Foundations of Physics (2020) 50:1645–1685
https://doi.org/10.1007/s10701-020-00391-z

Quantumness of States and Unitary Operations

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Received: 16 October 2019 / Accepted: 5 October 2020 / Published online: 30 October 2020

Abstract
This paper investigates various properties that may by possessed by quantum states, which are believed to be specifically "quantum" (entanglement, nonlocality, steerability, negative conditional entropy, non-zero quantum discord, non-zero quantum super discord and contextuality) and their opposites. It also considers their "absolute" counterparts in the following sense: a given state has a given property absolutely if after an arbitrary unitary transformation it still possesses it. The known relations between the listed properties and between their absolute counterparts are summarized. It is proven that the only two-qubit state that has zero quantum discord absolutely is the maximally mixed state. Finally, related conceptual issues concerning the terms "classical" and "quantum" are discussed.

Keywords Quantumness · Unitary operations · Quantum discord

1 Introduction

Quantum mechanics differs in various ways from classical physics because it is based on a different space of states. In quantum mechanics states are density operators on Hilbert space, which have various properties that may be labelled as "non-classical". The first aim of this paper is to review most prominent of these properties and relations between them. The second aim is to study how these properties (and their opposites) change under unitary operations performed on states possessing them. To this purpose, for each property X, its "absolute" version (the property of being absolutely X) will be introduced: a quantum state ρ is said to be an absolutely X state iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ has the property X. The importance of these absolute variants of properties comes from the physical meaning of unitary transformations and is explained in Sect. 2.2. The main new technical result of this paper is that the only two-qubit state that has zero quantum discord absolutely is the maximally mixed state $\frac{1}{4}\mathbb{1}_4$.

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The paper is organised as follows. Section 2 will provide the basic framework and fix the notation for the rest of the work. In Sect. 3 properties of quantum states discussed in the literature will be reviewed. In Sect. 4 relations between them will be analysed: whether possessing of one property by a given state implies possessing another property by the same state. In Sect. 5 the central notion of the paper will be introduced, namely possessing a given property in an absolute vs. non-absolute way. I will apply this notion to all properties listed in Sect. 3. Two main questions asked here will be as follows: Can a given property be possessed both in an absolute way and in a non-absolute way (by different classes of states)? If so, what are necessary and sufficient conditions for a state possessing a given property in an absolute way? In Sect. 6 relations between different absolute properties as well as between absolute properties and their "ordinary" counterparts will be investigated. Section 7 will discuss what exactly one can mean by "classical" when saying that the states of quantum mechanics have various non-classical features. And finally Sect. 8 will summarize the results.

The results presented here are of both theoretical and practical interest. From a theoretical point of view, the concept of possessing a given property absolutely gives us a better understanding of the nature of crucial properties of quantum states, by revealing their relativity (in some cases) to the choice of a basis in the Hilbert space. From a practical point of view, unitary operations can be used to transform a given state that lacks a property that is needed for a certain quantum computational task to another state that possesses this property. Therefore, it is important to know the class of states that allow for such a transformation in order to improve our capability of obtaining states useful for a certain quantum computational task from useless ones.

In the analyses of this paper I try to be neutral (as far as possible) with respect to the issues concerning the interpretation of quantum mechanics and the measurement problem (cf. footnote 2). The results reviewed and obtained here make sense (at least) for all interpretations of quantum mechanics that do not change the standard Hilbert space formalism; however, depending on the interpretation, the exact physical meaning of these results may differ. Some interpretational problems will arise, however, in Sect. 7, which has more conceptual character.

2 Framework

2.1 Quantum States

Let us denote the n-dimensional Hilbert space by $\mathcal{H}_n$ (we will restrict to $n < \infty$ and often consider only $n = 4$) and the space of all operators on $\mathcal{H}_n$ by $\tilde{\mathcal{H}}_n$. States of physical objects are assumed to be represented by density operators $\rho \in \tilde{\mathcal{H}}_n$, $\rho : \mathcal{H}_n \to \mathcal{H}_n$. In finite cases such operators can be represented by density matrices (denoted here by the same symbol). A state is called pure if there exist $|\psi\rangle \in \mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$.

A composite physical system consists of two or more subsystems. If this is the case, all the subsystems are represented by density operators defined on Hilbert spaces of appropriate dimensions and the whole system is represented by a density operator defined on the Hilbert space that is the tensor product of all those Hilbert spaces. For example, let us consider a system ρ^{AB} composed of two subsystems ρ^A and ρ^B, one of dimension n ($\rho^A : \mathcal{H}_n \to \mathcal{H}_n$) and one of dimension m ($\rho^B : \mathcal{H}_m \to \mathcal{H}_m$). Then a composite system is $\rho^{AB} : \mathcal{H}_n \otimes \mathcal{H}_m \to \mathcal{H}_n \otimes \mathcal{H}_m$. The subsystems can be obtained from a given composite system by taking partial trace of the respective density matrix: $\rho^A = \mathrm{Tr}_B \rho^{AB}$, $\rho^B = \mathrm{Tr}_A \rho^{AB}$.

2.2 Unitary Operations

One of the most important classes of quantum operations that can act on quantum states are unitary operators, which by definition satisfy $UU^\dagger = U^\dagger U = \mathbb{1}$, where $\mathbb{1}$ is an identity operator. Mathematically, unitary transformations are symmetries of Hilbert spaces—they leave invariant all their structure, in particular they do not change the values of inner product. Physically, they can be interpreted in two different ways. On the one hand, unitary transformations describe the time evolution of isolated quantum systems. On the other hand, they can be viewed as a change of a basis in a Hilbert space. Under the second interpretation, what changes is not the physical content of a state, but only our way of representing it (for a scrupulous argumentation that unitarily-related states are indeed physically equivalent at least in ordinary quantum mechanics see [50, pp. 24–29]).

These interpretations are not competitive: this distinction means only that two physically different operations (the change of a state in time, which is physically real and the change of a basis, which is only a formal manipulation) are represented by the same mathematical operation. The choice of an interpretation depends on a situation that is analysed and should be clear from the context.

For composite systems one can distinguish between local and global unitary operations. Local unitary operations have a form $U = U_A \otimes U_B$, where U_A and U_B are unitary operators that act independently on each subsystem. These operations cannot change relations between subsystems (such as those listed in Sects. 3.1–3.7; see Fact 5.1). Global unitary operations do not have this form and therefore intertwine both subsystems, possibly changing relations between them. Therefore, although unitary operations do not change the physical content of a state of a system, they may change a way in which the system is divided into subsystems; and under a new division, the relationships between subsystems may be different. This reveals a kind of relativity of entanglement and similar properties of quantum systems (namely, to the choice of a basis), which is, however, unsurprising: these properties capture relations between subsystems of a given system, so one may expect they will change with a new choice of subsystems (even if the whole system is the same). Interestingly, they are not totally free to change—the constraints to this will be investigated in Sect. 6.

2.3 Measurements

Another important type of quantum operations is quantum measurement. The standard notion of measurement is the so-called projective (or von Neumann) measurement.

Definition 2.1 Projective measurement (von Neumann measurement) is described by an observable, M, which is a hermitian operator on the state space of the observed system. The observable has a spectral decomposition

$$M = \sum_m m P_m, \tag{1}$$

where P_m is the projector onto the eigenspace of M with eigenvalue m. Projectors satisfy $\sum_m P_m = \mathbb{1}$ and $P_m P_{m'} = \delta_{mm'}$.

The von Neumann measurement "detects" the value of a given physical quantity (observable) of a quantum system in a given state and projects this state into the eigenstate of this observable associated with the measured value. This "detecting" is usually not deterministic in the sense that more than one value can be obtained in a given measurement with non-zero probability; but these values and probabilities are uniquely determined by the measured quantity and the state of the system. One can generalize this notion to the so-called generalized measurements, which are a mathematical representation of detectors with non-ideal efficiency, measurement outcomes that include additional randomness, measurements that give incomplete information, etc.

Definition 2.2 Generalised measurement (POVM—Positive Operator Valued Measure) is described by a collection of measurement operators M_m that satisfy $\sum_m M_m^\dagger M_m = \mathbb{1}$ (but are not necessarily projectors).

A special case of the generalised measurement is the weak measurement, first introduced in the paper [2]. With a view to its application in context of quantum super discord, instead of the original definition I will use the following one by Oreshkov and Brun [46]:

Definition 2.3 Weak measurement is given by a pair of operators:

$$P(\xi) = \sqrt{\frac{1-\tanh\xi}{2}}\Pi_1 + \sqrt{\frac{1+\tanh\xi}{2}}\Pi_2, \tag{2}$$

$$P(-\xi) = \sqrt{\frac{1+\tanh\xi}{2}}\Pi_1 + \sqrt{\frac{1-\tanh\xi}{2}}\Pi_2, \tag{3}$$

where Π_1 and Π_2 are two orthogonal projectors satisfying $\Pi_1 + \Pi_1 = \mathbb{1}\, and\, |\xi| \ll 1$ is the strength of the measurement.

One can also define a measurement performed on a single part of a composite system:

Definition 2.4 For a bipartite state ρ^{AB}, a local von Neumann measurement on the subsystem A is a family of one-dimensional orthogonal projections on the space of subsystem A, $\{\Pi_i^A\}$, such that $\sum_i \Pi_i^A = \mathbb{1}^A$.

Definition 2.5 For a bipartite state ρ^{AB}, a local weak measurement on the subsystem A is a pair of weak measurement operators acting on a subsystem A, $\{P^A(x), P^A(-x)\}$.

Analogous definitions can be formulated for the subsystem B.

2.4 Entropy of Quantum States

Classical probability of some random variable A can be described by a vector of probabilities $p(a)$ of obtaining the particular values a of this variable. For such vectors one can define entropy and some derivative notions, which measure its information content:

- Shannon entropy: $H(A) = -\sum_a p(a)\log p(a)$,
- Joint entropy: $H(A,B) = -\sum_{a,b} p(a,b)\log p(a,b)$,
- Conditional entropy: $H(A|B) = -\sum_{a,b} p(a,b)\log p(a|b)$,
- Mutual information: $I(A:B) = H(A) + H(B) - H(A,B)$.

One can show the following relations between conditional entropy on the one hand, and the joint entropy and entropies of the random variables considered separately on the other:

$$H(A|B) = H(A,B) - H(B), \tag{4}$$

$$H(B|A) = H(A,B) - H(A). \tag{5}$$

Quantum probability is encoded in density operators describing quantum systems. It is possible to define quantum analogues of classical entropies (see e.g. [44, ch. 11]). The analogy is rather straightforward, with the exception of conditional entropy, for which the definition is based on the relations (4) and (5). The analogue of Shannon entropy concerning a quantum state ρ is called von Neumann entropy $S(\rho)$. The derivative notions have the same names as in the classical case. Therefore, we obtain the following list of quantum entropies:

- von Neumann entropy: $S(\rho) = -\mathrm{Tr}(\rho\log\rho)$,
- Joint entropy: $S(\rho^{AB}) = -\mathrm{Tr}(\rho^{AB}\log\rho^{AB})$,
- Conditional entropy: $S(\rho^A|\rho^B) = S(\rho^{AB}) - S(\rho^B)$,
- Mutual information: $I(\rho^A : \rho^B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$.

In contrast to its classical counterpart, quantum conditional entropy can be less than zero. More specifically, its bounds are: $-S(B) \leq S(A|B) \leq S(A)$. However, one needs to remember that in the quantum case the basic entropy, that is, von Neumann entropy, is always positive and conditional entropy is only some algebraic combination of such entropies, not an entropy in the proper sense.

2.5 Fano-Bloch Decomposition

Any bipartite state ρ^{AB} of dimension $d_A \times d_B$ can always be represented in the so called Fano-Bloch form [14]:

$$\rho_{AB} = \frac{1}{d_A d_B}\left(\mathbb{1}_{AB} + \sum_{m=1}^{d_A^2-1} a_m \sigma_m^A \otimes \mathbb{1}_B + \sum_{n=1}^{d_B^2-1} \mathbb{1}_A \otimes b_n \sigma_n^B + \sum_{m=1}^{d_A^2-1}\sum_{n=1}^{d_B^2-1} t_{mn}\sigma_m^A \otimes \sigma_n^B\right), \tag{6}$$

where $a_m = \mathrm{Tr}\rho^{AB}(\sigma_m^A \otimes \mathbb{1}_B)$, $b_n = \mathrm{Tr}\rho^{AB}(\mathbb{1}_A \otimes \sigma_n^B)$ are Bloch vectors of reduced states ρ^A, ρ^B, $t_{mn} = \mathrm{Tr}\rho_{AB}(\sigma_m^A \otimes \sigma_n^B)$ is a correlation tensor and σ_m^A, σ_n^B are generalised Pauli matrices satisfying $\mathrm{Tr}(\sigma_m^i \sigma_n^i) = 2\delta_{mn}$, $\mathrm{Tr}(\sigma_n^i) = 0$, where $i = A, B$. This decomposition is often convenient and simplifies many reasonings.

