A Note on Penrose's Spin-Geometry Theorem and the Geometry of 'Empirical Quantum Angles'

László B. Szabados[1]

Abstract

In the traditional formalism of quantum mechanics, a simple direct proof of (a version of) the Spin Geometry Theorem of Penrose is given; and the structure of a model of the 'space of the quantum directions', defined in terms of elementary $SU(2)$-invariant observables of the quantum mechanical systems, is sketched.

Keywords Spin networks · Spin-Geometry Theorem · Quantum geometry · Empirical angle

1 Introduction

Penrose, in the pioneering work [1] (which appeared first already in 1966 as the appendix of his Adams Prize essay), suggested the so-called $SU(2)$ spin network as a simple model for the quantum spacetime (see also [2–5]). The key idea was that, unifying and splitting quantum mechanical systems and measuring the probabilities of occurrence of the various total (i.e. the j) values of angular momenta obtained in this procedure, an *empirical angle* between the angular momentum vectors can be *defined*; and the key result was that, in the large j (i.e. in the classical) limit, these angles tend to be angles between directions in the Euclidean 3-space.

The significance of this result is that the (conformal structure of the) 'physical 3-space' that we use as an *a priori* given 'arena' in which the physical objects are thought to be arranged and the interactions between them occur is *determined by the quantum physical systems themselves in the classical limit*. Its proof was based on combinatorial/graphical techniques. The key idea and a sketch of the proof appeared in [3, 6], but so far the detailed and complete proof has not been published [7]: the proof in [3] remained incomplete. Later, another (and *mathematically* different) version of the *same physical result* became known as the Spin Geometry Theorem [6].

✉ László B. Szabados
  lbszab@rmki.kfki.hu

1   Wigner Research Centre for Physics, P. O. Box 49, EU, Budapest 114 1525, Hungary

The proof of the latter version was based on the more familiar formalism of quantum mechanics. (For some more historical remarks, see also [3].)

However, soon after the appearance of the idea how the 'true' geometry of space(time) should be *defined in an operational way*, the emphasis was shifted from the systematic investigation of the *consequences* of the original ideas to the development of *mathematical theories* of possible *a priori quantum geometries*. The latter were only motivated by, but were not based directly on the original ideas. The twistor theory [8–10], and also canonical quantum theories of gravity (see e.g. Ref. [11]) are such promising mathematical models. In particular, in the latter, the area and volume [12] and also the angle [13], represented by appropriately regularized quantum operators, have been shown to be discrete; and, remarkably enough, the spin networks emerged as a basis of the states in this theory [14].

In the present note, we return to the original idea formulated in [1], and especially in [2, 3]: while e.g. the electrons or the electromagnetic field are *existing objects* (i.e. 'things'), by their primary definition, the spacetime and its points, the events, are *not*. The events are *phenomena*, and the spacetime, the set of them, is only a useful *notion* by means of which the laws of Nature can be formulated in a convenient, simple way. Thus, we also share the positivistic, Machian view (see [2, 3]) that spacetime and its geometry should be *defined* in an operational way *by existing material systems*.

Here, we adopt the idea of the algebraic formulation of quantum theory that the quantum system is specified completely if its *algebra of (basic) observables* (and, if needed, its representation) is fixed. Then the notion and all the structures of space/ spacetime should be defined *in terms of the observables* of the quantum physical subsystems of the Universe. In particular, the angles between 'directions' associated with quantum mechanical subsystems should also be introduced in this way even though the 'directions' themselves are *not* defined at all in the classical sense. Actually, these subsystems are chosen to be 'elementary' in the sense that the quantum observables are self-adjoint elements of the *enveloping algebra of the su(2) Lie algebra* of the angular momentum operators as the basic observables. We can use all the structures on this algebra, but, in addition to this, *no a priori* notion of space/ spacetime, as an 'arena' of events, is allowed to be used. It is this general strategy (but replacing $su(2)$ by the Lie algebra $e(3)$ of the Euclidean group) that we follow in [15] in deriving the *metrical* (rather than only the conformal) structure of the Euclidean three-space from elementary quantum systems.

As far as we know, no complete proof of the Spin Geometry Theorem, even the version given by Moussouris in [6], has been published. (The sketch of Moussouris's proof was summarized in [13].) The present note intends to make up this shortage, using an improved version of Moussouris's empirical angle between the angular momentum vectors of elementary quantum systems. We give a new proof of the theorem in the usual (but slightly more algebraic) formulation of quantum mechanics. In proving this theorem, no recoupling is needed. The present investigation also casts some light on the nature of the 'space of empirical quantum directions'.

In the next section we introduce the key notion, the improved version of the empirical angle between the angular momentum vectors of quantum systems. Based on this notion, in Sect. 3, we present the new proof and discuss the results. In Sect. 4

we sketch some key properties of a classical model of the geometry of empirical quantum angles.

2 The Empirical Angle

Let $S_\mathbf{i}$, $\mathbf{i} = 1, 2, \ldots, N$, be *classical mechanical* systems whose respective states are completely specified by the real 3-vectors $J_\mathbf{i}^a$, $a = 1, 2, 3$, called their angular momentum vectors. The length of these vectors is $|J_\mathbf{i}| := \sqrt{\delta_{ab} J_\mathbf{i}^a J_\mathbf{i}^b}$, and the *empirical angle* $\theta_{\mathbf{ik}}$ between $J_\mathbf{i}^a$ and $J_\mathbf{k}^a$ is defined (with range $[0, \pi]$) by

$$\cos\theta_{\mathbf{ik}} := \frac{\delta_{ab} J_\mathbf{i}^a J_\mathbf{k}^b}{|J_\mathbf{i}|\,|J_\mathbf{k}|}. \tag{1}$$

Clearly, both $|J_\mathbf{i}|$ and $\cos\theta_{\mathbf{ik}}$ are $SO(3)$-invariant. If the direction of the angular momentum vector $J_\mathbf{k}^a$ can be obtained from that of $J_\mathbf{i}^a$ by an $SO(3)$ rotation of angle $\beta_{\mathbf{ik}}$ in the plane spanned by $J_\mathbf{k}^a$ and $J_\mathbf{i}^a$, then, clearly, $\theta_{\mathbf{ik}} = \beta_{\mathbf{ik}}$. Thus, by measuring the $SO(3)$-*invariant observables* $\cos\theta_{\mathbf{ik}}$ we can recover the angle $\beta_{\mathbf{ik}}$ between the angular momentum vectors of the subsystems, defined *in the space of the classical observables*. Next we convey these ideas into the quantum theory in a systematic way. We will see that, for quantum systems, these two concepts of angle split with far reaching consequences.

