



# Bohmian Mechanics is Not Deterministic

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## Abstract

I argue that Bohmian mechanics (or any similar pilot-wave theory) cannot reasonably be claimed to be a deterministic theory. If one assumes the “quantum equilibrium distribution” provided by the wave function of the universe, Bohmian mechanics requires an external random oracle in order to describe the (Kolmogorov–Levin–Chaitin) algorithmic randomness properties of typical outcome sequences of long runs of repeated identical experiments (which provably follow from the Born rule). This oracle lies beyond the scope of Bohmian mechanics (or any deterministic extension thereof), including the impossibility of explaining the randomness property in question from “random” initial conditions. Thus the advantages of Bohmian mechanics over other interpretations of quantum mechanics, if any, must lie at an ontological level, and in its potential to derive the quantum equilibrium distribution and hence the Born rule.

**Keywords** Bohmian mechanics · Determinism · Randomness

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The claims in this paper follow from the general analysis of randomness in quantum mechanics in Landsman [1–3] and were first presented at a Harvard Foundations of Physics seminar (organized by Jacob Barandes), December 15, 2020, in a talk called ‘Determinism and undecidability: You can’t eat your cake and have it too.’ I am grateful to the audience for several critical questions, which have clarified my reasoning. Further improvements were made after a discussion at the Cambridge–LSE Philosophy of Physics Bootcamp, May 24, 2022. Finally, I am indebted to two anonymous referees for numerous constructive suggestions, which I took into account.

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## 1 Introduction

Bohmian mechanics is widely seen as a *deterministic* interpretation of quantum mechanics. This point was already made by Bohm himself, and has been repeated by his followers:<sup>1</sup>

My interpretation of the quantum theory describes all processes as basically causal and continuous.<sup>2</sup> [10, p. 175]

But in 1952 I saw the impossible done. It was in papers by David Bohm. Bohm showed explicitly how parameters could indeed be introduced, into nonrelativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. [11, p. 990]

Bohmian mechanics happens to be deterministic. A substantial success of Bohmian mechanics is the explanation of quantum randomness or Born's statistical law, on the basis of Boltzmann's principles of statistical mechanics, i.e. Born's law is not an axiom but a theorem in Bohmian mechanics. [12, p. 6]

My aim is to show that the alleged determinism of Bohmian mechanics is parasitical on some external random sampling mechanism ("oracle") the theory has to invoke in order to state the specific value of the hidden variable (i.e. position) in each experiment. Short of the above oracle, Bohmian mechanics by itself is not only unable to predict the outcome of individual experiments, but cannot even reproduce their provable (algorithmic) randomness properties (which follow from the Born rule). Furthermore, Bohmian mechanics cannot be extended by any deterministic theory so as to replace the random oracle, even if it is deemed acceptable that a deterministic theory fails to specify its initial conditions. Thus the performance of Bohmian mechanics and especially its level of determinism are similar to that of minimal versions of the Copenhagen interpretation, which also leaves the outcomes of experiments to a black box (arguably a more obscure one).

Partly to set the stage and partly to briefly discuss other aspects of determinism I will not go into here any further, let me summarize the formalism of Bohmian mechanics.<sup>3</sup> For simplicity, consider a system of  $N$  non-relativistic spinless particles with masses  $m_1, \dots, m_N$ , moving in  $\mathbb{R}^3$ , so that the classical configuration space is  $\mathbb{R}^{3N}$ . Suppose the classical Hamiltonian, defined on the associated phase space  $T^*(\mathbb{R}^{3N}) \cong \mathbb{R}^{6N}$ , is given by

<sup>1</sup> See Kaiser [4], Norsen [5], Greenstein [6], Bricmont [7], Bub [8], and Ryckman [9] for some history of Bohmian mechanics and the closely related de Broglie–Bohm pilot wave theory.

<sup>2</sup> Here 'causal' is synonymous to 'deterministic'. This is clear from a contrast Bohm draws on the same page: 'Thus, we are able in my interpretation to understand by means of a causal and continuous model just those properties of matter and light which seem most convincingly to require the assumption of discontinuity and incomplete determinism.' Similarly, Holland's well-known book *The Quantum Theory of Motion* is subtitled: *An Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics*.

<sup>3</sup> For details see e.g. Holland [13], Dürr and Teufel [12], Bricmont [14], and Goldstein [15].

$$h(p, q) = \sum_{k=1}^N \frac{p_k^2}{2m_k} + V(q); p = (p_1, \dots, p_N), q = (q_1, \dots, q_N). \tag{1.1}$$

The associated quantum theory starts from the Hilbert space  $H = L^2(\mathbb{R}^{3N})$ , on which one tries to define a Hamiltonian  $\hat{h} = -\sum_k \hat{\Delta}_k / (2m_k) + \hat{V}$  as a self-adjoint operator, where  $\hat{\Delta}_k$  is the Laplacian for the  $k$ th coordinate  $x_k = (x_k^1, x_k^2, x_k^3)$  in  $x = (x_1, \dots, x_N)$ , and  $\hat{V}$  is the potential seen as a multiplication operator. If this is done successfully,<sup>4</sup> by Stone’s theorem any  $\Psi \in H$  evolves unitarily for all  $t \in \mathbb{R}$  as a solution of the Schrödinger equation

$$\hat{h}\Psi_t = i\hbar \frac{\partial \Psi_t}{\partial t}, \tag{1.2}$$

where  $\Psi_0 = \Psi$ . This solution, in turn, gives a probability measure  $P_t$  on  $\mathbb{R}^{3N}$  via