2.6 Some Special Classes of Quantum States

Some classes of quantum states have a special status because they have particularly simple form (allowing for substantial simplifications in calculations), while still being non-trivial. Two examples relevant for our purposes are the Weyl states (that include the Bell states and the Werner states) and the Gisin states.

2.6.1 Weyl States

The first class of states that will be used here are the locally maximally mixed states, also known as the Weyl states. For $d_A = d_B = 2$ the Weyl states can be represented, up to local unitaries, as

$$\rho_{Weyl} = \frac{1}{4}\left(\mathbb{1}_{AB} + \sum_{n=1}^{3} \tilde{t}_n \sigma_n^A \otimes \sigma_n^B\right). \tag{7}$$

A special case of the Weyl states are the Werner states, introduced in [59], where they were used to prove that entanglement does not imply nonlocality (some entangled Werner states are local). The Werner states are convex combinations of one of the maximally entangled states and the maximally mixed state:

$$\rho_{\mathrm{Werner}} = w\rho_{\Psi^-} + \frac{1}{4}(1-w)\mathbb{1}_4 = \frac{1}{4}\left(\mathbb{1}_2 \otimes \mathbb{1}_2 - w\vec{\sigma} \otimes \vec{\sigma}\right), \tag{8}$$

where $\rho_{\Psi^-} = \frac{1}{4}\left(\mathbb{1}_4 - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$ is one of the Bell states. The particularly important fact about the Werner states is that they are parametrized by a single parameter w, which takes values $w \in [-\frac{1}{3}, 1]$. On the other hand, this family of states is theoretically nontrivial, as it encompasses some entangled states, the maximally mixed state and the spectrum of intermediate states.

2.6.2 Gisin States

Another interesting class of states, the Gisin states, was introduced in [20]. They have been used to prove the existence of the phenomenon of hidden nonlocality: some of the Gisin states which are local (do not violate CHSH inequality) lose this feature after applying a purely local operations, so called local filtering. The Gisin states are expressed by the following formula:

$$\rho_G(\lambda, \theta) = \lambda \rho_\theta + (1 - \lambda)\rho_{\text{top}}, \tag{9}$$

where

$$\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|, \quad |\psi_\theta\rangle = \sin\theta|01\rangle + \cos\theta|10\rangle, \tag{10}$$

$$\rho_{\text{top}} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \tag{11}$$

and the ranges of the parameters are $0 \le \theta \le \frac{\pi}{2}$,[1] $0 \le \lambda \le 1$.

Their eigenvalues are: $0, \frac{1-\lambda}{2}, \frac{1-\lambda}{2}, \lambda$. Observe, that these eigenvalues are dependent only on the parameter λ and are independent of the second parameter θ. This fact will be important later, in Sect. 6.1, where we will use these eigenvalues to check which of the Gisin states possess some absolute properties. The Gisin states are not the subset of the Weyl states, but these two classes of states have a non-empty intersection, as the Gisin states are locally maximally mixed for $\theta = \frac{\pi}{4}$.

3 Properties of Quantum States

Quantum states differ with each other in many ways and there are many formal tools to describe these differences. In this work the following properties will be considered: nonlocality/locality, steerability/unsteerability, entanglement/separability, negative/non-negative conditional entropy, non-zero/zero quantum discord (connected with the properties of being a classical–classical, quantum–classical and classical–quantum state), non-zero/zero quantum super discord (connected with the property of being a product state) and contextuality/noncontextuality. In each of these pairs one property is the opposite of the other, so a given state possesses exactly one property from each pair.

[1] Originally the Gisin states were defined only for $0 < \theta < \frac{\pi}{2}$, but the broader range of the parameter θ does not spoil their physicality and will be useful in our analysis of some properties and some absolute properties of the Gisin states in Sects. 4.1 and 6.1.

3.1 Separability vs. Entanglement

Let $\rho^{AB} \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$ be a bipartite state of an arbitrary dimension. It always belongs to at least one of the following three types of states:

Definition 3.1 A bipartite state ρ^{AB} is said to be a product state iff there exist ρ^A and ρ^B such that $\rho^{AB} = \rho^A \otimes \rho^B$. It is is called separable iff there exist states ρ^A_k and ρ^B_k and numbers $p_1, \dots, p_r$, $p_k > 0$, $\sum_{k=1}^r p_k = 1$ such that $\rho^{AB} = \sum_{k=1}^r p_k \rho^A_k \otimes \rho^B_k$ (a separable state is a convex combination of product states). If it is not separable, then it is called entangled.

Note that the above three notions gain their meaning only after the composed system is splitted into two subsystems, called A and B, so that the tensor product $\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$ is defined. Between the above properties the following relation holds: any product state is separable but (in general) not vice versa. For pure states, separability and being a product state coincide, but for mixed states they do not. Entanglement is the most important and most widely analysed feature of quantum states. It is recognised as a resource for most of quantum computation tasks, in the sense that it is responsible for specifically quantum effects in these tasks. For a review of current research concerning quantum entanglement see [27].

Definitions of separability and entanglement involve universal quantification over the set of density matrices and weights, so on the basis of this definition alone it is difficult to check whether a given state is separable or entangled. Therefore, it would be helpful to find some simpler criterions of separability and entanglement. Such criterions were found only for some special cases and a general criterion, working for any state of arbitrary dimension, is still not known. For pure bipartite states the criterion is given by the following theorem:

Criterion 3.1 *A state ρ^{AB} is separable iff the entropy of the reduced state is positive*, $S(\rho^A) > 0$ (*equivalently:* $S(\rho^B) > 0$).

For mixed bipartite states there is no universal criterion that gives necessary and sufficient conditions for separability. However, for Hilbert spaces of dimension 2×2 or 2×3 such conditions are given by the so-called PPT (Positive Partial Transpose) criterion. The following definition of partial transpose allows one to formulate the PPT criterion:

Definition 3.2 For a given state ρ, its partial transpose with respect to a subsystem B, $\rho^{\top_B}$, is given by $\langle m|\langle\mu|\rho^{\top_B}|n\rangle|\nu\rangle := \langle m|\langle\nu|\rho|n\rangle|\mu\rangle$. Analogously for a subsystem A.

Criterion 3.2 [25, 48] *A quantum state ρ of dimension* 2×2 *or* 2×3 *is separable iff* $\rho^{\top_B} \geq 0$ (*equivalently:* $\rho^{\top_A} \geq 0$).

3.2 Locality vs. Nonlocality

Some of the famous works in foundations of quantum mechanics concern quantum nonlocality. Nonlocality was discovered by Bell (see his collected papers in [5]) and later analysed by many others, including [11]. This phenomenon can be described as follows: the source produces a physical system, which is divided into two subsystems. They are send to two distant observers, called Alice and Bob. Upon receiving their subsystems, each observer performs a measurement on it. The measurement chosen by Alice is labeled x and its outcome is a. Similarly, Bob chooses measurement y and gets outcome b. The experiment is characterised by the joint probability distribution $p(a, b|x, y)$ of obtaining outcomes a and b when Alice and Bob choose measurements x and y, respectively. It turns out that the joint probability distribution predicted by quantum mechanics in general is not a product of probability distributions obtained by Alice and Bob considered separately: $p(a,b|x,y) \neq p(a|x)p(b|y)$, so these distributions are not independent, irrespectively of how large the distance between the observers is. One may wonder whether this independence is real or the quantum-mechanical description is incomplete and it is possible to introduce an additional factor, so called hidden variable, which enables one to describe the two subsystems as uncorrelated. The second option has been explored under the name of hidden variable models for quantum systems. In fact, possessing such a model is the defining condition for a state to be local.

Definition 3.3 A bipartite state ρ^{AB} is called local iff it can be described by a local hidden variable model, that is, there exists a hidden variable $\lambda \in \Lambda$ and a probability measure μ on the space Λ such that for every measurement choices x, y, one can reconstruct joint probability distribution $p(a, b|x, y)$ predicted by quantum mechanics from another probability distribution conditionalised on λ:

$$p(a,b|x,y) = \int_{\Lambda} d\lambda\, q(\lambda)p(a,b|x,y,\lambda), \tag{12}$$

where

$$p(a,b|x,y,\lambda) = p(a|x,\lambda)p(b|y,\lambda). \tag{13}$$

If a bipartite state is not local, it is called nonlocal.

Bell observed that the existence of a hidden variable model leads to some constraints on probabilities of the outcomes, which are known under the name of the Bell inequality. There are several versions of this inequality and the most popular is CHSH (Clauser–Horne–Shimony–Holt) inequality [11], which is more general than Bell's original one. Let us assume that $a, b \in \{-1, +1\}$ and define expectation value of joint measurement of values a and b with measurement choices x and y:

$$\langle a_x b_y \rangle = \sum_{a,b} ab\, p(a,b|x,y). \tag{14}$$

It can be proven that states that are nonlocal are precisely those states that violate the CHSH inequality:

Theorem 3.1 [11] *A bipartite state is nonlocal iff for some settings $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ it violates the CHSH inequality*

$$\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle \leq 2. \tag{15}$$

In general, Bell-type scenarios are characterised by three numbers: the number of subsystems, the number of possible measurements, and the number of outcomes of each measurement. Here we are interested in the CHSH inequality, which concerns the scenario with two subsystems, two measurements and two outcomes; therefore, it is the (2,2,2)-type Bell inequality. With different types of scenarios there are connected different types of nonlocality, but from now on we will use this term to denote only one particular kind of nonlocality, namely the CHSH-nonlocality.

The above results can also be formulated in terms of the formalism of density operators. We need to use the CHSH operator, defined as follows:

$$\mathfrak{B}_{\mathrm{CHSH}} := \vec{a} \cdot \vec{\sigma}^A \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma}^B + \vec{a}' \cdot \vec{\sigma}^A \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma}^B \tag{16}$$

and the Hilbert–Schmidt inner product: $(A|B)_{HS} := \mathrm{Tr}(A^\dagger B)$. Using the CHSH operator, one can show the analogue of CHSH inequality:

Theorem 3.2 *A bipartite state ρ^{AB} is nonlocal iff for some settings $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ it violates the inequality*

$$(2\mathbb{1} - \mathfrak{B}_{\mathrm{CHSH}}|\rho)_{HS} \geq 0, \tag{17}$$

where $\mathfrak{B}_{\mathrm{CHSH}}$ is given by Eq. (16).

As in the case of separability and entanglement, the definition of nonlocality involves a quantification over a large set. Thus, relying only on the definition, it is difficult to check whether a given state is local or not. Again, no universal criterion providing relatively simple necessary and sufficient conditions for nonlocality is known. However, for two-qubit states the following criterion has been found:

Criterion 3.3 (CHSH operator criterion [24]) *Let ρ be a density operator of a two-qubit state with correlation tensor $t = (t_{mn})$, defined in* (6)*, and let μ_1 and μ_2 be the two largest eigenvalues of $M_\rho = t^\top t$. The state is nonlocal iff*

$$\max_{\vec{a},\vec{a}',\vec{b},\vec{b}'} \langle \mathfrak{B}_{CHSH}\rangle = 2\sqrt{\mu_1 + \mu_2} > 2. \tag{18}$$

3.3 Quantum Steering

The concept of quantum steering was introduced in Schrödinger [53]. It captures the fact that one of the parties (A or B) can change the state of the other (B or A, respectively) by choosing a basis for local measurement (the state of the second party must collapse according to this choice).[2] If a bipartite state allows steering, it is called steerable. In contrast to nonlocality and entanglement, this property of quantum states is not symmetric between A and B (A's being steerable by B does not imply B's being steerable by A). The notion was defined mathematically in [60]. For a review of recent research on this topic see [9]. Similarly to nonlocality scenarios, steering scenarios can be characterised by a number of subsystems (here we restrict to two), a number of possible measurements and a number of their outcomes. The definition is also similar—instead of local hidden variables models, it uses the notion of local hidden state models, which can be described roughly as local hidden variable models for one subsystem only.