Let $\mathcal{S}_\mathbf{i}$, $\mathbf{i} = 1, 2, \ldots, N$, be *quantum mechanical* systems, whose (normalized) vector states $\phi_\mathbf{i}$ (or, in the bra-ket notation, $|\phi_\mathbf{i}\rangle$) belong, respectively, to the Hilbert spaces $\mathcal{H}_\mathbf{i}$. The general (e.g. mixed) states are density operators on them: $\rho_\mathbf{i} : \mathcal{H}_\mathbf{i} \to \mathcal{H}_\mathbf{i}$. The basic quantum observables are the angular momentum vector operators $\mathbf{J}_\mathbf{i}^a$ satisfying the familiar commutation relations $[\mathbf{J}_\mathbf{i}^a, \mathbf{J}_\mathbf{i}^b] = i\hbar\varepsilon^{ab}{}_c \mathbf{J}_\mathbf{i}^c$, where ε_{abc} is the alternating Levi-Civita symbol, and we lower and raise the Latin indices by the Kronecker delta δ_{ab} and its inverse.[1]

The action of $SO(3)$ (or rather of $SU(2)$) on this algebra is given by $\mathbf{J}_\mathbf{i}^a \mapsto (R^{-1})^a{}_b \mathbf{J}_\mathbf{i}^b$. If the $SU(2)$ matrix $U^A{}_B$ is parameterized by the familiar Euler angles (α, β, γ) according to

$$U^A{}_B = \begin{pmatrix} \exp\left(\frac{i}{2}(\alpha+\gamma)\right)\cos\frac{\beta}{2} & i\exp\left(-\frac{i}{2}(\alpha-\gamma)\right)\sin\frac{\beta}{2} \\ i\exp\left(\frac{i}{2}(\alpha-\gamma)\right)\sin\frac{\beta}{2} & \exp\left(-\frac{i}{2}(\alpha+\gamma)\right)\cos\frac{\beta}{2} \end{pmatrix}, \tag{2}$$

then the corresponding rotation matrix, $R^a{}_b = -\sigma^a_{AA'} U^A{}_B \bar{U}^{A'}{}_{B'} \sigma_b^{BB'}$, is

---

[1] Because of the natural action of $SO(3)$ (i.e. of $SU(2)$ in its vector representation) on the real 3-space of the basic quantum observables spanned by $\mathbf{J}^1$, $\mathbf{J}^2$ and $\mathbf{J}^3$, apart from an overall positive factor δ_{ab} is in fact a *naturally defined 3-metric*, which is proportional to the Killing–Cartan metric; and ε_{abc} is the corresponding *natural volume 3-form on this space*.

$$R^a{}_b = \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & -\sin\alpha\cos\gamma - \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\gamma + \sin\alpha\cos\beta\cos\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\sin\beta\cos\gamma \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix}.$$

$$(3)$$

Here $\sigma_a^{AA'}$ are the three non-trivial $SL(2,\mathbb{C})$ Pauli matrices (including the factor $1/\sqrt{2}$) according to the conventions of [16, 17]. (The minus sign in the expression of $R^a{}_b$ is a consequence of the convention that, in the present note, we lower and raise the Latin indices by the *positive definite* metric δ_{ab} and its inverse, respectively, rather than by the negative definite spatial part of the Minkowski metric. The spinor name indices $A, B, \dots$ are lowered and raised by the anti-symmetric Levi-Civita symbol ε_{AB} and its inverse.) For later use, note that $(R^{-1}(\alpha, \beta, \gamma))^a{}_b = (R(\pi - \gamma, \beta, \pi - \alpha))^a{}_b$.

Considering the systems $\mathcal{S}_1, \dots, \mathcal{S}_N$ to be a *single* system, the space of the vector (or pure) states of the resulting composite system will be $\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. Its elements are general linear combinations of the tensor products $\phi_1 \otimes \cdots \otimes \phi_N$ of the pure states of the subsystems, while a general state is given by a density operator $\rho : \mathcal{H} \to \mathcal{H}$. The operators $\mathbf{J}_\mathbf{i}^a$ define the operators $\mathbf{I}_1 \otimes \cdots \otimes \mathbf{J}_\mathbf{i}^a \otimes \cdots \otimes \mathbf{I}_N$ on $\mathcal{H}$, denoted for the sake of simplicity also by $\mathbf{J}_\mathbf{i}^a$. Here $\mathbf{I}_\mathbf{k}$ is the identity operator acting on $\mathcal{H}_\mathbf{k}$. With these notations, the operators $\mathbf{J}_\mathbf{i}^a \mathbf{J}_\mathbf{k}^b$ are well defined, and $\mathbf{J}_\mathbf{i}^a$ and $\mathbf{J}_\mathbf{k}^b$ are commuting if $\mathbf{i} \neq \mathbf{k}$.

