



# Three-Space from Quantum Mechanics

László B. Szabados<sup>1</sup>

Received: 13 March 2022 / Accepted: 9 August 2022 / Published online: 5 September 2022  
© The Author(s) 2022

## Abstract

The spin geometry theorem of Penrose is extended from  $SU(2)$  to  $E(3)$  (Euclidean) invariant elementary quantum mechanical systems. Using the natural decomposition of the total angular momentum into its spin and orbital parts, the *distance* between the centre-of-mass lines of the elementary subsystems of a classical composite system can be recovered from their *relative orbital angular momenta* by  $E(3)$ -invariant classical observables. Motivated by this observation, an expression for the ‘empirical distance’ between the elementary subsystems of a *composite quantum mechanical system*, given in terms of  $E(3)$ -invariant quantum observables, is suggested. It is shown that, in the classical limit, this expression reproduces the a priori Euclidean distance between the subsystems, though at the quantum level it has a discrete character. ‘Empirical’ angles and 3-volume elements are also considered.

**Keywords** Spin networks · Spin-Geometry Theorem · Quantum geometry · Empirical distance

## 1 Introduction

In 1966, Penrose suggested the so-called  $SU(2)$  spin network as a simple model for a quantum spacetime (which was published much later only in [1], see also [2, 3, 4]). In this model, *angles* between elementary quantum mechanical subsystems of the Universe were expressed in terms of  $SU(2)$  *Casimir invariants*, and, using combinatorial/graphical techniques, he showed that, in the classical limit, these angles tend to angles between directions in the Euclidean 3-space. The significance of this result is that the (conformal structure of the) ‘physical 3-space’ that we use as an a priori given ‘arena’ in which the physical objects are thought to be arranged and

---

Dedicated to András A. Márton, who is able to transform geometry, physics and philosophy into poetry: [www.martonaandras.hu](http://www.martonaandras.hu).

✉ László B. Szabados  
lbszab@rmki.kfki.hu

<sup>1</sup> Wigner Research Centre for Physics, 114, P. O. Box 49, Budapest 1525, Hungary

the interactions between them occur is *determined by the quantum physical systems themselves in the classical limit*. Later, this result was derived in [5] (and recently in [6]) in the usual framework of quantum mechanics; and, following [5], it became known as the Spin Geometry Theorem.

In [5], Moussouris investigated the structure of networks with other symmetry groups, in particular with the Poincaré group, and derived relations between various observables of such systems. However, as far as we know, recovering the *distance*, i.e. the *metric* (rather than only the *conformal structure* of the Euclidean 3-space/Minkowski vector space) from these models is still lacking.

According to the traditional view, the distance in a metric space is associated with a pair of *points*. Thus, the points are usually considered to be the primary concept. Nevertheless, from a positivistic, or rather a *Machian* view, the position in physical 3-space/spacetime in itself is meaningless, and it seems reasonable to speak only about *relative positions* of different elementary subsystems of the Universe. This view might suggest to consider the distance, or, in general the *relations* between points, to be more fundamental than the points; and that the points themselves should be *defined* by these relations.

In classical general relativity the spacetime geometry, and, in particular, the distance between the spacetime points, is determined by standard clocks, light rays and mirrors. However, at the fundamental level no such instruments exist and, also, the light rays are formed by infinitely many photons in special configurations. Thus, the *instrumentalist* approach of defining the spacetime and its properties, using e.g. some form of ‘quantum clocks’ (see e.g. [7]) at the *fundamental level*, does not seem to work.

The philosophy of the present investigations is the positivistic, Machian one of [1, 2, 3, 4] (that we also adopted in [6]), even from two points of view: first, no a priori notion of ‘physical 3-space’ or ‘spacetime’ is used. These notions should be *defined* in an operational way using only existing (material) quantum subsystems of the Universe. Second, we speak about angle and distance *only between two (elementary) subsystems* of the Universe, without introducing ‘directions’ and ‘points’ at all, e.g. in the form of some ‘position operator’. (For a well readable summary of the approaches based on the position operators, see [8].)

A potentially viable mathematical realization of such a program could be based on the use of the *algebra of (basic) quantum observables* in the *algebraic formulation* of quantum theory. We consider the quantum system to be specified completely if its *algebra of (basic) quantum observables* (and, if needed, its representation, too) is given. We think that the notion of space(time) and all of its properties should be *defined* in terms of the *observables of elementary systems*. Roughly speaking, an *elementary quantum system* is a system whose observables are the self-adjoint elements of the *universal enveloping algebra* of some ‘small’ Lie algebra, mostly of the symmetry group acting on the system, and the states of the system belong to the carrier space of some of its unitary, irreducible representations. From our point of view,  $su(2)$  and the Lie algebras  $e(3)$  and  $e(1, 3)$  of the Euclidean and the Poincaré group,  $E(3)$  and  $E(1, 3)$ , respectively, are particularly interesting. (This general idea of the ‘*G*-invariant elementary quantum mechanical systems’ is motivated by that of the Poincaré invariant systems of Newton and Wigner [9].) The observables belonging

to this ‘small’ Lie algebra will be called the *basic observables*. All the structures on the algebra of observables may be used in the construction, but additional extra structures may not. In particular, in the present investigations we use only the general *kinematical framework* of quantum mechanics, but e.g. no evolution equation for the quantum states.

The above strategy yields apparently paradoxical results: we have well defined distance between ‘points’ (or rather straight lines, as we will see) that are *not* defined at all in the quantum theory, just like the angles between ‘directions’ that are not defined either. The ‘true’ geometry of the physical 3-space/spacetime should ultimately be synthesized from the ‘empirical’ distances, angles, etc. introduced via the quantum observables.

As a first step in this strategy, in [6] we gave a simple proof of (a version of) the Spin Geometry Theorem by showing how the *conformal structure* of the Euclidean 3-space can be recovered in this algebraic framework from the basic quantum observables of  $SU(2)$ -invariant systems. The present paper provides one more step: we extend the Spin Geometry Theorem to recover the *metric* of the Euclidean 3-space by extending the symmetry group from  $SU(2)$  to the quantum mechanical Euclidean group  $E(3)$ . Here, by *quantum mechanical* Euclidean group we mean the semi-direct product of  $SU(2)$  and the group of translations in the Euclidean 3-space, rather than the *classical*  $E(3)$ , which is the semi-direct product of  $SO(3)$  and the group of translations.

Technically, the key idea is that the spin part of the total angular momentum of a *composite* system is built from the spin of the constituent subsystems and their *relative orbital angular momentum*; and the latter, in the traditional formulation, contains information on the distance between the subsystems. (In fact, already in the last but one sentence of [1], Penrose raised this possibility in Poincaré-invariant systems.) This idea can in fact be used to define the distance between the elementary Poincaré-invariant quantum mechanical systems as well. However, since the latter is technically different and considerably more complicated, results of those investigations will be published separately. This distance, expressed by  $E(3)$ -invariant classical observables of the composite system, is our key notion, and is called the *empirical distance*. The analogous notion in the quantum theory, defined exclusively in terms of the  $E(3)$ -invariant *basic quantum observables*, is motivated by this classical expression.

The main result of the present paper is that although in quantum theory this empirical distance has some discrete and highly non-Euclidean character, but in the classical limit it reproduces the a priori Euclidean distance between the classical centre-of-mass lines of point particles, i.e. of the straight lines in  $\mathbb{R}^3$ . This can be considered to be an extension of Penrose’s Spin Geometry Theorem from  $SU(2)$  to  $E(3)$ -invariant systems. Therefore, the *metric structure* of the Euclidean 3-space can be recovered from the quantum theory in the classical limit. Note, however, that in the present approach it is *the straight lines* rather than the points that emerged as the elementary objects in  $\mathbb{R}^3$ . This feature of the present approach is analogous to that of *twistor theory* [10, 11, 12], where, classically, the elementary objects are the (in general twisting) *null lines* in Minkowski spacetime with given vorticity or ‘twist’. The quantity here that is analogous to twist is the spin of the elementary systems.

Also, at the quantum level,  $E(3)$ -invariant ‘empirical angles’ and ‘empirical 3-volume elements’ are suggested, and it is shown that they reproduce the angles and 3-volume elements in  $\mathbb{R}^3$  in the classical limit.

In the next section we show how the distance of the centre-of-mass lines of  $E(3)$ -invariant elementary *classical* mechanical systems can be expressed by their  $E(3)$ -invariant classical observables. This expression will provide the basis of our empirical distance in the quantum theory. In Sect. 3, we define  $E(3)$ -invariant elementary quantum mechanical systems and summarize their key properties. In particular, we determine their centre-of-mass states, which are the closest analogs of those of the classical systems, which turn out to be given just by the spin weighted spherical harmonics. Section 4 is devoted to the calculation of the empirical distance, angles and 3-volume elements. We clarify their classical limit there. Some final remarks are given in Sect. 5. The paper is concluded with appendices, in which certain technical details that we used in the main part of the paper are presented.

In deriving the results we use complex techniques developed in general relativity. The related ideas, notations and conventions are mostly those of [10, 13], except that the signature of the spatial 3-metric is *positive*, rather than negative definite. We do not use abstract indices.

## 2 $E(3)$ -Invariant Classical Systems

### 2.1 The Definition of the Elementary Systems

A physical system will be called an  $E(3)$ -invariant *elementary* classical mechanical system if its states can be characterized *completely* by its linear momentum  $p^a$  and angular momentum  $J^{ab}$ ,  $a, b, \dots = 1, 2, 3$ ; under the action of  $SO(3)$ , they transform as a vector and anti-symmetric tensor, respectively, and, under the translation by  $\xi^a \in \mathbb{R}^3$ , as  $(p^a, J^{ab}) \mapsto (\tilde{p}^a, \tilde{J}^{ab}) := (p^a, J^{ab} + \xi^a p^b - \xi^b p^a)$ . Then

$$P^2 := \delta_{ab} p^a p^b, \quad W := \frac{1}{2} J^{ab} \epsilon_{abc} p^c \tag{2.1}$$

are invariant with respect to these transformations. Here  $\epsilon_{abc}$  is the Levi-Civita alternating symbol, and  $\delta_{ab}$  is the Kronecker delta. The space of the linear and angular momenta, endowed with the Lie products

$$\{p^a, p^b\} = 0, \quad \{p^a, J^{bc}\} = \delta^{ab} p^c - \delta^{ac} p^b, \tag{2.2}$$

$$\{J^{ab}, J^{cd}\} = \delta^{bc} J^{ad} - \delta^{bd} J^{ac} + \delta^{ad} J^{bc} - \delta^{ac} J^{bd} \tag{2.3}$$

form the Lie algebra  $e(3)$  of  $E(3)$ . Then  $P^2$  and  $W$  have vanishing Lie bracket both with  $p^a$  and  $J^{ab}$ , i.e. they are Casimir invariants. Lowering and raising the Latin indices will be defined by  $\delta_{ab}$  and its inverse, respectively.<sup>1</sup>

<sup>1</sup> Strictly speaking, the abstract Lie algebra  $e(3)$  is the semi-direct sum of a 3-dimensional commutative ideal and  $so(3) \approx su(2)$ ; and the linear and angular momenta belong to the *dual space* of these sub-Lie algebras. Hence, the *natural positive definite metric*  $\delta_{ab}$  on the commutative ideal (coming from the

If  $p^a = 0$ , then  $P = 0$  and  $W = 0$ . In this degenerate case the subsequent strategy to recover the metric structure of  $\mathbb{R}^3$  in terms of the basic observables does not seem to work. Thus, in the present paper, we consider only the  $p^a \neq 0$  case.

Next we form  $M^a := J^{ab}p_b$ , for which it follows that  $M_a p^a = 0$ , and the identity

$$P^2 J^{ab} = \epsilon^{abc} p_c W + M^a p^b - M^b p^a \tag{2.4}$$

holds. Since the first term on the right of (2.4) is translation invariant while  $M^a \mapsto \tilde{M}^a = M^a + (P^2 \delta_b^a - p^a p_b) \xi^b$ , this identity is usually interpreted as the decomposition of the angular momentum to its spin (or rather helicity) and orbital parts, and  $M^a$  as ( $P^2$ -times) the centre-of-mass vector of the system. Note that  $\delta_b^a - p^a p_b / P^2$  is the projection to the 2-plane orthogonal to  $p^a$ . Thus one can always find a 1-parameter family of translations, viz.  $\xi^a = -M^a / P^2 + u p^a$ ,  $u \in \mathbb{R}$ , which yields vanishing centre-of-mass vector,  $\tilde{M}^a = 0$ . The resulting total angular momentum is  $\tilde{J}^{ab} = \epsilon^{abc} p_c W / P^2$ , which is just the piece of the angular momentum that the Casimir invariant  $W$  represents. Thus the straight line  $q^a(u) := M^a / P^2 + u p^a$  is interpreted as the trajectory of the centre-of-mass point of the system, and will be called the system's *centre-of-mass line*. For the sake of brevity, we call an elementary system with given  $P$  and  $W$  a *single particle*.

Since  $M_a p^a = 0$ , the set of the pairs  $(p^a, M_a)$  is the cotangent bundle  $T^*S_P$  of the 2-sphere  $S_P := \{p^a \in \mathbb{R}^3 \mid P^2 := p^a p^b \delta_{ab} = \text{const}\}$  of radius  $P$  in the momentum space. Clearly, those and only those pairs  $(p^a, M_a)$  can be transformed to one another by an Euclidean transformation whose Casimir invariants  $P$  and  $W$  are the same. In particular,  $(p^a, M_a)$  can always be transformed into  $(p^a, 0)$  by an appropriate translation. This single particle state space,  $T^*S_P$ , is homeomorphic to the manifold of the directed straight lines in  $\mathbb{R}^3$ : a line  $L$  is fixed if its direction and any of its points are specified. Now, the direction of  $L$  is fixed by the unit vector  $p^a / P$ , while for a point of  $L$  we choose the point where  $L$  intersects the 2-plane containing the origin of  $\mathbb{R}^3$  and orthogonal to  $p^a / P$ . This latter is given by  $M^a / P^2$ . The scale on these lines is fixed by  $P$ .  $W$  is an additional structure on  $L$ : it fixes the component of the total angular momentum vector  $\frac{1}{2} \epsilon^{abc} J_{bc}$  in the direction  $p^a / P$ .

## 2.2 The Empirical Distance of Two Particles

The aim of the present subsection is to express the distance between any two straight lines of the Euclidean 3-space, considered to be the centre-of-mass lines of elementary classical systems, by  $E(3)$ -invariant classical observables.

Footnote 1 (continued)

Killing–Cartan metric on  $so(3)$  via the  $SU(2)$  group action on it) yields the metric  $\delta_{ab}$  on the classical momentum space and the whole tensor algebra over this space. Also, the *natural volume 3-form*  $\epsilon_{abc}$  on the momentum 3-space comes from the natural volume 3-form on the ideal. The transformation properties of the linear and angular momenta follow from the  $E(3)$  multiplication laws. Although, formally, the dual  $e(3)^*$  of  $e(3)$  is not Lie algebra, but it has a natural (linear) Poisson manifold structure with the bracket operation (2.2)–(2.3). With this structure  $e(3)^*$  is isomorphic with the Lie algebra  $e(3)$ . Thus, *all* these algebraic/geometric structures are manifestations of those on the *abstract* Lie algebra  $e(3)$ .

Let  $(p_i^a, J_i^{ab})$ ,  $i = 1, 2$ , characterize two elementary classical mechanical systems, and let us form their formal union with the linear and angular momenta  $p^a := p_1^a + p_2^a$  and  $J^{ab} := J_1^{ab} + J_2^{ab}$ , respectively. Then  $P^2$  and  $W$  for the composite system are defined in terms of  $p^a$  and  $J^{ab}$  according to the general rules (2.1). Since the Lie algebra of the basic observables of the composite system is the direct sum of the Lie algebras of those of the constituent subsystems, it is easy to check that  $p^a$  and  $J^{ab}$  satisfy the commutation relations (2.2)–(2.3), i.e. generate the Lie algebra  $e(3)$ ; and that  $P^2, P_1^2, P_2^2$  and also  $W, W_1, W_2$  are all commuting with  $p^a$  and  $J^{ab}$ . Hence,  $P_{12}^2 := \delta_{ab} p_1^a p_2^b = \frac{1}{2}(P^2 - P_1^2 - P_2^2)$  and  $W_{12} := W - W_1 - W_2$  are also commuting with  $p^a$  and  $J^{ab}$ . It is  $P_{12}^2$  and  $W_{12}$  (and their quantum mechanical version) that play fundamental role in the subsequent analysis.