$$dP_t(x) = \rho_t(x) d^{3N}x; \rho_t(x) := |\Psi_t(x)|^2, \tag{1.3}$$

where  $d^{3N}x := d^3x_1 \dots d^3x_N$  with  $d^3x_k = dx_k^1 dx_k^2 dx_k^3$ . Since Born [18], the role of  $P_t$  has been to provide probabilities for outcomes of measurements; e.g. if position is measured, then the probability of finding the collective  $N$ -particle position  $q$  in  $A \subset \mathbb{R}^{3N}$  at time  $t$  equals

$$\Pr_t(q \in A) = P_t(A). \tag{1.4}$$

In textbook (Copenhagen) quantum mechanics these probabilities do not have an ignorance interpretation (and hence are “irreducible”), because the particle positions are not just *unknown* before they are measured; they *do not even exist* prior to measurement.

Bohmian mechanics may be seen as a modification of either classical or quantum mechanics. In the former view it adds a (“pilot”) wave function  $\Psi_t(q)$  to the particle trajectories  $t \mapsto q(t)$ , whereas in the latter view it adds particle trajectories to the wave function. The wave function is supposed to satisfy the Schrödinger equation of quantum mechanics, seen however as a classical PDE (as opposed to an ODE in Hilbert space).<sup>5</sup> But the wave function only acts as a servant to the particles, in two completely different ways:

1. Once  $\Psi_t(q)$  is known by solving (1.2), the trajectories  $t \mapsto q(t)$  are found by solving

<sup>4</sup> In practice,  $\hat{h}$  is first defined on some safe domain like  $D = C_c^\infty(\mathbb{R}^{3N})$  and subsequently extended to a self-adjoint operator, but this procedure may be unique, non-unique, or even impossible, depending on the potential  $V$  (and, using configuration spaces  $Q$  different from  $\mathbb{R}^{3N}$ , on  $Q$ ). In the non-unique case (where  $\hat{h}$  has several self-adjoint extensions) one faces some kind of indeterminism Earman [16]; interesting examples of this phenomenon come from spacetime singularities [17]. Such “indeterminism” will be inherited by Bohmian mechanics. But I will ignore this on the understanding that situations where this happens can typically be resolved by providing *classical* boundary conditions.

<sup>5</sup> As pointed out by Arageorgis and Earman [19], in Bohmian mechanics the wave function  $\Psi$  is taken “on the nose”, whereas in quantum mechanics it is its equivalence class in  $L^2$  that defines the state.

$$\frac{dq_k(t)}{dt} = v_k(q(t), t); v_k(q(t), t) := \frac{\hbar}{m_k} \operatorname{Im} \left( \frac{\nabla_k \Psi_t(q(t))}{\Psi_t(q(t))} \right) = \frac{j_k(t, q(t))}{\rho_t(q(t))}, \quad (1.5)$$

where  $j_k = \rho v_k$  is the usual  $k$ th particle probability current of quantum mechanics,

$$j_k(t, x) = \frac{i\hbar}{2m_k} (\overline{\Psi_t(x)} \nabla_k \Psi_t(x) - \Psi_t(x) \nabla_k \overline{\Psi_t(x)}). \quad (1.6)$$

Here  $\nabla_k = (\partial/\partial x_k^1, \partial/\partial x_k^2, \partial/\partial x_k^3)$ . Provided  $\Psi_t$  satisfies Schrödinger equation (1.2), the current  $j_k$  is related to the probability density (1.3) by the conservation law

$$\partial_t \rho_t(x) + \sum_k \nabla_k \cdot j_k(t, x) = 0. \quad (1.7)$$

Though (1.5) is just a flow equation for a time-dependent vector field  $v$ , (1.2)–(1.5) form a coupled PDE-ODE system of a novel kind that has no counterpart in either classical or quantum mechanics. Although Stone’s theorem still applies to (1.2), the ensuing vector field  $v$  may be incomplete, in which case the trajectories  $q(t)$  do not exist for all time and all initial conditions. This adds a potential source of indeterminism to Bohmian mechanics that has no analogue in quantum mechanics but is unrelated to the “random-ish” kind of indeterminism discussed in this paper.<sup>6</sup>

2. Though considered real, the particle locations  $q(t)$  are mysteriously hidden *in principle* and can only be revealed by measurement. All we can know is their probability distribution, which at time  $t$  is given by the “quantum equilibrium measure” (1.3)–(1.4), just like in quantum mechanics. This move turns Bohmian mechanics into a special case of classical statistical mechanics, albeit with the unusual features that:
  - (a) The probabilities are defined on configuration space (rather than phase space).
  - (b) A consistent interpretation of these probabilities is elusive. On the one hand, the assumed reality of the particle positions and trajectories forces an ignorance interpretation: since at any time  $t$  all particles are supposed to actually *have* positions,  $\rho_t$  (which never describes these actual values) cannot be objective. But on the other hand, since these values cannot even be known in principle (until they are measured), the probabilities defy an ignorance interpretation and seem to be ontic or objective. See also Myrvold [22, Sect. 9.4]. This dilemma haunts Bohmian mechanics and may explain some of its ultimate indeterminism.

