



Symplectic Quantization II: Dynamics of Space–Time Quantum Fluctuations and the Cosmological Constant

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Abstract

The symplectic quantization scheme proposed for matter scalar fields in the companion paper (Gradenigo and Livi, [arXiv:2101.02125](https://arxiv.org/abs/2101.02125), 2021) is generalized here to the case of space–time quantum fluctuations. That is, we present a new formalism to frame the quantum gravity problem. Inspired by the *stochastic* quantization approach to gravity, *symplectic* quantization considers an explicit dependence of the metric tensor $g_{\mu\nu}$ on an additional time variable, named *intrinsic time* at variance with the coordinate time of relativity, from which it is different. The physical meaning of intrinsic time, which is truly a parameter and not a coordinate, is to label the sequence of $g_{\mu\nu}$ quantum fluctuations at a given point of the four-dimensional space–time continuum. For this reason symplectic quantization necessarily incorporates a new degree of freedom, the derivative $\dot{g}_{\mu\nu}$ of the metric field with respect to intrinsic time, corresponding to the conjugated momentum $\pi_{\mu\nu}$. Our proposal is to describe the quantum fluctuations of gravity by means of a symplectic dynamics generated by a generalized action functional $\mathcal{A}[g_{\mu\nu}, \pi_{\mu\nu}] = \mathcal{K}[g_{\mu\nu}, \pi_{\mu\nu}] - S[g_{\mu\nu}]$, playing formally the role of a Hamilton function, where $S[g_{\mu\nu}]$ is the standard Einstein–Hilbert action while $\mathcal{K}[g_{\mu\nu}, \pi_{\mu\nu}]$ is a new term including the kinetic degrees of freedom of the field. Such an action allows us to define an ensemble for the quantum fluctuations of $g_{\mu\nu}$ analogous to the microcanonical one in statistical mechanics, with the only difference that in the present case one has conservation of the generalized action $\mathcal{A}[g_{\mu\nu}, \pi_{\mu\nu}]$ and not of energy. Since the Einstein–Hilbert action $S[g_{\mu\nu}]$ plays the role of a potential term in the new *pseudo*-Hamiltonian formalism, it can fluctuate along the symplectic action-preserving dynamics. These fluctuations are the quantum fluctuations of $g_{\mu\nu}$. Finally, we show how the standard path-integral approach to gravity can be obtained as an approximation of the symplectic quantization approach. By doing so we explain how the integration over the conjugated momentum field $\pi_{\mu\nu}$ gives rise to a cosmological constant term in the path-integral approach.

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1 Introduction

A general consensus on a quantum gravity theory has still to be found, although very promising candidates such as string theory [2–5], loop quantum gravity [6–10], causal dynamic triangulations [11–13] and non-perturbative renormalization group approaches [14, 15] are already in the arena. In our opinion any attempt to correctly frame the study of quantum gravity must cope in first instance with the problem of *time* in general relativity. The *question of time* comes first. When we talk about “*dynamics*” of the gravitational field we refer to the possibility for the metric field $g_{\mu\nu}(\mathbf{p})$ to experience a sequence of changes at a given point $\mathbf{p} \in \mathcal{M}$ of the four-dimensional space–time manifold \mathcal{M} . The existence of such a sequence of changes is indeed very natural when thinking, for instance, to any numerical protocol which, starting from an arbitrary configuration of the field $g_{\mu\nu}(\mathbf{p})$, projects it by means of an iterative procedure onto a certain solution of the Einstein equation, the one corresponding to a given distribution of matter and energy. In this case the dynamics is the one of the algorithm iterative procedure and *time* is the number of iteration steps. It is evident that this “*time of the computer*”, which, as an evolution parameter, closely resembles the notion of time we have from classical and quantum mechanics, is totally different from the *coordinate time* of relativity, both special and general, which is simply a coordinate and not a parameter, as the name itself suggests. This fact was already well recognized within the attempt to frame general relativity into the Schrödinger representation of quantum mechanics by means of the Wheeler–DeWitt equation $\hat{H}\psi = 0$ [16, 19], where \hat{H} is the general-relativistic Hamiltonian of the gravitational field, ψ the world wave-function and the time derivative term, usually appearing in the Schrödinger equation, is set to zero. This is done precisely for the reason that in a relativistic context time is just a coordinate, not an evolution parameter as in quantum mechanics. The Wheeler DeWitt equation, consistently with Einstein equations, is just a *constraint* equation, not an evolution equation: it is the wave equation for the frozen block universe of Einstein, there is no time flowing. It is then not by a coincidence that the Wheeler–DeWitt canonical approach to quantum gravity gave rise to Loop Quantum Gravity (LQG) [6–10], a theory characterized by the absence of time at the microscopic level [21]. But here we are not interested in the microscopic degrees of freedom of space–time. We want to understand which sort of thing could be, physically and not just in a computer simulation, the *dynamics* of the field $g_{\mu\nu}$. The answer turns out to be quite simple: as *the dynamics* of $g_{\mu\nu}$ we can simply refer to the sequence of its quantum fluctuations. And as the corresponding *time* parameter we opt in favour of the *intrinsic time* introduced [1]: the *fifth* variable of symplectic quantization. Consider in fact the classical Einstein equations for pure gravity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the curvature scalar. One may think to a *gedanken* experiment where the field $g_{\mu\nu}$ is set to a very atypical configuration across the whole space–time manifold \mathcal{M} and then is let free to relax towards a solution of the Einstein equations by means of its quantum fluctuations. When we

talk about the dynamics of $g_{\mu\nu}$ we are of course not talking about a sequence of frames along a time-like trajectory in the space–time manifold but rather about the sequence of changes experienced by $g_{\mu\nu}$ due to quantum fluctuations at each point $\mathbf{p} \in \mathcal{M}$ of space–time. The existence of a dynamics for the gravitational field quantum fluctuations represents a compelling logical necessity as soon as one accepts the existence of quantum fluctuations themselves: there must be a process, be it stochastic or deterministic, which takes the field through their sequence. We propose here a generalization to gravity of the symplectic quantization procedure introduced in [1] for a scalar field theory. The basic idea is the same of [1]: we try to define a *pseudo*-microcanonical ensemble built on the conservation of an appropriately defined generalized action rather than on the conservation of energy. This generalized action, which formally plays the role of a Hamilton function, is the generator of a symplectic dynamics, the dynamics of quantum fluctuations.