Since  $P_1 P_2 \neq 0$ , by (2.1) and (2.4) we obtain

$$W_{12} = \frac{1}{2} J_1^{ab} \epsilon_{abc} p_2^c + \frac{1}{2} J_2^{ab} \epsilon_{abc} p_1^c = W_1 \frac{P_{12}^2}{P_1^2} + W_2 \frac{P_{12}^2}{P_2^2} + \left( \frac{M_1^a}{P_1^2} - \frac{M_2^a}{P_2^2} \right) \epsilon_{abc} p_1^b p_2^c. \tag{2.5}$$

Since our aim is to express  $M_1^a/P_1^2 - M_2^a/P_2^2$  from (2.5), for a moment we assume that the linear momenta of the constituent systems are linearly independent. This requirement is equivalent to the condition  $P_1^2 P_2^2 > P_{12}^4$ . In this case, the last term on the right in (2.5) is not zero, and then this equation can be solved for  $M_1^a/P_1^2 - M_2^a/P_2^2$ . This solution is

$$\frac{M_1^a}{P_1^2} - \frac{M_2^a}{P_2^2} = \frac{1}{P_1^2 P_2^2 - P_{12}^4} \left( W_{12} - W_1 \frac{P_{12}^2}{P_1^2} - W_2 \frac{P_{12}^2}{P_2^2} \right) \epsilon^a{}_{bc} p_1^b p_2^c + u_1 p_1^a + u_2 p_2^a \tag{2.6}$$

for arbitrary  $u_1, u_2 \in \mathbb{R}$ . Although its components in the 2-plane spanned by  $p_1^a$  and  $p_2^a$  are ambiguous, its component in the direction orthogonal to both  $p_1^a$  and  $p_2^a$ ,

$$d_{12} := \left( \frac{M_1^a}{P_1^2} - \frac{M_2^a}{P_2^2} \right) \frac{\epsilon_{abc} p_1^b p_2^c}{\sqrt{P_1^2 P_2^2 - P_{12}^4}} = \frac{1}{\sqrt{P_1^2 P_2^2 - P_{12}^4}} \left( W_{12} - W_1 \frac{P_{12}^2}{P_1^2} - W_2 \frac{P_{12}^2}{P_2^2} \right), \tag{2.7}$$

is well defined. Similarly, although under a translation  $M_1^a/P_1^2 - M_2^a/P_2^2$  changes as  $(M_1^a/P_1^2 - M_2^a/P_2^2) \mapsto (M_1^a/P_1^2 - M_2^a/P_2^2) - (p_1^a p_1^b / P_1^2 - p_2^a p_2^b / P_2^2) \xi_b$ , its component in the direction orthogonal both to  $p_1^a$  and  $p_2^a$  is invariant. Therefore, the vector  $d_{12}^a$  defined by the right hand side of (2.6) with  $u_1 = u_2 = 0$  is uniquely determined, it is orthogonal both to  $p_1^a$  and  $p_2^a$  and points from a uniquely determined point  $v_{21}$  of the straight line  $q_2^a(u) := q_2^a + u p_2^a$ ,  $u \in \mathbb{R}$ , to a uniquely determined point  $v_{12}$  of the straight line  $q_1^a(u) := q_1^a + u p_1^a$ ,  $u \in \mathbb{R}$ . Its physical dimension is length, and it is invariant with respect to the  $P_1 \mapsto \alpha P_1, P_2 \mapsto \beta P_2$  rescalings for any  $\alpha, \beta > 0$ . Thus  $d_{12}^a$  is the *relative position vector* of the first subsystem with respect to the second; and  $d_{12}$  is the (signed) *distance* between the *centre-of-mass lines* of the two constituent, elementary subsystems. In particular,  $d_{12} = 0$  holds precisely when the two centre-of-mass lines intersect each other. This can always be achieved by an appropriate

translation of *one of the two* subsystems, and hence  $d_{12}^a$  can always be characterized by such a translation.

If  $p_1^a$  and  $p_2^a$  are parallel, then, as we are going to show, the distance of the two corresponding centre-of-mass lines that do not coincide can be recovered as the limit of the distances between centre-of-mass lines with non-parallel tangents, i.e. as a limit of classical observables above. Since both the numerator and the denominator in (2.7) are vanishing if  $p_1^a$  and  $p_2^a$  are parallel, we should check that these tend to zero in the same order and that their quotient is finite and well defined. Since the two centre-of-mass lines are parallel but do not coincide, they lay in a 2-plane, and the vector  $M_1^a/P_1^2 - M_2^a/P_2^2$  is non-zero, tangent to this 2-plane and is orthogonal to  $p_1^a$  (and hence to  $p_2^a$ , too). Thus, there is a unit vector  $w^a$  which is orthogonal both to  $p_1^a$  and  $M_1^a/P_1^2 - M_2^a/P_2^2$ . Then let us consider the 1-parameter family of momenta  $p_2^a(\alpha) := P_2(\cos \alpha v^a + \sin \alpha w^a)$ , where  $v^a := p_1^a/P_1 = p_2^a/P_2$ . In the  $\alpha \rightarrow 0$  limit,  $p_2^a(\alpha) \rightarrow p_2^a$ . A straightforward calculation gives that  $P_1^2 P_2^2(\alpha) - P_{12}^2(\alpha) = P_1^2 P_2^2 \sin^2 \alpha$ ,  $\varepsilon^a{}_{bc} p_1^b p_2^c(\alpha) = P_1 P_2 \varepsilon^a{}_{bc} v^b w^c \sin \alpha$  and

$$\begin{aligned} W_{12}(\alpha) - W_1 \frac{P_{12}^2(\alpha)}{P_1^2} - W_2(\alpha) \frac{P_{12}^2(\alpha)}{P_2^2} &= \\ &= \frac{1}{2} \left( J_1^{ab} \varepsilon_{abc} w^c P_2 - J_2^{ab} \varepsilon_{abc} w^c P_1 \cos \alpha + J_2^{ab} \varepsilon_{abc} p_1^c \sin \alpha \right) \sin \alpha. \end{aligned}$$

Hence, the  $\alpha \rightarrow 0$  limit of  $d_{12}^a(\alpha)$  for the centre-of-mass lines with the tangents  $p_1^a$  and  $p_2^a(\alpha)$  is, indeed, the well defined finite value

$$\frac{1}{2} \left( \frac{J_1^{de} P_2 - J_2^{de} P_1}{P_1 P_2} \varepsilon_{def} w^f \right) \varepsilon^a{}_{bc} v^b w^c = \left( \frac{M_1^d}{P_1^2} - \frac{M_2^d}{P_2^2} \right) \varepsilon_{def} v^e w^f \varepsilon^a{}_{bc} v^b w^c.$$

Hence, the distance  $d_{12}$  is also well defined and finite. In this case, however, the points analogous to  $v_{12}$  and  $v_{21}$  above are undetermined.

To summarize: we found an alternative expression  $|d_{ij}|$  for the Euclidean distance,

$$D_{ij} := \inf \left\{ \sqrt{\delta_{ab}(q_i^a(u_i) - q_j^a(u_j))(q_i^b(u_i) - q_j^b(u_j))} \mid u_i, u_j \in \mathbb{R} \right\}, \quad (2.8)$$

between the centre-of-mass straight lines of any two elementary classical systems *in terms of E(3)-invariant basic observables of the composite system*. Since the latter is given by the *observables* of the composite system, we call  $d_{ij}$  the ‘empirical distance’ between the two subsystems. However, if we know the distance between any two straight lines in  $\mathbb{R}^3$ , then we can determine the Euclidean distance function, too. Hence, our empirical distance determines the metric structure of  $\mathbb{R}^3$ . We will see that the analogous empirical distance in quantum theory deviates from the a priori Euclidean distance  $D_{ij}$ , but in the classical limit the former reproduces the latter.

### 2.3 Empirical Angles and Volume Elements

In [6], we considered  $SO(3)$ -invariant elementary classical systems in which the only basic observable was the angular momentum vector  $J^a$ ; and we introduced an empirical angle between the angular momenta of two subsystems in a well defined and  $SO(3)$ -invariant way. However, in the present case the symmetry group is the Euclidean group  $E(3)$ , and although the analogously defined empirical angles would still be  $SO(3)$ -invariant, but they would *not* be invariant under translations. Thus, in  $E(3)$ -invariant classical systems, the empirical angles between the subsystems should be defined in terms of quantities that are invariant under translations and covariant under rotations.

The primary candidate for such quantities is the linear momentum. Thus we define the ‘empirical angle’  $\omega_{12}$  between  $p_1^a$  and  $p_2^a$  according to

$$\cos \omega_{12} := \frac{\delta_{ab} p_1^a p_2^b}{P_1 P_2} \tag{2.9}$$

with range  $\omega_{12} \in [0, \pi]$ . This angle is clearly  $E(3)$ -invariant, and if e.g.  $p_1^a/P_1$  can be obtained from  $p_2^a/P_2$  by a rotation with angle  $\beta_{12}$  in the 2-plane spanned by  $p_1^a$  and  $p_2^a$ , then  $\omega_{12} = \beta_{12}$ . Thus  $\omega_{12}$  reproduces the angles of the a priori Euclidean geometry of  $\mathbb{R}^3$  (see also [6]). However, as we will see, they split in the quantum theory, and they coincide only in the classical limit.

The natural volume 3-form  $\varepsilon_{abc}$  on the space of the translation generators in  $e(3)$  makes it possible to introduce the ‘empirical 3-volume element’ by the 3-volume of the tetrahedron spanned by three linear momenta,  $p_1^a$ ,  $p_2^a$  and  $p_3^a$ , of a three-particle system by

$$v_{123} := \frac{1}{3!} \varepsilon_{abc} \frac{p_1^a p_2^b p_3^c}{P_1 P_2 P_3}. \tag{2.10}$$

Just in the case of empirical angles, in quantum theory the corresponding empirical 3-volume element and the 3-volume element of the a priori Euclidean geometry of  $\mathbb{R}^3$  do not coincide. They do only in the classical limit (see also [6]). In Sect. 5 we raise the possibility of another notion of empirical angles and 3-volume elements, based on the relative position vectors  $d_{ij}^a$  of three-particle systems, rather than the linear momenta.

## 3 $E(3)$ -Invariant Elementary Quantum Mechanical Systems

### 3.1 The Definition and the Basic Properties of the Elementary systems

Adapting the idea of Poincaré-invariant *elementary* quantum mechanical systems of Newton and Wigner [9] to the present case, a *Euclidean-invariant elementary quantum mechanical system* will be defined to be a system whose states belong



to the carrier space of some *unitary, irreducible* representation of the (quantum mechanical) Euclidean group  $E(3)$ ; and, in this representation, the momentum and angular momentum tensor operators,  $\mathbf{p}^a$  and  $\mathbf{J}^{ab}$ , are the self-adjoint generators of the translations and rotations, respectively. These representations are labelled by a fixed value  $P^2 \geq 0$  and  $w$ , respectively, of the two Casimir operators

$$\mathbf{P}^2 := \delta_{ab}\mathbf{p}^a\mathbf{p}^b, \quad \mathbf{W} := \frac{1}{2}\epsilon_{abc}\mathbf{J}^{ab}\mathbf{p}^c. \tag{3.1}$$

Thus, the quantum mechanical operators will be denoted by boldface letters. Clearly, in the states belonging to such representations, the energy  $\mathbf{E}^2 = \mathbf{P}^2 + m^2$  or  $\mathbf{E} = \mathbf{P}^2/2m$  of the relativistic or non-relativistic systems of rest mass  $m$ , respectively, has a definite value. The commutators of  $\mathbf{p}^a$  and  $\mathbf{J}^{cd}$  are

$$[\mathbf{p}^a, \mathbf{p}^b] = 0, \quad [\mathbf{p}^a, \mathbf{J}^{cd}] = -i\hbar(\delta^{ac}\mathbf{p}^d - \delta^{ad}\mathbf{p}^c), \tag{3.2}$$

$$[\mathbf{J}^{ab}, \mathbf{J}^{cd}] = -i\hbar(\delta^{bc}\mathbf{J}^{ad} - \delta^{bd}\mathbf{J}^{ac} + \delta^{ad}\mathbf{J}^{bc} - \delta^{ac}\mathbf{J}^{bd}); \tag{3.3}$$

which are just the Lie brackets (2.2)–(2.3) with the  $p^a \mapsto \mathbf{p}^a$  and  $J^{ab} \mapsto (i/\hbar)\mathbf{J}^{ab}$  substitution.

Nevertheless,

$$\mathbf{M}_a := \mathbf{J}_{ab}\mathbf{p}^b$$

is *not* self-adjoint, because

$$\mathbf{M}_a^\dagger = \mathbf{p}^b\mathbf{J}_{ab} = \mathbf{M}_a + [\mathbf{p}^b, \mathbf{J}_{ab}] = \mathbf{M}_a + 2i\hbar\mathbf{p}_a.$$

Thus we form  $\mathbf{C}_a := \frac{1}{2}(\mathbf{M}_a + \mathbf{M}_a^\dagger) = \mathbf{M}_a + i\hbar\mathbf{p}_a$ , which is, by definition, the self-adjoint part of  $\mathbf{M}_a$ , and we consider this to be the centre-of-mass operator. The commutators of these operators can be derived from those for  $\mathbf{p}^a$  and  $\mathbf{J}^{ab}$  above:

$$[\mathbf{p}_a, \mathbf{C}_b] = -i\hbar(\delta_{ab}\mathbf{P}^2 - \mathbf{p}_a\mathbf{p}_b), \quad [\mathbf{C}_a, \mathbf{C}_b] = -i\hbar\mathbf{P}^2\mathbf{J}_{ab}. \tag{3.4}$$

As a consequence of the definitions,

$$\mathbf{P}^2\mathbf{J}_{ab} = \mathbf{C}_a\mathbf{p}_b - \mathbf{C}_b\mathbf{p}_a + \epsilon_{abc}\mathbf{p}^c\mathbf{W}, \tag{3.5}$$

i.e. the analog of (2.4) holds for the operators, too.

The unitary, irreducible representations of the quantum mechanical Euclidean group has been determined in various different forms (see e.g. [14–17]). The form that we use in the present paper is given in the appendices of [18, 19], and is based on the use of the complex line bundles  $\mathcal{O}(-2s)$  over the 2-sphere  $\mathcal{S}_P$  of radius  $P$  in the classical momentum 3-space, where  $2s \in \mathbb{Z}$ . Here  $\mathcal{O}(-2s)$  is the bundle of spin weighted scalars on  $\mathcal{S}_P$  with spin weight  $s$  (see e.g. [10]). The wave functions of the quantum system are square integrable cross sections  $\phi$  of  $\mathcal{O}(-2s)$ , and the Hilbert space of these cross sections will be denoted by  $\mathcal{H}_{P,s}$ , or simply by  $\mathcal{H}$ . Thus,  $\mathcal{S}_P$  is analogous to the mass shell of the Poincaré invariant systems (see e.g. [20]). The

radius  $P$  is fixed by the Casimir operator  $\mathbf{P}^2$ . The spin weight  $s$  of  $\phi$  is linked to the value of the other Casimir operator,  $\mathbf{W}$ , in the irreducible representation:  $w = \hbar Ps$ .

The  $SU(2)$  part of  $E(3)$  acts on  $\mathcal{S}_P$  as  $p^a \mapsto R^a_b p^b$ , where  $R^a_b$  is the rotation matrix determined by  $U^A_B \in SU(2)$  via  $R^a_b := -\sigma^{AA'}_b U^A_B \bar{U}^{A'B'} \sigma_b^{BB'}$ . Here  $\sigma^{AA'}$  are the three non-trivial  $SL(2, \mathbb{C})$  Pauli matrices (including the factor  $1/\sqrt{2}$ ), according to the conventions of [13]. Raising/lowering of the unprimed and primed spinor name indices,  $A, B$  and  $A', B'$ , are defined by the spinor metric  $\epsilon_{AB}$  and its complex conjugate, respectively. Then the action of  $SU(2)$  on the spin weighted function  $\phi$  is  $\phi(p^e) \mapsto \exp(-2is\lambda)\phi((R^{-1})^e_c p^c)$ ; while that of the translation by  $\xi^a$  is simply  $\phi(p^e) \mapsto \exp(-ip_a \xi^a/\hbar)\phi(p^e)$ . Here,  $\exp(i\lambda)$  is just the phase that appears in the action,  $\pi^A \mapsto \exp(i\lambda)\pi^A$ , of  $SU(2)$  on the spinor constituent  $\pi^A$  of  $p^a$  (see [18, 19]). This representation of  $E(3)$  is analogous to that of the (quantum mechanical) Poincaré group on the  $L_2$ -space of spinor fields on the mass shell [20].

The spin weighted spherical harmonics  ${}_s Y_{jm}$  with spin weight  $s$  and indices  $j = |s|, |s| + 1, \dots, m = -j, -j + 1, \dots, j$  are known to form an orthonormal basis in the space of the square integrable cross sections of  $\mathcal{O}(-2s)$  on the unit 2-sphere (see [13, 21]). Hence, the space that the functions  ${}_s Y_{jm}$  on  $\mathcal{S}_R$  span is precisely the carrier space of the unitary, irreducible representation of  $E(3)$  labelled by the Casimir invariants  $P > 0$  and  $w = \hbar Ps$ , in which  ${}_s Y_{jm}/P$  form an orthonormal basis with respect to the natural  $L_2$  scalar product.

In this representation, the action of  $\mathbf{p}_a$  and  $\mathbf{J}_a := \frac{1}{2}\epsilon_{abc}\mathbf{J}^{bc}$  on  $\phi$  is

$$\mathbf{p}_a \phi = p_a \phi, \tag{3.6}$$

$$\mathbf{J}_a \phi = P\hbar(m_a \delta' \phi - \bar{m}_a \delta \phi) + s\hbar \frac{P_a}{P} \phi, \tag{3.7}$$

where  $m^a$  and  $\bar{m}^a$  are the complex null tangents of  $\mathcal{S}_P$  and normalized by  $m_a \bar{m}^a = 1$ , and  $\delta$  and  $\delta'$  are the edth operators of Newman and Penrose [21]. The explicit form of the vectors  $p^a, m^a$  and  $\bar{m}^a$  as well as the operators  $\delta$  and  $\delta'$  in the complex stereographic coordinates  $(\zeta, \bar{\zeta})$  on  $\mathcal{S}_P$  is given in Appendix A.1. The first two terms together on the right of (3.7) give the orbital part of the angular momentum operator, denoted by  $\mathbf{L}_a$ , while the third its spin part. Since the spin weighted harmonics form a basis in  $\mathcal{H}_{P,s}$ , the basic observables  $\mathbf{p}_a$  and  $\mathbf{J}_a$  can also be given by their action on these harmonics, too. Their matrix elements are calculated in Appendix A.2.