In sum, although its role as a “pilot” in 1. gives the wave function  $\Psi_t$  some ontic status, its associated probability distribution  $\rho_t$  uncomfortably floats between

<sup>6</sup> For details see Berndl et al. [20], Teufel and Tumulka [21], and Arageorgis and Earman [19].

subjectivity and objectivity. According to Dürr et al. [23],  $\rho_t$  determines which particle configurations are deemed “typical” in the sense of “overwhelmingly probable”, but this shifts the burden of interpretation to the question what *that* is supposed to mean! See Sect. 4.

## 2 What Makes a Hidden Variable Theory Deterministic?

Since my analysis does not only apply to Bohmian mechanics and may perhaps even be clearer in a more general context, I momentarily take a broader perspective. A general hidden variable theory  $T$  underneath quantum mechanics, say in the specific context of some given well-understood and repeatable experimental setting, should provide at least:<sup>7</sup>

- A space  $\Lambda$  of hidden variables  $\lambda$ ; in Bohmian  $N$ -particle mechanics this is position space  $\Lambda = \mathbb{R}^{3N}$ , so that  $\lambda = q = (q_1, \dots, q_N)$  with  $q_k = (q_k^1, q_k^2, q_k^3)$ , as above.
- An assignment  $\Psi \mapsto P_\Psi$ , where  $\Psi$  is a quantum-mechanical state (not necessarily a wave function) describing the system, and  $P_\Psi$  is a probability measure on  $\Lambda$ . In Bohmian mechanics  $P_\Psi$  is the Born/“equilibrium” measure (1.3) at time  $t$ .
- Conditional probabilities  $P_\lambda(e | a)$  stating the probability of outcome  $e$  of the experiment on setting  $a$ , given the value  $\lambda$  of the hidden variable; in Bohmian mechanics this of course depends on the details of the experiment, which I need not go into.

The compatibility requirement between quantum mechanics and the given theory, then, is

$$P_\Psi(e | a) = \int_\Lambda dP_\Psi(\lambda) P_\lambda(e | a), \tag{2.1}$$

where  $P_\Psi(e|a)$  is the quantum-mechanical Born probability for outcome  $e$  given  $a$  and  $\Psi$ . In this formalism, a hidden variable theory  $T$  is called *deterministic* (just terminology!) if

$$P_\lambda(e | a) \in \{0, 1\} \text{ for any value of } \lambda, e, a. \tag{2.2}$$

That is, the value of  $\lambda$  *determines* the outcome of the experiment.<sup>8</sup> On this basis, exploring the bipartite (“Alice and Bob”) setting introduced by Einstein et al. [26], Bell [27] proved that the conjunction of the following properties is inconsistent:

1. *Determinism* in the (narrow) sense of (2.2) and preceding definitions;

<sup>7</sup> See Leifer [24] for a more detailed analysis of a setting like this. See also Bub [8] for history.  
<sup>8</sup> In Bohmian mechanics this is true if one knows the hidden positions exactly at the time of measurement. At earlier times, the pilot wave = quantum state  $\Psi$  is needed to make predictions, since  $\Psi$  determines the trajectories. See e.g. Barrett [25, Chap. 5], for a discussion of measurement in Bohmian mechanics.

2. *Quantum mechanics*, i.e. the compatibility rule (2.1);
3. *Locality* in a sense made precise in Bell [27];
4. *Free choice*, i.e. (statistical) independence of Alice's and Bob's settings  $a$  and  $b$  from each other and from the hidden variable  $\lambda$  (given the probability measure  $P_\Psi$ ).

There are four (minimal) ways out of this contradiction: just reject one of the assumptions!

1. The Copenhagen interpretation rejects determinism (I take this interpretation to be a "family resemblance" but all versions should agree on this point).
2. In arguing that the Born distribution (1.3) should be the end result of an equilibration process and hence did not always hold in the past, Valentini [28, 29] effectively departs from the Born rule and hence quantum mechanics. See also Sect. 4.
3. Bohmian mechanics is non-local [12, Chap. 10], [15, Goldstein, Sect. 13]. The question if this non-locality was really necessary motivated Bell [27].
4. Finally, 't Hooft [30] rejects free choice in his deterministic cellular automaton interpretation of quantum mechanics.<sup>9</sup> See also Hossenfelder and Palmer [31].

Restoring determinism in the quantum world is not the only goal, perhaps not even the most important goal of hidden variable theories. This applies to Einstein, and Bell (after whom I will quote similar words from Bricmont) continues the quotation in Sect. 1 with:

More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the "observer", could be eliminated. [11, p. 990].

While deterministic, the de Broglie–Bohm theory also accounts naturally for the apparent indeterminism of quantum phenomena. Finally, it explains the "active role" of measuring devices (...), so strongly emphasized by the Copenhagen school, but making it a consequence of the theory and not of some a priori philosophical doctrine. It also explains the nonlocal interactions inherent in quantum phenomena. What more could we ask for? [14, p. 19].

Having said this, much as one may read Bell [27] both as a proof that quantum mechanics is non-local in a suitable sense *and* as a proof that deterministic hidden variable theories (in the first sense defined below) are necessarily non-local (in a slightly different sense),<sup>10</sup> one may also look at Bohmian mechanics as an attempt to provide a realist ontology for quantum mechanics, *and* as an attempt to restore determinism.<sup>11</sup> These goals are closely related, since they are most easily achieved

<sup>9</sup> Bohmians do not explain the origin of the settings of the experiment, which are simply left out of the theory. In my view this weakens their case for determinism even further, but this is not my main point.