Next, for $\mathbf{i} \leq \mathbf{k}$, let us form the operator $\mathbf{J}_\mathbf{i} \cdot \mathbf{J}_\mathbf{k} := \delta_{ab} \mathbf{I}_1 \otimes \cdots \otimes \mathbf{J}_\mathbf{i}^a \otimes \cdots \otimes \mathbf{J}_\mathbf{k}^b \otimes \cdots \otimes \mathbf{I}_N : \mathcal{H} \to \mathcal{H}$. For $\mathbf{i} = \mathbf{k}$ this will be denoted simply by $(\mathbf{J}_\mathbf{i})^2$. Motivated by Eq. (1), we define *the empirical (quantum) angle* between the subsystems $\mathcal{S}_\mathbf{i}$ and $\mathcal{S}_\mathbf{k}$ in the pure tensor product state $\phi = \phi_1 \otimes \cdots \otimes \phi_N$ by

$$\cos\theta_{\mathbf{ik}} := \frac{\langle\phi|\mathbf{J}_\mathbf{i}\cdot\mathbf{J}_\mathbf{k}|\phi\rangle}{\sqrt{\langle\phi|(\mathbf{J}_\mathbf{i})^2|\phi\rangle}\sqrt{\langle\phi|(\mathbf{J}_\mathbf{k})^2|\phi\rangle}} = \frac{\langle\phi_\mathbf{i}|\mathbf{J}_\mathbf{i}^a|\phi_\mathbf{i}\rangle\,\delta_{ab}\,\langle\phi_\mathbf{k}|\mathbf{J}_\mathbf{k}^b|\phi_\mathbf{k}\rangle}{\sqrt{\langle\phi_\mathbf{i}|(\mathbf{J}_\mathbf{i})^2|\phi_\mathbf{i}\rangle}\sqrt{\langle\phi_\mathbf{k}|(\mathbf{J}_\mathbf{k})^2|\phi_\mathbf{k}\rangle}}. \quad (4)$$

Since the absolute value of the expression on the right is not greater than one, this can, in fact, be considered to be the cosine of some angle $\theta_{\mathbf{ik}}$; and for the range of this angle it seems natural to choose $[0, \pi]$.

Remarks:

1. $\cos\theta_{\mathbf{ik}}$ depends only on the states of $\mathcal{S}_\mathbf{i}$ and $\mathcal{S}_\mathbf{k}$, and it is independent of the states of the other subsystems. Moreover, using the transformation property $\mathbf{U}^\dagger \mathbf{J}^a \mathbf{U} = R^a{}_b \mathbf{J}^b$ of the angular momentum vector operator, it is straightforward to check that $\cos\theta_{\mathbf{ik}}$ is $SU(2)$-invariant.

2. Since for $\mathbf{i} = \mathbf{k}$ one has that $\mathbf{J}_\mathbf{i} \cdot \mathbf{J}_\mathbf{i} = \delta_{ab} \mathbf{J}_\mathbf{i}^a \mathbf{J}_\mathbf{i}^b$, which is just the Casimir operator $(\mathbf{J}_\mathbf{i})^2$ of $su(2)$ on $\mathcal{S}_\mathbf{i}$, $\cos\theta_{\mathbf{ii}} = 1$ holds. Thus the empirical angle between any angular momentum vector and itself is always zero, as it could be expected. In the rest of this note, we assume that $\mathbf{i} \neq \mathbf{k}$.

3. It is straightforward to define the angle between two subsystems even when the state of the composite system is an entangled state, $\phi = \sum_{i_1, \dots, i_N} c^{i_1 \cdots i_N} \phi_{i_1} \otimes \cdots \otimes \phi_{i_N}$, or when it is a general mixed state, represented by a density operator $\rho : \mathcal{H} \to \mathcal{H}$. In the first case, it is still defined by (4), while in the second by

$$\frac{\text{tr}\left(\rho\, \mathbf{J_i} \cdot \mathbf{J_k}\right)}{\sqrt{\text{tr}(\rho(\mathbf{J_i})^2)}\sqrt{\text{tr}(\rho(\mathbf{J_k})^2)}}.$$

   However, in these cases the states of the individual constituent subsystems would be *mixed*, and hence the interpretation of $\cos\theta_{\mathbf{ik}}$ in these cases would not be obvious. In addition, this angle might depend on the state of the other subsystems, too. Nevertheless, this extended notion of the empirical angle may provide the appropriate mathematical formulation of Penrose's 'ignorance factor' [1, 5] between $\mathcal{S_i}$ and $\mathcal{S_k}$.

4. The operator $\mathbf{J_i} \cdot \mathbf{J_k}$ was introduced by Moussouris in [6]. However, in the definition of the empirical angle according to him the states had to belong to *finite* dimensional representation spaces of *su*(2). In fact, the denominator in his definition is the *norm* of the unbounded operator $\mathbf{J_i} \cdot \mathbf{J_k}$, which is finite only on finite dimensional spaces. In our definition (4) the Hilbert spaces $\mathcal{H_i}$ and $\mathcal{H_k}$ are *not* required to be finite dimensional. Moreover, the geometric idea of angle given in the classical theory by (1) seems to be captured in the quantum theory more naturally if, in the denominator, the 'lengths' of the individual angular momentum vector operators in the given states are used, just according to (4), rather than the norm of $\mathbf{J_i} \cdot \mathbf{J_k}$. Indeed, $\cos\theta_{\mathbf{ii}}$ in the state $|j_\mathbf{i}, j_\mathbf{i}\rangle$ according to Moussouris would give $(j_\mathbf{i} + 1)/j_\mathbf{i}$, which is always *greater* than 1.