Definition 3.4 A bipartite state ρ^{AB} is said to be steerable from A to B iff there exists a measurement in Alice's part that produces an assemblage that does not admit a local hidden state model, that is, there exists no hidden variable $\lambda \in \Lambda$ and no probability measure μ on the space Λ such that

$$\sigma_{a|x} = \int_\Lambda d\lambda\ \mu(\lambda) p(a, x|\lambda) \rho^B_\lambda, \tag{19}$$

where $\sigma_{a|x} = p(a|x)\rho_{a|x}$, $\rho_{a|x} = \mathrm{Tr}_A(M_{a|x} \otimes \mathbb{1})\rho^{AB}/p(a|x)$ are Bob's states after Alice's measurement (with the setting x and the outcome a) and $p(a|x)$ are probabilities of these states.

The following theorem describes the relationship between steerability and nonlocality of the type (2,2,2):

Theorem 3.3 [19]) *If a two-qubit state ρ is steerable with CHSH-type measurements, i.e., with a set-up (2,2,2), then it violates the CHSH inequality.*

From this theorem it follows that in the (2, 2, 2) case nonlocality and steerability are equivalent. However, there are some states that are steerable with three measurements but not CHSH-nonlocal (for the examples see [9]). Necessary and sufficient conditions for steerability are in general not known. The exception is a two-qubit case, for which necessary and sufficient conditions are analysed in Nguyen and Vu [43] and Yu et al. [61]. For other dimensions there are some partial results, for example, many inequalities providing sufficient conditions for steerability are derived in Calvacanti et al. [8].

[2] What exactly this "changing" means, depends of course on our interpretation of quantum states in general. If they are interpreted in an instrumentalistic manner, as mere means for making predictions without any ontological significance, then this should not be understood as a real change in the world, but only as a change in the predictions concerning the state of the second party that are allowed by quantum mechanics. If instead one interprets both the quantum state and its collapse realistically, then this change is a real physical process that takes place in the world. In this paper I do not opt for any particular interpretation of quantum states but only analyse their properties that follow from the standard mathematical formalism of quantum mechanics.

3.4 Negative vs. Non-negative Conditional Entropy

As mentioned in Sect. 2.4, conditional entropy of a quantum state can be negative, which is impossible in the classical case. The physical meaning of such a phenomenon is analysed in [26]. Conditional entropy provides the answer to the following question: Given an unknown quantum state distributed over two systems, how much quantum communication is needed to transfer the full state to one system? If the conditional entropy is positive, its sender needs to communicate this number of quantum bits to the receiver; if it is negative, then the sender and the receiver instead gain the corresponding potential for future quantum communication. These intuitions are formalised in the so called quantum state merging protocol, whose details can be found in the mentioned paper.

3.5 Quantum Discord

Quantum discord was introduced in Ollivier and Zurek [45] as a new measure of quantum correlations that encompasses broader class of states than entanglement. The definition is as follows:

Definition 3.5 Quantum entropy of a state ρ^{AB} with respect to a measurement on the subsystem A, $\{\Pi_i^A\}$, is $S(\rho^B|\{\Pi_i^A\}) = \sum_i p_i S(\rho^{B|\Pi_i^A})$, where $p_i = \mathrm{Tr}((\Pi_i^A \otimes \mathbb{1}_B)\rho^{AB})$, and $\rho^{B|\Pi_i^A} = \mathrm{Tr}_A((\Pi_i^A \otimes \mathbb{1}_B)\rho^{AB})/p_i$.

Definition 3.6 Quantum discord of a state ρ^{AB} under a measurement on the subsystem A, $\{\Pi_i^A\}$, is the difference $D(B|A) := I(B:A) - J(B:A)$, where $I(B:A)$ is a mutual information defined in Sect. 2.4, $J(B:A) = max_{\{\Pi_i^A\}} J(B|\{\Pi_i^A\})$, $J(B|\{\Pi_i^A\}) := S(B) - S(B|\{\Pi_i^A\})$.

Classical counterparts of I and J coincide: $I_{cl}(A:B) := S(A) + S(B) - S(A,B) = S(B) - S(B|A) =: J_{cl}(B:A)$ and this is why this quantity has been called "discord". It has been argued Datta et al. [13] that a non-zero quantum discord of a given state indicates its usefulness for quantum computation, sometimes even in the absence of entanglement.

There is no general formula for computing quantum discord even for two-qubit states. Only results for special classes of states are available. There exist analytic results for Weyl states [40] and also for a broader class of states, the so-called X-states, that is, the states that have non-zero values only on their diagonal and anti-diagonal positions in the computational basis [1]. However, checking whether a given quantum state has zero discord is much easier than computing quantum discord in the case when it is non-zero. There exist some relatively simple criteria for checking whether a given bipartite state has zero discord, one of which will be used in this paper (for some other criteria see e.g. [12, 16]):

Criterion 3.4 [28] *A bipartite quantum state* $\rho^{AB} \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_A$ *has zero quantum discord*, $D(A|B) = 0$, *iff all the square blocks of its density matrix of dimension*

$d = \dim(\mathcal{H}_B)$ are normal matrices and commute with each other. For $D(B|A) = 0$ one needs to consider all the blocks of dimension $d = \dim(\mathcal{H}_A)$.

It can be proven [16] that the set of zero discord states is of measure zero. For comparison: separable pure states are of measure zero in the set of all pure states, but separable states have a positive measure in the set of all density matrices [62]. Therefore, there are significantly less state with zero quantum discord than states which are separable. This fact can be understood as an indication that quantum discord is able to "detect" more quantum correlations than the properties described earlier.

3.6 Classical–Classical, Quantum–Classical and Classical–Quantum States

One can define three interesting classes of states, which turn out to be strictly connected with the notion of quantum discord. These are classical–classical, quantum–classical and classical–quantum states, defined as follows (see e.g. [15]):

Definition 3.7 A state ρ^{AB} is called classical–classical iff it has a form $\rho^{AB} = \sum_{i,j} p^{AB}_{ij} \Pi^A_i \otimes \Pi^B_j$, where $\{p^{AB}_{ij}\}$ is a classical probability distribution, $\Pi^A_i := |i\rangle_A\langle i|$ and $\Pi^B_j := |j\rangle_B\langle j|$ are spectral projections of the reduced states $\rho^A = tr_B\rho^{AB}$ and $\rho^B = tr_A\rho^{AB}$, respectively, $\{|i\rangle\}$ and $\{|j\rangle\}$ are orthonormal bases for subsystems A and B, respectively.

Definition 3.8 A state ρ^{AB} is called classical–quantum iff it has a form $\rho^{AB} = \sum_i p^A_i \Pi^A_i \otimes \rho^B_i$.

Definition 3.9 A state ρ^{AB} is called quantum–classical iff it has a form $\rho^{AB} = \sum_j p^B_j \rho^A_j \otimes \Pi^B_j$.

It is useful to compare these definitions with Definition 3.1 of separable states. All of them postulate similar forms of states: they should be sums of the tensor products of the states of their subsystems. The difference lies in the details of the form of the states of the subsystems: sometimes we require that they should be spectral projections of the reduced states (in the case of classical–classical for both subsystems, in the case of quantum–classical and classical–quantum for one subsystem—called "classical"), and sometimes we do not impose on them any additional conditions (in the case of separable states for both subsystems, in the case of quantum–classical and classical–quantum for one subsystem—called "quantum").

The name "classical–classical" is justified by the fact that such a state is in some sense encoded in the classical probability distribution (although to reconstruct the state fully we need to know the related choice of projectors as well). In the case of classical–quantum and quantum–classical states, only one of the subsystems can be represented by a classical probability distribution, whereas the other is represented by a density matrix.

Another "classical" aspect of classical–classical states is that they are not perturbed by local von Neumann measurements on their subsystems $\Pi^A_i \otimes \Pi^B_j$ in the sense that $\sum_{i,j} \Pi_i^A \otimes \Pi_j^B \rho^{AB} \Pi_i^A \otimes \Pi_j^B = \rho^{AB}$ (see e.g. [38]), but only if we understand these measurements as non-selective, that is, not selecting a particular outcome; under the standard interpretation of measurement the states would of course be perturbed. Similarly, non-selective measurements on the subsystem A does not perturb the state of the subsystem B in classical–quantum states and vice versa for quantum–classical states.

One can prove the theorem connecting the above classes of states with quantum discord (see e.g. [6]):

Theorem 3.4 *The following equivalences hold:*

A bipartite state ρ^{AB} is classical–classical iff $D(A|B) = D(B|A) = 0$.

A bipartite state ρ^{AB} is classical–quantum iff $D(B|A) = 0$.

A bipartite state ρ^{AB} is quantum–classical iff $D(A|B) = 0$.

3.7 Quantum Super Discord

The notion of super quantum discord was introduced in [52]. It is similar to the notion of quantum discord—the only difference lies in the fact that it uses weak measurements (see Definition 2.3) instead of the standard von Neumann measurements (see Definition 2.1). One of the differences between these two concepts of a measurement (understood, again, as non-selective measurements) is that a von Neumann measurement on one subsystem destroys the entanglement, whereas after performing a weak measurement on one subsystem, the state may still remain entangled [52].

Super quantum discord measures the correlation in a state ρ^{AB} as seen by an observer who performs a weak measurement on one of the subsystems. Now, let us state a formal definition of super quantum discord with respect to the subsystem A [52]:

Definition 3.10 Quantum entropy of a state ρ^{AB} with respect to weak measurement on the subsystem A, $\{P^A(\pm\xi)\}$, is $S(\rho^B|\{P^A(\xi)\}) = p(\xi)S(\rho^{B|P^A(\xi)}) + p(-\xi)S(\rho^{B|P^A(-\xi)})$, where $p(\pm\xi) = \mathrm{Tr}((P^A(\pm\xi) \otimes \mathbb{1}_B)\rho^{AB})$, $\rho^{B|P^A(\pm\xi)} = \mathrm{Tr}_A((P^A(\pm\xi) \otimes \mathbb{1}_B)\rho^{AB})/p(\pm\xi)$

Definition 3.11 Quantum super discord of a state ρ^{AB} under a weak measurement on subsystem A, $\{P^A(\pm x)\}$, is the difference $D(B|A) := I(B:A) - J(B:A)$, where $J(B:A) = max_\xi J(B|\{P^A(\pm\xi)\})$, $J(B|\{P^A(\pm\xi)\}) := S(B) - S(B|\{P^A(\pm\xi)\})$.

In the above definitions ξ is fixed, so the sums contain only two components, for ξ and for $-\xi$. The following properties of super quantum discord will be interesting for us:

Theorem 3.5 [52] *For any bipartite state ρ^{AB}, the quantum super discord is greater than or equal to the quantum discord: $D_w(A|B) \geq D(A|B)$.*

Theorem 3.6 [39] *A bipartite state ρ^{AB} has zero super quantum discord $D_w(A|B) = D_w(B|A) = 0$ iff ρ^{AB} is a product state.*

In the case of quantum discord it is possible to have $D(A|B) = 0$ and $D(B|A) \neq 0$ or the other way around. In contrast, if a super quantum discord is zero with respect to one subsystem, it is also zero with respect to the other subsystem.

3.8 Contextuality vs. Noncontextuality

The last pair of properties to be analysed in this paper consists of contextuality and noncontextuality. In general, noncontextuality means that the measured value of any observable is independent on other observables that are measured jointly with it. Of course, we restrict only to observables that are compatible with a given observable (i.e., commuting with it), because otherwise they could not be measured jointly. There are two senses of contextuality: it can be understood as a state-independent property of a set of projectors [33] or as a state-dependent property, which is possessed by some states but not the others. In this paper I will be interested only in the second sense of contextuality. It can be formalised in terms of the nonexistence of a contextual hidden variable theory, which leads to a certain inequality (in analogy to nonlocality). There are many versions of this inequality with different numbers of projectors and the best known of them is KCBS (Klyachko–Can–Binicioğlu–Shumovsky) inequality Klyachko et al. [32]. First, let us formally define the notions of contextuality and noncontextuality:

Definition 3.12 A state $|\psi\rangle$ is noncontextual iff there exists a hidden variable $\lambda \in \Lambda$, a probability measure μ on the space Λ and a value assignment on observables that can be measured on ρ, i.e., a function $\nu : \mathbb{A} \times \Lambda \mapsto \mathbb{R}$ satisfying for any two commuting observables A, B:

1. $\nu(A + B|\lambda) = \nu(A|\lambda) + \nu(B|\lambda)$,
2. $\nu(AB|\lambda) = \nu(A|\lambda)\nu(B\lambda)$,
3. $\nu(\mathbb{1}|\lambda) = 1$,
4. $\nu(0|\lambda) = 0$,
5. $\langle\psi|A|\psi\rangle = \int_\Lambda \nu(A|\lambda)\mu(\lambda)d\lambda$.