<sup>10</sup> It is a miracle to me why the Bohmians so vehemently oppose the second reading Bricmont [7].

<sup>11</sup> Instead of as a hidden variable theory, Bohmian mechanics may also be seen as an observer-free non-collapse interpretation of quantum mechanics, see Bub [32]. Here the wave function  $\Psi$  is not so much

simultaneously by rewriting quantum mechanics as a classical theory, which is what Bohmian mechanics essentially tries to do.

However, the definition of determinism used above, which is typical for the hidden variable literature including Bohmian mechanics, strikes me as too weak, since nothing is said about individual experiments. Recall that for Born [18], collision theory marked a fundamental difference between classical and quantum mechanics in that: *the initial conditions being known in both cases*, the former could predict the outcome with certainty whereas the latter could not. Classical physics may be said to be “indeterministic” on account of its typical inability to provide or explain initial conditions (say for the solar system), but this is not the kind of indeterminism that concerned Born, or me. I will argue that in Bohmian mechanics, *even if the initial conditions (of either the universe or some relevant part thereof), i.e. the exact particle positions, were known*, no deterministic account of the very specific and provable random structure of typical outcome sequences of quantum-mechanical experiments can be given. To this end I will show that in order to actually acquire the mathematical information necessary to predict such outcome sequences the theory requires an external oracle *whose functioning cannot even in principle be described by any deterministic theory, including of course Bohmian mechanics itself*.

In particular, the randomness of these quantum-mechanical outcome sequences cannot be a consequences of whatever randomness in the initial conditions, as suggested by Dürr et al. [23]; see Sect. 4. Let me now try to make this argument more precise.

### 3 Is Bohmian Mechanics Deterministic?

First, as explained in Landsman [2, 3], in the narrow context of *deterministic* hidden variable theories, refined mathematical methods are available (namely from the theory of algorithmic randomness) that make an appeal to a bipartite setting unnecessary: since my argument does not rely on entanglement, we may simply work with a quantum coin toss (realized for example as a spin- $z$  measurement on a spin- $\frac{1}{2}$  particle in the state  $(1, 1)/\sqrt{2}$ , or optically using suitable polarizers). The settings, possible contexts, and quantum state of the experiments are then fixed to be the same for each experiment in a long run, so that only the hidden state  $\lambda$  (i.e. in Bohmian mechanics the particle positions  $q$ ) may change. Idealizing to an infinite run, one then simply has an outcome sequence

$$s : \mathbb{N} \rightarrow \{0, 1\},$$

such that  $e_n = s(n)$  is the outcome of experiment no.  $n$  (with the given settings etc.).

Although standard (Copenhagen) quantum mechanics refuses to say anything about the origin of each outcome  $s(n)$ , it does make very specific statistical

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Footnote 11 (continued)

seen as a “real” physical field (with mathematically of course is complex!) comparable with the electromagnetic field, but as a mathematical device that controls the modal properties of the theory.

predictions. These do not just concern the single-case probabilities for individual outcomes, which are given by the Born rule (and are 50–50 in the case at hand), but include far more refined claims *about the sequence  $s$  as a whole*, which go well beyond saying that the probability of each outcome is  $1/2$ . The basis of these more detailed predictions is the following theorem:<sup>12</sup>

**Theorem 1** *With respect to the standard “fair” Bernoulli probability measure on the space  $\underline{2}^{\mathbb{N}}$  of all binary sequences,<sup>13</sup> almost every outcome sequence  $s$  of an infinitely often repeated fair quantum coin toss is Kolmogorov–Levin–Chaitin random (i.e. 1-random).<sup>14</sup>*

This notion of randomness (in this case of binary sequences) arose in the 1960s when ideas from probability theory were combined with the theory of algorithms and computation (à la Turing). I already motivated its use in quantum theory (and elsewhere) in Landsman [3], and will return to this motivation in Sect. 4 below in connection with the notion of typicality used by the Bohmians. But here I use this specific concept of randomness because Theorem 1 is simply a theorem, whose relevance will become clear shortly.

I will show that Theorem 1 leads to insurmountable tension with determinism. First, note that in “deterministic” hidden variable theories the outcomes  $s$  factor through  $\Lambda$ , i.e.,

there are functions  $h : \mathbb{N} \rightarrow \Lambda$  and  $g : \Lambda \rightarrow \{0, 1\}$  such that

$$s = g \circ h. \quad (3.1)$$

- The existence of  $g$  expresses the idea that the value of  $\lambda$  *determines* the outcome of the experiment, cf. (2.2), which is *one* important sense in which hidden variable theories could try to be deterministic. The function  $g$  incorporates all details of the experiment that may affect its outcome, like the setting  $a$ , a possible context  $C$ , and the quantum state  $\Psi$ , *except the hidden variable  $\lambda$ , which gassumes as*

<sup>12</sup> This is Corollary 3.4.2 in Landsman [3], which is Corollary 4.2 of the version [arXiv:abs/2003.03554](https://arxiv.org/abs/2003.03554).

<sup>13</sup> This measure is the extension of the 50–50 probability for single bits to sequences [33, Sect. 8.2].