5. In the theory of canonical quantum gravity, Major [13] defined the angle *operator* acting on two edges of the spin network states, labelled by two *su*(2) Casimir invariants, say $j_\mathbf{i}$ and $j_\mathbf{k}$. That operator is $\mathbf{J_i} \cdot \mathbf{J_k}$ divided by the *norm* of $\mathbf{J_i}^a$ and of $\mathbf{J_k}^a$. Thus, Major's angle operator is the correct 'operator version' of (1) (and hence of (4)). Nevertheless, since $\mathbf{J_i}^a$ and $\mathbf{J_k}^a$ are *not* bounded, this angle operator is well defined only on *finite dimensional* Hilbert spaces.

In the present paper, we calculate the empirical angle between the subsystems only *in pure tensor product states of the composite system* according to (4). Thus, by the first remark above, it is enough to consider only two (and, at the end of Sect. 3, only three) subsystems. In the proof of the Spin Geometry Theorem, it will be enough to assume that these states are tensor products of eigenstates of the Casimir operators of the two subsystems, labelled by two Casimir invariants, say j_1 and j_2. Let $\{|j_1, m_1\rangle\}$ and $\{|j_2, m_2\rangle\}$ be the canonical angular momentum bases in the corresponding eigenspaces. Note that here m is only an index labeling the vectors of an orthonormal basis in the $2j+1$ dimensional carrier space of the unitary representation of *su*(2), but it does *not* refer to any Cartesian frame in the 'physical 3-space'. The basis $\{|j, m\rangle\}$ is chosen to be adapted to the actual choice for the components of the vector operator $\mathbf{J}^a$ *in the abstract space of the basic quantum observables*. We choose ϕ_1 and ϕ_2 simply to be $\mathbf{U}_1|j_1, m_1\rangle$ and $\mathbf{U}_2|j_2, m_2\rangle$, where the unitary operators $\mathbf{U}_1$ and $\mathbf{U}_2$ represent $SU(2)$ matrices of the form (2) with some Euler angles $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ in the given representations, respectively.

   Then, using $\mathbf{U}^\dagger \mathbf{J}^a \mathbf{U} = R^a{}_b \mathbf{J}^b$ and how the angular momentum operators act on the canonical bases, (4) yields

$$\cos\theta_{12} = \frac{\delta_{ab}R^a_{1c}R^b_{2d}\langle j_1,m_1|\mathbf{J}^c_1|j_1,m_1\rangle\langle j_2,m_2|\mathbf{J}^d_2|j_2,m_2\rangle}{\hbar^2\sqrt{j_1(j_1+1)j_2(j_2+1)}}$$

$$= \frac{m_1m_2}{\sqrt{j_1(j_1+1)j_2(j_2+1)}}\cos\beta_{12},$$

(5)

where, by (3), $\cos\beta_{12} := (R^{-1}_1R_2)_{33} = \cos\beta_1\cos\beta_2 + \cos(\gamma_1-\gamma_2)\sin\beta_1\sin\beta_2$.

Remarks:

1. The expression of $\cos\beta_{12}$ above is a simple consequence of the well known addition formulae for the Euler angles, which can be read off directly from (3), too. β_{12} is just the angle between the unit vectors $(R_1)^a{}_3$ and $(R_2)^a{}_3$, i.e. an angle between directions *in the 3-space of the basic quantum observables*. θ_{12} depends only on the *relative* orientations of the two subsystems.

2. (5) shows that, for $\beta_{12} \in [0,\pi/2)$, the empirical angle, θ_{12}, is *always greater* than β_{12}, and for $\beta_{12} \in (\pi/2,\pi]$ it is *always smaller* than β_{12}. $\theta_{12} = \beta_{12}$ precisely when $\beta_{12} = \pi/2$. For given j_1 and j_2, the range of $\cos\theta_{12}$ is the *whole* closed interval $[-\sqrt{j_1j_2/(j_1+1)(j_2+1)}, \sqrt{j_1j_2/(j_1+1)(j_2+1)}]$. If β_{12} is fixed, then the empirical angle is still not fixed and it can take different *discrete* values. Note that Planck's constant is canceled from its expression. θ_{12} tends to β_{12} *asymptotically* when $m_1 = j_1$, $m_2 = j_2$ and both j_1 and j_2 tend to infinity. It is this limit that is usually considered to be the classical limit of the spin systems (see e.g. [18]).

3. For $\beta_{12} = 0$, the empirical angle is given by $\cos\theta^0_{12} = m_1m_2/\sqrt{j_1(j_1+1)j_2(j_2+1)}$. Here, $m/\sqrt{j(j+1)}$ is just the cosine of the 'classical' angle between the angular momentum vector of length $\sqrt{j(j+1)}$ and its z-component with length m. Hence, for given $m_1m_2 \neq 0$ and $\beta_{12} = 0$, the greater the product m_1m_2, the smaller the angle θ^0_{12}, but it is *never* zero. Its minimum value corresponds to $\cos\theta^0_{12} = \cos\omega_1\cos\omega_2$, where $\cos\omega := \sqrt{j/(j+1)}$. Hence θ^0_{12} is greater than any of ω_1 and ω_2. θ^0_{12} tends to zero only asymptotically in the $m_1 = j_1 \to \infty$, $m_2 = j_2 \to \infty$ (classical) limit.

3 The Classical Limit and the Spin Geometry Theorem

By (5), the empirical angles θ_{ik} between the subsystems $\mathcal{S}_i$ and $\mathcal{S}_k$ of the composite system in the tensor product of the individual states $|\phi_i\rangle = U_i|j_i,j_i\rangle$ and $|\phi_k\rangle = U_k|j_k,j_k\rangle$, respectively, are given by

$$\cos\theta_{ik} = \sqrt{\frac{j_ij_k}{(j_i+1)(j_k+1)}}\cos\beta_{ik}.$$

(6)

Thus, as a consequence of the discussion at the end of the previous section, we immediately obtain the following statement:

Proposition 1 *For arbitrarily small $\epsilon > 0$ there is a positive integer J such that, in the states above for any $j_i, j_k > J$, $i, k = 1, \ldots, N$, $|\theta_{ik} - \beta_{ik}| < \epsilon$ holds.*

Thus, in the large j limit, the empirical angles θ_{ik} tend to the angles β_{ik} of the three dimensional Euclidean vector space of the basic quantum observables.