Definition 3.13 A state $|\psi\rangle$ is contextual iff it does not satisfy the noncontextuality condition.

The following theorem gives the necessary and sufficient conditions for the noncontextuality of a quantum state:

Theorem 3.7 [32] *A state $|\psi\rangle$ is noncontextual iff for any family of projectors P_0, P_1, P_2, P_3, P_4 such that each P_i commutes with P_{i+1} (where the sum should be understood modulo 5), the KCBS inequality holds:*

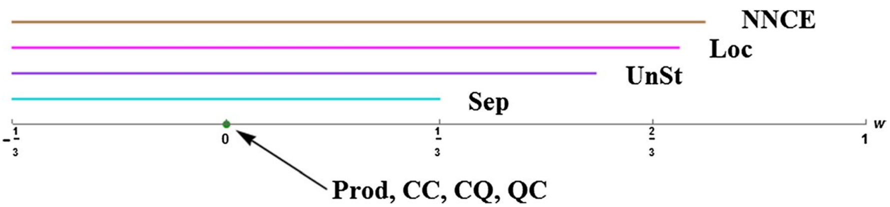


Fig. 1 Properties of the Werner states. The only product state (*Prod*) is the state with $w = 0$ (green point); it is also the only state with zero discord (and therefore classical–classical *CC*, classical–quantum *CQ* and quantum–classical *QC*). The other properties are: separability (*Sep*, blue), unsteerability (*UnSt*, violet), locality (*Loc*, purple) and non-negative conditional entropy (*NNCE*, brown) (Color figure online)

$$\langle\psi|(P_0 + P_1 + P_2 + P_3 + P_4)|\psi\rangle \leq 2. \tag{20}$$

According to the theorem, a state is contextual iff for some family of projectors satisfying conditions specified above, the KCBS inequality (20) is violated, that is, $\langle\psi|(P_0 + P_1 + P_2 + P_3 + P_4)|\psi\rangle > 2$. There are known examples both of states that are contextual and of states that are noncontextual. The simplest noncontextual state is the identity operator [31]. Examples of contextual states are provided in [29]; that paper contains also results of experimental tests that confirm the violation of the KCBS inequality.

4 Relations Between Different Properties

4.1 Relations for Special Classes of States

The previous section reviewed some properties of quantum states, which may be thought of as different ways of capturing their deviation from “classicality”. One may then ask a question whether these properties are really different from each other (in other words, whether the sets of states that possess them are different) and if so, what are the relations between them (do some of these properties imply some other properties). This section gives answers to these questions. First, let us look at the properties of two classes of states introduced in Sect. 2.6.

In Table 1 there are shown ranges of parameters for which the Werner states (8) and the Gisin states (9) are product states, have zero discord, are separable, unsteerable, local and have non-negative conditional entropy. They are also illustrated in Fig. 1 and 2. The results have been obtained with the use of Criteria: 3.2, 3.3 and 3.4. Most of these numerical results have already been presented in the literature: [59] (separability and locality for the Werner states), [55] (steerability for Werner states), [20] (separability and locality for the Gisin states), [17] (non-negative conditional entropy for the Werner states and for the Gisin states), [45] (quantum discord for the Werner states).

For the Werner states, the ranges of parameters are different with the exception of the product and zero discord states. However, the Gisin states are product states and have zero discord for different ranges of parameters. Therefore,

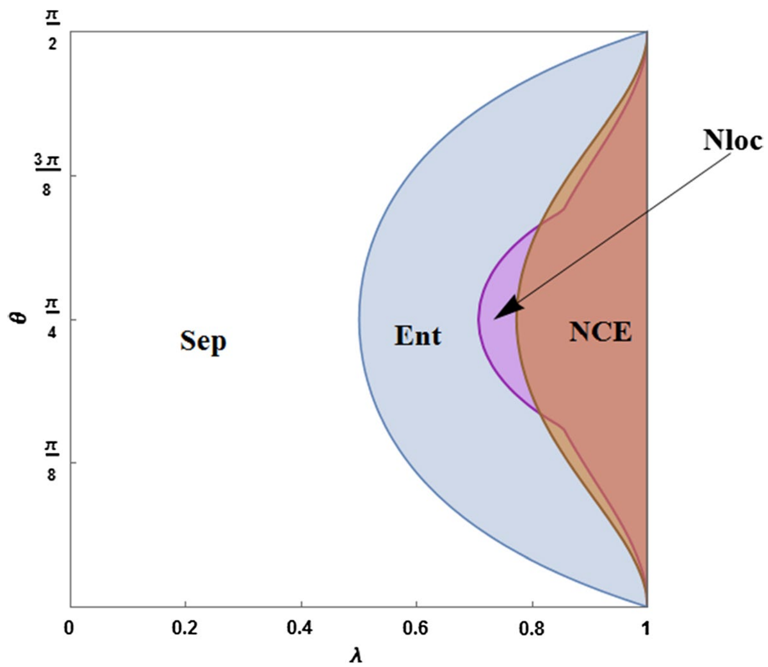


Fig. 2 Properties of the Gisin states (9) in the space of parameters λ, θ. The shaded regions are: *NLoc* nonlocal (violet), *NCE* negative conditional entropy (brown), *Ent* entangled (blue). The white region contains all and only separable Gisin states (*Sep*). Observe that there are states nonlocal and with negative conditional entropy, states local and with negative conditional entropy, as well as states nonlocal with non-negative conditional entropy, so there is clearly no implication between nonlocality and negative conditional entropy. This figure is inspired by a similar one in Friis [17] (Color figure online)

these two families of states are sufficient to distinguish between all the mentioned properties completely. From these results it follows that no two of the mentioned properties are equivalent.

4.2 General Relations for Bipartite States

We have seen that no two properties analysed here are equivalent. However, at least for the Werner states and the Gisin states there are some implications between them. One may ask whether these implications are specific to these classes of states or they hold in general. It turns out that in some cases the answer is positive and in some cases it is negative. For example, we have already seen that although for the Werner states negative conditional entropy implies nonlocality, this is no longer true for the Gisin states. The following theorem summarises what is known about these relations in the general case (see Fig. 3):

Theorem 4.1 *For any bipartite state ρ, the following implications hold:*

Table 1 Selected properties of the Werner states and the Gisin states in function of their parameters

	Werner states	Gisin states	
Parameter	w	θ	λ
Range of the parameter	$w \in [-\frac{1}{3}, 1]$	$\theta \in [0, \frac{\pi}{2}]$	$\lambda \in [0, 1]$
Product state	$w = 0$	$\theta \in \{0, \frac{\pi}{2}\}$	$\lambda = 1$
Zero discord	$w = 0$	θ arbitrary	$\lambda = 0$
		$\theta \in \{0, \frac{\pi}{2}\}$	λ arbitrary
Separable	$w \in [-\frac{1}{3}, \frac{1}{3}]$	$\lambda \cos^2\theta \geq 0 \,\&\, \lambda \sin^2\theta \geq 0 \,\&\, 1 - \lambda(1 + \sin(2\theta)) \geq 0 \,\&\, 1 - \lambda(1 - \sin(2\theta)) \geq 0$	
Unsteerable	$w \in [-\frac{1}{3}, \frac{1}{\sqrt{3}}]$	?	?
Local	$w \in [-\frac{1}{3}, \frac{1}{\sqrt{2}}]$	$\max\left\{\sqrt{\lambda^2 \sin^2(2\theta) + (1 - 2\lambda)^2}, \sqrt{2}\lambda \sin(2\theta)\right\} \leq 1$	
Non-negative conditional entropy	$w \in [-\frac{1}{3}, w_0]$, $w_0 \approx 0.7476$	$-2\frac{1-\lambda}{2}\log\left(\frac{1-\lambda}{2}\right) - \lambda\log\lambda + \left(\frac{1-\lambda}{2} + \lambda\cos^2\theta\right)\log\left(\frac{1-\lambda}{2}\lambda\cos^2\theta\right) + \left(\frac{1-\lambda}{2} + \lambda\sin^2\theta\right)\log\left(\frac{1-\lambda}{2} + \lambda\sin^2\theta\right) > 0$	

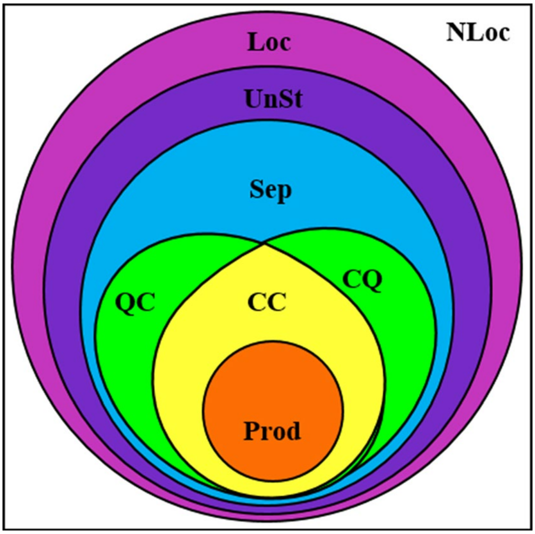


Fig. 3 Relations between properties of bipartite quantum states: *Prod* product states, *CC* classical–classical states, *CQ* classical–quantum states, *QC* quantum–classical states, *Sep* separable states, *UnSt* unsteerable states, *Loc* local states, *NLoc* nonlocal states. We have mentioned that the states with zero discord are of measure zero in the set of all density matrices, so the figure is out of scale

1. *ρ is nonlocal $\Rightarrow$ ρ is steerable,*
2. *ρ is steerable with n settings $\Rightarrow$ ρ is steerable with $n+1$ settings,*
3. *there exist $n \geq 2$ such that ρ is steerable with with n settings $\Rightarrow$ ρ is entangled,*
4. *ρ is entangled $\Rightarrow$ ρ is not classical–quantum $D(B|A) \neq 0$,*
5. *ρ is entangled $\Rightarrow$ ρ is not quantum–classical $D(A|B) \neq 0$,*
6. *ρ is not quantum–classical $\Rightarrow$ ρ is not classical–classical,*
7. *ρ is not classical–quantum $\Rightarrow$ ρ is not classical–classical,*
8. *ρ is not classical–classical $\Rightarrow$ ρ is not a product state, i.e. ρ has non-zero super quantum discord $D_w(A|B) \neq 0$, $D_w(B|A) \neq 0$.*

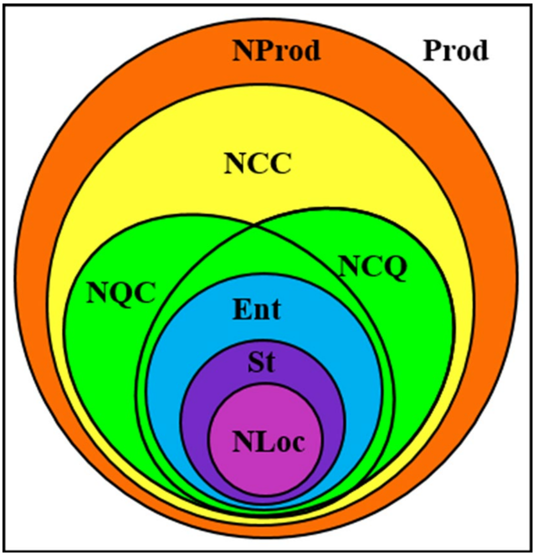
Proof

1. See Cavalcanti and Skrzypczyk [9].
2. If one has $n+1$ settings at the disposal and the method of steering a state by n settings, then one can perform this method with use of n from $n+1$ available settings.
3. See Cavalcanti and Skrzypczyk [9].
4. This follows from Definitions 3.1 and 3.8.
5. This follows from Definitions 3.1 and 3.9.
6. This follows from Definitions 3.7 and 3.9.
7. This follows from Definitions 3.7 and 3.8.
8. This is a consequence of Theorem 3.5 [52].

