On the Equivalence Principle and Relativistic Quantum Mechanics

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Abstract

Einstein's Equivalence Principle implies that the Lorentz force equation can be derived from a geodesic equation by imposing a certain (necessary) condition on the electromagnetic potential (Trzetrzelewski, EPL 120:4, 2018). We analyze the quantization of that constraint and find the corresponding differential equations for the phase of the wave function. We investigate these equations in the case of Coulomb potential and show that physically acceptable solutions do not exist. This result signals an inconsistency between Einstein's Equivalence Principle and Relativistic Quantum Mechanics at an atomic level.

Keywords Equivalence principle · Electrodynamics · Relativity · Quantum mechanics

1 Introduction

In our previous work [1] we considered a certain generalization of Einstein's elevator experiment when the elevator is charged. Due to the screening effect, an observer inside the elevator cannot detect the electromagnetic field that surrounds it. This, together with Einstein Equivalence Principle, implies that the observer may identify his trajectory with a geodesic line in some curved space–time, even though the trajectory is given by the Lorentz force law.

In the original formulation of Einstein's Equivalence Principle [2] one assumes a complete physical equivalence of a reference frame K and K' where K corresponds to a uniform gravitational field and K' has no gravitational field but is uniformly accelerated. This equivalence is only to be understood locally i.e. in an arbitrary small neighbourhood of the observer in K (and K') since uniform gravitational

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fields are only idealisations therefore imperfections of real uniform fields could in principle be detected in terms of the tidal forces.

The equivalence stated in this way is based on another equivalence between passive-gravitational mass m_G and inertial mass m_I of a body (also known as the Galilean Equivalence Principle) which has been confirmed by a number of experiments (see [3, 4] or [5] for a comprehensive review), most recently by the Eöt-Wash experiment [6] and the MICROSCOPE experiment [7, 8]. If m_G and m_I were not equal, one would be able to perform local experiments (such as dropping test bodies) in the accelerating frame K' which would produce differences between the outcomes of the same experiments performed in the reference K .

The charged elevator thought experiment may be considered as a specific case of Einstein's Equivalence Principle in which the source of the uniform acceleration of K' is given by the surrounding electromagnetic field. For that to work, it is also needful to assume that the observer is isolated from the charge. Finding the value of the electric field E that mimics the gravitational one in the above experiment is straightforward. If m and q are the mass and the charge of the elevator (we assume that the mass of the observer is negligible), then we must have $|q|E = mg$ where g is the value of the constant gravitational field. This would close the discussion if it were not for the fact that according to General Relativity, gravitational force is described completely by the metric tensor $g_{\mu\nu}$ and particles' trajectories are described by geodesic lines corresponding to $g_{\mu\nu}$. On the other hand, the only real field in the charged elevator experiment is the electromagnetic potential A'_μ . How can these two views coincide? This apparent conundrum could only be solved if there existed a metric that depends on the electromagnetic potential A'_μ , for which the geodesic equation results in the Lorentz force equation. That metric turns out to be of a very simple form [1]

$$g_{\mu\nu} = \eta_{\mu\nu} + k \frac{q^2}{m^2} A'_\mu A'_\nu, \quad (1)$$

where q and m are the charge and the mass of the body and k is some dimensionless constant (in $c = 1$ units). However to obtain the complete equivalence of equations of motion one must impose a constraint for the potential on the particle's trajectory as

$$qA'_\mu p^\mu = \frac{1}{k} m^2. \quad (2)$$

Only when such constraint is satisfied is the Lorentz force equation with A'_μ equivalent to the geodesic equation with metric (1).