$\mathbf{p}^a$  and  $\mathbf{J}^a$  are known to be  $SO(3)$  vector operators. Thus, if the unitary operator  $\mathbf{U}$  represents  $U^A_B \in SU(2)$  on  $\mathcal{H}_{P,s}$ , then

$$\mathbf{U}^\dagger \mathbf{p}^a \mathbf{U} = R^a_b \mathbf{p}^b, \quad \mathbf{U}^\dagger \mathbf{J}^a \mathbf{U} = R^a_b \mathbf{J}^b. \tag{3.8}$$

If  $\phi = \exp(-ip_a \xi^a/\hbar)\chi$ , then by (3.6) and (3.7)

$$\mathbf{p}^a \phi = \exp(-\frac{i}{\hbar} p_e \xi^e) \mathbf{p}^a \chi, \quad \mathbf{J}^a \phi = \exp(-\frac{i}{\hbar} p_e \xi^e) \left( \mathbf{J}^a \chi + \epsilon^a_{bc} \xi^b \mathbf{p}^c \chi \right). \tag{3.9}$$

<sup>2</sup> In [19] we defined the translation by multiplication by the phase factor  $\exp(ip_a \xi^a/\hbar)$ . Here, to be compatible with the standard sign convention, we changed the sign in the exponent to its opposite.

Hence, for  $\phi = \exp(-ip_a \xi^a / \hbar) \mathbf{U} \psi$ ,

$$\langle \phi | \mathbf{p}^a | \phi \rangle = R^a_b \langle \psi | \mathbf{p}^b | \psi \rangle, \quad \langle \phi | \mathbf{J}^a | \phi \rangle = R^a_b \langle \psi | \mathbf{J}^b | \psi \rangle + \varepsilon^a_{bc} \xi^b R^c_d \langle \psi | \mathbf{p}^d | \psi \rangle. \tag{3.10}$$

These are just the transformation laws of the classical basic observables under the action of the classical  $E(3)$ .

Finally, it is straightforward to derive the explicit form of the centre-of-mass operator:

$$\mathbf{C}^a \phi = i\hbar \left( P^2 m^a \delta' \phi + P^2 \bar{m}^a \delta \phi - p^a \phi \right). \tag{3.11}$$

This is also an  $SO(3)$  vector operator. A detailed discussion of the line bundle  $\mathcal{O}(-2s)$  and the derivation of these equations are also given in the appendices of [18, 19].

### 3.2 The Centre-of-Mass States

In this subsection we show that the spin weighted spherical harmonics form a distinguished basis among the orthonormal bases in  $\mathcal{H}_{P,s}$  in the sense that *they are adapted in a natural way to the basis in the abstract Lie algebra  $e(3)$  of the basic quantum observables* (but *not* to a Cartesian frame in the ‘physical 3-space’). In particular, they are just the eigenfunctions of the square of the centre-of-mass vector operator,  $\mathbf{C}_a \mathbf{C}^a$ , they are the critical points of the functional  $\phi \mapsto \langle \phi | \mathbf{C}_a \mathbf{C}^a | \phi \rangle$ , and the expectation value of the centre-of-mass vector operator in these states is zero. We will calculate the expectation value of our empirical distance and angle in the states that are obtained from these special ones by some  $E(3)$  transformation.

Using (3.11) and  $\delta p^a = m^a$  (see Appendix A.1), by integration by parts it is straightforward to form the centre-of-mass-square operator  $\mathbf{C}_a \mathbf{C}^a$ . It is

$$\mathbf{C}_a \mathbf{C}^a \phi = \hbar^2 P^2 \left( -P^2 (\delta \delta' + \delta' \delta) \phi + \phi \right). \tag{3.12}$$

Since the operators  $\mathbf{C}_a$  are self-adjoint,  $\mathbf{C}_a \mathbf{C}^a$  is a positive self-adjoint operator. Similar calculations yield the square of the total and the orbital angular momentum operators:

$$\mathbf{J}_a \mathbf{J}^a \phi = \hbar^2 \left( -P^2 (\delta \delta' + \delta' \delta) \phi + s^2 \phi \right), \quad \mathbf{L}_a \mathbf{L}^a \phi = -P^2 \hbar^2 (\delta \delta' + \delta' \delta) \phi.$$

Thus  $\mathbf{C}_a \mathbf{C}^a$ ,  $\mathbf{J}_a \mathbf{J}^a$  and  $\mathbf{L}_a \mathbf{L}^a$  deviate from one another only by a constant times the identity operator, and hence, in particular, their spectral properties are the same. Note also that  $(\delta \delta' + \delta' \delta)$  is just the metric Laplace operator on  $\mathcal{S}_P$ .

First we show that the critical points of the functional  $\phi \mapsto \langle \phi | \mathbf{C}_a \mathbf{C}^a | \phi \rangle$  on  $\mathcal{H}_{P,s}$  are just the combinations of the form  $\phi = \sum_m c^m_s Y_{jm}$  of spin weighted spherical harmonics  ${}_s Y_{jm}$  with given  $j$ . Using  $\langle \phi | \phi \rangle = 1$ , by integration by parts (3.12) gives

$$\langle \phi | C_a C^a | \phi \rangle = \hbar^2 P^2 \left( P^2 \int_{S_p} ((\bar{\partial}\bar{\phi})(\partial'\phi) + (\partial'\bar{\phi})(\bar{\partial}\phi)) dS_p + 1 \right).$$

The critical points of this functional with respect to the variations  $\delta\phi$  constrained by  $\langle \phi | \phi \rangle = 1$  are just the critical points of the functional  $\langle \phi | C_a C^a | \phi \rangle + \lambda \hbar^2 P^4 \langle \phi | \phi \rangle$  with respect to unconstrained variations, where  $\lambda$  is some real Lagrange multiplier. The vanishing of the variation of the latter functional yields

$$(\bar{\partial}\partial' + \partial'\bar{\partial})\phi = \lambda\phi. \tag{3.13}$$

Thus  $\lambda$  must be an eigenvalue of the Laplace operator  $\bar{\partial}\partial' + \partial'\bar{\partial}$  acting on spin weight  $s$  functions on  $S_p$ , and then the critical configurations are given by the corresponding eigenfunctions. These can be determined by expanding  $\phi$  as  $\phi = \sum_{j,m} c^{jm} {}_s Y_{jm}$  with complex constants  $c^{jm}$ , where  $j = |s|, |s| + 1, |s| + 2, \dots$  and  $m = -j, -j + 1, \dots, j$ ; and using the general formulae (A.7) how the operators  $\bar{\partial}$  and  $\partial'$  act on  ${}_s Y_{jm}$ . Substituting all these into (3.13), we find that  $\lambda = -P^{-2}(j^2 + j - s^2)$ . Hence,  $\lambda$  is linked to  $j$ , and the corresponding eigenfunctions have the form  $\phi = \sum_m c^m {}_s Y_{jm}$ . These eigenfunctions are the critical points of the functional  $\langle \phi | C_a C^a | \phi \rangle$ , and the corresponding critical values are

$$\langle \phi | C_a C^a | \phi \rangle = \hbar^2 P^2 (1 + j^2 + j - s^2) \geq \hbar^2 P^2 (1 + |s|). \tag{3.14}$$

The smallest of these corresponds to  $j = |s|$ , and the corresponding eigenfunction is  $\phi = \sum_m c^m {}_s Y_{|s|m}$ , which is *holomorphic* if  $s = -|s|$ , and it is *antiholomorphic* if  $s = |s|$  (see Appendix A.1). Forming the second variation of  $\langle \phi | C_a C^a | \phi \rangle + \lambda \hbar^2 P^4 \langle \phi | \phi \rangle$  at the critical points, one can see that the only minimum does, in fact, correspond to  $j = |s|$ , and all the other critical points are only inflection. Thus the right hand side of (3.14) is the sharp *strictly positive* lower bound for the expectation values of  $C_a C^a$ .

This analysis shows also that the spectrum of all the operators  $C_a C^a$ ,  $\mathbf{J}_a \mathbf{J}^a$  and  $\mathbf{L}_a \mathbf{L}^a$  is discrete (as it must be since  $\bar{\partial}\partial' + \partial'\bar{\partial}$  is an *elliptic* differential operator acting on cross sections of a vector bundle over a *compact* manifold), their eigenvalues, respectively, are

$$P^2 \hbar^2 (1 + j^2 + j - s^2), \quad \hbar^2 j(j + 1), \quad \hbar^2 (j^2 + j - s^2);$$

and the corresponding common eigenfunctions are of the form  $\sum_m c^m {}_s Y_{jm}$ . These imply, in particular, that the expectation value of any of these operators is not zero for  $s \neq 0$ .

By integration by parts and using how the edth operators act on  $p_a, m_a$  and  $\bar{m}_a$ , we can write

$$\begin{aligned} \langle \phi | C_a | \phi \rangle &= i\hbar P^2 \int_{S_p} \bar{\phi} (m_a \partial' \phi + \bar{m}_a \bar{\partial} \phi + \phi \bar{\partial} \bar{m}_a) dS_p = \\ &= i\hbar P^2 \int_{S_p} (\bar{\phi} m_a \partial' \phi - \phi \bar{m}_a \bar{\partial} \bar{\phi}) dS_p = i\hbar P^2 \int_{S_p} p_a (\phi \partial' \bar{\partial} \bar{\phi} - \bar{\phi} \bar{\partial} \partial' \phi) dS_p. \end{aligned}$$

Since  $\phi = \sum_m c^m {}_s Y_{jm}$  for some given  $j$ , by (A.7) we have that  $\bar{\phi}(\delta\delta'\phi) = -(j+s)(j-s+1)\bar{\phi}\phi/2P^2$ . Using this and its complex conjugate, finally we obtain

$$\langle \phi | C_a | \phi \rangle = 0. \tag{3.15}$$

Thus, the  $2j + 1$  dimensional subspaces in  $\mathcal{H}_{P,s}$  spanned by the harmonics  ${}_s Y_{jm}$  with given  $j$  are specified e.g. by  $C_a C^a$  (or, equivalently by  $\mathbf{J}_a \mathbf{J}^a$  or by  $\mathbf{L}_a \mathbf{L}^a$ ) in a natural way, while the basis in these subspaces, the index  $m$  is referring to, is linked to our choice for the basis in the sub-Lie algebra  $su(2) \subset e(3)$ .

To summarize,  $\{{}_s Y_{jm}/P\}$  is not only one of the many  $L_2$ -orthonormal bases in  $\mathcal{H}_{P,s}$ , but it is adapted in a natural way to the centre-of-mass operator, too. Although the expectation value of  $C_a$  is zero in any eigenstate of  $C_a C^a$ , the expectation value of  $C_a C^a$  can never be zero, even if  $s = 0$ . Its smallest expectation value, which is its smallest eigenvalue, cannot be made zero e.g. by any translation (in contrast to the classical case). It corresponds to  $j = |s|$ , and the corresponding eigenfunctions,  $\phi = \sum_m c^m {}_s Y_{|s|m}$ , are holomorphic if  $s = -|s|$ , and anti-holomorphic if  $s = |s|$ . We call these states the centre-of-mass states. These form a  $2|s| + 1$  dimensional subspace in  $\mathcal{H}_{P,s}$ , and these are the states of the  $E(3)$ -invariant elementary quantum mechanical systems that are the closest analogs of the states of the classical systems with vanishing centre-of-mass vector.

## 4 The Two-Particle System

### 4.1 The Quantum Observables of Two-Particle Systems

Let us consider two  $E(3)$ -invariant elementary quantum mechanical systems, whose basic quantum observables are  $\mathbf{p}_i^a$  and  $\mathbf{J}_i^{ab}$ ,  $i = 1, 2$ . These observables are self-adjoint operators on  $\mathcal{H}_i$ . The corresponding Casimir operators are denoted by  $\mathbf{P}_i^2$  and  $\mathbf{W}_i$ . The Hilbert space of the joint system is  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , and we can form the operators  $\mathbf{O}_1 \otimes \mathbf{I}_2, \mathbf{I}_1 \otimes \mathbf{O}_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$  for any  $\mathbf{O}_i : \mathcal{H}_i \rightarrow \mathcal{H}_i$ , where  $\mathbf{I}_i$  are the identity operators on the respective Hilbert spaces  $\mathcal{H}_i$ . Clearly,  $\mathbf{O}_1 \otimes \mathbf{I}_2$  and  $\mathbf{I}_1 \otimes \mathbf{O}_2$  are commuting. In particular,  $\mathbf{P}_1^2 \otimes \mathbf{I}_2, \mathbf{W}_1 \otimes \mathbf{I}_2, \mathbf{I}_1 \otimes \mathbf{P}_2^2$  and  $\mathbf{I}_1 \otimes \mathbf{W}_2$  are Casimir operators of the composite system.

Analogously to (3.1), we form

$$\begin{aligned} \mathbf{P}^2 &:= \delta_{ab} (\mathbf{p}_1^a \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{p}_2^a) (\mathbf{p}_1^b \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{p}_2^b) = \\ &= \mathbf{P}_1^2 \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{P}_2^2 + 2\delta_{ab} \mathbf{p}_1^a \otimes \mathbf{p}_2^b, \end{aligned} \tag{4.1}$$

$$\begin{aligned} \mathbf{W} &:= \frac{1}{2} \varepsilon_{abc} (\mathbf{J}_1^{ab} \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{J}_2^{ab}) (\mathbf{p}_1^c \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{p}_2^c) = \\ &= \mathbf{W}_1 \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{W}_2 + \frac{1}{2} \varepsilon_{abc} (\mathbf{p}_1^c \otimes \mathbf{J}_2^{ab} + \mathbf{J}_1^{ab} \otimes \mathbf{p}_2^c). \end{aligned} \tag{4.2}$$

Although  $\mathbf{W}$  does *not* commute with any of  $\mathbf{p}_1^a \otimes \mathbf{I}_2, \mathbf{I}_1 \otimes \mathbf{p}_2^a, \mathbf{J}_1^{ab} \otimes \mathbf{I}_2$  and  $\mathbf{I}_1 \otimes \mathbf{J}_2^{ab}$ , the observables of the *subsystems* in the algebra of observables of the composite system, and  $\mathbf{P}^2$  does *not* commute with  $\mathbf{J}_1^{ab} \otimes \mathbf{I}_2$  and  $\mathbf{I}_1 \otimes \mathbf{J}_2^{ab}$ , but both  $\mathbf{W}$  and  $\mathbf{P}^2$  *do* commute with  $\mathbf{p}_1^a \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{p}_2^a$  and  $\mathbf{J}_1^{ab} \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{J}_2^{ab}$ , i.e. with the linear and angular momentum operators of the *composite system*. Therefore, though  $\mathbf{P}^2$  and  $\mathbf{W}$  are *not* Casimir operators of the composite system, they *are* commuting with the generators of the symmetry group  $E(3)$ , i.e. they are  $E(3)$ -invariant. Moreover,  $[\mathbf{P}^2, \mathbf{W}] = 0$  also holds. Hence,

$$\mathbf{P}_{12}^2 := \frac{1}{2} \left( \mathbf{P}^2 - \mathbf{P}_1^2 \otimes \mathbf{I}_2 - \mathbf{I}_1 \otimes \mathbf{P}_2^2 \right) = \delta_{ab} \mathbf{p}_1^a \otimes \mathbf{p}_2^b, \tag{4.3}$$

$$\mathbf{W}_{12} := \mathbf{W} - \mathbf{W}_1 \otimes \mathbf{I}_2 - \mathbf{I}_1 \otimes \mathbf{W}_2 = \frac{1}{2} \epsilon_{abc} \left( \mathbf{p}_1^c \otimes \mathbf{J}_2^{ab} + \mathbf{J}_1^{ab} \otimes \mathbf{p}_2^c \right) \tag{4.4}$$

are also  $E(3)$ -invariant and  $[\mathbf{P}_{12}^2, \mathbf{W}_{12}] = 0$  holds. These operators characterize the relationship between the two subsystems in the composite system, and hence they will have particular significance for us.

Using the definitions above and the first of (3.4), the identity (3.5) yields

$$\left( \mathbf{P}_1^2 \otimes \mathbf{P}_2^2 \right) \mathbf{W}_{12} = \epsilon_{abc} \left( \mathbf{p}_1^b \otimes \mathbf{p}_2^c \right) \left( \mathbf{C}_1^a \otimes \mathbf{P}_2^2 - \mathbf{P}_1^2 \otimes \mathbf{C}_2^a \right) + \mathbf{P}_{12}^2 \left( \mathbf{P}_1^2 \otimes \mathbf{W}_2 + \mathbf{W}_1 \otimes \mathbf{P}_2^2 \right). \tag{4.5}$$

Let the two subsystems be elementary, characterized by the Casimir invariants  $(P_1, s_1)$  and  $(P_2, s_2)$ , respectively. Then by (3.6)  $\mathbf{P}_{12}^2$  is a multiplication operator on  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , and hence, for any  $\phi_1 \otimes \phi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ , (4.5) gives

$$\epsilon_{abc} p_1^b p_2^c \left( \frac{\mathbf{C}_1^a}{P_1^2} \otimes \mathbf{I}_2 - \mathbf{I}_1 \otimes \frac{\mathbf{C}_2^a}{P_2^2} \right) \phi_1 \otimes \phi_2 = \left( \mathbf{W}_{12} - \hbar \left( \frac{s_1}{P_1} + \frac{s_2}{P_2} \right) \mathbf{P}_{12}^2 \right) \phi_1 \otimes \phi_2. \tag{4.6}$$

This is analogous to the classical equation (2.5), and the operators on both sides are  $E(3)$ -invariant. Nevertheless, their physical dimension is momentum times angular momentum, rather than length. Thus, just as in the classical case (and motivated by (2.7)), we should consider the *component of  $\mathbf{C}_1^a \otimes \mathbf{I}_2 / P_1^2 - \mathbf{I}_1 \otimes \mathbf{C}_2^a / P_2^2$  in the direction  $\epsilon^a{}_{bc} p_1^b p_2^c$* .