□

The implications in Theorem 4.1 do not hold the other way around. For the opposites of these properties we have an analogous theorem (see Fig. 4):

Fig. 4 Relations between properties of bipartite quantum states: *NLoc* nonlocality, *St* steerability, *Ent* entanglement, *NCC* states that are not classical–classical, *NCQ* states that are not classical–quantum, *NQC* states that are not quantum–classical, *NonProd* states that are not product, *Prod* product states. As before, the figure is out of scale



Theorem 4.2 *For any bipartite state ρ the following implications hold:*

1. *ρ is a product state, i.e. ρ has zero super quantum discord $D_w(A|B) = D_w(B|A) = 0$ $\Rightarrow$ ρ is classical–classical, i.e. ρ has both quantum discords zero $D(A|B) = B(B|A) = 0$,*
2. *ρ is classical–classical $\Rightarrow$ ρ is both classical–quantum $D(B|A) = 0$ and quantum–classical $D(A|B) = 0$,*
3. *ρ is either classical–quantum or quantum–classical $\Rightarrow$ ρ is separable,*
4. *ρ is separable $\Rightarrow$ ρ is unsteerable (for any number of settings),*
5. *ρ is unsteerable with $n + 1$ settings $\Rightarrow$ ρ is unsteerable with n settings,*
6. *ρ is unsteerable (with any number of settings) $\Rightarrow$ ρ is local.*

Proof This theorem follows from Theorem 4.1 and the respective definitions. □

The property of negative/non-negative conditional entropy does not belong to the above hierarchy, because the following theorem holds:

Theorem 4.3 [17] *In general nonlocality does not imply negative conditional entropy and negative conditional entropy does not imply nonlocality.*

We can see that this is true by looking at Table 1 and Fig. 2. For the Werner states there is an implication from negative conditional entropy to nonlocality. However, this is only a special case, as there are nonlocal Gisin states with positive conditional entropy and Gisin states with negative conditional entropy that are local. While negative conditional entropy is not related to nonlocality, it does require entanglement:

Theorem 4.4 [10] *All states with negative conditional entropy are entangled.*

It is known that for pure states some of the analysed properties become equivalent:

Theorem 4.5 *If ρ is a pure bipartite state, the following equivalences hold: ρ is nonlocal $\Leftrightarrow$ ρ is steerable $\Leftrightarrow$ ρ is entangled $\Leftrightarrow$ ρ has negative conditional entropy. Equivalently: ρ is local $\Leftrightarrow$ ρ is not steerable $\Leftrightarrow$ ρ is separable $\Leftrightarrow$ ρ has non-negative conditional entropy.*

The only property, which has not been mentioned yet in the above theorems, is contextuality. It is different from the other properties because its definition does not rely on the division of a physical system into subsystems. In fact, there are contextual states even in 3 dimensions, whereas the other properties are defined for systems of dimension at least 4 (for composite systems the dimension cannot be a prime number). Therefore, for sure contextuality does not collapse to any other property described here. What is more, the set of contextual states does not contain and is not contained in any set of states possessing one of the other properties. However, one may ask what is the relation between contextuality and other properties in spaces where all of them are well-defined and, as far as the author knows, this relation has not been investigated.

5 Absolute Properties of Quantum States

Each of the properties defined in Sect. 3 can be possessed by a given quantum state "absolutely" or "non-absolutely" in the following sense. A property is possessed by a given state *absolutely* iff it is preserved under arbitrary unitary operation on that state; otherwise it is possessed *non-absolutely*. Therefore, the investigation of absolute vs. non-absolute properties reveals an interplay between properties that are constitutive for quantum nature of physical states on the one hand, and the crucial symmetry of quantum theory, namely, unitary symmetry. As we will see, in each pair of properties, usually exactly one of them can be possessed absolutely, for example, there exist states absolutely separable, but there are no states absolutely entangled (the only exception is contextuality and non-contextuality). In this section we will present necessary and sufficient conditions for a given state to possess a given property absolutely (as far as these conditions are known). We will start from observing that only global unitary transformations matter in this context:

Fact 5.1 *For any bipartite state ρ, none of the following properties of ρ: being a product state, entanglement/separability, locality/nonlocality, being classical–classical, being classical–quantum, being quantum–classical can be changed by performing local unitary transformations.*

Proof Observe that some of the properties introduced in Sect. 3 (being a product state, separability, being classical–classical, being classical–quantum and being quantum–classical) are defined by a state possessing a particular form, which cannot be changed by local unitary operations. For illustration, consider a separable state $\rho^{AB} = \sum_{k=1}^{r} p_k \rho_k^A \otimes \rho_k^B$. When applying to it a local unitary operation $U_A \otimes U_B$, we obtain $\rho'^{AB} = (U_A \otimes U_B) \sum_{k=1}^{r} p_k \rho_k^A \otimes \rho_k^B (U_A^\dagger \otimes U_B^\dagger) = \sum_{k=1}^{r} p_k (U_A \rho_k^A U_A^\dagger) \otimes (U_B \rho_k^B U_B^\dagger)$, which is again a separable state. For the same reason, being a product state is preserved. Concerning the preservation of being a classical–classical, classical–quantum or quantum–classical state, we need additionally the fact that any unitary operation applied to a projector gives again a projector.

If a state ρ^{AB} is nonlocal, then it violates the inequality (17), which is then also violated by $U_A \otimes U_B \rho^{AB} U_A \otimes U_B$:

$$
\begin{aligned}
&(2\mathbb{1} - \mathfrak{B}_{\text{CHSH}} | U_A \otimes U_B \rho^{AB} U_A^\dagger \otimes U_B^\dagger)_{HS} \\
&\quad = \operatorname{Tr}\Big(U_A \otimes U_B \rho^{AB} U_A^\dagger \otimes U_B^\dagger)(2\mathbb{1} - \vec{a} \cdot \vec{\sigma}^A \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma}^B - \vec{a}' \cdot \vec{\sigma}^A \\
&\quad \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma}^B)^\dagger \Big) \\
&\quad = \operatorname{Tr}\Big(\rho^{AB} \Big((U_A \otimes U_B)(2\mathbb{1} - \vec{a} \cdot \vec{\sigma}^A \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma}^B - \vec{a}' \cdot \vec{\sigma}^A \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma}^B) \\
&\quad (U_A^\dagger \otimes U_B^\dagger) \Big)^\dagger \Big) \\
&\quad = \operatorname{Tr}\Big(\rho^{AB} \Big((2\mathbb{1} - \vec{a} \cdot (U_A \vec{\sigma}^A U_A^\dagger) \otimes (\vec{b} + \vec{b}') \cdot (U_B \vec{\sigma}^B U_B^\dagger) \\
&\quad - \vec{a}' \cdot (U_A \vec{\sigma}^A U_A^\dagger) \otimes (\vec{b} - \vec{b}') \cdot (U_B \vec{\sigma}^B U_B^\dagger) \Big)^\dagger \Big) \\
&\quad = \operatorname{Tr}\Big(\rho^{AB} \Big((2\mathbb{1} - \vec{a} \cdot \vec{\sigma}^A \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma}^B - \vec{a}' \cdot \vec{\sigma}^A \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma}^B \Big)^\dagger \Big) \\
&\quad = (2\mathbb{1} - \mathfrak{B}_{\text{CHSH}} | \rho^{AB})_{HS} < 0,
\end{aligned}
$$

where the second equality follows from the cyclic property of trace and the fourth follows from the unitary invariance of Pauli matrices. If a state is local, then, by the same reasoning, this property is also not changed by local unitary operations. □

5.1 Absolute Separability

The definitions of separability and entanglement (Definition 3.1) assume a particular choice of a factorisation of the Hilbert space representing a system into the tensor product of Hilbert spaces representing its subsystems. As a consequence, a state that is entangled with respect to a given factorisation, can be separable with respect to another factorisation. Therefore, one can formulate the following definition of absolutely separable states:

Definition 5.1 [35] A bipartite state ρ is called absolutely separable iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ is separable.

The following three theorems describe the conditions under which a quantum state is absolutely separable:

Theorem 5.1 [56] *Any separable pure state can be transformed by a unitary operation into an entangled state and the other way around. Therefore, no pure separable states are absolutely separable.*

Theorem 5.2 [57] *If ρ is a mixed two-qubit state with an ordered spectrum $d_1 \geq d_2 \geq d_3 \geq d_4$, then ρ is absolutely separable iff*

$$d_1 - d_3 - 2\sqrt{d_2 d_4} \leq 0. \tag{21}$$

The next theorem provides a generalisation of this result for a higher dimension of the second subsystem, where the first subsystem remains 2-dimensional:

Theorem 5.3 [23] *for* 2×3 *case*, [30] *for the remaining cases) If ρ is a bipartite state of dimension $2 \times n$ (for arbitrary n) with an ordered spectrum $d_1 \geq d_2 \geq \cdots \geq d_{2n}$, then ρ is absolutely separable iff*

$$d_1 - d_{2n-1} - 2\sqrt{d_{2n-2} d_{2n}} \leq 0. \tag{22}$$

As we have seen, there is an asymmetry between separability and entanglement. Any state is separable in some basis (so there are no absolutely entangled states), but there are some states that are absolutely separable.

5.2 Absolute Locality

As in the previous case, one can formulate the following definition of an absolute version of the property of locality:

Definition 5.2 A bipartite state ρ is called absolutely local iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ is local.

The necessary and sufficient conditions for a state being absolutely local have been found:

Theorem 5.4 [18] *If ρ is a two-qubit state with an ordered spectrum $d_1 \geq d_2 \geq d_3 \geq d_4$, then ρ is absolutely local iff*

$$(2d_1 + 2d_2 - 1)^2 + (2d_1 + 2d_3 - 1)^2 \leq 1. \tag{23}$$

In the proof of this theorem the following lemma has been used (it will be also needed in our proof in Sect. 5.5):

Lemma 5.1 (Cartan decomposition of $SU(4)$, [34]) *Every matrix belonging to $SU(4)$ can be decomposed into two local unitary matrices $U_A \otimes U_B, V_A \otimes V_B$ and a global unitary matrix U_g in the so-called Cartan form:*

$$U = \left(U_A \otimes U_B\right) U_g \left(V_A \otimes V_B\right), \tag{24}$$

where $U_A, U_B, V_A, V_B \in SU(2)$ and U_g is given by

$$U_g = e^{-i(\lambda_1 \sigma_1 \otimes \sigma_1 + \lambda_2 \sigma_2 \otimes \sigma_2 + \lambda_3 \sigma_3 \otimes \sigma_3)} \tag{25}$$

with $\lambda_1, \lambda_2, \lambda_3 \in [0, 2\pi]$.

5.3 Absolutely Unsteerable States

Similarly to absolute locality, one can define absolute unsteerability:

Definition 5.3 A bipartite state ρ is called absolutely unsteerable iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ is unsteerable.

The necessary and sufficient conditions for a state being absolutely unsteerable are not known. However, there is in the literature the following partial result, concerning only steerability with three settings:

Theorem 5.5 [7] *If ρ is a two-qubit state with spectrum $d_1 d_2, d_3, d_4$, then ρ is absolutely unsteerable with three settings iff its eigenvalues satisfy*

$$3(d_1^2 + d_2^2 + d_3^2 + d_4^2) - 2(d_1 d_2 + d_1 d_3 + d_1 d_4 + d_2 d_3 + d_2 d_4 + d_3 d_4) \leq 1. \tag{26}$$

5.4 Absolute Non-negativity of Conditional Entropy

Similarly to the previous properties, non-negative conditional entropy can also be possessed absolutely:

Definition 5.4 [47] A bipartite state ρ is said to have absolutely non-negative conditional entropy iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ has non-negative conditional entropy.

The necessary and sufficient conditions for absolute non-negativity of conditional entropy are known:

Theorem 5.6 [47] *A two-qubit state ρ has absolutely non-negative conditional entropy iff $S(\rho) \geq 1$.*

5.5 Absolute Zero Quantum Discord

In contrast to the properties analysed before, quantum discord is quantitative, that is, to a given quantum state there is assigned its numerical value. To formulate an analogous problem as before, one can divide states into discordless (with quantum discord equal to zero) and states with non-zero quantum discord. Then one can ask in which cases the value of quantum discord is absolutely zero:

Definition 5.5 A bipartite state ρ is said to have zero quantum discord absolutely iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ has zero quantum discord.