<sup>14</sup> See Calude [34] or Downey and Hirschfeldt [35] for complete treatments, and Appendix B of Landsman [3] for a quick summary. Dasgupta [36] is intermediate between these extremes. This concept was first defined by Kolmogorov for binary *strings* (which are finite by convention) and was subsequently extended to (infinite) binary *sequences* by Levin and independently by Chaitin; whence the name ‘Kolmogorov–Levin–Chaitin’ randomness. The ‘1’ in the more technical name ‘1-randomness’ refers to a family of similar notions [35] whose use in physics should also be explored. Very briefly, and roughly: the *Kolmogorov complexity*  $K(\sigma)$  of a binary *string*  $\sigma$  is the length of “the” shortest computer program  $P$  that prints  $\sigma$  (and then halts). Then  $\sigma$  is *Kolmogorov random* if  $K(\sigma) = |\sigma| + O(\log |\sigma|)$ , which means that *the shortest computable description of  $\sigma$  is  $\sigma$  itself*. One may think of this as  $K(\sigma) \approx |\sigma|$ . A binary *sequence*  $x$  is *Kolmogorov–Levin–Chaitin random* if there exists  $c \in \mathbb{N}$  such that  $K(x_{|N}) \geq N - c$  for each  $N \in \mathbb{N}$ , where  $x_{|N}$  is the truncation of  $x$  to its first  $N$  bits. This definition justifies using infinite sequences as idealizations of long finite strings (where “long” means:  $N \gg c$  for given  $x$ ), in so far as randomness properties are concerned. I will freely do so in what follows.



its argument.<sup>15</sup> In this light, it would be more precise to write  $g(\lambda) = g_{(a,C,\Psi)}(\lambda)$ , but since we agreed that  $(a, C, \Psi)$  and whatever else is relevant for the outcome sequence is fixed throughout the entire run, for simplicity I just write  $g(\lambda)$ . In Bohmian mechanics the contextuality of  $g(q)$  via its hidden dependence on  $a$  and  $C$  is well known (e.g. Dürr and Teufel [12, Sect. 12.2.3]; Goldstein [15, Sect. 12]). Answering a F.A.Q., the existence of  $g$  therefore does not contradict the Kochen–Specker theorem.

- The function  $h$  gives the value of  $\lambda$  in experiment number  $n$  in a long run, for each  $n$ . Conceptually,  $h$  samples the probability measure  $P_\Psi$  relevant to the quantum-mechanical state in which the experiment is prepared (see Sect. 4 for further discussion). For example, in Bohmian mechanics  $h : \mathbb{N} \rightarrow \mathbb{R}^{3N}$  in principle picks an element

$$h(n) \equiv q(n) = (q_1(n), \dots, q_N(n)) \in \mathbb{R}^{3N}, \tag{3.2}$$

i.e., an  $N$ -particle configuration in  $\mathbb{R}^3$  (seen as the total space of the universe, assumed to contain  $N$  particles), for each  $n \in \mathbb{N}$ . This is vast, but as I will review in Sect. 4, under suitable independence and preparation assumptions the results of Dürr et al. [23] enable one to replace  $h : \mathbb{N} \rightarrow \mathbb{R}^{3N}$  by  $h' : \mathbb{N} \rightarrow \mathbb{R}^3$ .

**Theorem 2** *The sampling function  $h$  cannot be provided by Bohmian mechanics, or by some hitherto unknown extension of it that may reasonably be claimed to be deterministic.*

**Proof** Arguing by *reductio ad absurdum*, if  $h$  is provided by some deterministic theory  $T$ , right because  $T$  is supposed to be deterministic,  $h$  explicitly gives the values  $x_n = h(n)$  for each  $n \in \mathbb{N}$  (i.e. experiment no.  $n$ ). Since  $g$  is also given, this means that the sequence  $s = goh$  is described explicitly by some formula. By Chaitin’s second incompleteness theorem,<sup>16</sup> the outcome sequence cannot then be 1-random, against Theorem 1. □

Thus the question arises *what else provides  $h$* . In order to recover the predictions of quantum mechanics as meant in Theorem 1, i.e. not just the single-case Born probabilities but the (Kolmogorov–Levin–Chaitin) randomness properties of entire outcome sequences, the function  $h$  must sample either the Born measure  $P_\Psi$  on  $\Lambda = \mathbb{R}^{3N}$ , seen as the “quantum equilibrium distribution” of the universe, or the one-particle Born measure on  $\mathbb{R}^3$  provided by some effective wave function  $\psi$  (see above). Since  $g$  is supposed to be given by Bohmian mechanics, this implies that the randomness properties of  $s$  must entirely originate in  $h$ . This origin cannot be

<sup>15</sup> In more general measurements  $g$  takes values in the spectrum of the operator that is measured.

<sup>16</sup> This is Theorem B.4 in Landsman [3], both in the printed and the arXiv version. Briefly, this theorem states that even comprehensive mathematical theories (such as set theory) can compute only finitely many digits of a 1-random sequence  $x$ . See e.g. Calude [34], Theorem 8.7, which is stated for Chaitin’s famous random number  $\Omega$  but whose proof holds for any 1-random sequence. In view of Theorem B.1 (*loc. cit.*), also due to Chaitin, this argument applies equally well to long strings (cf. footnote 14).

deterministic, since in that case we are back to the contradictory scenario above. Hence  $h$  must come from, or indeed may be identified with, some unknown external random process in nature that Bohmian mechanics needs to invoke as a kind of random oracle—it cannot be overemphasized how strong the requirement of *random sampling* of the configuration space  $\mathbb{R}^{3N}$  with respect to the probability measure  $P_\Psi$  is; as shown above, this requirement excludes determinism (and, if different, computability).