As done in the companion paper [1] we introduce symplectic quantization for the gravitational field by recalling first the results on the *stochastic* quantization approach to gravity reported in [22]. This is because symplectic quantization stems from stochastic quantization. We need in fact to acknowledge that the “*problem of time*” in general relativity has been in a sense “already solved” within the stochastic quantization approach in general [23] and in particular in the application of these ideas to gravity [22]. In fact the true notion of gravitational field dynamics as the ordered sequence of its quantum fluctuations was already present in [22]. Unfortunately, at the time the stochastic quantization approach to gravity was proposed [22], the parameter of the stochastic process controlling the dynamics of $g_{\mu\nu}$ was just considered a fictitious variable. On the contrary, symplectic quantization promotes this fictitious time to be a true physical entity. Furthermore, symplectic quantization allows us to define a functional approach to quantum gravity [see, e.g., Eq. (25) below] which is well-defined irrespectively to the scale-invariance properties of the theory, in such a way that the *non-renormalizability* problem is apparently less compelling in the present context.

We start by discussing first the stochastic quantization approach to gravity in Sect. 2. Then in Sect. 3 we introduce symplectic quantization and in Sect. 4 we show how, within the symplectic quantization framework, the microcanonical partition function of a theory without cosmological constant can be mapped onto the path integral of an Einstein–Hilbert action *with* cosmological constant Λ . Sect. 5 is left for a concluding summary.

2 Stochastic Quantization

Since the proposal of symplectic quantization has been mainly inspired from stochastic quantization, let us start with a discussion of the latter, in particular of its application to the problem of quantum gravity [22]. The main goal of stochastic quantization is to introduce a stochastic equation for the metric field in terms of a fifth parameter τ , named *fictitious* time in [22] and termed by us *intrinsic time* in [1]:

$$\frac{\partial g_{\mu\nu}(x, \tau)}{\partial \tau} = - \int d^4 y G_{\mu\nu,\alpha\beta}(x, y, \tau) \frac{\delta S[\mathbf{g}]}{\delta g_{\alpha\beta}(y, \tau)} + \xi_{\mu\nu}(x, \tau), \quad (2)$$

where the symbol \mathbf{g} denotes the metric tensor and S is the Euclidean version of the Einstein–Hilbert action,

$$S[\mathbf{g}] = \frac{1}{2\kappa} \int d^4 x |g|^{1/2} R. \quad (3)$$

In Eq. (3) R is the Ricci curvature scalar, the symbol g on the right-hand side denotes the determinant of the metric tensor, $g = \det(g_{\mu\nu})$, $\kappa = 8\pi G/c^4$ is the Einstein gravitational constant and $\xi_{\mu\nu}(x, \tau)$ is a *white noise*:

$$\langle \xi_{\mu\nu}(x, \tau) \xi_{\alpha\beta}(y, \tau') \rangle = \delta(\tau - \tau') G_{\mu\nu,\alpha\beta}(x, y, \tau). \quad (4)$$

The symbol $G_{\mu\nu,\alpha\beta}(x, y, \tau)$ denotes the DeWitt supermetric in four-dimensional space–time, introduced in [19] with the purpose to define an invariant measure for the path-integral approach to quantum gravity. Its dependence on τ comes from the fact that $G_{\mu\nu,\alpha\beta}(x, y, \tau)$ is, as we are going to see, a function of $g_{\mu\nu}(x, \tau)$. By using the natural units $\hbar = c = 1$ and choosing the convention according to which the metric tensor $g_{\mu\nu}$ is dimensionless we have that in Eq. (3) the Einstein gravitational constant has the dimensions of a length squared, $[\kappa] = L^2$, while the Ricci curvature scalar has the dimensions of an inverse length squared, $[R] = L^{-2}$. The main result of [22] is the perturbative evaluation of metric field correlation amplitude in the $\tau \rightarrow \infty$ limit:

$$\lim_{\tau \rightarrow \infty} \langle g_{\mu\nu}(x, \tau) g_{\alpha\beta}(y, \tau) \rangle = \langle g_{\mu\nu}(x) g_{\alpha\beta}(y) \rangle \quad (5)$$

where the amplitude on the left hand side of Eq. (5) is computed averaging over the time-dependent probability distribution of the metric field obtained by solving the Fokker–Planck equation associated to Eq. (2). Not surprisingly an exact solution of this Fokker–Planck equation is out of reach and the results are only perturbative [22]. The problem of non-renormalizability is also not cured, since $g_{\mu\nu}$ is allowed to fluctuate at all scales. For gravity the only true advantage of stochastic quantization is to remove the need of gauge fixing procedures [23]: gauge orbits are naturally explored by the dynamics, the latter still yielding the euclidean path-integral measure in the $\tau \rightarrow \infty$ limit. The good term of comparison to understand why stochastic quantization removes the problem of gauge fixing is to consider the overdamped Langevin dynamics of a colloidal particle confined by a potential $U(x)$:

$$\dot{x} = -\frac{\partial U}{\partial x} + \eta(t), \quad (6)$$

where $\eta(t)$ is a white noise of amplitude $\langle \eta^2 \rangle = 2\beta^{-1}$. The “gauge invariance” hidden in equilibrium statistical mechanics amounts to the fact that any realization of the Gaussian noise $\eta(t)$ allows us to sample correctly the equilibrium distribution $e^{-\beta U(x)}$ in the $t \rightarrow \infty$ limit. Things work in the same manner for the euclidean path-integral measure of a field theory and stochastic quantization.

