}_h = 0$ and describes a single stationary asymptotically flat black hole; hence $\mathrm{rk}\, H_2(N;\mathbb{Z}) = 1$ (which apparently corresponds to the case that a "single" black hole is present). In this case the homology class of the "instantaneous" event horizon of the black hole as an (immersed) surface $i : \Sigma \looparrowright N$ represents a non-zero element $[\Sigma] \in H_2(N;\mathbb{Z})_{\mathrm{free}} \cong \mathbb{Z}$.

Then we proceed as follows: like the original volume-form $v_h \in \Omega^4(N)$, the induced 2 dimensional area-form $\sigma_h \in \Omega^2(\Sigma)$ of the "instantaneous" event horizon Σ is closed consequently it represents a class $[\sigma_h] \in H^2(\Sigma)$ which is not zero since the event horizon has finite area. Then exploiting singleness and stationarity, the "instantaneous" event horizon Σ is connected and its area $\mathrm{Area}_h(\Sigma) = \int_\Sigma \sigma_h$ is constant in time consequently we can suppose that the area form is proportional to the Hodge dual of the secondary Noether current with a time-independent constant. In other words with any choice $J_C \in \Omega^2(M)$ for the Hodge dual of the secondary Noether current the σ_h satisfies that const. $[i^*(*_h J_C)] = [\sigma_h] \in H^2(\Sigma) \cong \mathbb{R}$ therefore

$$\text{Entropy of the black hole in } (N,h) = \text{const.}\,\mathrm{Area}_h(\Sigma) = \text{const.}\int_\Sigma \sigma_h = \text{const.}\int_\Sigma *_h J_C$$
$$= \text{const.}\, Q_{C,[\Sigma]}$$

offering a natural way to normalize $Q_{C,[\Sigma]}$ to be *equal* to the entropy of the black hole i.e.

$$\text{Entropy of the black hole in } (N,h) = Q_{C,[\Sigma]} = \text{const.}\,\mathrm{Area}_h(\Sigma). \tag{7}$$

Accepting this choice of normalization therefore a natural physical interpretation of this abstract secondary or topological conserved quantity also emerges, namely: *if a 4 dimensional space-time (N, h) is a solution of the vacuum Einstein's equation with vanishing cosmological constant and describes a single stationary asymptotically flat black hole then the secondary Noether charge associated with the orientation-preserving diffeomorphism invariance of general relativity is not zero and as a secondary conserved quantity is equal to the entropy of the black hole* (cf. [7, 27]). Thus the point is that in this way black hole entropy is conceptionally connected with the invariance of the underlying differentiable manifold against its own diffeomorphisms.

We are now in a position to make our crucial observation: both the introduced set-theoretic entropy (4) previously (as the entropy of an abstract set in its manifold-macrostate) and the black hole entropy (7) here (as the Noether charge for the invariance of a stationary black hole space-time under diffeomorphisms) give rise to conserved quantities assigned to one and the same process namely the permutation of the points of differentiable manifolds by diffeomorphisms. This manifests itself in their common diffeomorphism invariance. However the two quantities, namely (4) and (7), even more resemble each other if the former is computed over a 3-manifold-with-boundary M, cast in a form of differences between areas via Theorem 2.3 yielding the two-sided inequality (5); and then this manifold-with-boundary M is inserted as a spacelike submanifold (M, g_1) into a space-time 4-manifold (N, h) containing a (not necessarily stationary or single) black hole such that $\partial M = \Sigma$ corresponds to the "instantaneous" event horizon whose area $\mathrm{Area}_h(\Sigma)$ is therefore equal to $\mathrm{Area}_1(\partial M)$. Thus (5) written as $\mathrm{Area}_0(\partial M) \leqq \mathrm{Area}_1(\partial M) \leqq \mathrm{Area}_0(\partial M) + S(M)$ qualitatively implies

$$\begin{aligned}&\text{Entropy of the black hole in } (N, h) \text{ with “instantaneous” event horizon } \partial M\\&\approx \text{const.} + \text{ Entropy of } X \text{ in its manifold-macrostate } M \subset N\end{aligned} \quad (8)$$

where const. $\approx \text{Area}_0(\partial M)$ with respect to some reference spacelike submanifold $(M, g_0) \subset (N, h)$ as in Sect. 2 throughout. Thus (8) can be regarded like the decomposition of the physical black hole entropy as mathematically appears in general relativity into the sum of an already recognized pure geometric term proportional to $\text{Area}_0(\partial M)$ and an unexpected other term proportional to $S(M)$ describing the inherent set-theoretic fuzziness of the mathematical model underlying general relativity. Since the right hand side of (8) contains the positive quantity (4) it follows that its left hand side (7) i.e. the entropy of a black hole cannot decrease in accord with Hawking's area theorem [11].