The choice of the dimensional constants q and m in (1) is made to cancel the dimension of the $A'_\mu A'_\nu$ term. This choice is convenient, however, it makes the metric observer dependent. On the other hand the thought experiment with the charged elevator does not imply that there should be a unique metric for all observers. It only requires the existence of such metric for a given observer. Moreover, one can find a universal metric for all observers, and hence avoid such concerns, by noting that since k in (1) is an arbitrary dimensionless number, and since the factor q^2/m^2

has the same dimension as the Newton constant G ($[G] = [q^2/m^2] = \text{g cm}$ in cgs units with $c = 1$), one can replace q^2/m^2 with G with no changes of the conclusions we have reached in [1]. If we do so, the metric and the necessary constraint for the electromagnetic potential will be

$$g_{\mu\nu} = \eta_{\mu\nu} + kGA'_\mu A'_\nu, \quad qA'_\mu p^\mu = \frac{q^2}{kG}.$$

However, in the remaining part of the paper we will use the original choice as in (1) and (2), keeping in mind that we can always substitute $k \rightarrow kGm^2/q^2$.

It is worth noting that extensions of the metric (1) involving first order derivatives of A'_μ

$$g_{\mu\nu} = \eta_{\mu\nu} + k\frac{q^2}{m^2}A'_\mu A'_\nu + l\frac{q^2}{m^3}\partial_{(\mu}A'_{\nu)}, \quad (3)$$

result in equations of motion equivalent in the leading order to the Landau–Lifshitz proposal of the radiation reaction force [1]. However in such case the counterpart of (2) is more complicated.

In this paper we investigate the consequences of (2) at the quantum level. We will write the operator counterpart of (2) and confront it with the Dirac equation. Clearly, one expects certain nontrivial constraints coming from quantum mechanics due to a standard argument: relativity and quantum mechanics imply the lower bound on the measurement of the position of a particle $\Delta x \geq \lambda_C/4\pi$ where $\lambda_C = 2\pi/m$ is the Compton wavelength [9]. In the elevator (charged or not) thought experiment we assume that the observer is the entire time inside the elevator. If however there is a fundamental limitation concerning the observer's position, we cannot say that this assumption is still valid for distances compared to the Compton wavelength. Therefore it is no longer clear if Einstein's Equivalence Principle is maintained below these scales because the corresponding experiment cannot even be performed. However, condition (2) is an equation of which the quantum counterpart clearly may be considered at any scale—in particular, in the case of ground state levels of the hydrogen atom which is the aim of this paper. We will show that quantum counterpart of (2) cannot be achieved in the relativistic treatment of any hydrogen-like atoms.

It should be stressed here that our approach to confront the equivalence principle with quantum mechanics is non-standard since it involves the electromagnetic field by design as well as its formulation is relativistically invariant (i.e. condition (2) is the same in all reference frames). On the other hand, in the existing literature [10–23], one is typically looking for an agreement between the equivalence principle and non-relativistic quantum theory with no electromagnetic field involved (although see [24, 25] for a discussion involving relativistic effects). A representative example for this matter is a neutral particle freely falling in a uniform gravitational field. In the framework of non-relativistic quantum mechanics one can easily show that there is full agreement between Einstein's Equivalence Principle and quantum theory, by considering the Schrödinger equation for a particle freely falling in a uniform and constant gravitational field. According to the equivalence principle this problem

should be equivalent to considering a free particle in the particle's rest frame. The equivalence is indeed achievable by a suitable choice of the wave-function phase difference between the two reference frames. That phase difference should be the source of quantum interference in a suitably prepared experiment, and indeed in one of the most remarkable experiments of this type, Colella, Overhauser and Werner [11–14, 21] detected quantum interference due to gravity, which agrees with the theoretical predictions.

2 Classical Constraints

Before considering quantum theory let us recall that (2) is compatible with gauge invariance i.e. it can be obtained from gauge transformation. If A_μ is an arbitrary potential then it follows that A'_μ defined as

$$A'_\mu := A_\mu + \partial_\mu \chi, \quad \chi = \frac{1}{q} \int_\gamma \left(\frac{m}{k} \sqrt{u^2} - q A_\mu u^\mu \right) d\tau, \quad (4)$$

where γ is the particle's worldline, satisfies (2). Therefore for any A_μ there exists a gauge phase χ such that A'_μ satisfies (2). However (2) is required by the equivalence principle and hence one concludes that the equivalence principle fixes the gauge marginally i.e. only on the trajectory γ . Outside of γ there is no requirement on A'_μ other than continuity conditions coming from equations of motion ($F'_{\mu\nu}$ has to be continuous everywhere). At the quantum level this constraint will be stronger as will be shown in the next section.