Let us now spend a couple of words on the bi-tensor density $G_{\mu\nu,\alpha\beta}(x, y, \tau)$ of Eq. (2), where it guarantees invariance of the equation under the action of diffeomorphisms in 3 + 1 dimensions. In particular, in the general case we have $G_{\mu\nu,\alpha\beta}(x, y, \tau) \sim \delta^{(4)}(x, y)$, so that the Langevin equation can be written as:

$$G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]\dot{g}_{\mu\nu}(x, \tau) = -\frac{\delta S[\mathbf{g}]}{\delta g_{\alpha\beta}(y, \tau)} + G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]\xi_{\mu\nu}(x, \tau), \tag{7}$$

where the bitensor density $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]$ is defined as [19, 22]:

$$G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)] = \frac{C}{2}|g|^{1/2} [g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + \lambda g^{\mu\nu}g^{\alpha\beta}], \tag{8}$$

with $\lambda \neq 1/2$ a dimensionless constant and the dependence on τ comes clearly by the one of $g_{\mu\nu}(x, \tau)$. The bitensor density in Eq. (8) represents a field of $d(d + 1)/2 \times d(d + 1)/2$ -dimensional matrices $\hat{G}[\mathbf{g}(x, \tau)]$ on the space-time manifold, each matrix having determinant [19]:

$$\det \hat{G}[\mathbf{g}(x, \tau)] = (-1)^{d-1} \left(1 + \frac{d\lambda}{2}\right) g^{\frac{1}{4}(4-d)(d+1)}. \tag{9}$$

Quite remarkably, the expression of $\det \hat{G}[\mathbf{g}(x, \tau)]$ is such that it turns out to be simply a constant *only* in $d = 4$:

$$\det \hat{G}[\mathbf{g}(x, \tau)] = -(1 + 2\lambda). \tag{10}$$

It is the above property of the determinant which guarantees the *triviality* of the path-integral measure for gravity [19]. In the framework of stochastic quantization no dependence on τ is assumed for the adimensional constant λ [22], so that we are going to do the same for the symplectic quantization approach to be introduced in Sect. 3. The importance of choosing a bitensor density $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]$ precisely of the form written in Eq. (8) was noticed already before the attempt to apply stochastic quantization to gravity [22] and was related to the definition of an invariant measure for gravitational path-integral [19]:

$$\mathcal{Z}_{\text{grav}} = \int e^{iS[\mathbf{g}]/\hbar} \mu[\mathbf{g}] \mathcal{D}g, \tag{11}$$

where $\mathcal{D}g = \prod_x \prod_{\mu\nu} g_{\mu\nu}(x)$ and where $\mu[\mathbf{g}]$ is a functional measure. If one, in facts, takes the functional measure $\mu[\mathbf{g}]$ to be invariant under the action of diffeomorphisms, i.e., of the form

$$\mu[\mathbf{g}] \propto \prod_x \det \hat{G}[\mathbf{g}(x)], \tag{12}$$

the choice of the operator $\hat{G}[\mathbf{g}(x)]$ made in Eq. (8) yields that $\det \hat{G}[\mathbf{g}(x)]$ is constant in $d = 4$. Accordingly, the functional measure $\mu[\mathbf{g}]$ is also constant. The dependence from τ of $\mathbf{g}(x, \tau)$ has been obviously dropped in the partition sum of Eq. (11) and in the expression of the measure, Eq. (12). The deep physical implications of

this apparently technical motivation for the choice of $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x)]$ will be fully highlighted in Sect. 4.

From the point of view of dimensional analysis, the requirement of homogeneity among all elements in Eq. (7) implies that $[G^{\mu\nu,\alpha\beta}] = L^{-1}$ in the natural units where $[\tau] = L$. As a consequence, the constant C in the definition of $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]$ carries the dimension of an inverse length-scale (inverse time-scale):

$$[C] = L^{-1}. \quad (13)$$

Only two comments are now in order before moving to the discussion of symplectic quantization. In our opinion the main breakthrough of stochastic quantization was the proposition of the Langevin equation in Eq. (2). In particular, stochastic quantization gave a clear indication of a plausible solution for the “*problem of time*” in general relativity. Quite unfortunately, since the additional time variable needed to this purpose was not regarded as a physical entity, but merely as a technical device (disappearing at the end of the calculation), this insightful approach was quite underrated. From the conceptual point of view, the main problems of stochastic quantization were related to the properties of $\xi_{\mu\nu}(x, \tau)$, only fixed by the requirement of overall self-consistency for the theory and apparently lacking any physical interpretation. In fact, quite obviously, the presence of $\xi_{\mu\nu}(x, \tau)$ in Eq. (2) does not represent the action of a thermostat. In conclusion, any possible deep conceptual implication of the stochastic quantization approach to gravity was ruled out for the reason that the *fifth* time variable was regarded just a fictitious one, a fact related to the lack of a physical interpretation for $\xi_{\mu\nu}(x, \tau)$. On the technical side the theory was working, but not remarkably better than other proposals. In particular the worst plague of quantum gravity, the non-renormalizability problem, was still there. There have been some very interesting recent progresses on the stochastic quantization approach to gravity, where higher order derivatives with respect to the fictitious time were added to the Langevin equation for $g_{\mu\nu}$ and where it is clearly alluded to some physical relevance of the fictitious time [20]. Nevertheless, as long as the presence of noise is a cornerstone of the approach proposed [20], the interpretative problems of the fictitious time remain. We will show in the next section that both conceptual and technical problems related to noise completely disappear in the context of symplectic quantization, which is a deterministic process. Furthermore, since the value of the generalized action is fixed and the cornerstone of the approach is an action preserving symplectic dynamics, the non-renormalizability problem of gravity is possibly less severe in this context than in the Euclidean path-integral approach. The reason is that alongside with symplectic quantization one introduces a sort of microcanonical measure which is well defined irrespectively to whether the theory is or not renormalizable.