However, still we should check that, in this limit, the uncertainties do not grow. In fact, we show that these uncertainties tend to zero. First we calculate the square of the standard deviation of $\mathbf{J}_1 \cdot \mathbf{J}_2$. The expectation value of its square is

$$
\begin{aligned}
\langle \phi_1 \otimes \phi_2 | (\mathbf{J}_1 \cdot \mathbf{J}_2)^2 | \phi_1 \otimes \phi_2 \rangle &= \langle \mathbf{J}_1^a \phi_1 | \mathbf{J}_1^c \phi_1 \rangle \delta_{ab} \delta_{cd} \langle \mathbf{J}_2^b \phi_2 | \mathbf{J}_2^d \phi_2 \rangle \\
&= (R_1^{-1} R_2)_{ab} (R_1^{-1} R_2)_{cd} \langle j_1, j_1 | \mathbf{J}_1^a \mathbf{J}_1^c | j_1, j_1 \rangle \langle j_2, j_2 | \mathbf{J}_2^b \mathbf{J}_2^d | j_2, j_2 \rangle.
\end{aligned}
\tag{7}
$$

Since the only non-zero matrix elements of $\mathbf{J}^a \mathbf{J}^b$ in the states $|j,j\rangle$ are

$$
\langle j,j | \mathbf{J}^1 \mathbf{J}^1 | j,j \rangle = \langle j,j | \mathbf{J}^2 \mathbf{J}^2 | j,j \rangle = \frac{1}{2} \hbar^2 j,
$$

$$
\langle j,j | \mathbf{J}^1 \mathbf{J}^2 | j,j \rangle = -\langle j,j | \mathbf{J}^2 \mathbf{J}^1 | j,j \rangle = \frac{i}{2} \hbar^2 j,
$$

$$
\langle j,j | \mathbf{J}^3 \mathbf{J}^3 | j,j \rangle = \hbar^2 j^2,
$$

(7) takes the form

$$
\begin{aligned}
\langle \phi_1 \otimes \phi_2 | (\mathbf{J}_1 \cdot \mathbf{J}_2)^2 | \phi_1 \otimes \phi_2 \rangle =& \frac{1}{4} \hbar^4 j_1 j_2 (((R_1^{-1} R_2)_{11})^2 + ((R_1^{-1} R_2)_{12})^2 + ((R_1^{-1} R_2)_{21})^2 + ((R_1^{-1} R_2)_{22})^2) \\
&+ \frac{1}{2} \hbar^4 (j_1)^2 j_2 (((R_1^{-1} R_2)_{31})^2 + ((R_1^{-1} R_2)_{32})^2) \\
&+ \frac{1}{2} \hbar^4 j_1 (j_2)^2 (((R_1^{-1} R_2)_{13})^2 + ((R_1^{-1} R_2)_{23})^2) + \hbar^4 (j_1)^2 (j_2)^2 ((R_1^{-1} R_2)_{33})^2 \\
&+ \frac{1}{2} \hbar^4 j_1 j_2 ((R_1^{-1} R_2)_{12} (R_1^{-1} R_2)_{21} - (R_1^{-1} R_2)_{11} (R_1^{-1} R_2)_{22}).
\end{aligned}
$$

Using the explicit form (3) of the rotation matrix, a lengthy but elementary calculation yields that

$$
\begin{aligned}
\langle \phi_1 \otimes \phi_2 | (\mathbf{J}_1 \cdot \mathbf{J}_2)^2 | \phi_1 \otimes \phi_2 \rangle =& \frac{1}{4} \hbar^4 j_1 j_2 (1 + \cos^2 \beta_{12}) + \frac{1}{2} \hbar^4 (j_1)^2 j_2 (1 - \cos^2 \beta_{12}) \\
&+ \frac{1}{2} \hbar^4 j_1 (j_2)^2 (1 - \cos^2 \beta_{12}) + \hbar^4 (j_1 j_2)^2 \cos^2 \beta_{12} - \frac{1}{2} \hbar^4 j_1 j_2 \cos \beta_{12}.
\end{aligned}
$$

Hence, the square of the standard deviation of $\mathbf{J}_1 \cdot \mathbf{J}_2$ in the state $\phi_1 \otimes \phi_2$ is

$$
\begin{aligned}
(\Delta_\phi (\mathbf{J}_1 \cdot \mathbf{J}_2))^2 &= \langle \phi | (\mathbf{J}_1 \cdot \mathbf{J}_2)^2 | \phi \rangle - (\langle \phi | \mathbf{J}_1 \cdot \mathbf{J}_2 | \phi \rangle)^2 \\
&= \frac{1}{2} \hbar^4 j_1 j_2 \left(\frac{1}{2} (1 - \cos \beta_{12})^2 + (j_1 + j_2) \sin^2 \beta_{12} \right).
\end{aligned}
$$

Since the state $\phi_1 \otimes \phi_2$ is an eigenstate both of $(\mathbf{J}_1)^2$ and $(\mathbf{J}_2)^2$, finally we obtain that the square of the uncertainty of $\cos \theta_{ik}$ in the state $\phi = \phi_1 \otimes \cdots \otimes \phi_N$, defined by the first equality below, is

$$
(\Delta_\phi \cos \theta_{ik})^2 := \frac{(\Delta_\phi (\mathbf{J}_i \cdot \mathbf{J}_k))^2}{\langle \phi | (\mathbf{J}_i)^2 | \phi \rangle \langle \phi | (\mathbf{J}_k)^2 | \phi \rangle} = \frac{1}{4} \frac{(1 - \cos \beta_{ik})^2 + 2(j_i + j_k) \sin^2 \beta_{ik}}{(j_i + 1)(j_k + 1)}.
\tag{8}
$$

For given $j_{\mathbf{i}}$ and $j_{\mathbf{k}}$ this uncertainty is zero precisely at $\beta_{\mathbf{ik}} = 0$, and it takes its maximal value at $\cos \beta_{\mathbf{ik}} = 1/(1 - 2(j_{\mathbf{i}} + j_{\mathbf{k}}))$. However, independently of $\beta_{\mathbf{ik}}$, this uncertainty tends to zero if $j_{\mathbf{i}}, j_{\mathbf{k}} \to \infty$. With this conclusion we have proven the next statement.