In this section we will use Criterion 3.4 [28] of zero quantum discord to determine the necessary and sufficient conditions for a two-qubit state to have zero quantum discord absolutely. For a two-qubit system represented by a density matrix

$$\rho = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{27}$$

with blocks

$$A = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \quad B = \begin{pmatrix} r_{13} & r_{14} \\ r_{23} & r_{24} \end{pmatrix}, \quad C = \begin{pmatrix} r_{31} & r_{32} \\ r_{41} & r_{42} \end{pmatrix}, \quad D = \begin{pmatrix} r_{33} & r_{34} \\ r_{43} & r_{44} \end{pmatrix} \tag{28}$$

this criterion means that the following equalities must be satisfied:

$$[A, A^\dagger] = [B, B^\dagger] = [C, C^\dagger] = [D, D^\dagger] = 0 \tag{29}$$

and

$$[A, B] = [A, C] = [A, D] = [B, C] = [B, D] = [C, D] = 0. \tag{30}$$

The first simplification follows from the fact that every density matrix is hermitian and therefore can be diagonalized by some unitary matrix. If a given state has zero discord absolutely, in particular it has zero discord after diagonalization (because diagonalizing matrix belongs to the class of unitary matrices). Therefore, each equivalence class of states has a representative that is a diagonal matrix and to find the class of all states with absolute zero discord it suffices to restrict to the class of diagonal density matrices.

Recall from Sect. 5.2 that every $SU(4)$ matrix can be decomposed into a local part and a special global matrix U_g given by (25). One idea is to act with the global unitary matrix (24) in its most general form and then solve the equations that follow from the conditions (29) and (30). However, these equations are rather complicated, so from the practical point of view it is better to divide our task into two steps. In the first step, we will act on arbitrary diagonal density matrix with

U_g only, obtaining necessary conditions for having zero discord absolutely. As we will see, the result will be a one-parameter family of states. In the second step, we will act on this family with U in its general form (including local parts and the special global part). It turns out that it is possible to make a further simplification by putting some parameters in local matrices to zero and such a less general form is sufficient to eliminate all potential candidates for being absolutely zero discord state with one exception—the maximally mixed state $\frac{1}{4}\mathbb{1}_4$. It is easy to see that this state indeed has zero discord absolutely because after an arbitrary unitary transformation it remains unchanged.

Let us perform the first step. Consider an arbitrary diagonal density matrix

$$\rho_d = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 1 - d_1 - d_2 - d_3 \end{pmatrix}, \tag{31}$$

where $d_1, d_2, d_3 \in \mathbb{R}$. The unitary matrix U_g can be written in the Cartan form (25), which simplifies to

$$U_g = \begin{pmatrix} e^{-i\lambda_3}\cos(\lambda_1 - \lambda_2) & 0 & 0 & -ie^{-i\lambda_3}\sin(\lambda_1 - \lambda_2) \\ 0 & e^{i\lambda_3}\cos(\lambda_1 + \lambda_2) & \sin(\lambda_1 + \lambda_2)(\sin\lambda_3 - i\cos\lambda_3) & 0 \\ 0 & \sin(\lambda_1 + \lambda_2)(\sin\lambda_3 - i\cos\lambda_3) & e^{i\lambda_3}\cos(\lambda_1 + \lambda_2) & 0 \\ -ie^{-i\lambda_3}\sin(\lambda_1 - \lambda_2) & 0 & 0 & e^{-i\lambda_3}\cos(\lambda_1 - \lambda_2) \end{pmatrix}. \tag{32}$$

Under the action of U_g the state ρ_d is transformed as follows:

$$\begin{aligned} \rho'_d &= U_g^\dagger \rho_d U_g \\ &= \frac{1}{2}\begin{pmatrix} (2d_1 + d_2 + d_3 - 1)C_- - d_2 - d_3 + 1 & 0 & 0 & i(2d_1 + d_2 + d_3 - 1)S_- \\ 0 & (d_2 - d_3)C_+ + d_2 + d_3 & i(d_2 - d_3)S_+ & 0 \\ 0 & -i(d_2 - d_3)S_+ & (d_2 - d_3)C_+ + d_2 + d_3 & 0 \\ -i(2d_1 + d_2 + d_3 - 1)S_- & 0 & 0 & -(2d_1 + d_2 + d_3 - 1)C_- - d_2 - d_3 + 1 \end{pmatrix}, \end{aligned} \tag{33}$$

where $S_+ = \sin(2\lambda_1 + 2\lambda_2)$, $S_- = \sin(2\lambda_1 + 2\lambda_2)$, $C_+ = \cos(2\lambda_1 + 2\lambda_2)$, $C_- = \cos(2\lambda_1 - 2\lambda_2)$.

To this transformed state ρ'_d we apply conditions (29) and (30). Three of them are always satisfied: $[A, A^\dagger] = [D, D^\dagger] = [A, D] = 0$. The rest gives us equations for eigenvalues of ρ_d, which have the following solutions: $d_1 = d_2 = d_3 = \frac{1}{4}$ and $d_1 = \frac{1}{2} - d_2, d_3 = d_2$. This gives us the following necessary condition: If a two-qubit state with eigenvalues $d_1, d_2, d_3, 1 - d_1 - d_2 - d_3$ has zero discord absolutely, then its eigenvalues satisfy the following relation:

$$d_1 = \frac{1}{2} - d_2, d_3 = d_2 \tag{34}$$

or some of its permutations.

Now, let us perform the second step. Any unitary matrix belonging to $SU(2)$ can be parameterised in the following way:

$$U_{\text{loc}} = \begin{pmatrix} e^{i\alpha}\cos\phi & e^{i\beta}\sin\phi \\ -e^{-i\beta}\sin\phi & e^{-i\alpha}\cos\phi \end{pmatrix}. \tag{35}$$

Each of the matrices $U_{\text{A}}, U_{\text{B}}, V_{\text{A}}, V_{\text{B}}$ has this form, so the whole matrix U given by (24) contains four independent matrices of the type (35). We will add to each parameter α, β, ϕ indices associated with matrices $U_{\text{A}}, U_{\text{B}}, V_{\text{A}}, V_{\text{B}}$, so, for example,

$$U_{\text{A}} = \begin{pmatrix} e^{i\alpha_{\text{UA}}}\cos\phi_{\text{UA}} & e^{i\beta_{\text{UA}}}\sin\phi_{\text{UA}} \\ -e^{-i\beta_{\text{UA}}}\sin\phi_{\text{UA}} & e^{-i\alpha_{\text{UA}}}\cos\phi_{\text{UA}} \end{pmatrix} \tag{36}$$

and similarly for $U_{\text{B}}, V_{\text{A}}$ and V_{B}.

Consider the case $\alpha_{\text{UA}} = \alpha_{\text{UB}} = \alpha_{\text{VA}} = \alpha_{\text{VB}} = \beta_{\text{UA}} = \beta_{\text{UB}} = \beta_{\text{VA}} = \beta_{\text{VB}} = \phi_{\text{UB}} = \phi_{\text{VA}} = 0$ (only ϕ_{UA} and ϕ_{VB} are non-zero). We apply a transformation of this type to our state (31) satisfying (34). From the conditions (29) and (30) we again obtain the set of equations constraining d_2, the only solution of which is the state $\frac{1}{4}\mathbb{1}_4$. This means that we do not need to consider a more general U (with all parameters potentially non-zero), because our special case already restricts the class of 4×4 density matrices to one element, of which we know that it has zero discord absolutely. Therefore the following theorem holds:

Theorem 5.7 *The only two-qubit state that has zero discord absolutely is the maximally mixed state* $\frac{1}{4}\mathbb{1}_4$.

Taking into account Theorem 3.4, one can define absolute versions of being classical–classical, classical–quantum and quantum–classical:

Definition 5.6 A bipartite state ρ is called absolutely classical–classical/classical–quantum/quantum–classical iff for any unitary operator U, the state $\rho' = U\rho U^{\dagger}$ is classical–classical/classical–quantum/quantum–classical, respectively.

The criteria for belonging to these classes of states are the same as for having vanishing discord: for absolutely quantum–classical states $D(A|B)$ must vanish absolutely, for absolutely classical–quantum states $D(B|A)$ must vanish absolutely and for absolutely classical–classical states both of these conditions must be satisfied. Of course, as a corollary to the previous theorem, the following holds for two-qubit states:

Theorem 5.8 *The only two-qubit state that is absolutely classical–classical is the maximally mixed state* $\frac{1}{4}\mathbb{1}_4$.

The only two-qubit state that is absolutely classical–quantum is the maximally mixed state $\frac{1}{4}\mathbb{1}_4$.

The only two-qubit state that is absolutely quantum–classical is the maximally mixed state $\frac{1}{4}\mathbb{1}_4$.

5.6 Absolute Zero Quantum Super Discord

Similarly to the case of quantum discord, we can define absolute version of having zero quantum super discord:

Definition 5.7 A bipartite state ρ is said to have zero quantum super discord absolutely iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ has zero quantum super discord.

Theorem 3.6 [39] implies that having zero super discord absolutely is equivalent to being absolutely product, where the last property is defined as follows:

Definition 5.8 A bipartite state ρ is said to be an absolutely product state iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ is a product state.

From Theorems 3.5 [52] and 5.7 we can conclude that

Theorem 5.9 *The only two-qubit state that has zero quantum super discord absolutely is the maximally mixed state* $\frac{1}{4}\mathbb{1}_4$.

From the relation between zero super discord and being a product state it follows, as a corollary, that

Theorem 5.10 *The only two-qubit state that is absolutely product is the maximally mixed state* $\frac{1}{4}\mathbb{1}_4$.

5.7 Absolute Contextuality and Noncontextuality

Contextuality and non-contextuality is the only pair of properties analysed here such that both elements of the pair have non-trivial absolute counterparts.

Definition 5.9 A bipartite state ρ is called absolutely contextual/absolutely noncontextual iff for any unitary operator U, the state $\rho' = U\rho U^\dagger$ is contextual/noncontextual, respectively.

Contextuality and noncontextuality are different from other properties considered in this paper because they do not refer to the division of system into subsystems. Therefore, we can expect that the distinction between local and global unitary operations does not matter for preserving these properties. In fact, one can prove an even stronger result:

Theorem 5.11 *Contextuality and noncontextuality are always absolute, that is, if a given state* ρ *(of arbitrary dimensionality) is contextual, then it is also absolutely contextual and if it is noncontextual, then it is also absolutely noncontextual.*

Table 2 Selected absolute properties of the Werner states and the Gisin states in function of their parameters

	Werner states	Gisin states	
Parameters	w	θ	λ
Range of the parameter	$w \in [-\frac{1}{3}, 1]$	$\theta \in [0, \frac{\pi}{2}]$	$\lambda \in [0, 1]$
Absolutely product state	$w = 0$	Never	Never
Absolutely zero discord	$w = 0$	Never	Never
Absolutely separable	$w \in [-\frac{1}{3}, \frac{1}{3}]$	Never	Never
Absolutely unsteerable	$w \in [-\frac{1}{3}, \frac{1}{\sqrt{3}}]$	θ arbitrary	$\lambda \in [0, \frac{2}{3}]$
Absolutely local	$w \in [-\frac{1}{3}, \frac{1}{\sqrt{2}}]$	θ arbitrary	$\lambda \in [0, \frac{1}{\sqrt{2}}]$
Absolutely non-negative conditional entropy	$w \in [-\frac{1}{3}, w_1]$, $w_1 \approx 0.7476$	θ arbitrary	$\lambda \in [0, \lambda_1]$, $\lambda_1 \approx 0.7729$

Proof Suppose that for a given ρ there exist projectors P_i, $i = 1, \ldots, 5$ such that $P_i P_{i+1} = 0$ and $\mathrm{Tr}(\rho P) > 2$, where $\mathfrak{P} = \sum_{i=1}^{5} P_i$. Consider the rotation of ρ by an arbitrary unitary matrix U: $\rho' = U\rho U^\dagger$. It can be shown that for this new state there also exist projectors that are witnesses for the violation of the KCBS inequality: it suffices to take $P'_i = UP_iU^\dagger$, whose sum is $\mathfrak{P}' = \sum_{i=1}^{5} P'_i = U\mathfrak{P}U^\dagger$. These projectors satisfy the condition of orthogonality $P'_i P'_{i+1} = UP_iU^\dagger UP_{i+1}U^\dagger = UP_iP_{i+1}U^\dagger = 0$. For this choice of projectors, the state ρ violates the KCBS inequality: $\mathrm{Tr}(\rho'\mathfrak{P}') = \mathrm{Tr}(U\rho U^\dagger U\mathfrak{P}U^\dagger) = \mathrm{Tr}(U\rho\mathfrak{P}U^\dagger) = \mathrm{Tr}(U^\dagger U\rho\mathfrak{P}) = \mathrm{Tr}(\rho\mathfrak{P}) > 2$. □

One may wonder why do we not have a similar argument for nonlocality, as both contextuality and nonlocality consist of violation of a certain inequality for some choice of an appropriate operator ($\mathfrak{B}_{CHSH}$ or $\mathfrak{P}$, respectively). The difference lies in the dissimilarity of the criteria imposed on these operators. The CHSH operator must have a certain structure given by Eq. (16). This structure can be spoilt by global unitary operations, whereas, as we have seen, the conditions defining $\mathfrak{P}$ are still satisfied after an arbitrary unitary operation.