3 Symplectic Quantization

The key idea of symplectic quantization is to replace the stochastic dynamics of Eq. (2) with a deterministic one. This choice, which is motivated and explained in great detail in [1], comes from the following argument. The Langevin equation of stochastic quantization is a fictitious stochastic dynamics which allows one to sample, for asymptotically long times, field configurations with the Boltzmann-like weight appearing in the euclidean version of the path-integral for gravity. In [1] it has been shown that the “canonical” ensemble which is in that way implicitly defined by stochastic quantization can, and sometimes has to, be replaced with a *fixed-action “microcanonical” ensemble*, for which an appropriate generalized action has to be drawn [1]. This generalized action is obtained by adding the kinetic terms containing the derivative of the field with respect to intrinsic time, usually absent in any standard field theoretic action S , which usually contains derivatives only with respect to space–time coordinates. Soon after the first appearance of the present work as a preprint on public repositories we knew [17] of a previous attempt to build a generalized action for quantum fields containing momenta with respect to a fifth time variable: the functional approach to field theory proposed by De Alfaro, Fubini and Furlan in 1983 [18]. The additional kinetic term therein was build according to a logic similar to the one followed here, but making different choices on the technical level (vierbeins V^a_μ were used instead of $g_{\mu\nu}$ and the DeWitt supermetric was not considered). Then, most importantly, in [18] the symplectic dynamics and the microcanonical ensemble introduced in [1] and generalized here to the case of gravity were never considered nor alluded and the fifth time variable was *not* considered a physical one. Coming back to the present proposal, we have that for gravity the *kinetic* term corresponding to the one of Eq. (20) in [1] has to be an object quadratic in $\dot{g}_{\mu\nu}$, where the dot now indicates derivative with respect to intrinsic time τ . At variance with [1], where the only constraint on the new kinetic term was to be a Lorentz scalar, in the case of gravity we need it to be a scalar with respect to the action of diffeomorphisms. As the key ingredient to define a *kinetic* term fulfilling this requirement we used the DeWitt supermetric $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]$ [19]. Taking inspiration from the analysis of [1] we propose a generalized action for the gravitational field where $S[\mathbf{g}]$ is treated as a potential term:

$$\mathcal{A}[\mathbf{g}, \dot{\mathbf{g}}] = \mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}] - S[\mathbf{g}], \tag{14}$$

with

$$\mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}] = \frac{1}{2\kappa_g} \int d^4x |g|^{1/2} \dot{g}_{\mu\nu}(x, \tau) G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)] \dot{g}_{\alpha\beta}(x, \tau), \tag{15}$$

where we have introduced the dimensionless supermetric $\mathcal{G}^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)] = G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]/C$. Due to the properties of the supermetric the expression in Eq. (15) clearly behaves as a scalar under the action of diffeomorphisms. $S[\mathbf{g}]$ is the Einstein–Hilbert action. In particular $S[\mathbf{g}]$ is the *original* Einstein–Hilbert action for a Lorentzian space–time, not the Euclidean version of the action. In fact, as explained in full detail in [1], symplectic quantization allows one to

define the probability of configurations directly in a space–time with Minkowskian signature, with no need to consider a Wick rotation to the Euclidean theory. For consistency we introduced the constant κ_g with dimensions $[\kappa_g] = L^2$. If we regard the generalized action $\mathcal{A}[\mathbf{g}, \dot{\mathbf{g}}]$ as *pseudo*-Hamiltonian which generates the evolution in intrinsic time τ , in Eq. (14) the Einstein–Hilbert action $S[\mathbf{g}]$ clearly plays in the role of a potential term, since it contains no intrinsic time derivative terms. In order to dissipate any possible doubt, let us stress that intrinsic time τ is a parameter, a sort of *internal* degree of freedom of the metric field, so that the kinetic term $\mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}]$ has to be a scalar under the action of diffeomorphisms in $3 + 1$ dimensions, *not* in $3 + 2$ dimensions. We are not adding extra-dimensions to space–time. In order to introduce Hamilton equations it is convenient to rewrite $\mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}]$ in terms of the conjugated momentum

$$\pi^{\mu\nu}(x, \tau) = \frac{1}{\kappa_g} G^{\mu\nu, \alpha\beta}[\mathbf{g}(x, \tau)] \dot{g}_{\alpha\beta}(x, \tau). \tag{16}$$

We thus have

$$\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}] = \frac{\kappa_g}{2} \int d^4x |g|^{1/2} \pi^{\mu\nu}(x, \tau) G_{\mu\nu, \alpha\beta}[\mathbf{g}(x, \tau)] \pi^{\alpha\beta}(x, \tau), \tag{17}$$

or, what is equivalent,

$$\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}] = \frac{\kappa_g}{2} \int d^4x |g|^{1/2} \pi_{\mu\nu}(x, \tau) G^{\mu\nu, \alpha\beta}[\mathbf{g}(x, \tau)] \pi_{\alpha\beta}(x, \tau), \tag{18}$$

where, to get Eq. (17) from Eq. (16) and Eq. (15), we have used the identity $\mathcal{G}_{\mu\nu, \alpha\beta} \mathcal{G}^{\mu\nu, \alpha\beta} = d(d + 1)/2$ and to get Eq. (18) we have raised/lowered indices with the help of the metric tensor, taking always advantage of the identity $g_{\mu\nu} g^{\mu\nu} = d$:

$$\begin{aligned} \mathcal{G}_{\mu'\nu', \alpha'\beta'} &= \mathcal{G}^{\mu\nu, \alpha\beta} g_{\mu\mu'} g_{\nu\nu'} g_{\alpha\alpha'} g_{\beta\beta'} \\ \pi_{\mu'\nu'} &= \pi^{\mu\nu} g_{\mu\mu'} g_{\nu\nu'}. \end{aligned} \tag{19}$$

In Eq. (17) constants coming from the contraction of tensor indices have been adsorbed into the definition of κ_g . Since the supermetric is now dimensionless, $[\mathcal{G}^{\mu\nu, \alpha\beta}] = 0$, and in natural units intrinsic time has dimensions of length we have that the intrinsic time derivative of the metric field has dimensions

$$[\dot{g}_{\mu\nu}] = L^{-1}, \tag{20}$$

while its conjugated momentum $[\pi^{\mu\nu}] = [\mathcal{G}^{\mu\nu, \alpha\beta} \dot{g}_{\alpha\beta} / \kappa_g]$ has dimensions

$$[\pi^{\mu\nu}] = L^{-3} \tag{21}$$

Let us notice that, despite having the same physical dimension, the Einstein gravitational constant κ and the new dimensional constant κ_g appearing in $\mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}]$ do not need to be the same. And in fact we will see they are different constants. We also assume, consistently with the stochastic quantization approach proposed in [22], that

the adimensional constant λ appearing in the definition of $G^{\mu\nu,\alpha\beta}[\mathbf{g}(x, \tau)]$ is a constant with respect to intrinsic time τ .