The choice of the metric (1) is unique up to the dimensionless parameter k , which can be seen as follows. The metric should depend on A'_μ and not its derivatives because 1st order derivatives of A'_μ in the metric would produce 2nd order derivatives in the corresponding geodesic equation which are not present in the Lorentz force. Therefore the most general choice of the metric is

$$g_{\mu\nu} = f(A'^2) \eta_{\mu\nu} + h(A'^2) A'_\mu A'_\nu, \quad (5)$$

where $A'^2 = \eta^{\mu\nu} A'_\mu A'_\nu$ and where f and h are such that $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ in the $A'_\mu \rightarrow 0$ limit. The f factor can be removed by introducing new co-ordinates y^μ such that $dy^\mu = \sqrt{f} dx^\mu$. Therefore in the remaining part we will set $f = 1$ with no loss. The equations of motion following from the variation of the action $S = -m \int \sqrt{g_{\mu\nu} u^\mu u^\nu}$ will become

$$\dot{u}_\mu = A'_\nu u^\nu h F'_{\mu\alpha} u^\alpha + h_\mu, \quad (6)$$

with

$$h_\mu := -(A'_\nu u^\nu h) A'_\mu + (A'_\nu u^\nu)^2 h' A'^\alpha \partial_\mu A'_\alpha, \quad (7)$$

where u^μ are particles' 4-velocities and where for simplicity we used reparametrization invariance and set $g_{\mu\nu}u^\mu u^\nu = \text{const}$. Compatibility with the Lorentz force requires that

$$A'_\nu u^\nu h = \frac{q}{m}, \quad h_\mu = 0. \quad (8)$$

Equations (7) and (8) both imply that $h'A'^\alpha \partial_\mu A'_\alpha = 0$. Therefore either $h' = 0$ or we have $A'^\alpha \partial_\mu A'_\alpha = 0$ which implies $(A'^2)' = 0$ and hence $A'^2 = \text{const}$. Both cases imply that $h = \text{const}$. and therefore imply that the metric (1) is general for our problem.

Although in this paper we only consider Abelian gauge fields, it is clear the one may in general consider non-Abelian extensions such as

$$g_{\mu\nu} = \eta_{\mu\nu} + k_1 \text{Tr}(B_\mu B_\nu) + k_2 \text{Tr}(B_\mu B_\nu B_\rho B^\rho) + k_3 \text{Tr}(B_\mu B_\rho B_\nu B^\rho) + (\mu \leftrightarrow \nu),$$

where B_μ is a matrix belonging to some Lie algebra. Let us note however that the equivalence principle, in the context used in this paper, cannot be considered for non-Abelian gauge fields because the corresponding charged elevator experiment, with colour charge, cannot be performed.

3 Quantum Constraints

Let us write (2) in terms of the generalized momentum for the charged particle $\pi_\mu = p_\mu + qA'_\mu$. A quantum counterpart of that constraint will be

$$\frac{1}{2}q\{\pi_\mu - qA'_\mu, A'^\mu\}\psi' = \frac{m^2}{k}\psi', \quad \pi_\mu = -i\partial_\mu, \quad (9)$$

where ψ' is the particle's wave-function and $\{\cdot, \cdot\}$ is the anti-commutator. The relation between the primed and unprimed quantities is as usual

$$A'_\mu = A_\mu + \partial_\mu \chi, \quad \psi' = e^{iq\chi}\psi, \quad (10)$$

so that for both, the Dirac equation has the same form

$$\gamma^\mu(\pi_\mu - qA'_\mu)\psi = -m\psi, \quad \gamma^\mu(\pi_\mu - qA'_\mu)\psi' = -m\psi'.$$

In our notation the pair (A'_μ, ψ') satisfies the constraint (9) i.e. a quantum counterpart of (2). Therefore, given an arbitrary field A_μ we are looking for a scalar function χ such that the gauge transformation (10) results in the pair (A'_μ, ψ') . The existence of such χ is a necessary condition for the equivalence principle to hold at the quantum level.