The time has come to complete the circle of our arguments. On the mathematical side, the formal concept of the arithmetical continuum (or the set $\mathbb{R}$ of real numbers) contains a tacit uncertainty or fuzziness in the sense that the effective identification of the arithmetical continuum with its individual disjoint constituents, the *points*, cannot be carried out (our interpretation of Theorem 2.1). On the physical side, the nowadays accepted mathematical formalization of our intuitive concept of the spatial or temporal continuum in terms of the arithmetical continuum lifts the purely formal—and concerning its Leibnizian monadistic origin, metaphysical—entity, the same *point* again, to an ontological level. We may then ask ourselves whether or not this sort of description of space-time in a mathematical model of a physical theory introduces a similar uncertainty or fuzziness into the physical theory. Let us formulate our question more carefully. In the modern understanding by a *physical theory* one means a two-level description of a certain class of natural phenomena: the theory possesses a *syntax* provided by its mathematical core structure and a *semantics* which is the meaning i.e. interpretation of the bare mathematical model in terms of physical concepts. Consider a physical theory whose semantics contains a description of space and time (like general relativity) and its syntax uses the arithmetical continuum to mathematically model the thing which corresponds to the space-time continuum at the semantical level of the theory (like general relativity). Then we may ask whether or not the uncertainty or fuzziness recognized at the syntactical level of the physical theory (introduced by the utilization of the arithmetical continuum) shows up at the semantical level of the physical theory too. To answer this for a given physical theory, one has to search among those physical concepts which describe uncertainty, fuzziness, or *disorder* at the semantical level and check their counterparts at the syntactical level. Of course the basic physical concept of this kind is the *entropy*.

Therefore, in this context, one can be concerned whether or not entropy within classical general relativity, appearing in its semantics in the form of black hole entropy [2, 11], simply comes from its syntax i.e. has a pure mathematical origin only (hence probably not corresponding to any "objective" thing in the world)? This suspicion is also supported in some extent by the several controversial (geometrical, thermodynamical, quantum and information theoretic, etc.) properties of one and the same thing, the black hole entropy. Our analysis of the black hole entropy formula in general relativity culminating in its formal factorization (8) into the sum of a geometric term and a pure set-theoretic term points at least *in part* towards a set-theoretic origin. That is, even if the geometric term in (8) indeed corresponds to the "physical part" of black hole entropy, the next term could be a "pure mathematical" or more precisely a "pure set-theoretic" contribution only. This could be an example how certain physical statements within the physical theory of general relativity get "contaminated" by the underlying mathematical model akin to the situation in quantum field theory [1].

However, beyond the "balanced" interpretation above, if one prefers one can read (8) in two extreme ways as well, going into exactly the opposite directions as follows. The first is that black hole entropy is of pure mathematical origin without any physical content hence e.g. the long-sought as well as quite problematic physical degrees of freedom responsible for black hole entropy (far from being complete cf. e.g. [23]) would in fact be not physical at all but would simply coincide with the purely "mathematical degrees of freedom" of the *point constituents* of the arithmetical continuum used to formulate general relativity mathematically. We have to acknowledge that as long as the physical origin of black hole entropy is not confirmed by the experimental discovery of e.g. black hole radiation [12, 25] or other thermal phenomena, this possibility cannot be *a priori* refuted. The second extreme reading is that black hole entropy is of pure physical origin without any mathematical content which arises if one rather wishes to accept the physical origin of black hole entropy (and e.g. looks forward its experimental discovery). Then (8) can be interpreted as an argument for the "physical origin" (cf. Heisenberg uncertainity) of what we have called the inherent fuzziness or uncertainty within the set of real numbers [20, 21]. This interpretation then could explain the expected independence of the set-theoretic entropy (4) of any axiomatic system S which is in accordance with the universal character of Theorem 2.1.