This is just (4.6) divided by  $\sqrt{P_1^2 P_2^2 - (\delta_{ab} p_1^a p_2^b)^2}$ , the length of  $\epsilon_{abc} p_1^b p_2^c$ , and we could consider the operator

$$\frac{\epsilon_{abc} p_1^b p_2^c}{\sqrt{P_1^2 P_2^2 - (\delta_{ab} p_1^a p_2^b)^2}} \left( \frac{1}{P_1^2} \mathbf{C}_1^a \otimes \mathbf{I}_2 - \frac{1}{P_2^2} \mathbf{I}_1 \otimes \mathbf{C}_2^a \right). \tag{4.7}$$

This is a well defined, self-adjoint and  $E(3)$ -invariant operator, which is analogous to the classical expression (2.7). However, in contrast to the classical case, the coefficient under the square root sign in the denominator is *not* constant. Hence it could be difficult to use this expression e.g. in the calculation of the expectation values. To cure this difficulty, using

$$\left(\frac{d^k}{dx^k} \frac{1}{\sqrt{1-x}}\right)(0) = \frac{1}{2} \frac{3}{2} \frac{5}{2} \cdots \frac{2k-1}{2} = \frac{1}{2^k} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-1) \cdot (2k)}{2 \cdot 4 \cdots (2k)} = \frac{1}{2^{2k}} \frac{(2k)!}{k!},$$

for the Taylor expansion of  $1/\sqrt{P_1^2 P_2^2 - (\delta_{ab} P_1^a P_2^b)^2}$  we obtain

$$\frac{1}{\sqrt{P_1^2 P_2^2 - (\delta_{ab} P_1^a P_2^b)^2}} = \frac{1}{P_1 P_2} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \frac{(2k)!}{(k!)^2} \left(\delta_{ab} \frac{P_1^a P_2^b}{P_1 P_2}\right)^{2k}. \tag{4.8}$$

Using this and recalling that  $\mathbf{P}_{12}^2$  is a multiplication operator, the above candidate for the ‘distance operator’ becomes an expression of the *positive powers* of  $\mathbf{W}_{12}$  and  $\mathbf{P}_{12}^2$ . Nevertheless, now the infinite series makes the application of the resulting expression difficult in practice. Thus, although in principle this might yield a well defined *operator* for the distance of the two subsystems, and certainly it would be worth studying this, in the present paper we choose a different strategy and look for only the ‘empirical distance’. This is the one that we followed in [6] in defining the angle between the angular momentum vectors of  $SU(2)$ -invariant elementary quantum mechanical systems.

### 4.2 The Empirical Distance

Based on equation (4.6) and the discussion above, we define the *empirical distance of the two  $E(3)$ -invariant elementary quantum mechanical systems* (characterized by their Casimir invariants  $(P_1, s_1)$  and  $(P_2, s_2)$ ) in their states  $\phi_1$  and  $\phi_2$ , respectively, by

$$d_{12} := \frac{\langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12} - \hbar(s_1/P_1 + s_2/P_2) \mathbf{P}_{12}^2 | \phi_1 \otimes \phi_2 \rangle}{\sqrt{P_1^2 P_2^2 - \langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^4 | \phi_1 \otimes \phi_2 \rangle}}. \tag{4.9}$$

$d_{ij}$  can, in fact, be defined in any state of the composite system consisting of any number of elementary systems,  $\mathbf{i}, \mathbf{j} = 1, \dots, N$ , represented by a general density operator  $\rho : \mathcal{H}_{P_1, s_1} \otimes \cdots \otimes \mathcal{H}_{P_N, s_N} \rightarrow \mathcal{H}_{P_1, s_1} \otimes \cdots \otimes \mathcal{H}_{P_N, s_N}$ , not only in pure tensor product states of a bipartite system. However, if the density operator represents a pure vector state which is an *entangled* state of the constituent systems, or if it is a genuine *mixed* state of the composite system, then the state of the constituent systems would necessarily be *mixed*; moreover the empirical distance  $d_{ij}$  would depend on the state of the subsystems other than the  $\mathbf{i}$ ’s and the  $\mathbf{j}$ ’s. Hence, in these cases the interpretation of  $d_{ij}$  would not be obvious. Therefore, in the present paper, we assume that the states of the composite system are *tensor products of pure vector states* of the constituent systems; and hence, without loss of generality, the composite system could be assumed to consist only of two subsystems.

In this subsection, we calculate  $d_{12}$  using (4.9) and discuss its properties at the genuine quantum level. The classical limit will be considered in subsection 4.4.

Let us write  $\phi_1 = \exp(-ip_{1e} \xi_1^e / \hbar) \mathbf{U}_1 \psi_1$ , where  $\mathbf{U}_1$  is the unitary operator on  $\mathcal{H}_{P_1, s_1}$  representing an  $SU(2)$  matrix  $U_{1B}^A$  and  $\xi_1^e$  is a translation. Or, in other words,

$\phi_1$  is considered to be obtained from the state  $\psi_1$  by some  $E(3)$  transformation. The state  $\phi_2$  is assumed to have the analogous form. Then by (4.4), (3.8) and (3.10) the first term in the numerator in (4.9) is

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12} | \phi_1 \otimes \phi_2 \rangle &= (\xi_1^a - \xi_2^a) \varepsilon_{abc} (R_1)^b{}_d \langle \psi_1 | \mathbf{p}_1^d | \psi_1 \rangle (R_2)^c{}_e \langle \psi_2 | \mathbf{p}_2^e | \psi_2 \rangle + \\ &+ (R_1^{-1} R_2)_{ab} \left( \langle \psi_1 | \mathbf{p}_1^a | \psi_1 \rangle \langle \psi_2 | \mathbf{J}_2^b | \psi_2 \rangle + \langle \psi_1 | \mathbf{J}_1^a | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^b | \psi_2 \rangle \right). \end{aligned} \tag{4.10}$$

In a similar way, the relevant factor in the second term of the numerator and the non-trivial term in the denominator, respectively, are

$$\langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^2 | \phi_1 \otimes \phi_2 \rangle = (R_1^{-1} R_2)_{ab} \langle \psi_1 | \mathbf{p}_1^a | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^b | \psi_2 \rangle; \tag{4.11}$$

$$\langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^4 | \phi_1 \otimes \phi_2 \rangle = (R_1^{-1} R_2)_{ab} (R_1^{-1} R_2)_{cd} \langle \psi_1 | \mathbf{p}_1^a \mathbf{p}_1^c | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^b \mathbf{p}_2^d | \psi_2 \rangle. \tag{4.12}$$

Thus, it is only  $\xi_1^a - \xi_2^a$ , i.e. only the *relative* ‘position’ of the two systems, that matters in the empirical distance. We will see that, in a similar way, it is only the *relative* ‘orientation’ of the two subsystems that matters. Next we specify the states  $\psi_1$  and  $\psi_2$ .

If  $\psi = {}_s Y_{jm} / P$ , then by (A.19), (A.22), (A.25) and (A.28)–(A.30)

$$\langle \psi | \mathbf{p}^a | \psi \rangle = \delta_3^a P \frac{ms}{j(j+1)}, \quad \langle \psi | \mathbf{J}^a | \psi \rangle = \delta_3^a \hbar m. \tag{4.13}$$

Thus, roughly speaking, in the state  ${}_s Y_{jm} / P$  both the linear and angular momenta point in the ‘z-direction’ (with respect to the basis in the *momentum space*). Using (A.18)–(A.26), a direct calculation gives that

$$\begin{aligned} \langle {}_s Y_{jm} | \mathbf{p}^1 \mathbf{p}^1 | {}_s Y_{jm} \rangle &= \langle {}_s Y_{jm} | \mathbf{p}^2 \mathbf{p}^2 | {}_s Y_{jm} \rangle = \frac{P^4}{j(j+1)(2j-1)(2j+3)} \left( -3s^2 m^2 + \right. \\ &\left. + j(j+1)(s^2 + m^2) + j(j+1)(j^2 + j - 1) \right), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \langle {}_s Y_{jm} | \mathbf{p}^3 \mathbf{p}^3 | {}_s Y_{jm} \rangle &= \frac{P^4}{j(j+1)(2j-1)(2j+3)} \left( 6s^2 m^2 - \right. \\ &\left. - 2j(j+1)(s^2 + m^2) + j(j+1)(2j^2 + 2j - 1) \right); \end{aligned} \tag{4.15}$$

and that all the other components of  $\langle {}_s Y_{jm} | \mathbf{p}^a \mathbf{p}^b | {}_s Y_{jm} \rangle$  are vanishing. (4.14) and (4.15) imply that  $\delta_{ab} \langle {}_s Y_{jm} | \mathbf{p}^a \mathbf{p}^b | {}_s Y_{jm} \rangle = P^4$ , as it should be. These expectation values may appear to be singular when  $j = 0$  or  $1/2$ , but these are not. In fact, in these cases  $s = 0$  and  $|s| = 1/2$ , and hence  $j = n$  and  $j = 1/2 + n$ ,  $n = 0, 1, 2, \dots$ , respectively. Writing these into (4.14) and (4.15) and then substituting  $n = 0$ , we obtain that these are  $P^4/3$  in both cases.

Choosing both  $\psi_1$  and  $\psi_2$  in the above way, and substituting (4.13) into (4.10) and (4.11), we find, respectively, that



$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12} | \phi_1 \otimes \phi_2 \rangle &= (\xi_1^a - \xi_2^a) \epsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 P_1 P_2 \frac{s_1 s_2 m_1 m_2}{j_1(j_1 + 1)j_2(j_2 + 1)} + \\ &+ (R_1^{-1} R_2)_{33} \hbar m_1 m_2 \left( \frac{P_1 s_1}{j_1(j_1 + 1)} + \frac{P_2 s_2}{j_2(j_2 + 1)} \right), \\ \langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^2 | \phi_1 \otimes \phi_2 \rangle &= (R_1^{-1} R_2)_{33} P_1 P_2 \frac{s_1 s_2 m_1 m_2}{j_1(j_1 + 1)j_2(j_2 + 1)}. \end{aligned}$$

Hence, the numerator of (4.9) is

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12} - \hbar \left( \frac{s_1}{P_1} + \frac{s_2}{P_2} \right) \mathbf{P}_{12}^2 | \phi_1 \otimes \phi_2 \rangle &= \\ &= P_1 P_2 \left\{ (\xi_1^a - \xi_2^a) \epsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 + \right. \\ &\left. + (R_1^{-1} R_2)_{33} \hbar \left( \frac{j_2(j_2 + 1) - s_2^2}{s_2 P_2} + \frac{j_1(j_1 + 1) - s_1^2}{s_1 P_1} \right) \right\} \frac{s_1 s_2 m_1 m_2}{j_1(j_1 + 1)j_2(j_2 + 1)}. \end{aligned} \tag{4.16}$$

If  $s_1 s_2 m_1 m_2 = 0$ , then this is zero. In this case at least one of the expectation values  $\langle \phi_1 | \mathbf{p}_1^a | \phi_1 \rangle$  and  $\langle \phi_2 | \mathbf{p}_2^a | \phi_2 \rangle$  is vanishing (see the first of (4.13)). This case is analogous to the classical situation when  $p_1^a = 0$  or  $p_2^a = 0$ , and that we excluded from our investigations (see the second paragraph in subsection 2.1). It might be worth noting that (4.16) is just ( $P_1 P_2$  times) the expectation value of the distance operator (4.7) in the zeroth approximation according to the expansion (4.8).

Using  $\langle {}_s Y_{jm} | \mathbf{p}^1 \mathbf{p}^1 | {}_s Y_{jm} \rangle = \langle {}_s Y_{jm} | \mathbf{p}^2 \mathbf{p}^2 | {}_s Y_{jm} \rangle$ , we obtain that, in the states above, (4.12) takes the form

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^4 | \phi_1 \otimes \phi_2 \rangle &= \left( (R_1^{-1} R_2)_{11} \right)^2 + \left( (R_1^{-1} R_2)_{12} \right)^2 + \left( (R_1^{-1} R_2)_{21} \right)^2 + \\ &+ \left( (R_1^{-1} R_2)_{22} \right)^2 \langle \psi_1 | \mathbf{p}_1^1 \mathbf{p}_1^1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^1 \mathbf{p}_2^1 | \psi_2 \rangle + \\ &+ \left( \left( (R_1^{-1} R_2)_{13} \right)^2 + \left( (R_1^{-1} R_2)_{23} \right)^2 \right) \langle \psi_1 | \mathbf{p}_1^1 \mathbf{p}_1^1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^3 \mathbf{p}_2^3 | \psi_2 \rangle + \\ &+ \left( \left( (R_1^{-1} R_2)_{31} \right)^2 + \left( (R_1^{-1} R_2)_{32} \right)^2 \right) \langle \psi_1 | \mathbf{p}_1^3 \mathbf{p}_1^3 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^1 \mathbf{p}_2^1 | \psi_2 \rangle + \\ &+ \left( (R_1^{-1} R_2)_{33} \right)^2 \langle \psi_1 | \mathbf{p}_1^3 \mathbf{p}_1^3 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^3 \mathbf{p}_2^3 | \psi_2 \rangle. \end{aligned} \tag{4.17}$$

If the  $SU(2)$  matrix  $U^A{}_B$  is parameterized by the familiar Euler angles  $(\alpha, \beta, \gamma)$  according to

$$U^A{}_B = \begin{pmatrix} \exp\left(\frac{i}{2}(\alpha + \gamma)\right) \cos(\beta/2) & i \exp\left(-\frac{i}{2}(\alpha - \gamma)\right) \sin(\beta/2) \\ i \exp\left(\frac{i}{2}(\alpha - \gamma)\right) \sin(\beta/2) & \exp\left(-\frac{i}{2}(\alpha + \gamma)\right) \cos(\beta/2) \end{pmatrix}, \tag{4.18}$$

then the corresponding rotation matrix is

$$R^a_b = \begin{pmatrix} \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & -\sin \alpha \cos \gamma - \cos \alpha \cos \beta \sin \gamma & \sin \beta \sin \gamma \\ \cos \alpha \sin \gamma + \sin \alpha \cos \beta \cos \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma & -\sin \beta \cos \gamma \\ \sin \alpha \sin \beta & \cos \alpha \sin \beta & \cos \beta \end{pmatrix}. \tag{4.19}$$

This yields, in particular, that

$$\cos \beta_{12} := (R_1^{-1}R_2)_{33} = \cos \beta_1 \cos \beta_2 + \cos(\gamma_1 - \gamma_2) \sin \beta_1 \sin \beta_2; \tag{4.20}$$

and that  $\beta_{12}$  is just the angle between the unit vectors  $(R_1)^a_3$  and  $(R_2)^a_3$ , too:  $\cos \beta_{12} = \delta_{ab}(R_1)^a_3(R_2)^b_3$ . Also, the combinations of the matrix elements  $((R_1^{-1}R_2)_{ab})^2$  in (4.17) are all expressions of  $\cos^2 \beta_{12}$  alone:

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{P}^4_{12} | \phi_1 \otimes \phi_2 \rangle &= (1 + \cos^2 \beta_{12}) \langle \psi_1 | \mathbf{p}^1_1 \mathbf{p}^1_1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}^1_2 \mathbf{p}^1_2 | \psi_2 \rangle + \\ &+ (1 - \cos^2 \beta_{12}) \langle \psi_1 | \mathbf{p}^1_1 \mathbf{p}^1_1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}^3_2 \mathbf{p}^3_2 | \psi_2 \rangle + \\ &+ (1 - \cos^2 \beta_{12}) \langle \psi_1 | \mathbf{p}^3_1 \mathbf{p}^3_1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}^1_2 \mathbf{p}^1_2 | \psi_2 \rangle + \\ &+ \cos^2 \beta_{12} \langle \psi_1 | \mathbf{p}^3_1 \mathbf{p}^3_1 | \psi_1 \rangle \langle \psi_2 | \mathbf{p}^3_2 \mathbf{p}^3_2 | \psi_2 \rangle. \end{aligned} \tag{4.21}$$

Since the length of the vector  $\epsilon^a_{bc}(R_1)^b_3(R_2)^c_3$  in (4.16) is  $\sin \beta_{12}$ , (4.21) shows that  $d_{12}$  depends only on the  $\beta_{12}$ , i.e. on the *relative* ‘orientation’ of the two constituent systems, rather than the individual Euler angles  $(\alpha_1, \beta_1, \gamma_1)$  and  $(\alpha_2, \beta_2, \gamma_2)$ .