## 4 Discussion

Theorem 2 allows a sharp comparison with Copenhagen quantum mechanics:

- In Bohmian mechanics the outcome sequence  $s = g \circ h$  factors through its hidden variable space  $\mathbb{R}^{3(N)}$  and the corresponding sampling function  $h : \mathbb{N} \rightarrow \mathbb{R}^{3(N)}$  is its source of randomness, lying outside the theory as a black box or external “oracle”.
- In Copenhagen quantum mechanics there is no such factorization of  $s$ ; here, the unanalyzed black box and source of randomness is the measurement process itself.

So although in Bohmian mechanics the source of indeterminism has been shifted compared to its place in Copenhagen quantum mechanics, this source certainly has not been removed.

As especially stressed by Dürr et al. [23], Dürr and Teufel [12], and Goldstein [15], Bohmian mechanics tries (or: Bohmians try) to answer all such worries by an appeal to *typicality*. To this end, their main weapon is what they call the *quantum equilibrium hypothesis*, which I will now try to summarize.<sup>17</sup> In the analysis reviewed in Sect. 1 we take  $\Psi$  to be the wave function of the universe, seen as a system of  $N$  particles moving in  $\mathbb{R}^3$ , where  $N$  is very large (like  $N = 10^{80}$ ). We now take  $M$  large enough to describe the number of measurements in a long run, but small compared to  $N$  (think of  $M = 10^{10}$ ), and accordingly split  $x = (y, z)$  with  $y = (x_1, \dots, x_M)$  and  $(z = x_{M+1}, \dots, x_N)$ . Similarly, we write  $q = (r, s)$  for the actual particle positions.<sup>18</sup> The *conditional wave function*

$$\Psi^s(y) = \frac{\Psi(y, s)}{\|\Psi(\cdot, s)\|_{L^2(\mathbb{R}^{3M})}}; \|\Psi(\cdot, s)\|_{L^2(\mathbb{R}^{3M})} = \sqrt{\int d^{3M}y |\Psi(y, s)|^2} \quad (4.1)$$

<sup>17</sup> I try to follow Dürr and Teufel [12, Chap. 11], who however omit the phase factor in (4.6). See also Norsen [37, Sect. 5], whose derivation should be adjusted in the same way, and Myrvold [22, Sect. 9.4].

<sup>18</sup> Seen as random variables on the probability space  $\mathbb{R}^{3N}$ ,  $q$  consists of the coordinate functions  $q_k^i(x) = x_k^i$  for  $k = 1, \dots, N$  and  $i = 1, 2, 3$ , and similarly  $r_k^i(y, z) = y_k^i$  for  $k = 1, \dots, M$  and  $i = 1, 2, 3$ , etc.

of the  $M$ -particle system then trivially gives the conditional Born probabilities for  $r$ , i.e.<sup>19</sup>

$$dP_{\Psi}(r | s) = dP_{\psi_s}(r). \tag{4.2}$$

Now take some fixed wave function  $\psi_M \in L^2(\mathbb{R}^{3M})$ , i.e.  $\psi$  is a function of  $y \in \mathbb{R}^{3M}$ , where

$$\|\psi_M\|_{L^2(\mathbb{R}^{3M})} = 1, \tag{4.3}$$

and assume that the “environmental” particle positions  $s \in \mathbb{R}^{3(N-M)}$  decouple in that

$$\Psi(y, s) = \psi_M(y)\Phi(s), \tag{4.4}$$

for some  $\Phi \in L^2(\mathbb{R}^{3(N-M)})$ , whilst the  $r$  variables have autonomous Bohmian dynamics:  $\psi_M$  should satisfy a Schrödinger equation with a potential  $V(y)$  not containing  $z$ , and the flow equations (1.5) for  $r$  do not contain  $s$ . The factorization (4.4) then lasts as long as we use  $\psi_M$  (i.e. during the repeated measurements to be described below), and we have

$$\Psi^s(y) = e^{i\alpha(s)}\psi_M(y); e^{i\alpha(s)} = \frac{\Phi(s)}{|\Phi(s)|}. \tag{4.5}$$

Thus  $\exp(i\alpha(s))$  is a phase factor.<sup>20</sup> Hence for any  $y \in \mathbb{R}^{3M}$ ,  $(y, s) \in \mathbb{R}^{3N}$  lies in the set

$$\{\Psi \cong \psi_M\} = \{(y, z) \in \mathbb{R}^{3N} \mid \Psi^z(y) = e^{i\alpha(z)}\psi_M(y) \text{ for some } \alpha : \mathbb{R}^{3(N-M)} \rightarrow \mathbb{R}\}. \tag{4.6}$$

We may finally “trace out” (i.e. marginalize) over all such  $s$  to arrive at the formula

$$dP_{\Psi}(r \mid \Psi \cong \psi_M) = dP_{\psi_M}(r). \tag{4.7}$$

If we finally assume that  $\psi_M(y) = \psi(y_1) \cdots \psi(y_M)$  for some normalized  $\psi \in L^2(\mathbb{R}^3)$ , then

$$P_{\Psi}(r \in A \mid \Psi \cong \prod_{k=1}^M \psi_k) = \int_A d^{3M}y |\psi(y_1)|^2 \cdots |\psi(y_M)|^2, \tag{4.8}$$

for any (measurable)  $A \subset \mathbb{R}^{3M}$ , where  $\psi_k(y_k) = \psi(y_k)$  for the given single-particle wave function  $\psi$ . This means that under the stated independence assumptions (which one should be able to satisfy in setting up a repeated single-particle measurement whose quantum-mechanical description involves identical initial conditions for each experiment but whose Bohmian description allows for differences in the particle locations  $r$  in each case), the probability distributions of the given  $M$ -particle system induced by the quantum equilibrium distribution for the wave function of

<sup>19</sup> This is really a shorthand for  $P_{\Psi}(r \in A \mid s) = P_{\psi_s}(A) = \int_A d^{3M}y |\Psi^s(y)|^2$ , where  $A \subset \mathbb{R}^{3M}$ .