Appendix: A Temporal Approach to General Relativity

In this closing section we make an attempt to introduce temporality into general relativity by "recycling" the conceptional and technical apparatus introduced in Sects. 2 and 3. Quickly acknowledging that our efforts are certainly unsatisfactory yet, there is no doubt that we face a deep problem here taking into account the current paradoxical situation that although general relativity is the most advanced presently known physical theory about the structure of physical space and time, it is in fact a timeless theory due to its huge, namely diffeomorphism invariance (cf. [26, Appendix E]). For an excellent summary of the "problem of time" in general relativity see [3, Chapter 3]. Hence a reconciliation is certainly necessary.

However before sinking in technical details let us make some general remarks concerning the complex, diverse tension between the so-called manifest time and physical time. As we have recalled in Sect. 1 and elaborated in Sects. 2 and 3 in detail the division of the spatio-temporal continuum into disjoint points is conceptionally problematic. Encounters with quantum phenomena (in the form of *Heisenberg*'s uncertainty relations or various *EPR* phenomena, for instance) dictate that the physical space cannot be the simple union of extensionless disjoint empty “places”; likewise and probably even more embarassingly psycho-physiological evidences (cf. e.g. [3, Chapters 8 and 9] and the hundreds of related references therein) indicate that physical time cannot be the simple union of our “nows”. Indeed, nothing in the outer physical world seems to correspond to a (conscious) observer’s inner now-experience. As *Carnap* recalls a personal conversation ([19, pp. 37–38]):

> Once Einstein said that the problem of the Now worried him seriously. He explained that the experience of the Now means something special for man, something essentially different from the past and the future, but that this important difference does not and cannot occur within physics. That this experience cannot be grasped by science seemed to him a matter of painful but inevitable resignation. I remarked that all that occurs objectively can be described in science; on the one hand the temporal sequence of events is described in physics; and, on the other hand, the peculiarities of man’s experiences with respect to time, including his different attitude towards past, present, and future, can be described and (in principle) explained in psychology. But Einstein thought that these scientific descriptions cannot possibly satisfy our human needs; that there is something essential about the Now which is just outside of the realm of science.

What is so essential about the “now”? One can effectively analyse the content of a conscious observer’s now-experience for instance within the framework of *Husserl*’s phenomenology (see e.g. [15]). Phenomenologically (or rather very roughly) speaking it resembles a thick whirling cloud of strongly interacting elements whose primary representatives are the immediately given sensations of outer or inner events, various intentions, reflections and (spontaneously) recalled memories from the past constituting the “now” itself; however these primary contents have already been the sources for further primary sensations, the targets of further iterated intentions, reflections or have been simply shifting towards the past as memories, etc., etc., i.e. are already secondary contents as well carrying the characters of the past; in this way furnishing the now-experience with a sort of persistent standing-flowing feature. Or as *Weyl* (who actually studied philosophy under *Husserl* in his young ages) probably more clearly puts it [29, p. 92]:

> What I am conscious of is for me both a being-now and, in its essence, something which, with its temporal position, slips away. In this way there arises the persisting factual existent, something ever new which endures and changes in consciousness. What disappears can reappear; not, of course, as an experience which I have over again, but as content of an (accurate) memory, having become something past. In the objective picture which I form of the course of the life , such a past thing is to be opposed to what presently is as something earlier. So we can gather the following concerning objectively presented time:

1. an individual point in it is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a "point of transition" (which, of course, can in no way be understood mathematically);
2. it is due to this essence of time (and not to contingent imperfections in our medium) that a fixed time-point cannot be exhibited in any way, that always only an *approximate*, never an *exact* determination is possible.

Thus apparently at least in part the essential about the now as an experience is its standing-flowing, slipping-away or let us say its persisting incoming-outgoing character. This latter characterization might be more clearly formulated by saying that the "now" equally shares the properties of the future and the past; and if the future is fundamentally different in its nature from the past then the "now" is indeed something very difficult to grasp.