Denoting the denominator in (4.9) by  $P_1P_2D$  and substituting (4.14) and (4.15) into (4.21), a lengthy but straightforward calculation gives that

$$\begin{aligned} D^2 &= 1 - \frac{1}{(2j_1 - 1)(2j_1 + 3)(2j_2 - 1)(2j_2 + 3)} \left( 5j_1(j_1 + 1)j_2(j_2 + 1) + \right. \\ &+ (s_1^2 - 4)j_2(j_2 + 1) + (s_2^2 - 4)j_1(j_1 + 1) - 3s_1^2s_2^2 + 3 \left. \right) \\ &- \frac{(j_1(j_1 + 1) - 3s_1^2)(j_2(j_2 + 1) - 3s_2^2)}{(2j_1 - 1)(2j_1 + 3)(2j_2 - 1)(2j_2 + 3)} \left( \frac{m_1^2}{j_1(j_1 + 1)} + \frac{m_2^2}{j_2(j_2 + 1)} - \frac{3m_1^2m_2^2}{j_1(j_1 + 1)j_2(j_2 + 1)} \right) \\ &- \frac{(j_1(j_1 + 1) - 3s_1^2)(j_1(j_1 + 1) - 3m_1^2)(j_2(j_2 + 1) - 3s_2^2)(j_2(j_2 + 1) - 3m_2^2)}{j_1(j_1 + 1)(2j_1 - 1)(2j_1 + 3)j_2(j_2 + 1)(2j_2 - 1)(2j_2 + 3)} \cos^2 \beta_{12}. \end{aligned} \tag{4.22}$$

If  $j_1$  and  $j_2$  take their *smallest* value, viz.  $j_1 = |s_1|$  and  $j_2 = |s_2|$ , i.e. when  $\psi_1$  and  $\psi_2$  are *centre-of-mass states* (see subsection 3.2), then this expression reduces to

$$D^2 = \frac{2}{3} + \frac{(3m_1^2 - j_1(j_1 + 1))(3m_2^2 - j_2(j_2 + 1))}{3(j_1 + 1)(2j_1 + 3)(j_2 + 1)(2j_2 + 3)} (1 - 3 \cos^2 \beta_{12}). \tag{4.23}$$

If at least one of  $j_1 = |s_1|$  and  $j_2 = |s_2|$  is 0 or 1/2, then  $D^2 = 2/3$ , and hence, in particular, it is not zero and it does not depend on  $\beta_{12}$ . For small spins the dependence of  $D^2$  on  $\beta_{12}$  is weak. The higher the spins  $s_1$  and  $s_2$ , the closer the  $D^2$  to zero for  $|m_1| = j_1, |m_2| = j_2$  and  $\cos^2 \beta_{12} = 1$ . Nevertheless, we show that  $D^2$  is *strictly positive* for any finite  $|m_1| \leq j_1 = |s_1|$  and  $|m_2| \leq j_2 = |s_2|$  and any angle  $\beta_{12}$ .

Suppose, on the contrary, that  $D^2 = 0$ , i.e. that for some  $m_1, m_2$  and  $\beta_{12}$

$$2(j_1 + 1)(j_2 + 1)(2j_1 + 3)(2j_2 + 3) = (3m_1^2 - j_1(j_1 + 1))(3m_2^2 - j_2(j_2 + 1))(3 \cos^2 \beta_{12} - 1) \tag{4.24}$$

holds. Since the left hand side is positive,  $3 \cos^2 \beta_{12} \neq 1$ ,  $3m_1^2 \neq j_1(j_1 + 1)$ ,  $3m_2^2 \neq j_2(j_2 + 1)$  and  $j_1 \neq 0 \neq j_2$  must hold. First, let us suppose that  $-1 \leq 3 \cos^2 \beta_{12} - 1 < 0$ . Then by (4.24)  $3m_1^2 > j_1(j_1 + 1)$  and  $3m_2^2 < j_2(j_2 + 1)$  or  $3m_1^2 < j_1(j_1 + 1)$  and  $3m_2^2 > j_2(j_2 + 1)$  follow. Let us consider the first case. Then, also by (4.24),

$$\begin{aligned} 2(j_1 + 1)(j_2 + 1)(4j_1j_2 + 6(j_1 + j_2) + 9) &\leq -(3m_1^2 - j_1(j_1 + 1))(3m_2^2 - j_2(j_2 + 1)) \\ &< 3m_1^2j_2(j_2 + 1) - 3m_2^2(3m_1^2 - j_1(j_1 + 1)) - j_1j_2(j_1 + 1)(j_2 + 1) < 3m_1^2j_2(j_2 + 1) \\ &\leq 3j_1^2j_2(j_2 + 1). \end{aligned}$$

This implies  $8j_1^2j_2 < 2(j_1 + 1)(4j_1j_2 + 6j_1 + 6j_2 + 9) \leq 3j_1^2j_2$ , which is a contradiction. The proof is similar if  $3m_1^2 < j_1(j_1 + 1)$  and  $3m_2^2 > j_2(j_2 + 1)$ . Next suppose that  $0 < 3 \cos^2 \beta_{12} - 1 \leq 2$ . Then (4.24) implies that

$$(j_1 + 1)(j_2 + 1)(4j_1j_2 + 6j_1 + 6j_2 + 9) \leq (3m_1^2 - j_1(j_1 + 1))(3m_2^2 - j_2(j_2 + 1)),$$

and that either  $3m_1^2 > j_1(j_1 + 1)$  and  $3m_2^2 > j_2(j_2 + 1)$  or  $3m_1^2 < j_1(j_1 + 1)$  and  $3m_2^2 < j_2(j_2 + 1)$ . In the first case this yields

$$\begin{aligned} (j_1 + 1)(j_2 + 1)(3j_1j_2 + 6j_1 + 6j_2 + 9) &\leq 9m_1^2m_2^2 - 3m_1^2j_2(j_2 + 1) - 3m_2^2j_1(j_1 + 1) \\ &\leq 9m_1^2m_2^2 - 3m_1^2j_2^2 - 3m_2^2j_1^2 \leq 3m_1^2m_2^2 - 3m_1^2(j_2^2 - m_2^2) - 3m_2^2(j_1^2 - m_1^2) \\ &\leq 3m_1^2m_2^2 \leq 3j_1^2j_2^2, \end{aligned}$$

which is a contradiction. In the second case,

$$\begin{aligned} (j_1 + 1)(j_2 + 1)(3j_1j_2 + 6j_1 + 6j_2 + 9) &\leq 9m_1^2m_2^2 - 3m_1^2j_2(j_2 + 1) - 3m_2^2j_1(j_1 + 1) \\ &< 9m_1^2m_2^2 - 9m_1^2m_2^2 - 9m_2^2m_1^2 = -9m_1^2m_2^2, \end{aligned}$$

which is also a contradiction. We expect that the denominator  $D$ , given by (4.22), is not zero even in the general case when  $j_1 > |s_1|$  and  $j_2 > |s_2|$ .

The other extreme case is when both  $j_1$  and  $j_2$  tend to infinity. Now there are three sub-cases: when  $m_1$  and  $m_2$  remain bounded, and when one of them, say  $m_1$ , or both tend to infinity with  $j_1$  and  $j_2$ . As equations (4.13) show, in all these cases the expectation value of the linear momenta tends to zero, but in the first case the expectation value of the angular momenta remain finite; in the second the expectation value of  $\mathbf{J}_1^a$  diverges but that of  $\mathbf{J}_2^a$  remains finite; while in the third the expectation value of  $\mathbf{J}_1^a$  and  $\mathbf{J}_2^a$  diverges. By (4.22) these limits of  $D^2$  are

$$\frac{11}{16} - \frac{1}{16} \cos^2 \beta_{12}, \frac{5}{8} + \frac{1}{8} \cos^2 \beta_{12}, \frac{3}{4} - \frac{1}{4} \cos^2 \beta_{12}, \tag{4.25}$$

respectively. These are independent of the spins, and none of them is zero.

Therefore, the empirical distance  $d_{12}$  between the elementary systems is well defined, finite or zero, at least in the states obtained from centre-of-mass states

by  $E(3)$  transformations, and also in the  $j_1, j_2 \rightarrow \infty$  limit, and depends only on the relative ‘position’ and ‘orientation’ of the subsystems. Next we discuss the resulting expression of  $d_{12}$  in these two extreme cases.

In the first case, i.e. when  $j_1 = |s_1|$  and  $j_2 = |s_2|$ ,

$$d_{12} = \frac{1}{D} \left( (\xi_1^a - \xi_2^a) \epsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 + \hbar \left( \frac{s_1}{|s_1| P_1} + \frac{s_2}{|s_2| P_2} \right) \cos \beta_{12} \right) \frac{s_1 s_2}{|s_1 s_2| (|s_1| + 1)(|s_2| + 1)}, \tag{4.26}$$

where now  $D$  is given by (4.23), and the components of the unit vectors  $(R_1)^a{}_3$  and  $(R_2)^a{}_3$  can be read off from (4.19). The first term in the brackets can be zero when these unit vectors are parallel,  $(R_1)^a{}_3 = \pm (R_2)^a{}_3$ , i.e. when  $\beta_{12} = 0$  or  $\pi$ , or when  $\xi_1^a - \xi_2^a$  is zero or at least it lays in the 2-plane spanned by  $(R_1)^a{}_3$  and  $(R_2)^a{}_3$ . This term can be arbitrarily large, depending on  $\xi_1^a - \xi_2^a$ . This does not contain Planck’s constant and the Casimir invariants  $P_1$  and  $P_2$ , and gives the ‘classical part’ of the distance, being analogous to the last term on the right of equation (2.5).

The second term in the brackets, being proportional to  $\hbar$ , is a genuine quantum correction to the classical part.  $d_{12}$  depends on  $P_1$  and  $P_2$  only through this term. This can be zero only if  $\beta_{12} = \pi/2$ ; and, for  $\beta_{12} \neq \pi/2$ , only in the *very exceptional case* when  $P_1 = P_2$  and  $\text{sign}(s_1) = -\text{sign}(s_2)$ , i.e. if one of  $\psi_1$  and  $\psi_2$  is holomorphic and the other is anti-holomorphic.

Even if  $\xi_1^a - \xi_2^a$ ,  $(R_1)^a{}_3$  and  $(R_2)^a{}_3$  are given,  $d_{12}$  is not fixed: it depends on the discrete ‘quantum numbers’  $m_1$  and  $m_2$  of the actual states in an essential way. In particular, for  $s_1 = s_2 = 1/2$  (4.26) gives

$$d_{12} = \pm \frac{1}{3\sqrt{6}} \left( (\xi_1^a - \xi_2^a) \epsilon_{abc} (R_1)^a{}_3 (R_2)^c{}_3 + \hbar \frac{P_1 + P_2}{P_1 P_2} \cos \beta_{12} \right).$$

Its ‘classical part’ is less than one-sixth of the distance between the two classical point particles with the same  $((R_1)^a{}_b, \xi_1^a)$  and  $((R_2)^a{}_b, \xi_2^a)$ .

If  $\cos \beta_{12} = \pm 1$ , then the first term between the brackets in (4.26) vanishes, and  $d_{12}$  becomes an expression of the Casimir invariants  $(P_1, s_1)$ ,  $(P_2, s_2)$  of the elementary systems and the discrete quantum numbers  $m_1$  and  $m_2$  alone. So the distance in this case is ‘universal’, it is of purely quantum mechanical origin, and, apart from the very exceptional case above, non-zero. Thus, at the quantum level, the expression for  $d_{12}$  is well defined, in contrast to the classical case when, by the discussion of subsection 2.2, the distance between two centre-of-mass lines with parallel linear momenta could be recovered only as a limit.

As we concluded above,  $d_{12}$  is well defined also in the other extreme case when  $s_1$  and  $s_2$  are fixed but  $j_1, j_2 \rightarrow \infty$ . Now we determine the distance in this case explicitly. As (4.16) shows, in the first two cases considered in (4.25) the empirical distance  $d_{12}$  tends to zero, while in the third (i.e. when  $|m_1| = j_1, |m_2| = j_2 \rightarrow \infty$ ) it tends to

$$2\hbar \left( \frac{s_1}{P_2} + \frac{s_2}{P_1} \right) \frac{\cos \beta_{12}}{\sqrt{3 - \cos^2 \beta_{12}}}.$$

In particular, this limit is independent of  $\xi_1^a - \xi_2^a$ , and  $d_{12}$  reduces to the quantum correction. Nevertheless, this extreme case corresponds to a rather exotic situation, since by (4.13) the expectation value of the angular momenta tend to infinity while that of the linear momenta to zero.

As we noted, in the states with the choice for  $\psi_1, \psi_2$  above all the expectation values are vanishing for  $s_1 = 0, s_2 = 0$ . To get non-zero results in this case, more general states with  $\psi = \sum_{j,m} c^{jm} Y_{jm}$  should be considered, because the matrix elements  $\langle {}_s Y_{j\pm 1,n} | \mathbf{p}^a | {}_s Y_{j,m} \rangle$  are not all zero even for  $s = 0$  (see Appendix A.2). However, the explicit form of the resulting expectation values are much more complicated.

### 4.3 The Empirical Angles and Volume Elements

Dictated by the classical formula (2.9), we define the ‘empirical angle’ between the linear momenta of two elementary subsystems (characterized by  $(P_1, s_1)$  and  $(P_2, s_2)$ , respectively) in their pure tensor product state  $\phi = \phi_1 \otimes \phi_2$  by

$$\cos \omega_{12} := \frac{\langle \phi | \delta_{ab} \mathbf{p}_1^a \otimes \mathbf{p}_2^b | \phi \rangle}{\sqrt{\langle \phi | \mathbf{P}_1^2 | \phi \rangle} \sqrt{\langle \phi | \mathbf{P}_2^2 | \phi \rangle}} = \frac{\langle \phi_1 | \mathbf{p}_1^a | \phi_1 \rangle \delta_{ab} \langle \phi_2 | \mathbf{p}_2^b | \phi_2 \rangle}{P_1 P_2} \tag{4.27}$$

with range  $\omega_{12} \in [0, \pi]$ . Also, motivated by the classical expression (2.10), we define the ‘empirical 3-volume element’ for three elementary systems in the pure tensor product state  $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$  by

$$v_{123} := \frac{1}{3!} \epsilon_{abc} \frac{\langle \phi | \mathbf{p}_1^a \otimes \mathbf{p}_2^b \otimes \mathbf{p}_3^c | \phi \rangle}{\sqrt{\langle \phi | \mathbf{P}_1^2 | \phi \rangle} \sqrt{\langle \phi | \mathbf{P}_2^2 | \phi \rangle} \sqrt{\langle \phi | \mathbf{P}_3^2 | \phi \rangle}}. \tag{4.28}$$

Since these quantities are built only from the momentum operators, and the momentum operators are invariant with respect to translations, it is enough to evaluate these expressions only in the states of the form  $\phi = \mathbf{U}\psi$ .

Thus, if  $\phi_1 = \mathbf{U}_1 \psi_1$  and  $\phi_2 = \mathbf{U}_2 \psi_2$ , then

$$\begin{aligned} \cos \omega_{12} &= \frac{1}{P_1 P_2} \langle \psi_1 | \mathbf{U}_1^\dagger \mathbf{p}_1^a \mathbf{U}_1 | \psi_1 \rangle \delta_{ab} \langle \psi_2 | \mathbf{U}_2^\dagger \mathbf{p}_2^b \mathbf{U}_2 | \psi_2 \rangle = \\ &= \frac{1}{P_1 P_2} (R_1^{-1} R_2)_{ab} \langle \psi_1 | \mathbf{p}_1^a | \psi_1 \rangle \langle \psi_2 | \mathbf{p}_2^b | \psi_2 \rangle. \end{aligned} \tag{4.29}$$

In particular, if  $\psi_1 = {}_{s_1} Y_{j_1, m_1} / P_1$  and  $\psi_2 = {}_{s_2} Y_{j_2, m_2} / P_2$ , then

$$\cos \omega_{12} = s_1 s_2 \frac{m_1 m_2}{j_1(j_1 + 1)j_2(j_2 + 1)} \cos \beta_{12}. \tag{4.30}$$

The angle  $\omega_{12}$  has the same qualitative properties that the empirical angle  $\theta_{12}$  has in the  $SU(2)$ -invariant systems [6]. In particular, for given  $s_1$  and  $s_2$  and angle  $\beta_{12}$ , the empirical angle  $\omega_{12}$  is still not fixed, that may take different discrete values. Moreover,  $\omega_{12}$  is never zero even if  $\beta_{12} = 0$ , and is never  $\pi$  even if  $\beta_{12} = \pi$ . With given  $s_1$  and  $s_2$  the empirical angle  $\omega_{12}$  takes its minimal value in the special centre-of-mass

states when  $m_1 = j_1 = |s_1|$  and  $m_2 = j_2 = |s_2|$ , and  $\beta_{12} = 0$ . In particular, for  $|s_1| = |s_2| = 1/2$  this angle is  $\omega_{12}^{min} \approx 83.62^\circ$ , while for  $|s_1| = |s_2| = 1$  it is  $\approx 75.52^\circ$ . The maximal value of  $\omega_{12}$  is  $\omega_{12}^{max} = \pi - \omega_{12}^{min}$ . This minimal/maximal value tends to zero/ $\pi$  only in the  $|s_1|, |s_2| \rightarrow \infty$  limit. For a classical model of the ‘geometry of the quantum directions’, see Section 4 of [6].

The evaluation of the empirical 3-volume element in the analogous states is similar. By the first of (4.13) we obtain

$$v_{123} = \frac{s_1 s_2 s_3 m_1 m_2 m_3}{j_1(j_1 + 1)j_2(j_2 + 1)j_3(j_3 + 1)} \frac{1}{3!} \epsilon_{abc} (R_1)^a{}_3 (R_2)^b{}_3 (R_3)^c{}_3. \tag{4.31}$$

The second factor in (4.31) is just the Euclidean 3-volume of the tetrahedron spanned by the unit vectors  $(R_1)^a{}_3, (R_2)^a{}_3$  and  $(R_3)^a{}_3$ . Hence, even for given  $(R_1)^a{}_3, (R_2)^a{}_3$  and  $(R_3)^a{}_3$ , the empirical 3-volume element takes different *discrete* values, depending on the ‘quantum numbers’ of the states of the constituent subsystems. By (4.31)  $v_{123}$  is *always* smaller than the Euclidean 3-volume element, even if all the  $j$ ’s take their minimal, and all the  $m$ ’s take their maximal value, viz.  $m = j = |s|$ .