Proposition 2 *For arbitrarily small $\epsilon > 0$ there is a positive integer J such that, in the states above for any $j_{\mathbf{i}}, j_{\mathbf{k}} > J$, $\mathbf{i}, \mathbf{k} = 1, \dots, N$, $\Delta_\phi \cos \theta_{\mathbf{ik}} < \epsilon$ hold.*

With the Propositions 1 and 2 at hand we have already given a simple proof of (a version of) the Spin Geometry Theorem in the traditional framework of quantum mechanics:

Theorem *Let $\mathcal{S}$ be composed of the quantum mechanical systems $\mathcal{S}_1, \dots, \mathcal{S}_N$. Then there is a collection of pure tensor product states of $\mathcal{S}$, $\phi_1(j_1) \otimes \cdots \otimes \phi_N(j_N)$ indexed by an N-tuple $(j_1, \dots, j_N)$ of non-negative integers or half-odd-integers, such that, in the $j_1, \dots, j_N \to \infty$ limit, the empirical angles between these subsystems converge with asymptotically vanishing uncertainty to angles between directions of the three dimensional Euclidean vector space.*

Remarks:

1. Since for any given $\mathbf{i}$ and $j_{\mathbf{i}}' \neq j_{\mathbf{i}}$ the states $|j_{\mathbf{i}}, j_{\mathbf{i}}\rangle$ and $|j_{\mathbf{i}}', j_{\mathbf{i}}'\rangle$ belong to orthogonal subspaces of the Hilbert space $\mathcal{H}_{\mathbf{i}}$ of all the pure states of the system $\mathcal{S}_{\mathbf{i}}$, the states $\mathbf{U}_1|j_1, j_1\rangle \otimes \cdots \otimes \mathbf{U}_N|j_N, j_N\rangle \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ labelled by different N-tuples, say $(j_1, j_2, \dots, j_N)$ and $(j_1', j_2, \dots, j_N)$, are orthogonal to one another. Hence, the sequence of the states $\phi_1(j_1) \otimes \cdots \otimes \phi_N(j_N) \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ does *not* converge to any normalized state in the strong topology of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. (In the weak topology, it converges to zero.) Therefore, there is *no* quantum state of the system which would represent the above classical limit $j_1, \dots, j_N \to \infty$. It is not clear whether or not one can find *actual states* in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, analogous e.g. to the so-called canonical coherent states of the Heisenberg systems and which could also be interpreted as the composite system's 'most classical state' (see e.g. [19]), in which the empirical angles would coincide with those of the three dimensional Euclidean vector space.
2. One novelty of the analysis behind the above version of the Spin Geometry Theorem is that it is based on a concept of empirical angle, viz. that given by (4), which is well defined not only asymptotically (like that in the version of the Spin Geometry Theorem proven in [6]), but *even at the genuine quantum level*. Thus, in the present approach, some non-trivial aspect of the *quantum geometry* defined by the quantum mechanical systems is already shown up. (We discuss this issue a bit more in the next section.) The other novelty is that it gives *explicitly* a sequence of states which provides the correct, expected classical limit.
3. It might be worth noting that *mathematically* the Theorem stated in [1–5], the version proven in [6] and the version above are *not* equivalent. Nevertheless, their *physical content*, viz. that the conformal structure of the Euclidean 3-space can

be recovered in the classical limit from quantum observables, is the same. Hence we can consider them only different *versions* of the *same* physical theorem.

4. According to expectations of certain recent investigations (see e.g. [20–22]), the geometry of 3-space/spacetime emerges from the *entanglement* of the *states* of the quantum subsystems of the Universe. However, the results of the present work do not seem to support this expectation, at least in the quantum *mechanical* approximation of the quantum world. In fact, the Spin Geometry Theorem could successfully be proven using only *pure tensor product states* of the subsystems. The entanglement of the *states* of the subsystems was not needed. On the other hand, the quantum operator $\mathbf{J_i} \cdot \mathbf{J_k}$, by means of which the empirical angles were defined, is *a sum of the products* of observables, $\mathbf{J_i^1 J_k^1} + \mathbf{J_i^2 J_k^2} + \mathbf{J_i^3 J_k^3}$, which structure is analogous to that of the entangled states. But, in contrast to local quantum *field theory*, in quantum *mechanics* there is no locality: the quantum operators 'feel' the whole wave function on the *entire* momentum/configuration space. Therefore, the entanglement can be considered to be built already into the structure of the *quantum mechanical observables of the composite system*, by means of which the geometry of 3-space/spacetime can be defined in an operational way. The *states* do not need to be entangled.