Let us now write the constraint (9) as

$$T_{A'}\psi' = 0, \quad T_{A'} := -\frac{1}{2}iq(\partial A') - iqA' \cdot \partial - q^2 A'^2 - \frac{m^2}{k}, \quad (11)$$

where we introduced the operator $T_{A'}$. This operator does not involve any gamma matrices and acts on each component of the spinor ψ . Therefore it is also true that $T_{A'}^* \bar{\psi}' = 0$ where $T_{A'}^*$ is a complex conjugate of $T_{A'}$ and $T_{A',q}^* = T_{A',-q}$ (we also have $T_{A'}^* \psi'^* = 0$ but it is more convenient to use $T_{A'}^* \bar{\psi}' = 0$). Substituting (10) into (11) we obtain

$$\begin{aligned} 0 = T_{A'} \psi' &= e^{iq\chi} \left(T_A - q^2 A \cdot \partial \chi - iq \partial \chi \cdot \partial - \frac{i}{2} q \partial^2 \chi \right) \psi \\ \Rightarrow T_A \psi &= \left(q^2 A \cdot \partial \chi + iq \partial \chi \cdot \partial + \frac{i}{2} q \partial^2 \chi \right) \psi. \end{aligned} \quad (12)$$

Multiplying the above equation and its complex conjugate, by $\bar{\psi}$ or ψ and then taking their sum or difference respectively we find two real equations

$$(\partial^2 \chi) \bar{\psi} \psi + (\partial A) \bar{\psi} \psi + (A + \partial \chi) \cdot \partial (\bar{\psi} \psi) = 0, \quad (13)$$

and

$$\begin{aligned} & - \left(\frac{m^2}{k} + q^2 A^2 \right) \bar{\psi} \psi \\ & = q^2 A \cdot (\partial \chi) \bar{\psi} \psi + \frac{iq}{2} (A + \partial \chi) \cdot [\bar{\psi} \partial \psi - (\partial \bar{\psi}) \psi], \end{aligned} \quad (14)$$

where we substituted the definition of T_A . Introducing a 4-vector

$$J_\mu = q(A_\mu + \partial_\mu \chi) \bar{\psi} \psi, \quad (15)$$

we observe that (13) and (14) considerably simplify

$$\partial^\mu J_\mu = 0, \quad (16)$$

$$- \frac{m^2}{k} (\bar{\psi} \psi)^2 = J \cdot \left[q A \bar{\psi} \psi + \frac{i}{2} (\bar{\psi} \partial \psi - (\partial \bar{\psi}) \psi) \right]. \quad (17)$$

Hence J_μ satisfies the continuity equation and an additional linear (algebraic) constraint. Equations (16) and (17) are central in this manuscript. Accordingly, the phase of the corresponding wave function, i.e. of $\psi \exp(iq\chi)$, is also constrained since that phase is the only one that satisfies the condition (9). However, we note that (16) and (17) are only necessary conditions for (12) to hold, and may not be sufficient. A good example to see this more clearly is to consider left-handed (or right-handed) Dirac spinors. In Weyl basis they are $\psi^T = (\psi_L^T, 0)$, where ψ_L^T is the transpose of the two component left-handed Weyl spinor. With this choice we have $\bar{\psi} \psi = 0$ and hence $J_\mu = 0$ and therefore (16) and (17) are satisfied trivially with no constraints on χ . However non-trivial constraints on χ are still there at the level of Eq. (12).

We observe that (17) contains the phase part of the Dirac current $j_\mu^D = \bar{\psi} \gamma_\mu \psi$ i.e. j_μ^P in the Gordon decomposition

$$mj_\mu^D = j_\mu^P + j_\mu^S,$$

$$j_\mu^P = \frac{i}{2}[\bar{\psi}\partial_\mu\psi - (\partial_\mu\bar{\psi})\psi] + qA_\mu\bar{\psi}\psi, \quad j_\mu^S = \frac{i}{2}\partial^\nu(\bar{\psi}\gamma_{\nu\mu}\psi),$$

where j_μ^S is the spin part. This observation may prove useful in attempts to solve (16) and (17) for general A_μ , in the following way. Since j_μ^P and j_μ^S satisfy the continuity equation independently, we may consider the Ansatz

$$J_\mu = Aj_\mu^D + Bj_\mu^S,$$

where A and B are some constants. The above Ansatz automatically satisfies Eq. (16) but whether (17) is also satisfied needs to be checked independently.