#### 4.4 The Classical Limit of the Empirical Geometrical Quantities

Traditionally, the classical limit of an  $SU(2)$ -invariant system is defined to be the limit in which  $m = j \rightarrow \infty$  (see e.g. [22]). However, by  $|m| \leq j$  and (4.13), in the present  $E(3)$ -invariant case, the expectation value of the linear momentum tends to zero unless the spin  $s$  also tends to infinity; and this expectation value can tend to a large macroscopic value if  $P$  is also growing appropriately. Thus, formally, the classical limit of the  $E(3)$ -invariant systems should be defined to be the limit in which  $|s| = m = j \rightarrow \infty$  and  $P \rightarrow \infty$ . Moreover, if we expect that the expectation value of the linear and angular momenta tend to the corresponding large classical value in the same order, then by (4.13) we should require that asymptotically  $P = pj + O(1/j)$  holds for some positive  $p$ . We use this latter condition in the calculation of the classical limit of the uncertainty of the empirical distance. Note that by  $j = |s|$  the states in such a sequence are obtained from *centre-of-mass states* by some  $E(3)$  transformations. In the present subsection, we determine this limit of the empirical distances and their uncertainty, and also that of the angles and 3-volume elements. We find that these are just those in the Euclidean 3-space.

The  $s_1 = m_1 = j_1, \pm s_2 = m_2 = j_2 \rightarrow \infty$  limit of the numerator in the expression (4.26) of the empirical distance is

$$\pm (\xi_1^a - \xi_2^a) \epsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 \pm \hbar \frac{P_1 \pm P_2}{P_1 P_2} \cos \beta_{12};$$

while, by (4.23), the same limit of its denominator is  $\sin \beta_{12}$ . But the latter is just the magnitude of  $\epsilon^a{}_{bc} (R_1)^b{}_3 (R_2)^c{}_3$ , and hence, apart from the quantum correction, for  $\beta_{12} \neq 0, \pi$  this gives just the classical (signed) distance between the centre-of-mass lines of the two classical point particles characterized by  $((R_1)^a{}_b, \xi_1^a)$  and  $((R_2)^a{}_b, \xi_2^a)$ , respectively. If, in addition,  $P_1, P_2 \rightarrow \infty$ , then the quantum correction goes away, and the whole expression reduces to the classical empirical distance.

Since  $d_{12}$  is *not* the expectation value of some quantum observable, its standard deviation/variance cannot be defined in the standard manner. Nevertheless, its uncertainty can be introduced by importing the idea from experimental physics how the error of a quantity, built from experimental data, is defined. Namely, if  $Q = Q(q^1, \dots, q^n)$  is a differentiable function of its variables and, in a series of measurements, we obtain the mean values  $\bar{q}^\alpha$  and errors  $\delta q^\alpha$  of the quantities  $q^\alpha$ ,  $\alpha = 1, \dots, n$ , then the mean value and error of  $Q$  are defined to be  $\bar{Q} := Q(\bar{q}^1, \dots, \bar{q}^n)$  and

$$\delta Q := \left| \frac{\partial Q}{\partial q^1}(\bar{q}^\alpha) \right| \delta q^1 + \dots + \left| \frac{\partial Q}{\partial q^n}(\bar{q}^\alpha) \right| \delta q^n,$$

respectively. Hence, since the empirical distance has the structure

$$d_{12} = \frac{\langle \phi | \mathbf{A}_{12} | \phi \rangle}{\sqrt{\langle \phi | \mathbf{B}_{12} | \phi \rangle}}$$

with  $\mathbf{A}_{12} := \mathbf{W}_{12} - \hbar(s_1/P_1 + s_2/P_2)\mathbf{P}_{12}^2$  and  $\mathbf{B}_{12} := P_1^2 P_2^2 - \mathbf{P}_{12}^4$ , it seems natural to define the *uncertainty* of  $d_{12}$  in the state  $\phi = \phi_1 \otimes \phi_2$  to be

$$\delta_\phi d_{12} := \left( \frac{\Delta_\phi \mathbf{A}_{12}}{|\langle \phi | \mathbf{A}_{12} | \phi \rangle|} + \frac{1}{2} \frac{\Delta_\phi \mathbf{B}_{12}}{\langle \phi | \mathbf{B}_{12} | \phi \rangle} \right) |d_{12}|. \tag{4.32}$$

Here

$$(\Delta_\phi \mathbf{A}_{12})^2 = \langle \phi | (\mathbf{A}_{12})^2 | \phi \rangle - (\langle \phi | \mathbf{A}_{12} | \phi \rangle)^2$$

and

$$(\Delta_\phi \mathbf{B}_{12})^2 = \langle \phi | (\mathbf{B}_{12})^2 | \phi \rangle - (\langle \phi | \mathbf{B}_{12} | \phi \rangle)^2,$$

the two familiar variances. We are going to show that, in the classical limit defined above, both terms between the brackets tend to zero. Since  $|d_{12}|$  in this limit is bounded, this means that the uncertainty  $\delta_\phi d_{12}$  also tends to zero.

Since the subsequent calculations are quite lengthy but elementary, we do not provide all the details. We indicate only the key steps. First, let us consider  $\langle \phi | (\mathbf{A}_{12})^2 | \phi \rangle = \langle \phi | (\mathbf{W}_{12}^2 - 2\hbar(j_1/P_1 \pm j_2/P_2)\mathbf{P}_{12}^2 \mathbf{W}_{12} + \hbar^2(j_1/P_1 \pm j_2/P_2)^2 \mathbf{P}_{12}^4) | \phi \rangle$ . The expectation values in

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12}^2 | \phi_1 \otimes \phi_2 \rangle &= \langle \phi_1 | \mathbf{p}_1^a \mathbf{p}_1^b | \phi_1 \rangle \delta_{ac} \delta_{bd} \langle \phi_2 | \mathbf{J}_2^c \mathbf{J}_2^d | \phi_2 \rangle + \langle \phi_1 | \mathbf{p}_1^a \mathbf{J}_1^b | \phi_1 \rangle \delta_{ac} \delta_{bd} \langle \phi_2 | \mathbf{J}_2^c \mathbf{p}_2^d | \phi_2 \rangle \\ &+ \langle \phi_1 | \mathbf{J}_1^a \mathbf{p}_1^b | \phi_1 \rangle \delta_{ac} \delta_{bd} \langle \phi_2 | \mathbf{p}_2^c \mathbf{J}_2^d | \phi_2 \rangle + \langle \phi_1 | \mathbf{J}_1^a \mathbf{J}_1^b | \phi_1 \rangle \delta_{ac} \delta_{bd} \langle \phi_2 | \mathbf{p}_2^c \mathbf{p}_2^d | \phi_2 \rangle \end{aligned}$$

can be calculated by using (3.9). These are

$$\begin{aligned} \langle \phi | \mathbf{p}^a \mathbf{p}^b | \phi \rangle &= R^a_c R^b_d \langle \psi | \mathbf{p}^c \mathbf{p}^d | \psi \rangle, \\ \langle \phi | \mathbf{p}^a \mathbf{J}^b | \phi \rangle &= R^a_c R^b_d \langle \psi | \mathbf{p}^c \mathbf{J}^d | \psi \rangle + R^a_c \varepsilon^b_{ef} \xi^e R^f_d \langle \psi | \mathbf{p}^c \mathbf{p}^d | \psi \rangle, \\ \langle \phi | \mathbf{J}^a \mathbf{J}^b | \phi \rangle &= R^a_c R^b_d \langle \psi | \mathbf{J}^c \mathbf{J}^d | \psi \rangle + R^a_c \varepsilon^b_{ef} \xi^e R^f_d \overline{\langle \psi | \mathbf{p}^d \mathbf{J}^c | \psi \rangle} + \\ &\quad + \varepsilon^a_{ef} \xi^e R^f_c R^b_d \langle \psi | \mathbf{p}^c \mathbf{J}^d | \psi \rangle + \varepsilon^a_{ef} \xi^e R^f_c \varepsilon^b_{gh} \xi^g R^h_d \langle \psi | \mathbf{p}^c \mathbf{p}^d | \psi \rangle, \end{aligned}$$

where some of the expectation values with  $\psi = \pm_j Y_{jj}/P$  have already been given by (4.14) and (4.15), while the others can be calculated by (A.18)–(A.26) and (A.28)–(A.30). These are

$$\begin{aligned} \langle \psi | \mathbf{p}^a \mathbf{J}^b | \psi \rangle &= \pm \frac{\hbar P j}{2(j+1)} \left( \delta_1^a \delta_1^b + \delta_2^a \delta_2^b + 2j \delta_3^a \delta_3^b + i(\delta_1^a \delta_2^a - \delta_2^a \delta_1^a) \right), \\ \langle \psi | \mathbf{J}^a \mathbf{J}^b | \psi \rangle &= \frac{1}{2} \hbar^2 j \left( \delta_1^a \delta_1^b + \delta_2^a \delta_2^b + 2j \delta_3^a \delta_3^b + i(\delta_1^a \delta_2^a - \delta_2^a \delta_1^a) \right). \end{aligned}$$

Using (4.20) and that, for large  $j$ , asymptotically  $P = pj + O(1/j)$ , we obtain

$$\begin{aligned} \frac{1}{P_1^2 P_2^2} \langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12}^2 | \phi_1 \otimes \phi_2 \rangle &= \left( (\xi_1^a - \xi_2^a) \varepsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 \right)^2 + \\ &\quad + 2\hbar \left( \frac{1}{p_1} \pm \frac{1}{p_2} \right) (\xi_1^a - \xi_2^a) \varepsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 \cos \beta_{12} + \\ &\quad + \hbar^2 \left( \frac{1}{p_1} \pm \frac{1}{p_2} \right)^2 \cos^2 \beta_{12} + O\left(\frac{1}{j_1}\right) + O\left(\frac{1}{j_2}\right). \end{aligned} \tag{4.33}$$

The calculation of the expectation value of the other terms is similar, and for them we obtain

$$\begin{aligned} &\frac{1}{P_1^2 P_2^2} \langle \phi_1 \otimes \phi_2 | -2\hbar \left( \frac{j_1}{P_1} \pm \frac{j_2}{P_2} \right) \mathbf{P}_{12}^2 \mathbf{W}_{12}^2 \\ &\quad + \hbar^2 \left( \frac{j_1}{P_1} \pm \frac{j_2}{P_2} \right)^2 \mathbf{P}_{12}^4 | \phi_1 \otimes \phi_2 \rangle = \\ &\quad - 2\hbar \left( \frac{j_1}{P_1} \pm \frac{j_2}{P_2} \right) (\xi_1^a - \xi_2^a) \varepsilon_{abc} (R_1)^b{}_3 (R_2)^c{}_3 \cos \beta_{12} \\ &\quad - \hbar^2 \left( \frac{j_1}{P_1} \pm \frac{j_2}{P_2} \right)^2 \cos^2 \beta_{12} + O\left(\frac{1}{j_1}\right) + O\left(\frac{1}{j_2}\right). \end{aligned} \tag{4.34}$$

Comparing this with (4.33) and recalling that the first term on the right of (4.33) is just the classical limit of the square of  $\langle \phi_1 \otimes \phi_2 | \mathbf{W}_{12} - \hbar(j_1/P_1 \pm j_2/P_2) \mathbf{P}_{12}^2 | \phi_1 \otimes \phi_2 \rangle / P_1 P_2$ , we find that the first term in the brackets in (4.32) is of order  $O(1/j_1) + O(1/j_2)$ .

Using (4.21), (4.14) and (4.15), we immediately obtain the asymptotic form of the expectation value of the first two terms in  $\mathbf{B}_{12}^2 / P_1^4 P_2^4 = 1 - 2\mathbf{P}_{12}^4 / P_1^2 P_2^2 + \mathbf{P}_{12}^8 / P_1^4 P_2^4$ . It is



$$\frac{1}{P_1^4 P_2^4} \langle \phi_1 \otimes \phi_2 | P_1^4 P_2^4 - 2P_1^2 P_2^2 \mathbf{P}_{12}^4 | \phi_1 \otimes \phi_2 \rangle = 1 - 2 \cos^2 \beta_{12} + O\left(\frac{1}{j_1}\right) + O\left(\frac{1}{j_2}\right).$$

Thus, we should calculate only the asymptotic form of

$$\begin{aligned} \langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^8 | \phi_1 \otimes \phi_2 \rangle &= \langle \phi_1 | \mathbf{p}_1^a \mathbf{p}_1^b \mathbf{p}_1^c \mathbf{p}_1^d | \phi_1 \rangle \delta_{ae} \delta_{bf} \delta_{cg} \delta_{dh} \langle \phi_2 | \mathbf{p}_2^e \mathbf{p}_2^f \mathbf{p}_2^g \mathbf{p}_2^h | \phi_2 \rangle = \\ &= \langle \psi_1 | \mathbf{p}_1^a \mathbf{p}_1^b \mathbf{p}_1^c \mathbf{p}_1^d | \psi_1 \rangle (R_1^{-1} R_2)_{ae} (R_1^{-1} R_2)_{bf} (R_1^{-1} R_2)_{cg} (R_1^{-1} R_2)_{dh} \langle \psi_2 | \mathbf{p}_2^e \mathbf{p}_2^f \mathbf{p}_2^g \mathbf{p}_2^h | \psi_2 \rangle \end{aligned}$$

with  $\psi_1 = {}_{j_1} Y_{j_1 j_1} / P_1$  and  $\psi_2 = {}_{\pm j_2} Y_{j_2 j_2} / P_2$  when  $j_1, j_2 \rightarrow \infty$ . Since by (A.18)–(A.26)

$$\langle {}_{\pm j} Y_{jj} | \mathbf{p}^3 | {}_{\pm j} Y_{jj} \rangle = \pm P^3 \frac{j}{j+1} + O\left(\frac{1}{j}\right)$$

and all the other matrix elements of the form  $\langle {}_{\pm j} Y_{jj} | \mathbf{p}^a | {}_{\pm j} Y_{lm} \rangle$  fall off as  $1/\sqrt{j}$  for large  $j$ , we have that

$$\langle {}_{\pm j} Y_{jj} | \mathbf{p}^a \mathbf{p}^b \mathbf{p}^c \mathbf{p}^d | {}_{\pm j} Y_{jj} \rangle = P^6 \frac{j^4}{(j+1)^4} \delta_3^a \delta_3^b \delta_3^c \delta_3^d + O\left(\frac{1}{\sqrt{j}}\right).$$

Thus, by (4.20)

$$\frac{1}{P_1^4 P_2^4} \langle \phi_1 \otimes \phi_2 | \mathbf{P}_{12}^8 | \phi_1 \otimes \phi_2 \rangle = \cos^4 \beta_{12} + O\left(\frac{1}{\sqrt{j_1}}\right) + O\left(\frac{1}{\sqrt{j_2}}\right).$$

Hence

$$\begin{aligned} \frac{1}{P_1^4 P_2^4} \langle \phi_1 \otimes \phi_2 | \mathbf{B}_{12}^2 | \phi_1 \otimes \phi_2 \rangle &= \sin^4 \beta_{12} + O\left(\frac{1}{\sqrt{j_1}}\right) + O\left(\frac{1}{\sqrt{j_2}}\right) = \\ &= \frac{1}{P_1^4 P_2^4} \left( \langle \phi_1 \otimes \phi_2 | \mathbf{B}_{12} | \phi_1 \otimes \phi_2 \rangle \right)^2 + O\left(\frac{1}{\sqrt{j_1}}\right) + O\left(\frac{1}{\sqrt{j_2}}\right), \end{aligned}$$

which yields that the second term in the brackets in (4.32) is also vanishing in the  $j_1, j_2 \rightarrow \infty$  limit.

Therefore, as a summary of the results of the above calculations, we have proven the following statement:

**Theorem** *Let  $L_1, \dots, L_N$  be straight lines in  $\mathbb{R}^3$  such that no two of them are parallel. Then there are  $E(3)$ -invariant elementary quantum mechanical systems  $\mathcal{S}_1, \dots, \mathcal{S}_N$  and a sequence of their pure quantum states  $\phi_{1k}, \dots, \phi_{Nk}$ ,  $k \in \mathbb{N}$ , indexed by pairs  $(s_{1k}, P_{1k}), \dots, (s_{Nk}, P_{Nk})$  of their Casimir invariants, respectively, such that, in the  $(|s_1|, P_1), \dots, (|s_N|, P_N) \rightarrow \infty$  limit, the magnitude  $|d_{ij}|$  of the empirical distances tend with asymptotically vanishing uncertainty to the Euclidean distances  $D_{ij}$  between the straight lines  $L_i$  and  $L_j$ , given by (2.8), for any  $\mathbf{i}, \mathbf{j} = 1, \dots, N$ .*

Thus, the metric structure of the Euclidean 3-space could be recovered in the classical limit from appropriate quantum observables of Euclidean invariant elementary quantum mechanical systems.

Following the same strategy, the calculation of the classical limit of the empirical angles and 3-volume elements is quite straightforward: the  $m_1 = j_1 = s_1$ ,  $m_2 = j_2 = s_2 \rightarrow \infty$  limit of the empirical angles  $\omega_{12}$ , given by (4.30), is  $\beta_{12}$ ; and the analogous limit of the 3-volume element given by (4.31) is just the Euclidean 3-volume element. These results, together with the Theorem above, provide an extension of the Spin Geometry Theorem of Penrose [1–4] from  $SU(2)$  to  $E(3)$ -invariant systems.