A close look at Eq. (18) reveals that the new generalized action, which we may also call the *pseudo-Hamiltonian* of symplectic quantization, is non-separable because the DeWitt supermetric depends on the metric field $\mathbf{g}(x, \tau)$. This *non-separability* of the pseudo-Hamiltonian is the most remarkable difference between the symplectic quantization of gravity and that of a prototypical matter field discussed in [1]. We will see in Sect. 4 that this fact has very interesting consequences. For the moment let us write down the Hamilton equations which, according to the symplectic quantization scenario, govern the dynamics of the gravitational field quantum fluctuations:

$$\begin{aligned} \dot{g}^{\mu\nu}(x, \tau) &= \frac{\delta \mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]}{\delta \pi_{\mu\nu}(x, \tau)} \\ \dot{\pi}^{\mu\nu}(x, \tau) &= -\frac{\delta \mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]}{\delta g_{\mu\nu}(x, \tau)} \end{aligned} \tag{22}$$

The quantity $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ is fixed by the choice of initial conditions, due to the symplectic nature of dynamics. This means that the quantum fluctuations of the gravitational field, which are nothing but the fluctuations of the “*potential*” term $S[\mathbf{g}]$ driven by the action-preserving dynamics of Eq. (22), are only those compatible with the conservation of $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$.

If then, rather than following the dynamics of the metric field fluctuations, one wants to sum other them by assuming the “*bona fide*” ergodicity of the dynamics in Eq. (22), the appropriate statistical ensemble must be chosen. As it has been already outlined in the companion paper [1], for a relativistic field theory the most appropriate choice is that of a microcanonical-type of ensemble, defined with respect to the generalized action $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$, rather than a canonical-type one, which would correspond to the standard path-integral approach. As we did for the scalar field theory in [1], let us point out the main and only assumption at the basis of the *microcanonical ensemble* for the quantum fluctuations of space–time:

“*All configurations of the tensorial fields $\mathbf{g}(x)$ and $\boldsymbol{\pi}(x)$ which correspond to the same value of the generalized action $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ are realized with identical probability.*”

This is the microcanonical postulate for the metric field in the general relativistic context. Clearly this generalized microcanonical ensemble has nothing to do with temperature or thermal fluctuations, since the quantity which is conserved is action, not energy. For this ensemble the partition function reads as:

$$\Omega(A) = \int \mathcal{D}g \mathcal{D}\pi \delta(A - \mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]), \tag{23}$$

where $\mathcal{D}g$ can be simply taken as $\mathcal{D}g = \prod_x \prod_{\mu\nu} g_{\mu\nu}(x)$, consistently with the choice of the functional measure $\mu[\mathbf{g}]$ shown in Eq. (12), while $\mathcal{D}\pi$ is:

$$\mathcal{D}\pi = \prod_x \prod_{\mu\nu} \left[\kappa_g^{3/2} d\pi_{\mu\nu}(x) \right] = (\kappa_g^V)^{3/2} \prod_x \prod_{\mu\nu} d\pi_{\mu\nu}(x), \tag{24}$$

where the dimensional constant κ_g has been added as a factor for each infinitesimal volume element of functional integration in order to keep the partition function dimensionless, while the symbol V in κ_g^V denotes the invariant integration volume.

Let us stress that the partition function in Eq. (23) is well defined irrespectively to the renormalizability of the theory. The expression may look formal, but it is not. As already outlined in [1], the advantage of an expression like the one in Eq. (23) is to be a functional integral with a clear probabilistic interpretation, at variance with the Feynman path integral. Then, similarly to the path integral, one has the problem of functional integration, but nothing more. In particular, while the theoretical justification of path integrals in quantum field theory comes only “*à posteriori*” as long as they allow to compute finite amplitudes, and to this purpose renormalizability is crucial in order for a theory to be well defined, this is not the case for the partition function in Eq. (23), which comes from first principles and has a clear physical interpretation on its own. Within symplectic quantization the fundamental level of description of the systems is assumed to be, as in classical statistical mechanics, the microscopic symplectic dynamics. Then, under the assumption of ergodicity for the latter, one can write a partition function such as the one in Eq. (23). Quantum fluctuations are controlled in first place by the dynamics of Eq. (22), from which the microcanonical partition function in Eq. (23) follows. There might be “*pathological*” configurations which satisfy the fixed-action constraint by putting a very large, but compensating, amount of action respectively in the “kinetic”, $\mathcal{K}[\mathbf{g}, \dot{\mathbf{g}}]$, and in the “potential”, $S[\mathbf{g}]$, terms of the generalized action. But this is not a problem, since within this approach the physical consistency of the theory does not rely on renormalizability. From such a perspective, where the scale-invariance property of the theory is not compelling as usual, the problem of functional integration can be easily solved by putting the theory on a lattice (recipes for this have been devised in the past [24]) and by sampling configurations with probability

$$P_A(\mathbf{g}, \boldsymbol{\pi}) = \frac{1}{\Omega(A)} \delta(A - \mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]). \quad (25)$$

The key point is that the probability density in Eq. (25) comes directly from the original theory with no need of Wick-rotating to Euclidean path-integrals. The rotation from real to imaginary time has in fact no physical motivation other than allowing us to work with a well-defined probability measure. Such a measure allows for non-perturbative approaches, for instance to cast strong-coupling expansions on the lattice for non-abelian gauge theories as high-temperature expansions in statistical mechanics [25]. But the whole justification of working with Euclidean field theory comes only “*à posteriori*”: it is thus very appealing to have a functional approach to field theory which already in real time is well defined from the point of view of probability. Moreover, recent results pertaining precisely the stochastic quantization approach to gravity [20], showed that in certain circumstances (e.g., during inflation) it is not even possible to rotate the Euclidean theory back to the original quantum field theory. Therefore, in particular for gravity, a formalism allowing us to write functional integrals well defined probabilistically already in Minkowski space seems particularly useful.