We proceed further along these lines because it is a basic experience that the future and the past are indeed very different: for example, one cannot remember the future but can remember the past. In an early paper [28] *von Weizsäcker* suggests that this is because the *future* consists of physical *possibilities* but the *past* consists of physical *facts* (this is a refinement of the homogeneous timeless concept, the physical *event* of relativity theory) and calls this dichotomy as the *chronologicality of time*. Thus the appropriate mathematical theory to model the open future would be probability theory while to model the fixed past would be geometry. Therefore, accepting this, in order to grasp the phenomenon of the "now" mathematically one should seek a mathematical structure in which probability and geometry are in conjunction. It is very interesting that a structure of this kind exists and in fact is well-known: a *Riemannian manifold*. Indeed, a (compact oriented) Riemannian manifold both accommodates a geometric structure by its metric and a Kolmogorov probability space by the normalized positive volume-form of the same metric. It comes as a further surprise that exactly the same mathematical structure, the (pseudo-)Riemannian manifold plays a central role in formulating general relativity mathematically. Therefore it seems that without effort and in fact in a very conservative way we can reach our aim how to reconcile temporality and general relativity by revisiting the role of the Riemannian structure in this theory.

But if we want probability theory to enter the game too then the first question is how to interpret physically the probabilities here? The space-time continuum is modeled by a Lorentzian 4-manifold (N, h) in general relativity and its points are physically interpreted as physical events (without further specification at this level of generality). The relative positions between these points i.e. the physical events then can be examined with the tools of Riemannian geometry. In particular a Riemannian sub-3-manifold $(M, g) \subset (N, h)$ can be identified with the space in an "instant" and its points are the right-now-occuring physical events in general relativity which on the one hand admit usual characterization in terms of the Riemannian structure (M, g) such as lengths, area, curvature, etc. describing their factual or "standing" properties. In addition consider the Kolmogorov probability space $(M, \mathscr{A}_g, \frac{1}{V_g}\mu_g)$ induced by (M, g) too and if $A \in \mathscr{A}_g$ put $p_g(A) := \frac{1}{V_g}\int_A \mu_g$. If

$x_0 \in M$ is a distinguished physical event then its sharp localization is not possible; thus in accord with the **Assumption–mathematical form** we put

Assumption–physical form. *The number* $0 \leqq p_g(A) \leqq 1$*is the probability that a distinguished right-now-occuring physical event* $x_0 \in M$*appears in the spatial region* $A \subseteqq M$.

Thus, on the other hand, the same right-now-occuring physical events admit further characterization in terms of the probability structure $(M, \mathscr{A}_g, \frac{1}{V_g}\mu_g)$ hence one can talk about their appearance probabilities in spatial regions, their expectation values, etc. too capturing their possibilitial or "flowing" properties as well.

Let (N, h) be a globally hyperbolic 4 dimensional space-time; then recall (cf. e.g. [26, Chapter 10]) that there exists a spacelike sub-3-manifold $M \subset N$ called a *Cauchy surface* with induced Riemannian metric $g := h|_M$ such that $(N, h) = D^-(M, g) \cup D^+(M, g)$ i.e. (N, h) can be written as the union of the past and future domain of dependencies of (M, g) and obviously $(M, g) = D^-(M, g) \cap D^+(M, g)$.

Definition 4.1 Let (N, h) be a globally hyperbolic Lorentzian 4-manifold or called simply a space-time and assume that it admits a Cauchy surface $M \subset N$ which is compact and oriented. (This implies that if $g := h|_N$ denotes the induced Riemannian metric then (M, g) is complete moreover if $V_g := \int_M \mu_g$ is its volume then $0 < V_g < +\infty$.) Consider the induced decomposition $(N, h) = D^-(M, g) \cup D^+(M, g)$.

The triple

$$\left(D^-(M, g), (M, g), \left(M, \mathscr{A}_g, \frac{1}{V_g}\mu_g\right)\right)$$

is a chronological space-time where $D^-(M, g) \subset (N, h)$ is a Lorentzian 4-manifold equal to the past domain of dependence of (M, g) within (N, h) and is called the past, (M, g) is a compact oriented Riemannian 3-manifold and called the presence and $\left(M, \mathscr{A}_g, \frac{1}{V_g}\mu_g\right)$ is the Kolmogorov probability measure space induced by (M, g) and called the future.