### 5 Final Remarks

The relative position vectors  $d_{ij}^a$ ,  $\mathbf{i}, \mathbf{j} = 1, 2, 3$ , of a three-particle system also make it possible to define a notion of angle that is *different* from that we considered in subsections 2.3 and 4.3:

$$\cos \varpi_{12,32} := \frac{\delta_{ab} d_{12}^a d_{32}^b}{|d_{12}| |d_{32}|} = \frac{\delta_{ab} (\epsilon^a_{cd} p_1^c p_2^d) (\epsilon^b_{ef} p_3^e p_2^f)}{|\epsilon^a_{bc} p_1^b p_2^c| |\epsilon^a_{bc} p_3^b p_2^c|} = \frac{P_2^2 P_{13}^2 - P_{12}^2 P_{23}^2}{\sqrt{P_1^2 P_2^2 - P_{12}^4} \sqrt{P_2^2 P_3^2 - P_{23}^4}}$$

with range  $\varpi_{12,32} \in [0, \pi]$  defines the angle *between the relative position vectors* pointing from the second subsystem’s centre-of-mass line to that of the first and the third subsystems, respectively. The angles  $\varpi_{23,13}$  and  $\varpi_{21,31}$  are defined analogously. Although at the classical level this angle coincides with the Euclidean one, the analogous empirical angle in the quantum theory deviates from  $\omega_{ij}$ . Another concept of the empirical 3-volume element could also be introduced, as the volume of the tetrahedron spanned by the three vectors  $\epsilon^a_{bc} p_1^b p_2^c$ . Thus, at the fundamental, quantum level there might not exist unique, a priori obvious analog of the classical geometrical notions, like angle, distance or 3-volume element. In addition to the requirement of their correct behaviour in the classical limit (and their ‘naturalness’ and ‘usefulness’), can we have some selection rule to choose one from the various possibilities?

The states of the composite system by means of which the correct classical limit of the various empirical geometrical quantities could be derived are *pure tensor product states*, built from the pure vector states of the elementary subsystems. Thus, in deriving these, we did *not* need to use entangled states of the composite system. But then, if the subsystems are independent, how can one obtain the distance, angle, etc. between them? The answer is that the entanglement of the subsystems can be considered to be *already built into the structure of the observables of the composite system*. In fact, the operators by means of which the empirical distances, angles and 3-volume elements are defined have the structure  $\mathbf{W}_{12} = \frac{1}{2} \delta_{ab} (\mathbf{p}_1^a \mathbf{J}_2^b + \mathbf{J}_1^a \mathbf{p}_2^b)$ ,  $\mathbf{P}_{12}^2 = \delta_{ab} \mathbf{p}_1^a \mathbf{p}_2^b$  and  $\epsilon_{abc} \mathbf{p}_1^a \mathbf{p}_2^b \mathbf{p}_3^c$ , respectively. These observables of the *composite system* are ‘entanglements’ of the *observables of the subsystems*. The *states* of the subsystems do not need to be entangled.

As we already noted in subsection 4.2, if the state of the composite system is mixed or entangled, then the interpretation of the empirical distance (and of the angles and 3-volume elements, too) is not obvious, and e.g. the distance between

two subsystems may depend on the state of other subsystems. In this case, the resulting distances cannot be expected to be compatible with the structure of any metric space. Thus, by assuming that the state of the composite system is a pure tensor product state we implicitly assumed that the subsystems are independent, and hence interact with one another weakly. If, however, the subsystems are inextricably entangled (e.g. since they are very strongly interacting with one another), then the ‘quantum geometry’ defined by such systems may not be expected even to resemble to the Euclidean geometry *at all*. The Euclidean structure of the classical ‘physical 3-space’ that we see appears to be defined only by the independent, very weakly interacting subsystems of the Universe.

## A. Appendix

### A.1 Complex Coordinates and the Line Bundles $\mathcal{O}(-2s)$ over $S_p$

In the complex stereographic coordinates  $(\zeta, \bar{\zeta})$  on  $S_p$ , defined by  $\zeta := \exp(i\varphi) \cot(\theta/2)$  in terms of the familiar spherical polar coordinates  $(\theta, \varphi)$ , the Cartesian components of the ‘position vector’  $p^a$  in the classical momentum space and the complex null tangent  $m^a$ , respectively, are

$$p_a = P \left( \frac{\bar{\zeta} + \zeta}{1 + \zeta\bar{\zeta}}, i \frac{\bar{\zeta} - \zeta}{1 + \zeta\bar{\zeta}}, \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}} \right), \quad m_a = \frac{1}{\sqrt{2}} \left( \frac{1 - \zeta^2}{1 + \zeta\bar{\zeta}}, i \frac{1 + \zeta^2}{1 + \zeta\bar{\zeta}}, \frac{2\zeta}{1 + \zeta\bar{\zeta}} \right). \tag{A.1}$$

These imply that  $p^a \varepsilon_{abc} = -iP(m_b \bar{m}_c - \bar{m}_b m_c)$ , where  $\bar{m}_a$  is the complex conjugate of  $m_a$ . Also in these coordinates, the line element of the metric and the corresponding area element on  $S_p$ , respectively, are

$$dh^2 = \frac{4P^2}{(1 + \zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}, \quad dS_p = -\frac{2iP^2}{(1 + \zeta\bar{\zeta})^2} d\zeta \wedge d\bar{\zeta}. \tag{A.2}$$

These are just the metric and area element inherited from the metric and volume element of momentum 3-space, respectively. The complex null vectors  $m^a$  and  $\bar{m}^a$  are unique up to a phase as they are (1, 0) and (0, 1) type vectors, respectively, in the natural complex structure of  $S_p \approx S^2$  (see e.g. [10]). As a differential operator,  $m^a$  is given by

$$m^a \left( \frac{\partial}{\partial p^a} \right) = \frac{1}{\sqrt{2}P} (1 + \zeta\bar{\zeta}) \left( \frac{\partial}{\partial \bar{\zeta}} \right). \tag{A.3}$$

The contraction of the complex null vectors,  $m^a$  and  $\bar{m}^a$ , as well as of the ‘position vector’  $p^a$  in the momentum space with the Pauli matrices can also be expressed by the vectors  $\{o^A, i^A\}$  of the (normalized) Newman–Penrose spinor basis:  $m^a \sigma_a^{AA'} = -o^A \bar{i}^{A'}$ ,  $\bar{m}^a \sigma_a^{AA'} = -i^A \bar{o}^{A'}$  and  $p^a \sigma_a^{AA'} = P(i^A \bar{i}^{A'} - o^A \bar{o}^{A'})/\sqrt{2}$ . The

components of the vectors of the Newman–Penrose basis in the spinor basis  $\{O^A, I^A\}$  associated with the Cartesian vector basis are

$$o^A = \frac{-i}{\sqrt{1 + \zeta\bar{\zeta}}} \begin{pmatrix} \zeta \\ 1 \end{pmatrix}, \quad r^A = \frac{-i}{\sqrt{1 + \zeta\bar{\zeta}}} \begin{pmatrix} 1 \\ -\bar{\zeta} \end{pmatrix}.$$

Note that this basis is well defined only on  $S_p$  minus its ‘north pole’, which is the domain of the coordinate system  $(\zeta, \bar{\zeta})$ , as well as of the complex null vectors  $m^a$  and  $\bar{m}^a$ .

A scalar  $\phi$  is said to have spin weight  $s = \frac{1}{2}(p - q)$  if under the rescaling  $\{o^A, r^A\} \mapsto \{\lambda o^A, \lambda^{-1} r^A\}$ ,  $\lambda \in \mathbb{C} - \{0\}$ , the scalar  $\phi$  transforms as  $\phi \mapsto \lambda^p \bar{\lambda}^q \phi$  (see [10, 13]). The bundle of such scalars is denoted by  $\mathcal{O}(-2s)$ . The complex line bundle  $\mathcal{O}(-2s)$  is globally trivializable precisely when  $s = 0$ ; otherwise it has a twist. The domain of the coordinate system  $(\zeta, \bar{\zeta})$  is a local trivialization domain for  $\mathcal{O}(-2s)$  for any  $s$ . For a detailed discussion of the line bundles  $\mathcal{O}(-2s)$ , see e.g. [10, 13].

The edth and edth-prime operators of Newman and Penrose [21] acting on spin weighted functions, e.g. on the cross section  $\phi$  of  $\mathcal{O}(-2s)$ , can be defined by

$$\delta\phi = \frac{1}{\sqrt{2P}} \left( (1 + \zeta\bar{\zeta}) \frac{\partial\phi}{\partial\bar{\zeta}} + s\zeta\phi \right), \quad \delta'\phi = \frac{1}{\sqrt{2P}} \left( (1 + \zeta\bar{\zeta}) \frac{\partial\phi}{\partial\zeta} - s\bar{\zeta}\phi \right); \tag{A.4}$$

and hence for their commutator we obtain that  $(\delta\delta' - \delta'\delta)\phi = -(1/P^2)s\phi$ . It is not difficult to check that  $\delta p^a = m^a$ ,  $\delta m^a = 0$  and  $\delta' m^a = -p^a/P^2$ .

A purely algebraic introduction of the *spin weighted spherical harmonics*, given in [13], is based on the comparison of the appropriate symmetrized products of the vectors of the Cartesian spinor basis  $\{O_A, I_A\}$  and those of the Newman–Penrose spinor basis  $\{o_A, l_A\}$  adapted to the unit sphere (and given explicitly by equation (4.15.98) of [13]):

$${}_s Y_{jm} := N_{s,j,m} O_{(A_1} \cdots O_{A_{j-m}} I_{A_{j-m+1}} \cdots I_{A_{2j}}) o^{A_1} \cdots o^{A_{j+s}} l^{A_{j+s+1}} \cdots l^{A_{2j}}, \tag{A.5}$$

where the coefficient  $N_{s,j,m}$  is

$$N_{s,j,m} := (-)^{j+m} \sqrt{\frac{2j+1}{4\pi}} \frac{(2j)!}{\sqrt{(j-m)!(j+m)!(j-s)!(j+s)!}}, \tag{A.6}$$

and  $2s \in \mathbb{Z}$ ,  $j = |s|, |s| + 1, |s| + 2, \dots$  and  $m = -j, -j + 1, \dots, j$ . This choice of the normalization factor yields that the spherical harmonics  ${}_0 Y_{jm}$  coincide with the standard expressions for the ordinary spherical harmonics  $Y_{jm}$ . The action of the edth operators on the harmonics  ${}_s Y_{jm}$  is

$$\delta_s Y_{jm} = -\frac{1}{\sqrt{2P}} \sqrt{(j+s+1)(j-s)} {}_{s+1} Y_{jm}, \quad \delta'_s Y_{jm} = \frac{1}{\sqrt{2P}} \sqrt{(j-s+1)(j+s)} {}_{s-1} Y_{jm}. \tag{A.7}$$

The harmonics  ${}_s Y_{jm}$  form an orthonormal basis in the space of the spin weighted functions with spin weight  $s$  with respect to the  $L_2$  scalar product on the unit 2-sphere (see e.g. [13]).

A spin weighted scalar  $\phi$  is called *holomorphic* if  $\delta' \phi = 0$ , and *anti-holomorphic* if  $\delta \phi = 0$ . It is known (see e.g. [10, 13]) that  $\dim \ker \delta = \dim \ker \delta' = 0$  for  $s < 0$  and  $s > 0$ , respectively; and  $\dim \ker \delta = \dim \ker \delta' = 2|s| + 1$  for  $s \geq 0$  and  $s \leq 0$ , respectively. By (A.7) these kernels are spanned by the special spherical harmonics  ${}_s Y_{|s|m}$ . Hence the spin weight of the *holomorphic* cross sections is non-positive,  $s = -|s|$ , while that of the *anti-holomorphic* ones is non-negative,  $s = |s|$ . They form  $2|s| + 1$  dimensional subspaces in  $\mathcal{H}_{P,s}$ .

**A.2 The Evaluation of  $\langle {}_s Y_{kn} | \mathbf{p}_a | {}_s Y_{jm} \rangle$  and  $\langle {}_s Y_{kn} | \mathbf{J}_a | {}_s Y_{jm} \rangle$**

(A.5) implies that in the product of two spherical harmonics,  ${}_s Y_{j_1 m_1} {}_s Y_{j_2 m_2}$ , the difference of the total number of the  $I_A$  and of the  $O_A$  spinors is  $2(m_1 + m_2)$ , and the difference of the total number of the  $o^A$  and of the  $t^A$  spinors is  $2(s_1 + s_2)$ . Hence, the spin weight of  ${}_s Y_{j_1 m_1} {}_s Y_{j_2 m_2}$  is  $s_1 + s_2$ , and in its expansion in terms of spin weighted spherical harmonics only the harmonics of the form  ${}_{s_1+s_2} Y_{j(m_1+m_2)}$  appear. Thus, there are constants  $C(s_1, j_1, m_1; s_2, j_2, m_2 | j)$  such that

$${}_s Y_{j_1 m_1} {}_s Y_{j_2 m_2} = \sum_j C(s_1, j_1, m_1; s_2, j_2, m_2 | j) {}_{s_1+s_2} Y_{j(m_1+m_2)}, \tag{A.8}$$

where, as one can show,  $\max\{|j_1 - j_2|, |s_1 + s_2|\} \leq j \leq j_1 + j_2$ . These constants are analogous to the (inverse) of the usual Clebsch–Gordan coefficients, and hence these may also be called the (inverse) C-G coefficients.

By (A.1) and the explicit expression of the ordinary spherical harmonics in the coordinates  $(\zeta, \bar{\zeta})$ , the components of the linear momentum are

$$p_a = P \sqrt{\frac{2\pi}{3}} \left( {}_0 Y_{1-1} - {}_0 Y_{11}, i({}_0 Y_{1-1} + {}_0 Y_{11}), \sqrt{2} {}_0 Y_{10} \right). \tag{A.9}$$

Hence, to determine the matrix elements  $\langle {}_s Y_{kn} | \mathbf{p}_a | {}_s Y_{jm} \rangle$ , we need to calculate the expansion (A.8) only for  ${}_0 Y_{1n} {}_s Y_{jm}$ .

This calculation is based on (A.5), in which, following [13], we introduce the notations  $Z(j, m)_{A_1 \dots A_j} := O_{(A_1} \dots O_{A_{j-m}} I_{A_{j-m+1}} \dots I_{A_{2j})}$  and  ${}_s Z(j, m) := (N_{s,j,m})^{-1} {}_s Y_{jm}$ . Then for any  $M_A$  and any totally symmetric spinor  $Z_{A_1 \dots A_j}$  the complete algebraically irreducible decomposition of their product is

$$Z_{A_1 \dots A_j} M_A = Z_{(A_1 \dots A_j)} M_A + \frac{1}{2j+1} \epsilon_{A_1 A} M^B Z_{BA_2 \dots A_j} + \dots + \frac{1}{2j+1} \epsilon_{A_j A} M^B Z_{A_1 \dots A_{j-1} B}.$$

Applying this formula to  $I_A$  and  $Z(j, m)_{A_1 \dots A_j}$ , we obtain

$$\begin{aligned}
 Z(j, m)_{A_1 \dots A_{2j}} I_A &= Z(j + \frac{1}{2}, m + \frac{1}{2})_{AA_1 \dots A_{2j}} + \\
 &+ \frac{1}{2j + 1} \epsilon_{A_1 A} I^B O_{(B} O_{A_2} \dots O_{A_{j-m}} I_{A_{j-m+1}} \dots I_{A_{2j}}) + \dots + \\
 &+ \frac{1}{2j + 1} \epsilon_{A_{j-m} A} I^B O_{(A_1} \dots O_{A_{j-m-1}} O_B I_{A_{j-m+1}} \dots I_{A_{2j}}) = \\
 &= Z(j + \frac{1}{2}, m + \frac{1}{2})_{AA_1 \dots A_{2j}} + \\
 &+ \frac{1}{(2j + 1)2j} \left\{ (j - m) \epsilon_{A_1 A} O_{(A_2} \dots O_{A_{j-m}} I_{A_{j-m+1}} \dots I_{A_{2j}}) + \dots + \right. \\
 &\left. + (j - m) \epsilon_{A_{2j} A} O_{(A_1} \dots O_{A_{j-m-1}} I_{A_{j-m}} \dots I_{A_{2j-1}}) \right\} = \\
 &= Z(j + \frac{1}{2}, m + \frac{1}{2})_{AA_1 \dots A_{2j}} - \frac{j - m}{2j + 1} \epsilon_{A(A_1} Z(j - \frac{1}{2}, m + \frac{1}{2})_{A_2 \dots A_{2j})}.
 \end{aligned}
 \tag{A.10}$$

In a similar way

$$Z(j, m)_{A_1 \dots A_{2j}} O_A = Z(j + \frac{1}{2}, m - \frac{1}{2})_{AA_1 \dots A_{2j}} + \frac{j + m}{2j + 1} \epsilon_{A(A_1} Z(j - \frac{1}{2}, m - \frac{1}{2})_{A_2 \dots A_{2j})}.
 \tag{A.11}$$