The problem of gravity non-renormalizability becomes more compelling when we try to change the statistical ensemble of quantum fluctuations, that is, when we try to transform the microcanonical type of partition sum in Eq. (23) into a canonical one. For the same reason explained in [1], namely the non-definiteness of a relativistic action sign, the integral transformation which allows us to change ensemble must be a Fourier transform:

$$Z_{\text{grav}} = \int dA e^{-iA/\hbar} \Omega(A) = \int \mathcal{D}\pi \mathcal{D}g e^{-i\hbar A[g,\pi]}. \tag{26}$$

For an expression like the one in Eq. (26) we are back to the old problem: it makes sense as long as the theory it is renormalizable. Nevertheless now there are two intermediate steps which could be forbidden for physical reasons: first, the assumption that an equilibrium measure, like the one in Eq. (23), really exists, assumption which is based on the ergodicity of dynamics in Eq. (22); second, the assumption that the ensemble with a *hard* constraint on the action, Eq. (23), and the “canonical” ensemble of Eq. (26), characterized by a *soft* constraint, are equivalent. If we believe that in most cases of interest ergodicity holds, we are then left with the hypothesis that for gravity the two ensembles just mentioned are not equivalent. Something not at all surprising, if true, since the lack of equivalence of statistical ensembles is not uncommon in statistical mechanics. Moreover, precisely the gravitational potential in the non-relativistic regime is a well-known case where inequivalence of statistical ensembles for thermal fluctuations takes place [27]. In particular, the non equivalence of thermal ensembles for gravitating systems comes from the *long-range* nature of the gravitational interaction [27], which makes energy non-extensive.

In summary, while it is standard knowledge that different statistical ensembles are not equivalent for the large-scale thermal fluctuations of gravity [27], in the present work we also put forward the hypothesis that a similar “non-equivalence” holds even for the statistical ensembles of quantum fluctuations at high energy. In this perspective, as soon as consensus will be gathered around one of the high-energy regulators proposed for a quantum theory of gravity (e.g., loop quantum gravity [9, 10]), this finding will also have an impact on the problem of statistical ensembles inequivalence for the quantum fluctuations of gravity. But, since a general agreement on such a UV regulator has not been reached so far, we rely here on the most *agnostic*, at present, hypothesis: the *pseudo*-microcanonical ensemble of Eq. (23) and the path-integral of Eq. (26) are not equivalent. In what follows the path-integral approach to gravity will be thus regarded just as a low-energy approximation of the theory.

In [1] we have shown that the *pseudo*-microcanonical partition function for a relativistic scalar field reduced to the Feynman path-integral of the corresponding field theory, after Fourier transforming with respect to the action and integrating over momenta. In what follows we are going to do the same for the *pseudo*-microcanonical partition function of Eq. (23).

4 Cosmological Constant

We expect that the main goal of symplectic quantization will be represented by the possibilities it opens up to describe in a consistent manner the *dynamics* of the gravitational field in all situations where this can be relevant. For instance, we have in mind the non-equilibrium aspects of the quantum fluctuations relaxational dynamics in inflationary cosmology [30–34]. We also expect that symplectic quantization will allow us to give a simple and unequivocal definition in the framework of quantum cosmology of concepts typical of statistical systems such as *non-equilibrium dynamics* and *irreversibility*. For example, we expect symplectic quantization to be the good formalism to go beyond the Einstein's frozen-block universe scenario, i.e., the good formalism to represent an *evolutionary* dynamics of the universe [28]. But right now we leave these speculations as a matter for future investigations and we present an analysis more limited in scope. We want to show how the new kinetic degrees of freedom introduced in the context of symplectic quantization are directly related to the appearance of a cosmological constant term in Einstein equations passing via the path-integral approach to gravity. In particular we want to show that the symplectic quantization of pure gravity *without* a cosmological constant term produces, by simply integrating over the conjugated momenta $\pi_{\mu\nu}(x)$, a theory of gravity *with* a cosmological constant Λ . This derivation of Λ is therefore still quantum in nature but is *different* from the usual one, where Λ is interpreted as the vacuum energy of matter fields: here Λ is solely related to intrinsic properties of pure gravity, in particular to the fact that $g_{\mu\nu}$ has its own dynamics. Let us stress that the procedure outlined here, though providing a new physical interpretation and derivation of the cosmological constant, is in first instance motivated by the necessity to relate the new kinetic degrees of freedom of the field, $\pi_{\mu\nu}(x)$, so far unobserved, to known physics.

The procedure which allows us to relate the momenta $\pi_{\mu\nu}(x)$ to the cosmological constant requires to transform the partition function in Eq. (23) from the hard-constraint ensemble, where the value of $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ is fixed with a Dirac delta, to the ensemble where the constraint is soft, for instance by means of an integral (Fourier) transform:

$$\begin{aligned} \mathcal{Z}_{\text{grav}}(z) &= \int dA e^{-izA} \Omega(A) = \int \mathcal{Dg} \mathcal{D}\pi e^{-iz\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]} \\ &= \int \mathcal{Dg} \mathcal{Z}_{\text{kin}}[z, \mathbf{g}] e^{izS[\mathbf{g}]}, \end{aligned} \quad (27)$$

where we have defined

$$\mathcal{Z}_{\text{kin}}[z, \mathbf{g}] = \int \mathcal{D}\pi e^{-iz\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]}. \quad (28)$$

The integration over momenta is technically quite easy, since the pseudo-Hamiltonian $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ depends quadratically on them, but it is at the same time non-trivial due to the non-separable nature of $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$. This non-separability comes from the dependence of the kinetic term $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$ on the metric itself, a characteristic which is

typical of the symplectic quantization of gravity and not of ordinary matter fields. By choosing $z = 1$ (in natural units, which means $z = \hbar^{-1}$), one can write:

$$\begin{aligned} Z_{\text{kin}}[1, \mathbf{g}] &= \int \mathcal{D}\pi(x) e^{-i\frac{\kappa_g}{2} \int d^4x |g|^{1/2} \pi_{\mu\nu}(x) \mathcal{G}^{\mu\nu,\alpha\beta}[\mathbf{g}(x)] \pi_{\alpha\beta}(x)} \\ &= \exp\left(-\frac{i}{2\kappa_g^2} \int d^4x |g|^{1/2} \log[\det \hat{\mathcal{G}}[\mathbf{g}(x)]]\right). \end{aligned} \tag{29}$$

Intermediate steps of the not difficult calculation yielding the right hand side of Eq. (29) are reported in the Appendix.