A chronological space-time $\left(D^-(M_i, g_i), (M_i, g_i), \left(M_i, \mathscr{A}_i, \frac{1}{V_i}\mu_i\right)\right)$ with $i = 0$ precedes the other one with $i = 1$ if there exists an isometric embedding $D^-(M_0, g_0) \subseteqq D^-(M_1, g_1)$ of their pasts as Lorentzian 4-manifolds and they are equivalent if both precede each other i.e. if both $D^-(M_0, g_0) \subseteqq D^-(M_1, g_1)$ and $D^-(M_1, g_1) \subseteqq D^-(M_0, g_0)$ holds hence $D^-(M_0, g_0)$ and $D^-(M_1, g_1)$ are isometric.

Remark

1. We conclude with a few comments on this definition. Compared with the usual definition of a space-time as a whole and first of all *timeless* entity in general relativity, in a chronological space-time this timeless structure has been broken

up from the point of view of an observer into a more familiar past-presence-future *temporal* structure. This is achieved simply by replacing the $\left(D^-(M,g),(M,g),D^+(M,g)\right)$ decomposition of (N, h) with $\left(D^-(M,g),(M,g),\left(M,\mathscr{A}_g,\frac{1}{V_g}\mu_g\right)\right)$. As a subtlety of the definition note that the present (M, g) does not determine the past $D^-(M,g)$ because to obtain it one needs the *a priori* knowledge on the ambient space-time (N, h). However at least in the case of a *compact oriented* Cauchy surface M (a purely technical requirement) the present (M, g) does uniquely detemine the future $\left(M,\mathscr{A}_g,\frac{1}{V_g}\mu_g\right)$; this follows from a theorem of Moser [17] stating that the space $(M,\frac{1}{V_g}\mu_g)$ is independent of the particular (M, g) providing the unit-volume density by its normalized volume-form $\frac{1}{V_g}\mu_g$. This directional (in)dependence is therefore the manifestation of a sort of temporal openness of a chronological space-time towards the future.

2. Nevertheless, in spite of this future-openness, a globally hyperbolic space-time uniquely determines its associated chronological space-time. More precisely, the equivalence of two chronological space-times as defined here implies that their pasts $D^-(M_0,g_0)$ and $D^-(M_1,g_1)$ are isometric; moreover by its compactness the Cauchy surface M_0 is diffeomorphic to the other one M_1 hence even their futures are isomorphic as probability spaces again by Moser's theorem [17]. Consequently this definition meets the demand for introducing temporality into general relativity in a covariant i.e. diffeomorphism-invariant way, found to be non-fullfillable in general cf. e.g. [3, Sections 2.3–2.6]. However to gain this covariance the technical assumption that the Cauchy surface M be compact as in Definiton 4.1 is essential. For example, there exists a space-time diffeomorphic to $\mathbb{R}^4$ admitting Cauchy surfaces which are even not homeomorphic [18]. One of them is the standard $\mathbb{R}^3$ while the other is a so-called Whithead space [30]: an open contractible 3-manifold W which is not homeomorphic (hence not diffeomorphic) to $\mathbb{R}^3$; there are uncountable many pairwise non-homeomorphic Whitehead spaces but it is known [16] that every Whithead space W satisfies that the product $W\times\mathbb{R}$ is always diffeomorphic to $\mathbb{R}^4$. Consequently, given this space-time, its two associated chronological space-times of this sort might have non-isomorphic futures.
3. This framework permits one to exhibit a mathematical model for coming into existence. One can say that within a given chronological space-time $\left(D^-(M,g_0),(M,g_0),\left(M,\mathscr{A}_0,\frac{1}{V_0}\mu_0\right)\right)$ the *occurence* of a new geometry (M,g_1) is a chronological space-time $\left(D^-(M,g_1),(M,g_1),\left(M,\mathscr{A}_1,\frac{1}{V_1}\mu_1\right)\right)$, if exists, such that $\left(D^-(M,g_0),(M,g_0),\left(M,\mathscr{A}_0,\frac{1}{V_0}\mu_0\right)\right)$ precedes $\left(D^-(M,g_1),(M,g_1),\left(M,\mathscr{A}_1,\frac{1}{V_1}\mu_1\right)\right)$. The Kullback–Leibler entropy of (M,g_1) relative to (M,g_0) as defined in Theorem 2.2 then measures the lacking of knowledge on the geometry (M,g_1) from the viewpoint of the present geometry (M,g_0). This necessarily implies that (M,g_1) is in the future of (M,g_0) for any spacelike section in the past of (M,g_0) always satisfies $(M,g_{-1})\subset D^-(M,g_0)$ hence is explicitly known already.

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