4. Using the natural volume 3-form ε_{abc} on the algebra *su*(2) of the basic quantum observables, a further potentially interesting geometric notion, viz. the 'empirical 3-volume elements' can be introduced. In the state $\phi \in \mathcal{H}$ spanned by three subsystems this is defined by

$$v_{\mathbf{ijk}} := \frac{1}{3!}\varepsilon_{abc} \frac{\langle\phi|\mathbf{J_i}^a\mathbf{J_j}^b\mathbf{J_k}^c|\phi\rangle}{\sqrt{\langle\phi(\mathbf{J_i})^2|\phi\rangle}\sqrt{\langle\phi(\mathbf{J_j})^2|\phi\rangle}\sqrt{\langle\phi(\mathbf{J_k})^2|\phi\rangle}}. \tag{9}$$

Then, in the tensor product of the states of the form $\mathbf{U}|j, m\rangle$ with the unitary operator $\mathbf{U}$ representing some $SU(2)$ matrix $U^A{}_B$ of the form (2), this expression gives

$$v_{\mathbf{ijk}} = \frac{m_\mathbf{i} m_\mathbf{j} m_\mathbf{k}}{\sqrt{j_\mathbf{i}(j_\mathbf{i}+1)j_\mathbf{j}(j_\mathbf{j}+1)j_\mathbf{k}(j_\mathbf{k}+1)}} \frac{1}{3!}\varepsilon_{abc}(R_\mathbf{i})^a{}_3 (R_\mathbf{j})^b{}_3 (R_\mathbf{k})^c{}_3, \tag{10}$$

where $R^a{}_b$ is the rotation matrix (3) corresponding to $U^A{}_B$, and the second factor in (10) is just the Euclidean 3-volume of the tetrahedron spanned by the unit vectors $(R_\mathbf{i})^a{}_3$, $(R_\mathbf{j})^a{}_3$ and $(R_\mathbf{k})^a{}_3$. Thus, even if $m_\mathbf{i} = j_\mathbf{i}$, $m_\mathbf{j} = j_\mathbf{j}$ and $m_\mathbf{k} = j_\mathbf{k}$, $v_{\mathbf{ijk}}$ is *always smaller* than its Euclidean counterpart: the former is only *conformal* to the latter, and it tends to the Euclidean 3-volume only in the $m_\mathbf{i} = j_\mathbf{i}, m_\mathbf{j} = j_\mathbf{j}, m_\mathbf{k} = j_\mathbf{k} \to \infty$ limit.

4 A Classical Model of the 'Space of the Quantum Directions'

In [5], Penrose (quoting Aharonov, too) raises the possibility in the context of the quantum mechanical double slit experiment that the 'true' geometry that the electron 'sees' might be different from the 'classical' geometry of the two slits. Motivated by this idea, we may ask 'What kind of geometry should we have if we want to arrange the "empirical" geometric notions and quantities introduced via the observables of the quantum systems?' In particular, what could be the geometry in which the empirical angles and 3-volume elements, defined by the composite quantum system $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_N$, are angles and 3-volume elements?

To illustrate by a classical model how this 'true' (conformal) geometry might look like, for the sake of simplicity suppose that all the subsystems have the same total angular momentum j. Let us assume that the empirical angles are angles between pairs of unit vectors of (and the empirical 3-volumes are 3-volumes of tetrahedra formed by triplets of unit vectors in) some n dimensional real vector space. This vector space is modeled by $\mathbb{R}^n$, which is endowed by some positive definite metric $G_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, n$.

By (5) the empirical angle between the 'directions' of two subsystems *cannot be smaller* than $\theta^{min} = \arccos(j/(j+1))$ and *cannot be greater* than $\theta^{max} = \arccos(-j/(j+1)) = \pi - \theta^{min}$. (For example, for $j = 1/2$ these bounds are $\approx 70.53°$ and $\approx 109.47°$; and for $j = 1$ these are $60°$ and $120°$, respectively.) Clearly, for any given N, these empirical angles can always be arranged in $\mathbb{R}^n$ for large enough but finite n. Nevertheless, the existence of the bounds θ^{min} and θ^{max} provides a lower bound for n. (For a given state $\phi_1 \otimes \cdots \otimes \phi_N$, the optimal value of n might be determined by a procedure analogous to that in the so-called sphere packing problem [23], see below.) Let the unit vector $V_{\mathbf{i}}^\alpha$ represent the 'direction' associated with the $\mathbf{i}$th subsystem in this space. Then let us draw two solid cones in $\mathbb{R}^n$ with $V_{\mathbf{i}}^\alpha$ as their common axis, their vertices at the origin, and with the opening angle θ^{min} and θ^{max}, respectively, such that the cone with opening angle θ^{max} contains the cone with opening angle θ^{min}. Then the result that the empirical angle between the 'directions' of any two subsystems cannot be greater than θ^{max} and cannot be smaller than θ^{min} implies that the *inner cone* with axis $V_{\mathbf{i}}^\alpha$ and that with axis $V_{\mathbf{k}}^\alpha$ intersect each other *at most* along one line in their lateral surface, and their *outer cone* intersect each other *at least* along one line in their lateral surface.

Let $p_{\mathbf{i}}$ be the point of the unit sphere S^{n-1} in $\mathbb{R}^n$ that the unit vector $V_{\mathbf{i}}^\alpha$ defines, let $C_{\mathbf{i}}$ denote the intersection of the cone with axis $V_{\mathbf{i}}$ and opening angle θ^{max} with the unit sphere, and let $c_{\mathbf{i}}$ be the intersection of the inner cone with the unit sphere. Thus, $C_{\mathbf{i}}$ and $c_{\mathbf{i}}$ are concentric spherical caps with the common centre $p_{\mathbf{i}}$ on S^{n-1}. Then the resulting spherical caps $C_{\mathbf{i}}$ and $C_{\mathbf{k}}$, $\mathbf{i} \neq \mathbf{k}$, *must* intersect each other at least in one point, but the corresponding inner spherical caps $c_{\mathbf{i}}$ and $c_{\mathbf{k}}$ *may* intersect each other at most in one point. Therefore, the set of the empirical angles between any two subsystems considered in Sect. 2 may be represented by the set of these configurations of the points $p_{\mathbf{i}}$ and the corresponding pairs $(C_{\mathbf{i}}, c_{\mathbf{i}})$, $\mathbf{i} = 1, \ldots, N$. There are *no distinguished directions* in this space, i.e. the space

does *not* have any naive lattice structure, *but the angle between any two directions cannot be arbitrarily small or arbitrarily large*.