We need now to recall a quite remarkable coincidence: *only in four space–time dimensions* it happens that the determinant of the bi-tensor $\hat{\mathcal{G}}[\mathbf{g}(x)]$ is constant across the whole space–time manifold [19, 22]:

$$\det \hat{\mathcal{G}}[\mathbf{g}(x)] = -1 - 2\lambda. \tag{30}$$

In [22] it is proposed the value $\lambda = -1$ for the adimensional constant in order to have physical consistency of the stochastic quantization procedure. In the case of symplectic quantization we find convenient to ask for a small deviation from this value, i.e., we propose

$$\lambda = -\left(1 + \frac{\epsilon_\Lambda^2}{2}\right), \tag{31}$$

where we take ϵ_Λ , which is the only free parameter of the new theory, as a small positive number, $\epsilon_\Lambda < 1$. The choice of the subscript Λ in ϵ_Λ will be immediately clear.

We have achieved an interesting result: the integration over the kinetic degrees of freedom of the metric field represents a possible way to derive the existence of a cosmological term in Einstein equations.

In fact defining

$$2\Lambda = \frac{\kappa}{\kappa_g^2} \log \det \hat{\mathcal{G}}[\mathbf{g}(x)] \tag{32}$$

and plugging into Eq. (30) the definition of λ given in Eq. (31) we get

$$2\Lambda = \frac{\kappa}{\kappa_g^2} \log[1 + \epsilon_\Lambda^2] \simeq \epsilon_\Lambda^2 \frac{\kappa}{\kappa_g^2}. \tag{33}$$

The choice made for the parameter λ in Eq. (31) is therefore consistent with a positive cosmological constant, $\Lambda > 0$, and we can finally recover the path-integral for the Einstein–Hilbert action of pure gravity *with* a cosmological constant term:

$$\begin{aligned} \mathcal{Z}_{\text{grav}} &= \int \mathcal{D}g(x) \mathcal{D}\pi(x) e^{-iA[\mathbf{g}, \boldsymbol{\pi}]} \\ &= \int \mathcal{D}g(x) \exp\left(i \frac{1}{2\kappa} \int d^4x |g|^{1/2} [R - 2\Lambda]\right). \end{aligned} \quad (34)$$

The derivation of Eq. (34) from the *pseudo*-microcanonical partition function in Eq. (23) is the main result of this paper: the integration over the kinetic degrees of freedom of the field $\boldsymbol{\pi}(x)$ give rise to a cosmological constant term.

We want to recall at this point that in order to define the symplectic quantization ensemble for the quantum fluctuations of gravity we have introduced only one new physical constant, κ_g , the one appearing in the definition of $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$ in Eq. (17). The constant λ appearing in the definition of the DeWitt supermetric $\mathcal{G}^{\mu\nu, \alpha\beta}[\mathbf{g}(x)]$ is in fact just a dimensionless number. Now, by reverse engineering the relation between constants in Eq. (33), we can see that κ_g is not really a new object but can be written as a combination of known constants:

$$\kappa_g \simeq \sqrt{\frac{\kappa}{2\Lambda}}. \quad (35)$$

That is, rather than saying that the dynamics of $g_{\mu\nu}(x)$, which is controlled by κ_g , give rise to a cosmological constant term, we can see the relation between constants from the point of view of Eq. (35): it is Λ that determines the typical scale for the symplectic dynamics of $\mathbf{g}(x, \tau)$. In order to make this clear one can rewrite the kinetic term $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$, taking advantage of Eq. (33), as:

$$\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}] = \frac{1}{2\varepsilon_\Lambda} \sqrt{\frac{2\Lambda}{\kappa}} \int d^4x |g|^{1/2} \dot{g}_{\mu\nu}(x, \tau) \mathcal{G}^{\mu\nu, \alpha\beta}[\mathbf{g}(x)] \dot{g}_{\alpha\beta}(x, \tau) \quad (36)$$

It is easy to see that the expression in Eq. (36) is dimensionless, as it has to be, since $[\varepsilon_\Lambda] = [\mathcal{G}] = [g] = 0$, $[\Lambda] = L^{-2}$, $[\kappa] = L^2$ and $[\dot{g}_{\mu\nu}] = L^{-1}$.

In conclusion we have shown how to obtain from the microcanonical partition function of symplectic quantization, Eq. (23), the standard Feynman path-integral for gravity [26], showing that the cosmological constant term in Einstein equations is a fingerprint of the existence of *kinetic* degrees of freedom for $g_{\mu\nu}$. The theory proposed is therefore predictive: it describes a new phenomenon, the existence of an intrinsic dynamics for the field $\mathbf{g}(x, \tau)$, and relates a feature of this new phenomenon, i.e., the characteristic time-scale of the gravitational field quantum fluctuations dynamics, to known physical constants: the Einstein gravitational constant κ and the cosmological constant Λ , which has to be taken as an input from observational data. In the new theory there is only one dimensionless parameter let free to be fixed, the constant ε_Λ appearing in the definition of the DeWitt supermetric $\mathcal{G}^{\mu\nu, \alpha\beta}[\mathbf{g}(x)]$.