6 Relations Between Different Absolute Properties

6.1 Relations for Special Classes of States

In Table 2 there are shown ranges of parameters for which the Werner states (8) and the Gisin states (9) are absolutely product states, have zero quantum discord absolutely, are absolutely separable, absolutely unsteerable, absolutely local and have non-negative conditional entropy absolutely. These results have been obtained with the use of the following Theorems: 5.2 [57], 5.4 [18], 5.5 [7], 5.6 [47], 5.7 and 5.9. Note that for the Gisin states the parameter θ does not matter in this context; this is because possessing absolute properties depends only on the eigenvalues of the state and for the Gisin states the eigenvalues depend only on the parameter λ

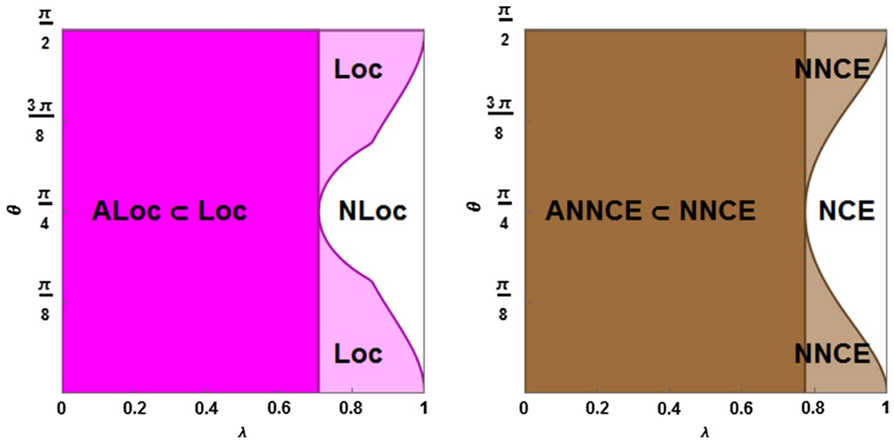


Fig. 5 Properties of the Gisin states. Left figure: *ALoc* absolutely local states (dark purple), *Loc* local states (dark purple and light purple), *NLoc* nonlocal states (white). Right figure: *ANNCE* absolutely non-negative conditional entropy (dark brown), *NNCE* non-negative conditional entropy (light brown and dark brown), *NCE* negative conditional entropy (white) (Color figure online)

(see Sect. 2.6.2). Most of these numerical results have already been presented in the literature: [18] (absolute separability and absolute locality for the Werner states, absolute locality for the Gisin states), [47] (absolutely non-negative conditional entropy for the Werner states), [7] (absolute unsteerability for the Werner states and the Gisin states).

When we compare these results with Table 1 as well as with Fig. 1 and 2, we can observe that for the Werner states there is no difference between possessing a given property and possessing a given property absolutely. In contrast, for the Gisin states the ranges of parameters are changed in all the cases.[3] Therefore, the equivalence between the properties and the respective absolute properties holds only for very special families of states (such as the Werner states) and in general is not true. The comparison between the properties and the respective absolute properties for Gisin states is illustrated in Fig. 5.

The results for the Werner states and the Gisin states allow one to distinguish between almost all of the absolute properties analysed here: only being an absolutely product state and having zero quantum discord absolutely are impossible to distinguish (and as we have seen in Sect. 5.6, in fact they are generally equivalent).

[3] We do not know the ranges of the parameters for which the Gisin states are unsteerable. However, we know that the set of unsteerable states must contain the set of separable states (Theorem 4.2) and that the set of absolutely unsteerable states must be contained in the set of absolutely local states (Theorem 6.2). As a consequence, if the sets of Gisin unsteerable states and of Gisin absolutely unsteerable states were the same, the set of separable Gisin states would be contained in the set of absolutely local Gisin states. As there are Gisin states that are separable but not absolutely local, the sets of unsteerable states and absolutely unsteerable states are not the same.

6.2 General Relations for Bipartite States

The relations between absolute versions of the properties are very much similar to the relations between “ordinary” versions of these properties, as summarised in the following theorem:

Theorem 6.1 *Assume that for any quantum state possessing the property A implies possessing the property B. It follows that if a given state ρ has the property A absolutely, then it also has the property B absolutely.*

Proof Assume that ρ has the property A absolutely and that for any quantum state, possessing the property A implies possessing the property B. Let us transform the state ρ by some unitary operator U, obtaining $\rho' = U\rho U^\dagger$. Because ρ has the property A absolutely, ρ' must have the property A. Therefore, from the implication, ρ' must have the property B. As the unitary operator U was arbitrary, it follows that ρ has the property B absolutely. □

From Theorem 6.1 it follows that the analogue of Theorem 4.2 holds for absolute versions of the respective properties (see Fig. 6):

Theorem 6.2 *For any bipartite state ρ, the following implications hold:*

1. *ρ is a product state absolutely, i.e. ρ has zero super quantum discord $D_w(A|B) = D_w(B|A) = 0$ absolutely $\Rightarrow$ ρ is absolutely classical–classical, i.e. ρ has both quantum discords zero $D(A|B) = B(B|A) = 0$ absolutely,*
2. *ρ is absolutely classical–classical $\Rightarrow$ ρ is both absolutely classical–quantum $D(B|A) = 0$ and absolutely quantum–classical $D(A|B) = 0$,*
3. *ρ is either absolutely classical–quantum or absolutely quantum–classical $\Rightarrow$ ρ is absolutely separable,*
4. *ρ is absolutely separable $\Rightarrow$ ρ is absolutely unsteerable (with any number of settings),*
5. *ρ is absolutely unsteerable with $n + 1$ settings $\Rightarrow$ ρ is absolutely unsteerable with n settings,*
6. *ρ is absolutely unsteerable (with any number of settings) $\Rightarrow$ ρ is absolutely local.*

The implication from absolute separability to absolute locality has been already noted in Roy et al. [49] and Ganguly et al. [18]. The relation between the above properties and absolute non-negativity of conditional entropy is less understood. Its relation with absolute separability is known:

Theorem 6.3 [47] *The class of absolutely separable two-qubit states forms a proper subset of the class of two-qubit states that have non-negative conditional entropy absolutely.*

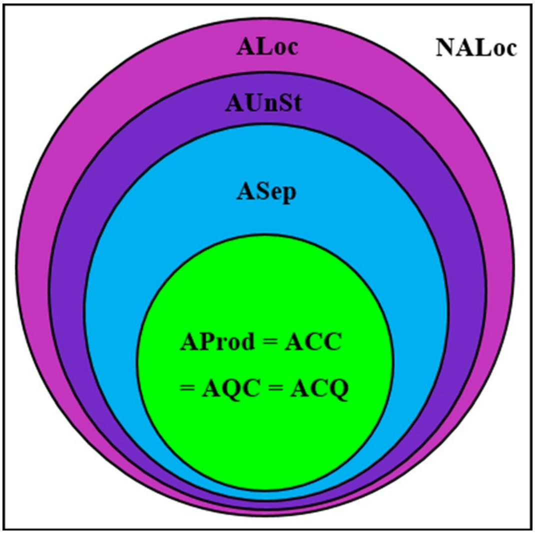


Fig. 6 Relations between different absolute properties: *AProd* absolutely product states, *ACC* absolutely classical–classical states (with both discords equal zero absolutely), *ACQ* absolutely classical–quantum states, *AQC* absolutely quantum–classical states, *ASep* absolutely separable states, *AUnSt* absolutely unsteerable states, *ALoc* absolutely local states. Note that the green circle denotes only one point ($\frac{1}{4}\mathbb{1}_4$), so the figure is out of scale

Being a subset follows from the Theorem 6.1. Being a proper subset follows from the fact, that there exist two-qubit states that have non-negative conditional entropy absolutely but are not absolutely separable. Examples of such states are Gisin states for $\lambda \in [0, \lambda_1]$, $\lambda_1 \approx 0.7729$ and arbitrary θ (see Table 2).

The relation between absolutely local two-qubit states and two-qubit states that have non-negative conditional entropy absolutely is in general not known. For the Werner states absolute separability implies absolute non-negative conditional entropy, as the Werner states are absolutely separable for $w \leq \frac{1}{3}$ and have non-negative conditional entropy absolutely for $w \leq w_e$, where w_e is the solution of the equation $3(1 - w_e)\log(1 - w_e) + (1 + 3w_e)\log(1 + 3w_e) = 4$ and its numerical value is $w_e \approx 0.7476$ [47].

7 The Meaning of "Classical" as Opposed to "Quantum"

In the introduction I have made an ambiguous claim that quantum mechanics differs when compared with classical physics. The ambiguity comes from the fact that "classical physics" can be given at least three different meanings in this context: (non-relativistic[4]) classical mechanics, classical statistical mechanics, or classical probability theory (which is not, strictly speaking, a physical theory, but rather a broader mathematical framework that may be applied in various fields). Let us discuss which of these options fits best the analyses of this paper, that is, to which theory belong the "classical states", compared to which our quantum states reveal some non-classical properties.

4 Because we are only dealing with non-relativistic quantum mechanics here.

Of course, for any of the mentioned classical theories, one can find properties that are possessed by quantum states analysed in this paper, but are not possessed by the states of that given theory. Therefore, at the purely formal level, the states of any of these theories can be compared to quantum states analysed here. However, it seems to me that this is not the most natural reading of the claim that a quantum theory reveals some specifically quantum effects that are absent at the classical level. Such a claim seems to make most sense if one considers the same system or the same phenomenon and two different descriptions of it: one classical and one quantum. Only in such a situation quantum theory and classical theory can be regarded as saying different things about the same subject. In the light of this observation, I understand the question posed at the end of the previous paragraph as asking something like the following: which of the classical theories is such that the states of this theory can be used to describe physical systems of some type and the quantum states analysed in this paper can also be used to describe physical systems of the same type (in the same respects), and using the latter description provides us with predictions concerning these systems that are specifically quantum and cannot be obtained by using the former description. With respect to these classical states our quantum states can be said to reveal some non-classical properties and they will be called classical counterparts of our quantum states. The non-classical properties, to recall, include the properties analysed in this paper, that is: entanglement, nonlocality, steerability, negative conditional entropy, non-zero quantum discord, non-zero quantum super discord and contextuality. Because the results reviewed and obtained in this paper are not easily generalizable, I will consider the question concerning classical counterparts only with respect to the states explicitly analysed here, namely, quantum states belonging to finitely-dimensional Hilbert spaces.

Let us start with the first of the mentioned options. Quantum mechanics is typically compared with classical mechanics. In this approach, the main difference between the quantum and classical case is that classical states determine the results of measurements of physical quantities (observables) uniquely, whereas quantum states determine only probabilities of measurements results, which for many choices of a quantum state and an observable are non-trivial, that is, different from 0 and 1. This approach is present in various comparisons between quantum and classical, as well as in many analyses of quantum–classical correspondence (for example, the one based on the Ehrenfest theorem, which relates quantum expectation values to the predictions of classical mechanics). This seems to be the most common point of view and may be found in many textbooks (e.g. [54]), as well as in popular presentations of the subject.

Among the properties analysed here, entanglement is perhaps regarded as non-classical mostly in the sense of being absent in classical mechanics. The phenomenon of entanglement is possible because of the fact that the quantum state space for a system composed of two or more subsystems is the tensor product of the state spaces for its subsystems. In classical mechanics, the state space for a composite system is a Cartesian product of the state spaces for its subsystems, so the phenomenon of entanglement cannot occur (all states are product states).

Another approach is to compare quantum mechanics with classical statistical mechanics rather than with classical mechanics. Among the classics of quantum mechanics, Max Born expressed the conviction that this is a more adequate point of view:

> It is misleading to compare quantum mechanics with the deterministically formulated classical mechanics; instead, one should first reformulate the classical theory, even for a single particle, in an indeterministic, statistical manner. After that some of the distinctions between the two theories disappear, [while] others emerge with great clarity. (Born, quoted in Mehra and Rechenberg [42, p. 1200])

Among modern textbooks, this point of view is explored, for example, by Ballentine [3, pp. 388–405] and Landsman [36]. Indeed, classical statistical mechanics is more similar to quantum mechanics in that both theories are probabilistic, whereas classical mechanics is not. The states in classical statistical mechanics are probability distributions over states in the sense of classical mechanics. This enables to make comparisons between classical and quantum states that do not make sense when on the "classical" side one chooses classical mechanics: for example, those concerning entropy of the states. Entropy provides us with the information about the uncertainty concerning the system, given its state. If there is no such uncertainty (which is the case in classical mechanics), the entropy is 0 (or, alternatively, the states of classical mechanics, being non-probabilistic objects, do not have well-defined entropy at all). In both quantum mechanics and classical statistical mechanics there are states with non-trivial entropies, so it makes more sense to compare these two theories in this respect.