These two are the key formulae on which the present calculation of the (inverse) C-G coefficients is based. In particular, using the technique of complete algebraic irreducible decomposition of the various spinors, the repeated application of these formulae yields

$$\begin{aligned}
 Z(j, m)_{A_1 \dots A_{2j}} I_A I_B &= Z(j + 1, m + 1)_{ABA_1 \dots A_{2j}} - \\
 &- \frac{j - m}{2(j + 1)} \left( \epsilon_{A(A_1} Z(j, m + 1)_{A_2 \dots A_{2j})B} + \epsilon_{B(A_1} Z(j, m + 1)_{A_2 \dots A_{2j})A} \right) - \\
 &- \frac{(j - m)(j - m - 1)}{2j(2j + 1)} \epsilon_{A(A_1} Z(j - 1, m + 1)_{A_2 \dots A_{2j-1}} \epsilon_{A_{2j})B},
 \end{aligned}
 \tag{A.12}$$

$$\begin{aligned}
 Z(j, m)_{A_1 \dots A_{2j}} I_A O_B &= Z(j + 1, m)_{ABA_1 \dots A_{2j}} - \frac{1}{2} \epsilon_{AB} Z(j, m)_{A_1 \dots A_{2j}} + \\
 &+ \frac{m}{2(j + 1)} \left( \epsilon_{A(A_1} Z(j, m)_{A_2 \dots A_{2j})B} + \epsilon_{B(A_1} Z(j, m)_{A_2 \dots A_{2j})A} \right) + \\
 &+ \frac{(j - m)(j + m)}{2j(2j + 1)} \epsilon_{A(A_1} Z(j - 1, m)_{A_2 \dots A_{2j-1}} \epsilon_{A_{2j})B},
 \end{aligned}
 \tag{A.13}$$

$$\begin{aligned}
 Z(j, m)_{A_1 \dots A_{2j}} O_A O_B &= Z(j + 1, m - 1)_{ABA_1 \dots A_{2j}} + \\
 &+ \frac{j + m}{2(j + 1)} \left( \epsilon_{A(A_1} Z(j, m - 1)_{A_2 \dots A_{2j})B} + \epsilon_{B(A_1} Z(j, m - 1)_{A_2 \dots A_{2j})A} \right) - \\
 &- \frac{(j + m)(j + m - 1)}{2j(2j + 1)} \epsilon_{A(A_1} Z(j - 1, m - 1)_{A_2 \dots A_{2j-1}} \epsilon_{A_{2j})B}.
 \end{aligned}
 \tag{A.14}$$

Since  $Z(\frac{1}{2}, \frac{1}{2})_A = I_A$ ,  $Z(\frac{1}{2}, -\frac{1}{2})_A = O_A$ ,  $Z(1, 1)_{AB} = I_A I_B$ ,  $Z(1, 0)_{AB} = I_A O_B$  and  $Z(1, -1)_{AB} = O_A O_B$ , equations (A.10)–(A.14), together with (A.5) and (A.6), already give

$$\begin{aligned}
 {}_0Y_{1,1} {}_sY_{j,m} &= \sqrt{\frac{3}{8\pi}} \frac{1}{j+1} \sqrt{\frac{(j-s+1)(j+s+1)(j+m+1)(j+m+2)}{(2j+1)(2j+3)}} {}_sY_{j+1,m+1}^- \\
 &\quad - \sqrt{\frac{3}{8\pi}} \frac{s}{j(j+1)} \sqrt{(j+m+1)(j-m)} {}_sY_{j,m+1}^- \\
 &\quad - \sqrt{\frac{3}{8\pi}} \frac{1}{j} \sqrt{\frac{(j-s)(j+s)(j-m-1)(j-m)}{(2j-1)(2j+1)}} {}_sY_{j-1,m+1}^-;
 \end{aligned}
 \tag{A.15}$$

$$\begin{aligned}
 {}_0Y_{1,0} {}_sY_{j,m} &= \sqrt{\frac{3}{4\pi}} \frac{1}{j+1} \sqrt{\frac{(j+1+s)(j+1-s)(j+1+m)(j+1-m)}{(2j+1)(2j+3)}} {}_sY_{j+1,m}^+ \\
 &\quad + \sqrt{\frac{3}{4\pi}} \frac{sm}{j(j+1)} {}_sY_{j,m}^+ \\
 &\quad + \sqrt{\frac{3}{4\pi}} \frac{1}{j} \sqrt{\frac{(j-s)(j+s)(j+m)(j-m)}{(2j-1)(2j+1)}} {}_sY_{j-1,m}^+;
 \end{aligned}
 \tag{A.16}$$

$$\begin{aligned}
 {}_0Y_{1,-1} {}_sY_{j,m} &= \sqrt{\frac{3}{8\pi}} \frac{1}{j+1} \sqrt{\frac{(j+1+s)(j+1-s)(j-m+1)(j-m+2)}{(2j+1)(2j+3)}} {}_sY_{j+1,m-1}^+ \\
 &\quad + \sqrt{\frac{3}{8\pi}} \frac{s}{j(j+1)} \sqrt{(j+m)(j-m+1)} {}_sY_{j,m-1}^- \\
 &\quad - \sqrt{\frac{3}{8\pi}} \frac{1}{j} \sqrt{\frac{(j+s)(j-s)(j+m)(j+m-1)}{(2j-1)(2j+1)}} {}_sY_{j-1,m-1}^-;
 \end{aligned}
 \tag{A.17}$$

where recall that, on the right hand sides, the spherical harmonics  ${}_sY_{j,m}$  are vanishing if  $j < |s|$  or  $j < |m|$ . (In these formulae, to avoid confusion, we inserted a comma between the indices  $j$  and  $m$  of  ${}_sY_{j,m}$ .)

Then, using (A.9) and (A.15)–(A.17), we find that the only non-zero matrix elements of  $\mathbf{p}_a$  are

$$\begin{aligned}
 \langle {}_sY_{j+1,n} | \mathbf{p}_1 | {}_sY_{j,m} \rangle &= \frac{P^3}{2(j+1)} \sqrt{\frac{(j+s+1)(j-s+1)}{(2j+1)(2j+3)}} \\
 &\quad \times \left( \sqrt{(j-m+1)(j-m+2)} \delta_{n,m-1} - \sqrt{(j+m+1)(j+m+2)} \delta_{n,m+1} \right),
 \end{aligned}
 \tag{A.18}$$

$$\begin{aligned}
 \langle {}_sY_{j,n} | \mathbf{p}_1 | {}_sY_{j,m} \rangle &= \frac{P^3 s}{2j(j+1)} \\
 &\quad \times \left( \sqrt{(j+m)(j-m+1)} \delta_{n,m-1} + \sqrt{(j-m)(j+m+1)} \delta_{n,m+1} \right),
 \end{aligned}
 \tag{A.19}$$

$$\begin{aligned} \langle {}_s Y_{j-1,n} | \mathbf{p}_1 | {}_s Y_{j,m} \rangle &= \frac{P^3}{2j} \sqrt{\frac{(j+s)(j-s)}{(2j-1)(2j+1)}} \\ &\times \left( \sqrt{(j-m)(j-m-1)} \delta_{n,m+1} - \sqrt{(j+m)(j+m-1)} \delta_{n,m-1} \right); \end{aligned} \tag{A.20}$$

$$\begin{aligned} \langle {}_s Y_{j+1,n} | \mathbf{p}_2 | {}_s Y_{j,m} \rangle &= i \frac{P^3}{2(j+1)} \sqrt{\frac{(j+s+1)(j-s+1)}{(2j+1)(2j+3)}} \\ &\times \left( \sqrt{(j+m+1)(j+m+2)} \delta_{n,m+1} + \sqrt{(j-m+1)(j-m+2)} \delta_{n,m-1} \right), \end{aligned} \tag{A.21}$$

$$\begin{aligned} \langle {}_s Y_{j,n} | \mathbf{p}_2 | {}_s Y_{j,m} \rangle &= i \frac{P^3 s}{2j(j+1)} \\ &\times \left( \sqrt{(j+m)(j-m+1)} \delta_{n,m-1} - \sqrt{(j-m)(j+m+1)} \delta_{n,m+1} \right), \end{aligned} \tag{A.22}$$

$$\begin{aligned} \langle {}_s Y_{j-1,n} | \mathbf{p}_2 | {}_s Y_{j,m} \rangle &= -i \frac{P^3}{2j} \sqrt{\frac{(j+s)(j-s)}{(2j-1)(2j+1)}} \\ &\times \left( \sqrt{(j+m)(j+m-1)} \delta_{n,m-1} + \sqrt{(j-m)(j-m-1)} \delta_{n,m+1} \right), \end{aligned} \tag{A.23}$$

$$\langle {}_s Y_{j+1,n} | \mathbf{p}_3 | {}_s Y_{j,m} \rangle = \frac{P^3}{j+1} \sqrt{\frac{(j+s+1)(j-s+1)(j+m+1)(j-m+1)}{(2j+1)(2j+3)}} \delta_{n,m}, \tag{A.24}$$

$$\langle {}_s Y_{j,n} | \mathbf{p}_3 | {}_s Y_{j,m} \rangle = \frac{P^3 m s}{j(j+1)} \delta_{n,m}, \tag{A.25}$$

$$\langle {}_s Y_{j-1,n} | \mathbf{p}_3 | {}_s Y_{j,m} \rangle = \frac{P^3}{j} \sqrt{\frac{(j+s)(j-s)(j+m)(j-m)}{(2j-1)(2j+1)}} \delta_{n,m}. \tag{A.26}$$

Thus, the subspaces spanned by  ${}_s Y_{j,m}$  with given  $s$  and  $j$  are not invariant under the action of the momentum operators; and while  $\mathbf{p}_3$  does not change the index  $m$ ,  $\mathbf{p}_1 \pm i\mathbf{p}_2$  increases/decreases the value of  $m$ .

Next, we calculate the matrix elements of the angular momentum operator using  $m^a = \delta p^a$ ,  $\bar{m}^a = \delta' p^a$ , equations (A.7) and the expression (3.7) for the angular momentum vector operator. By integration by parts we obtain



$$\begin{aligned}
 \langle {}_s Y_{k,n} | \mathbf{J}_a | {}_s Y_{j,m} \rangle &= \hbar P \int_{S_P} \overline{{}_s Y_{k,n}} \left( (\delta p_a) (\delta' {}_s Y_{j,m}) - (\delta' p_a) (\delta {}_s Y_{j,m}) + s \frac{P_a}{P^2} {}_s Y_{j,m} \right) dS_P = \\
 &= \hbar P \int_{S_P} \left( -\overline{(\delta' {}_s Y_{k,n})} p_a (\delta' {}_s Y_{j,m}) + \overline{(\delta {}_s Y_{k,n})} p_a (\delta {}_s Y_{j,m}) + \right. \\
 &\quad \left. + \overline{{}_s Y_{k,n}} p_a (\delta' \delta {}_s Y_{j,m} - \delta \delta' {}_s Y_{j,m}) + s \frac{P_a}{P^2} \overline{{}_s Y_{k,n}} {}_s Y_{j,m} \right) dS_P = \\
 &= \frac{\hbar}{2P} \left\{ \sqrt{(k+s+1)(k-s)} \sqrt{(j+s+1)(j-s)} \langle {}_{s+1} Y_{k,n} | \mathbf{p}_a | {}_{s+1} Y_{j,m} \rangle - \right. \\
 &\quad \left. - \sqrt{(k-s+1)(k+s)} \sqrt{(j-s+1)(j+s)} \langle {}_{s-1} Y_{k,n} | \mathbf{p}_a | {}_{s-1} Y_{j,m} \rangle + \right. \\
 &\quad \left. + 4s \langle {}_s Y_{k,n} | \mathbf{p}_a | {}_s Y_{j,m} \rangle \right\}.
 \end{aligned}
 \tag{A.27}$$

Hence, the matrix elements of the angular momentum vector operator are simple expressions of those of the linear momentum. Using (A.18)–(A.26) and (A.27), we find that the only non-zero matrix elements of  $\mathbf{J}_a$  are

$$\begin{aligned}
 \langle {}_s Y_{j,n} | \mathbf{J}_1 | {}_s Y_{j,m} \rangle &= \frac{1}{2} \hbar P^2 \left( \sqrt{(j+m)(j-m+1)} \delta_{n,m-1} + \right. \\
 &\quad \left. + \sqrt{(j-m)(j+m+1)} \delta_{n,m+1} \right),
 \end{aligned}
 \tag{A.28}$$

$$\begin{aligned}
 \langle {}_s Y_{j,n} | \mathbf{J}_2 | {}_s Y_{j,m} \rangle &= \frac{i}{2} \hbar P^2 \left( \sqrt{(j+m)(j-m+1)} \delta_{n,m-1} - \right. \\
 &\quad \left. - \sqrt{(j-m)(j+m+1)} \delta_{n,m+1} \right),
 \end{aligned}
 \tag{A.29}$$

$$\langle {}_s Y_{j,n} | \mathbf{J}_3 | {}_s Y_{j,m} \rangle = \hbar P^2 m \delta_{n,m}.
 \tag{A.30}$$

These are precisely the well known matrix elements of the angular momentum operator in quantum mechanics. In particular, these are independent of the spin weight of the spherical harmonics.

**Acknowledgements** Thanks are due to Jörg Frauendiener and Paul Tod for their useful remarks on the determination of the inverse Clebsch–Gordan coefficients; to László Fehér for the clarification of the algebraic structures on the dual of  $e(3)$ ; to Péter Vecsernyés for the numerous discussions on the algebraic formulation of quantum theory; and to Ted Jacobson for the discussion on the role of entanglement in the emergence of the geometry of the 3-space/spacetime from quantum mechanics.

**Funding** Open access funding provided by ELKH Wigner Research Centre for Physics.

**Declarations**

**Conflict of interest** There is no conflict of interest that could be relevant to the content of this article. No funds, grants or support was received.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons

licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

1. Penrose, R.: Combinatorial quantum theory and quantized directions. In: Houghston, L.P., Ward, R.S. (eds.) *Advances in Twistor Theory*. Pitman Publishing Ltd, London (1979)
2. Penrose, R.: Angular momentum: an approach to combinatorial spacetime. In: Bastin, T. (ed.) *Quantum Theory and Beyond*. Cambridge University Press, Cambridge (1971)
3. Penrose, R.: On the nature of quantum geometry. In: Klauder, J. (ed.) *Magic Without Magic*. Freeman, San Francisco (1972)
4. Penrose, R.: Theory of quantized directions. In: *Collected Works*. vol. 1, pp. 769–800. Oxford University Press, Oxford (2010)
5. Moussouris J.P.: *Quantum Models of Spacetime Based on Recoupling Theory*, PhD dissertation, Oxford University. <https://ora.ox.ac.uk/objects/uuid:6ad25485-c6cb-4957-b129-5124bb2adc67> (1984). Accessed Jan 2022
6. Szabados L.B.: A note on Penrose's Spin-Geometry Theorem and the geometry of 'empirical quantum angles'. *Found. Phys.* **52**, 96 (2022). <https://doi.org/10.1007/s10701-022-00616-3>
7. Salecker, H., Wigner, E.P.: Quantum limitations of the measurement of space-time distances. *Phys. Rev.* **109**, 571–577 (1958). <https://doi.org/10.1103/PhysRev.109.571>
8. Bacry, H.: Localizability and space in quantum mechanics. In: *Lecture Notes in Physics*, vol. 308. Springer, Berlin (1988)
9. Newton, T.D., Wigner, E.P.: Localized states for elementary systems. *Rev. Mod. Phys.* **21**, 400–406 (1949). <https://doi.org/10.1103/RevModPhys.21.400>
10. Hugget, S.A., Tod, K.P.: *An Introduction to Twistor Theory*, London Mathematical Society Student Texts 4, 2nd edn. Cambridge University Press, Cambridge (1994)
11. Penrose, R., MacCallum, M.A.H.: Twistor theory: an approach to the quantisation of fields and spacetime. *Phys. Rep.* **6**, 241–316 (1972). [https://doi.org/10.1016/0370-1573\(73\)90008-2](https://doi.org/10.1016/0370-1573(73)90008-2)
12. Penrose, R., Rindler, W.: *Spinors and Spacetime*, vol. 2. Cambridge University Press, Cambridge (1986)
13. Penrose, R., Rindler, W.: *Spinors and Spacetime*, vol. 1. Cambridge University Press, Cambridge (1984)
14. Rno, J.S.: Clebsch-Gordan coefficients and special functions related to the Euclidean group in three-space. *J. Math. Phys.* **15**, 2042–2047 (1974). <https://doi.org/10.1063/1.1666579>
15. Tung, W.-K.: *Group Theory in Physics*. World Scientific, Singapore (1985)
16. Kowalski, K., Rembieliński, J.: Quantum mechanics on a sphere and coherent states. *J. Phys. A: Math. Gen.* **33**, 6035–6048 (2000). <https://doi.org/10.1088/0305-4470/33/34/309>
17. Silva, R.A., Jacobson, T.: Particle on the sphere: group-theoretic quantization in the presence of a magnetic monopole. *J. Phys. A: Math. Theor.* **54**, 235303 (2021). <https://doi.org/10.1088/1751-8121/abf961>
18. Szabados, L.B.: An odd feature of the 'most classical' states of  $SU(2)$  invariant quantum mechanical systems, [arXiv: 2106.08695](https://arxiv.org/abs/2106.08695) [gr-qc]
19. Szabados, L.B.: The 'most classical' states of Euclidean invariant elementary quantum mechanical systems, [arXiv: 2111.11876](https://arxiv.org/abs/2111.11876) [quant-ph]
20. Streater, R.F., Wightman, A.S.: *PCT, Spin and Statistics, and All That*. W. A. Benjamin, INC, New York (1964)
21. Newman, E.T., Penrose, R.: Note on the Bondi-Metzner-Sachs group. *J. Math. Phys.* **7**, 863–879 (1966). <https://doi.org/10.1063/1.1931221>
22. Wigner, E.P.: *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*. Academic Press, New York (1959)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.