5 Conclusions

Let us summarize here the main logical steps explaining how symplectic quantization, first proposed for a scalar field in [1], works for gravity. The most relevant novelty of the approach is the claim that fields, in particular the metric field $\mathbf{g}(x)$ in the present case, depend on an additional variable τ with physical dimensions of time, i.e. we have $\mathbf{g}(x, \tau)$. This variable, here referred to as *intrinsic time* τ , is not an additional coordinate of the space–time continuum: τ is the parameter which controls the sequence of quantum fluctuations at each point of the space–time manifold and is different from the coordinate time $x^0 = ct$. *Intrinsic time* τ , which indeed already appeared as a fictitious variable in the stochastic quantization approach [22, 23], is raised here to its full dignity of physical time. In the framework of symplectic quantization it is therefore possible to consistently define a *dynamics* for the quantum fluctuations of $\mathbf{g}(x, \tau)$. This dynamics, which is deterministic, is generated by a generalized action of the kind $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}] = \mathcal{K}[\mathbf{g}, \boldsymbol{\pi}] - S[\mathbf{g}]$. While $S[\mathbf{g}]$, which can be taken as any general relativistic action for the gravitation field (e.g., Einstein–Hilbert, but the approach works for modified gravity theories as well [29]), plays the role of a *potential* term, we have that $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$ is a *kinetic* term, controlling the rate of change of $\mathbf{g}(x, \tau)$ with respect to intrinsic time. The term $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$ appears for the first time in the present paper, up to our knowledge. The knowledge of $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ allows then us to write down the symplectic dynamics in intrinsic time generated by the Hamilton equations in (22). The fluctuations of the term $S[\mathbf{g}]$ along the action-preserving dynamics represent the quantum fluctuations of the gravitational field. By then assuming a sort of ergodicity for this symplectic dynamics, a fact which is far from trivial and might not be true in some extreme conditions (e.g., close to cosmological singularities and black holes), it is then possible to define a *pseudo-microcanonical* ensemble based on the hypothesis that all the configurations of the fields $\mathbf{g}(x)$ and $\boldsymbol{\pi}(x)$ corresponding to the same value of the generalized action $A = \mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ have identical probability. This is the only and main assumption of the paper. Remarkably, within the framework of symplectic quantization the *non-renormalizability* problem of gravity is much more limited in scope. In particular, it does not cause problems to the definition of the hard-constraints ensemble for quantum fluctuations, Eq. (23). Finally, by considering a sort of *low-energy* approximation where the ensemble defined by a *hard* constraint on the functional $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ is replaced with the ensemble characterized by a *soft* constraint on $\mathcal{A}[\mathbf{g}, \boldsymbol{\pi}]$ (e.g., by means of an integral transform), we have shown that the integration over the momentum field $\boldsymbol{\pi}(x)$ gives rise to a cosmological constant in a theory of pure gravity initially free from it. We have therefore shown that within the framework of symplectic quantization the appearance of a cosmological constant Λ in Einstein equations is still a quantum effect, but it solely related to the intrinsic properties of the metric field $\mathbf{g}(x)$, in particular to its *rate of change* with respect to intrinsic time expressed by the momentum field $\boldsymbol{\pi}(x)$. Quite remarkably, exploiting the definition of the DeWitt supermetric $\mathcal{G}^{\mu\nu,\alpha\beta}[\mathbf{g}(x)]$ to build the kinetic term $\mathcal{K}[\mathbf{g}, \boldsymbol{\pi}]$ of symplectic quantization, Λ turns out to be constant across the whole space–time manifold only in $d = 4$.

Thanks to the additional *intrinsic time* τ variable of the symplectic quantization approach we have thus been able to reincorporate the most natural notion of *time* as

a flux of events in a quantum theory of gravity. We also provided a *new* interpretation of the cosmological constant Λ and hopefully defined a consistent framework for the study of cosmological problems where an explicit dependence on time is needed. Finally, we want to stress that the symplectic quantization of gravity, though in some sense closer to the canonical approach to quantum gravity (it does not require any sort of supersymmetry or extra-dimensions), does not make any claim on the microscopic degree of freedom of the gravitational field and can be therefore equally well compatible with different microscopic theories.

Appendix

Appendix A: Gaussian Functional Integral

The result of the Gaussian functional integral in Eq. (29) can be obtained as follows. The first step is to introduce the dimensionless integration variable

$$\hat{\pi}_{\mu\nu}(x) = \kappa_g^{3/2} \pi_{\mu\nu}(x) \quad (37)$$

so that the functional integral to be calculated reads as

$$\mathcal{Z}(1) = \int \prod_x \prod_{\mu\nu} d\hat{\pi}_{\mu\nu}(x) \exp\left(-\frac{1}{2\kappa_g^2} \int d^4x |g|^{1/2} \hat{\pi}_{\mu\nu}(x) \mathcal{G}^{\mu\nu,\alpha\beta}[g(x)] \hat{\pi}_{\alpha\beta}(x)\right), \quad (38)$$

where, for convenience, we have considered the Euclidean version of the path integral. It is then natural to introduce the invariant and dimensionless infinitesimal volume element

$$d\mu(x) = \frac{d^4x |g|^{1/2}}{\kappa_g^2}, \quad (39)$$

and simply write

$$\begin{aligned} \mathcal{Z}(1) &= \int \prod_x \prod_{\mu\nu} d\hat{\pi}_{\mu\nu}(x) \exp\left(-\frac{1}{2} \int d\mu(x) \hat{\pi}_{\mu\nu}(x) \mathcal{G}^{\mu\nu,\alpha\beta}[g(x)] \hat{\pi}_{\alpha\beta}(x)\right) \\ &= \int \prod_{\mathbf{r}} \prod_{\mu\nu} d\hat{\pi}_{\mu\nu}(\mathbf{r}) \exp\left(-\frac{1}{2} \sum_{\mathbf{r}} \hat{\pi}_{\mu\nu}(\mathbf{r}) \mathcal{G}^{\mu\nu,\alpha\beta}[g(\mathbf{r})] \hat{\pi}_{\alpha\beta}(\mathbf{r})\right) \\ &= \prod_{\mathbf{r}} \frac{1}{\sqrt{\det[\mathcal{G}(\mathbf{r})]}} = \exp\left(-\frac{1}{2} \sum_{\mathbf{r}} \log \det[\hat{\mathcal{G}}(\mathbf{r})]\right) \\ &= \exp\left(-\frac{1}{2\kappa_g^2} \int d^4x |g|^{1/2} \log \det[\hat{\mathcal{G}}(x)]\right). \end{aligned} \quad (40)$$

The only subtlety of the calculation in Eq. (40) above is that we have discretized the path integral on a disordered lattice (a lattice consistent with local Lorenz invariance) such that each point \mathbf{r} is surrounded by a unitary invariant volume. The original continuum measure is restored in the last line, at the end of the calculation, so that the particular choice of the discretization should be irrelevant.

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