This second realization of the classical model of the 'space of the quantum directions' makes it possible, at least in principle, to determine the minimal dimension n in which the directions of N subsystems, each with spin j, can be arranged: this is the minimal dimension for which N pairs of concentric $(n-1)$ dimensional spherical caps, or rather balls, with given radii can be packed into the unit sphere S^{n-1} satisfying the above conditions. This is a version of the 'sphere packing problem' of [23].

Requiring that the 3-volume elements determined by $V_{\mathbf{i}}^{\alpha}$, $V_{\mathbf{j}}^{\alpha}$ and $V_{\mathbf{k}}^{\alpha}$ in $\mathbb{R}^n$ be just $v_{\mathbf{ijk}}$ for the composite system, (10) might suggest to choose the metric $G_{\alpha\beta}$ to be conformal to the Euclidean one, $G_{\alpha\beta} = (j/(j+1))\delta_{\alpha\beta}$.

Declarations

Conflict of Interest There is no conflict of interest that could be relevant to the content of this article.

References

1. Penrose, R.: Combinatorial quantum theory and quantized directions. In: Houghston, L.P., Ward, R.S. (eds.) Advances in Twistor Theory. Pitman Publishing Ltd, London ISBN: 0-8224-8448-X (1979)
2. Penrose, R.: Theory of quantized directions, (handwritten manuscript, 1967), https://math.ucr.edu/home/baez/penrose/Penrose-TheoryOfQuantizedDirections.pdf (1967)
3. Penrose, R.: Theory of quantized directions. In: Collected Works, vol. 1, pp. 769–800. Oxford University Press, Oxford. ISBN-13: 978-0199219445. ISBN-10: 0199219443 (2010)
4. Penrose, R.: Angular momentum: an approach to combinatorial spacetime. In: Bastin, T. (ed.)Quantum Theory and Beyond. Cambridge University Press, Cambridge ISBN 9780521115483 (1971)

5. Penrose, R.: On the nature of quantum geometry. In: Klauder, J. (ed.) Magic Without Magic. Freeman, San Francisco. ISBN 0-7167-0337-8 (1972)
6. Moussouris, J.P. : Quantum Models of Spacetime Based on Recoupling Theory, PhD dissertation, Oxford University https://ora.ox.ac.uk/objects/uuid:6ad25485-c6cb-4957-b129-5124bb2adc67 (1984)
7. Penrose, R.: Private communication (2022)
8. Penrose, R., MacCallum, M.A.H.: Twistor theory: an approach to the quantisation of fields and space-time. Phys. Rep. **6**, 241–316 (1972). https://doi.org/10.1016/0370-1573(73)90008-2
9. Bailey, N.T., Baston, R.J. (eds.): Twistors in Mathematics and Physics. London Math. Soc. Lecture Note Series 156, Cambridge University Press, Cambridge. ISBN: 9781107325821, https://doi.org/10.1017/CBO9781107325821 (1990)
10. Atiyah, M., Dunajski, M., Mason, L.: Twistor theory at fifty: from contour integrals to twistor strings. Proc. R. Soc. A **473**, 20170530 (2017). https://doi.org/10.1098/rspa.2017.0530
11. Rovelli, C., Smolin, L.: Loop representation of quantum general relativity. Nucl. Phys. B **331**, 80–152 (1990). https://doi.org/10.1016/0550-3213(90)90019-A
12. Rovelli, C., Smolin, L.: Discreteness of area and volume in quantum gravity. Nucl. Phys. B **422**, 593–619 (1995). https://doi.org/10.1016/0550-3213(95)00150-Q
13. Major, S.A.: Operators for quantized directions. Class. Quantum Grav. **16**, 3859–3877 (1999). https://doi.org/10.1088/0264-9381/16/12/307
14. Rovelli, C., Smolin, L.: Spin networks and quantum gravity. Phys. Rev. D **52**, 5743–5759 (1995). https://doi.org/10.1103/PhysRevD.52.5743
15. Szabados, L.B.: Three-space from quantum mechanics, arXiv: 2203.04827 [quant-ph]
16. Penrose, R., Rindler, W.: Spinors and Spacetime, vol. 1. Cambridge University Press, Cambridge. ISBN-10: 0521337070. ISBN-13: 9780521337076 (1984)
17. Hugget, S.A., Tod, K.P.: An Introduction to Twistor Theory, London Mathematical Society Student Texts vol. 4, 2nd edn. Cambridge University Press, Cambridge. ISBN-10: 0521456894. ISBN-13: 9780521456890 (1994)
18. Wigner, E.P.: Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra. Academic Press, New York ISBN-10: 0127505504, ISBN-13: 978-0127505503 (1959)
19. Szabados, L.B.: An odd feature of the 'most classical' states of $SU(2)$ invariant quantum mechanical systems, arXiv: 2106.08695 [gr-qc]
20. Van Raamsdonk, M.: Building up spacetime with quantum entanglement. Gen. Relativ. Gravit. **42**, 2323–2329 (2010). https://doi.org/10.1007/s10714-010-1034-0
21. Jacobson, T.: Entanglement equilibrium and the Einstein equation. Phys. Rev. Lett. **116**, 201101 (2016). https://doi.org/10.1103/PhysRevLett.116.201101
22. Cao, C.J., Carroll, S.M., Michalakis, S.: Space from Hilbert space: recovering geometry from bulk entanglement. Phys. Rev. D **95**, 024031 (2017). https://doi.org/10.1103/PhysRevD.95.024031
23. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups. Springer, New York (1993). https://doi.org/10.1007/978-1-4757-2249-9