This second approach (comparing quantum mechanics with classical statistical mechanics) seems to be more promising in our case (because many of our comparisons involve probability-based concepts), but it is also not fully adequate. There are some general interpretative problems (which will be only indicated here) and some problems specific for the issues discussed here (which are more important from the point of view of this paper).

Let us start with the former group of problems. First, the states of classical statistical mechanics are often interpreted as describing an ensemble of systems, whereas the states of quantum mechanics are interpreted as representing a single system. If this is correct, then the comparison of quantum mechanics states with classical statistical mechanics states would make no sense (unless we are interested only in purely formal properties of the theories, in abstraction from their physical meaning), because they do not represent the same objects, so they cannot be regarded as saying different things about the same physical phenomena. Some authors, however, avoid this discrepancy by treating quantum states also as describing ensembles. For example, Sakurai in his textbook ([51], p. 24) writes: "to determine probability [of a particular measurement result on a given state] empirically, we must consider a great number of measurements performed on an ensemble—that is, a collection of identically prepared physical systems". This observation leads him to the conclusion that quantum states (both pure and mixed) represent ensembles rather than single (one- or multi-particle) systems. On the other hand, the proponents of Boltzmannian (as opposed to Gibbsian) approach to statistical mechanics interpret this theory as being about individual systems rather than ensembles of systems (see e.g. [21]). If one of these approaches is correct, then quantum and classical theory can be claimed to describe physical systems of the same type and our problem will be resolved, but this issue is far from being non-controversially established.

Second, probabilities of classical statistical mechanics are commonly interpreted as accounting for our lack of knowledge about the exact state of the system, whereas probabilities of quantum mechanics are usually thought of to be an objective feature of the world, revealing the indeterministic nature of quantum processes. Again, if this is correct, then the comparison of states of these theories (in our sense of comparison, explained at the beginning of this section) would make no sense, because the states of the two theories represent something ontologically entirely different. However, both these claims have been challenged, that is, there are authors who interpret probabilities in classical statistical mechanics in an objective manner (e.g. [22, 37]) as well as authors who interpret quantum probabilities in an epistemic way (such as the proponents of Quantum Bayesianism, see e.g. Baeyer [4])[5]; see also Wallace [58, pp. 211–213] for the claim that the controversy concerning the meaning of probabilities in classical statistical mechanics should be resolved with the reference to quantum mechanics and McCoy [41] for the similarities in interpretative problems of quantum mechanics and classical statistical mechanics. Therefore, we have some potential ways out of our second problem, but, as previously, the issue is highly debatable. The two mentioned interpretative problems, although perplexing, do not settle the issue against what I called here the second approach (namely, treating the states of classical statistical mechanics as the classical counterparts of quantum states, compared to which the latter reveal some non-classical properties), although much more would need to be said for its defense.

More importantly for the context of this paper, there are some problems specific to the class of states under consideration here, which are all associated with finite-dimensional Hilbert spaces. (These are not problems of a general importance, but only obstacles for treating the states of classical statistical mechanics as classical counterprats of these particular quantum states.) The state space of classical mechanics has the cardinality of the set of real numbers, so the states of classical statistical mechanics built upon it must be continuous probability distributions, which cannot be represented as probability vectors (cf. the beginning Sect. 2.4).[6] Such probability distributions form an infinitely-dimensional space. However, this paper is restricted to quantum states that belong to finite-dimensional Hilbert spaces and it is difficult to generalize its results to infinite-dimensional cases (this would

[5] The main thesis of Quantum Bayesianism, also called QBism, as formulated by Baeyer [4, p. 131] is as follows: "The principal thesis of QBism is simply this: quantum probabilities are numerical measures of personal degrees of belief." (Perhaps to be more precise, one should replace in this formulation "degrees of belief" by "rational degrees of belief".) This does not automatically entail that the "real" state of the physical system is itself determinate and only unknown to us; but the main point is that the states of the theory represent states our knowledge, not objective states of the physical system.

[6] By "states" of classical statistical mechanics I do not mean here the states of the underlying classical–mechanical description (which are finite-dimensional, namely, $6N$-dimensional for an N-particle system), but continuous probability distributions over these states (which themselves belong to an infinite-dimensional space). This is because it is these probability distributions, not classical–mechanical states, that are considered here to be (candidate) counterparts of quantum-mechanical states.

require changing even the definitions of the basic notions we are using, such as separability and entanglement, see e.g. [27, pp. 917–921]). Therefore, there is a difference in dimensionality here: quantum states considered in this paper belong to finite-dimensional Hilbert spaces, whereas states of classical statistical mechanics belong to infinite-dimensional spaces. Because of this difference in dimensionality, quantum states associated with finite-dimensional Hilbert spaces cannot be straightforwardly compared with states of classical statistical mechanics. Therefore, the states of classical statistical mechanics cannot be regarded as a classical counterpart of quantum states belonging to finite-dimensional Hilbert spaces (i.e., these quantum states cannot be conceived as more refined descriptions of the same physical phenomena as are described by these classical statistical states).

This is less surprising if one notices that what these quantum states of finite dimension represent are particles with spin, which is a physical quantity that does not appear in classical physics at all. To represent positions and momenta, which have classical counterparts (and which are what classical mechanics, as well as classical statistical mechanics deals with), one needs to move to an infinite-dimensional Hilbert space. Therefore, even if finite-dimensional states of quantum mechanics and the states of classical statistical mechanics (i.e., probability distributions) describe the same physical systems, they describe them in different respects, as they take into account different physical quantities.

Our last option is that quantum states of finite-dimensional Hilbert spaces can be compared with classical discrete probability distributions and it seems that indeed this is the approach chosen more or less explicitly by many authors in the field. For example, in quantum information literature, where classical and quantum information is compared, it is assumed that the classical counterpart of a (finite-dimensional) quantum state is a classical (discrete) probability distribution, because classical information theory is formulated in terms of such distributions (see e.g. [44, pp. 500–527]).

Our third approach, contrasting (finite-dimensional) quantum mechanics with classical probability theory, seems to fit best some of the analyses of this paper, but still there are some complications. The comparison of entropies of quantum and classical states (Sect. 3.4) indeed assumes about the latter only their agreement with classical probability theory; the same concerns the notion of quantum discord, which is defined in terms of entropy. The properties of being classical–classical, classical–quantum and quantum–classical (Sect. 3.6) capture the extent to which a quantum state can or cannot be encoded by a classical (discrete) probability distribution, so they also fit ideally with the third approach.

To see more clearly why comparing our quantum states with discrete classical probability distributions makes more sense than comparing them with the states of classical mechanics (at least for some non-classical properties on our list), consider the case of negative conditional entropy. It is called a non-classical property of quantum states not because the states of classical mechanics, being non-probabilistic objects, have zero or ill-defined entropy, but because classical probability vectors have always non-negative conditional entropy. Therefore, when we analyse the meaning of the claim "negative conditional entropy is a non-classical property", by "classical" in the expression "non-classical" we cannot mean "belonging to/being related to classical mechanics", but "belonging to/being related to classical probability theory". In contrast, as observed earlier, when we analyse the meaning of the claim "entanglement is a non-classical property", by "classical" in the expression "non-classical" we can reasonably mean (and in fact, we usually mean) "belonging to/being related to classical mechanics". This is because the concept of a product state could be defined for classical mechanics and it can be shown that all its compiste states are product states, which implies that they are not entangled.

One may object that discrete classical probability distributions do not describe particles with spin, so they also do not to satisfy our conditions for being a classical counterpart of finite-dimensional quantum states and we should simply conclude that there is no such counterpart. A possible answer is that the common type of physical systems that could be described by both classes of states are information-carrying systems. This is not in conflict with the claim that our quantum states describe particles with spin, because any actual physical system belongs to different types under different descriptions and a group of particles with spin can be considered as an information-carrying system. Quantum mechanics, from this point of view, reveals non-classical properties of information-carrying systems that are absent in classical information theory that is based on classical probability theory.

What about other properties on our list? Are they also non-classical in the sense of being not allowed by classical probability theory? The definitions of nonlocality and steerability also refer to the concept of classical probability distribution, but they additionally use some independence conditions (indicated by the word "local" in the phrase "local hidden variable model"), whose motivation comes from physical assumptions rather than from classical probability theory itself. A similar situation is with contextuality, which is defined by the non-existence of hidden variable models of a different type. Therefore, in the case of these three properties (nonlocality, steerability, and noncontextuality), their non-classicality is not merely a deviation from what is allowed by classical probability theory, but from the latter supplemented by some additional assumptions. And if one were to look for the justification of the assumption of locality in some theory of classical physics, this would be relativistic classical mechanics rather than non-relativistic classical mechanics, as the latter does not forbid nonlocal phenomena, which makes the issue even more complicated. Therefore, when we analyse the meaning of the claim "nonlocality is a non-classical property", the term "classical" is related to a combination of classical probability theory and some principles of relativistic provenience (i.e., a rather hybrid object, not belonging to any theory actually used in physics).

We need to conclude that various properties of quantum states labelled in the literature as "non-classical" are "non-classical" in not exactly the same sense. Quantum states may be compared with states of different classical theories and the conclusions vary depending on the particular choice. There is no single classical theory such that its states can be used to describe some physical system, which can also be described (in the same respects) by quantum states analysed in this paper and relative to which these quantum states reveal all the non-classical properties listed in this paper. Classical mechanics is not a good candidate, because some of these properties (namely, those based on probabilistic concepts) do not make sense within it. Classical statistical mechanics is also not a good candidate, because its probability distributions are infinitely-dimensional, whereas our quantum states are finitely dimensional and, relatedly, they describe different phenomena (classical statistical mechanics concerns positions and momenta, which can be captured only be means of infinitely-dimensional Hilbert spaces, whereas finitely-dimensional Hilber spaces analysed here describe spins). Finally, classical probability theory seems to be the best candidate in our case, because its states are probabilistic and some of them are finite-dimensional. However, it is not a physical theory, strictly speaking, and it is our point of comparison only in the case of some properties (e.g. negative conditional entropy). For some other properties (e.g. nonlocality and the like) the point of comparison are not states of some classical theory, but some hybrid objects, which combine principles originating from different theories.

8 Summary

This paper provides a review of some properties of quantum states, which express the dissimilarity of quantum mechanics from classical physics, relations between these properties, and their behaviour under unitary transformations. It also contributes some results that are believed to be new:

- The only two-qubit state that has zero quantum discord absolutely is the maximally mixed state $\mathbb{1}_4$—see Theorem 5.7;
- The only two-qubit state that has zero quantum super discord absolutely is the maximally mixed state $\mathbb{1}_4$—see Theorem 5.9;
- Contextuality and noncontextuality are always absolute, that is, if a given state ρ (of arbitrary dimensionality) is contextual, then it is also absolutely contextual and if it is noncontextual, then it is also absolutely noncontextual—see Theorem 5.11;

With regard to specific classes of states, for the Gisin states, the range of parameters for which they are product, zero quantum discord, absolutely product and absolutely zero quantum discord have been determined—see Table 1 in Sect. 4.1 and Table 2 in Sect. 6.1.

The presented results clearly do not exhaust the topic. For example, the theorems concerning absolute zero quantum discord and absolute product states have been proven only for two-qubit states. The conjecture that they also hold for higher dimensions seems to be natural. Given the parametrisation of unitary matrix for the dimension $m \times n$, one can extend the method used in the proof of Theorem 5.7 to check the conjecture for the $m \times n$ case. However, the number of equations to solve will be large and, what is worse, this method could not be used to confirm the conjecture in its full generality. Therefore, some other methods would be needed.

Acknowledgements I would like to thank Karol Życzkowski, who was a supervisor of my MSc thesis in physics, entitled “Quantum correlations in composite systems under global unitary operations” (arXiv:1912.08285), on which most of the content of this paper is based. His many helpful comments enabled me to improve my work. During working on this paper, I was supported by the National Science Centre grant Preludium 2017/25/N/HS1/00705.

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