<sup>20</sup> Like in quantum mechanics, also in Bohmian mechanics wave functions differing by a phase should be identified, since both the velocity field (1.5) and the probability distribution (1.3) are the same.

the universe  $\Psi$  equals the Born distribution. Eq. (4.8) is what Dürr et al. [23] call the *quantum equilibrium hypothesis*, which, then, is actually a theorem (albeit under restrictive independence assumptions):<sup>21</sup>

We prove that for *every* initial  $\Psi$ , this agreement with the predictions of the quantum formalism is obtained for *typical*—i.e., for the overwhelming majority of—choices of initial  $q$ . And the sense of typicality here is with respect to the only mathematically natural—because equivariant—candidate at hand, namely, quantum equilibrium. Thus, on the universal level, the physical significance of quantum equilibrium is as a measure of typicality, and the ultimate justification of the quantum equilibrium hypothesis is, as we shall show, in terms of the statistical behavior arising from a typical initial configuration [23, p. 858].

While (4.8) does support the Bohmian mechanics formalism and relates it to experiment, it does not materially affect my discussion in Sect. 3. In that context, all the quantum equilibrium hypothesis shows is that in the Bohmian description of repeated experiments based on identical single-particle states with wave function  $\psi$ , instead of sampling from the probability distribution of all particles in the universe given by  $|\Psi|^2$  one may sample from the single-particle probability distribution  $|\psi|^2$ . But this leads to exactly the same problems.

Restricting ourselves to quantum coin tosses, as before, and assuming with the Bohmians that every measurement can ultimately be reduced to position measurements, Eq. (4.8) is simply replaced by the 50-50 Born probabilities used in Sect. 3. The quantum equilibrium hypothesis/theorem then states that, provided the initial particle configuration of the universe is typical, so is the outcome sequence of an endlessly repeated quantum coin toss. This matches the “almost every” in the statement of Theorem 1, and in this experiment we may therefore identify the “typicality” of an outcome sequence with its 1-randomness.

The idea that, at least in the context of Theorem 1 (and, I believe, also in the context of classical statistical physics),<sup>22</sup> 1-randomness (= Kolomogorov–Levin–Chaitin randomness) is the correct notion of typicality for binary sequences has been argued by Landsman [3], (2022), and should be uncontroversial. It is much more *precise* than the notion of “overwhelmingly probable” used e.g. in Dürr et al. [23], Dürr and Teufel [12], and elsewhere, and is much *stronger* than (Borel) normality,<sup>23</sup> which is also sometimes used in the literature on Bohmian mechanics e.g. Callender [41]. However, Borel normal sequences may be computable,<sup>24</sup> and hence despite

<sup>21</sup> It would appear to be more natural to me to use the name “quantum equilibrium hypothesis” to the postulate that the  $N$  particles comprising the material content of the universe  $\mathbb{R}^3$  were, and hence are, distributed according to the Born probabilities  $|\Psi|^2$ , where  $\Psi$  is the wave function of the universe.

<sup>22</sup> For a start, see Grünwald and Vitányi [38]. What is lacking so far is, among others, the connection between algorithmic randomness and large deviation theory [39, 40].

<sup>23</sup> A sequence  $s$  is *Borel normal* if each possible finite string  $\sigma$  in  $s$  has (asymptotic) frequency  $10^{-|\sigma|}$  (so that each digit 0, ..., 9 occurs 10% of the time, each block 00 to 99 occurs 1% of the time, etc.).

<sup>24</sup> In base 10 the simplest example is *Champernowne’s number* 01234567891011121314151617181920 ..., which can be shown to be Borel normal. The decimal expansion of  $\pi$  is conjectured to be Borel normal.

their abundance and attractive statistical properties they are useless in the context of quantum mechanics, whose randomness properties are much stronger than normality, as Theorem 1 shows.<sup>25</sup>

The identification of “typicality” with “overwhelming probability” is always predicated on a background measure, which needs to be justified. For a repeated quantum (or even classical) coin flip this background measure declares that each binary string (of some given length) or sequence is equally likely to occur. In particular, a sequence like 10110000101111 is as probable as 00000000000000. The intuition that the former is typical in a probabilistic sense while the latter is not, applies to the coarse-graining obtained by counting zeros and ones; indeed, (6, 8) (i.e. a sequence with 6 zeros and 8 ones) is typical whereas (14, 0) is not. The notion of 1-randomness captures this difference already at the level of the sequences themselves, without any need for coarse-graining; though it occurs with probability one on the space of binary sequences, the concept of 1-randomness is not itself defined probabilistically (but algorithmically). What seems lacking in Bohmian mechanics is a similar concept of typicality of  $N$ -particle configurations that is defined intrinsically, and subsequently can be shown to occur with probability one relative to the background measure provided by the equilibrium distribution  $|\Psi|^2$  defined by the wave function of the universe  $\Psi$ . Defining typicality *by* this background measure would, in the absence of coarse graining, be similar to saying that all binary strings or sequences are typical—for they all have the same probability relative to the pertinent background measure, as just pointed out. This makes it hard to understand what could be meant by ‘*typical* choices of  $q$ ’ (in the above quotation); note once again that appealing to an ‘overwhelming majority’ only makes sense after coarse graining. In this light, consider the Bohmian ideology that:

For a universe governed by Bohmian mechanics (...), given the initial wave function and the initial positions of all particles, *everything* is completely determined and nothing whatsoever is actually random. (...) The origin of the randomness in the results of quantum measurements lies in random initial conditions, in our ignorance of the complete description of the system of interest—including the apparatus—of which we know only the wave function. Dürr et al. [23, pp. 846, 844]

In my view, this ideology rests on an equivocation in which the initial conditions of the universe (simplified as some  $N$ -particle configuration in  $\mathbb{R}^{3N}$ ) are simultaneously treated as uniquely given and hence singular (though inaccessible in principle) and yet subject to some probability distribution with respect to which they are supposed to be typical.<sup>26</sup>

Random initial conditions are drawn from a sample space. The (unique) result of this draw induces a binary outcome string or sequence  $h$  for a repeated quantum coin toss for which some specific  $M$ -particle subsystem has been used. Choosing a different  $M$ -particle subsystem would probably give a different outcome sequence  $h'$ , but

<sup>25</sup> Any 1-random sequence is Borel normal [34, Corollary 6.32.]

<sup>26</sup> To make things worse, this distribution is also unknown in practice, since its source  $\Psi$  is unknown.

in Bohmian mechanics both are ultimately *determined* by the initial  $N$ -particle configuration. Yet according to quantum mechanics, as mimicked by Bohmian mechanics via (4.8) (and more generally by its aim to preserve all statistical results of quantum mechanics), *each* of these sequences is 1-random *by itself*, cf. Theorem 1. By Theorem 2 this randomness cannot be a consequence of the randomness of the initial conditions, since in Bohmian mechanics these are *fixed* and everything afterwards is supposed to be deterministic. In other words, the above claim that ‘The origin of the randomness in the results of quantum measurements lies in random initial conditions’ cannot be upheld, since there is no way for the “randomness” in initial conditions to influence measurement results  $h$  and hence “cause” *their* randomness.<sup>27</sup>

In comparison, take classical coin tossing. Here, the initial conditions are (to a good approximation) the vertical velocity  $v$  and rate of spin  $\omega$  of the coin at the time it is launched; see Diaconis et al. [42] or Diaconis and Skyrms [43]. This time, the (apparent) randomness in an outcome sequence is genuinely “caused” by randomness in the initial conditions, which are sampled by varying wrist and thumb movements, and hence ultimately by some sort of brain process. Due to the extreme sensitivity of the outcome to the initial conditions, almost any non-sharply peaked probability distribution on the space of initial conditions leads to a 50-50 distribution on the binary outcome space.<sup>28</sup>

Nonetheless, as in the case of quantum mechanics (see Theorems 1 and 2) the outcome sequence would only be 1-random if the ultimate sampling process in the brain were indeterministic (in which case it might be quantum mechanical). Indeed, by the same reasoning as in the proof of Theorem 2, any kind of determinism blocks 1-randomness. In particular, sequences produced by coin tossing machines are not 1-random and I would not be surprised if those produced by humans aren’t either. But this is no problem, since few scientist would maintain that the probabilities in coin tossing are fundamental; in contrast, the Born probabilities *are* regarded as exact and fundamental, also by the Bohmians.

This discussion suggests a way out, however. Theorems 1 and 2 assume the exact validity of the Born rule. But in my view no probability measure used in mathematical physics is ever exact, because the physical origin of the measure lies in the way it is sampled.<sup>29</sup> Thus I expect the origin of the Born measure to lie in some underlying theory from which quantum mechanics is emergent à la Butterfield [47]. This theory may well be deterministic—*provided the Born rule is an approximation*. The program of Bohm and Vigier [48] and Valentini [28, 29], who try to *derive* quantum equilibrium from equilibration, is in this spirit see also Norsen [37] and Myrvold [22], especially because it accepts departures from the Born rule (also cf. Sect. 2). The only disagreements I would have with Valentini [29] is that I would not describe

<sup>27</sup> I am grateful to Joanna Luc for questions in this direction at the Cambridge–LSE PoP Bootcamp.

<sup>28</sup> In fact, there is a slight 1% bias for the coin to land the way it started (see references in main text). A Galton board provides a similar example with a richer outcome space [12, Sect. 4.1.1].

<sup>29</sup> Boltzmann’s program of justifying the microcanonical ensemble by ergodic theory [44, 45] is paradigmatic; “typical” behaviour would be dynamically generated via time averages. The Bohmians disagree with this goal Lazarovici and Reichert [46], but this is not the disagreement in the main text.

the Bohmian arguments based on typicality as ‘circular’ but as *inconclusive* (see above), and that I would prefer the underlying theory not to be the de Broglie-Bohm pilot-wave theory but something really new. Whatever this new theory may be, its sampling process should be washed out, as in the “method of arbitrary functions” in probability theory [22, 49]. Indeed, this is also what happens in examples like coin tossing (even if it is deterministic).

**Data Availability** The data for this article consist of the reasoning and the references. All references are easily found either on the internet or, in case of books, in standard academic libraries. Readers who have any trouble finding a reference are welcome to contact the author.

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