4 Free Particle

Equations (16) and (17) should hold for arbitrary A_μ i.e. for any A_μ there should exist χ such that Eqs. (16) and (17) are satisfied. In particular we can take A_μ to a very small level. By continuity there should exist a solution for a free particle $A_\mu = 0$, even though in such case the thought experiment with charged elevator is trivial.

Let us start with the continuity equation. There are several expressions in the Dirac theory that satisfy (16). An obvious choice is the phase part of the Dirac current

$$J_\mu = C[\bar{\psi}\partial_\mu\psi - (\partial_\mu\bar{\psi})\psi]. \quad (18)$$

Not only does it satisfy the continuity equation but such expression also appears in the second equation i.e. in (17).

The solution of the Dirac equation of a free particle with four-momentum p_μ can be written as $\psi = ue^{-ip \cdot x}$ where u is a constant spinor, in which case we have

$$\bar{\psi}\partial_\mu\psi - (\partial_\mu\bar{\psi})\psi = -2ip_\mu\bar{\psi}\psi,$$

and therefore condition (17) implies $C = i/2k$. Now, returning to the definition of the current (15), we find that

$$J_\mu = q\bar{\psi}\psi\partial_\mu\chi \implies \chi = \frac{1}{kq}p \cdot x.$$

As mentioned previously, this does not yet show that Eq. (12) is satisfied. The LHS of (12) with $A_\mu = 0$ is $T_A\psi = -\frac{m^2}{k}\psi$ while the RHS becomes $\frac{m^2}{k}\psi$ when using $\chi = \frac{1}{kq}p \cdot x$. Therefore the solution is satisfied only for a massless case.

5 Semi-classical Limit

For the purposes of the semi-classical limit let us introduce the Planck constant $\hbar$ and consider the wave function of the following form

$$\psi = e^{-iS/\hbar} u, \quad S = S_0 + \hbar S_1 + \dots, \quad (19)$$

where u is a slowly-varying bispinor [9]. It is well-known that substitution of (19) to the “square” of the Dirac equation results in the Hamilton–Jacobi equation for the 0th order term S_0 . Explicitly

$$\begin{aligned} & \left[(\pi_\mu - qA_\mu)(\pi^\mu - qA^\mu) - m^2 - \frac{i}{2} q\hbar F_{\mu\nu} \gamma^{\mu\nu} \right] \psi = 0 \\ & \rightarrow [(\partial_\mu S + qA_\mu)(\partial^\mu S + qA^\mu) - m^2 + O(\hbar)] \psi = 0, \end{aligned}$$

where $\pi_\mu = -i\partial_\mu$. It is worth noting that the 0th-order equation does not involve the spin matrices, they appear only starting from the 1st order terms. This is of course expected since the fermion spin is essentially a quantum concept and hence should not appear in the $\hbar \rightarrow 0$ limit.

Let us now see what the corresponding semi-classical limit of Eqs. (16) and (17) is (introduction of $\hbar$ in these equations is by rescaling $\partial_\mu \rightarrow \hbar\partial_\mu$ and $\chi \rightarrow \chi/\hbar$). The continuity equation (16) becomes

$$q\bar{u}u\partial^\mu(A_\mu + \partial_\mu\chi) + q(A_\mu + \partial_\mu\chi)\partial^\mu(\bar{u}u) = 0,$$

which, given that u is slowly-varying, implies a Lorenz gauge condition $\partial^\mu A'_\mu = 0$ for $A'_\mu = A_\mu + \partial_\mu\chi$. The second Eq. (17) becomes

$$-\frac{m^2}{k} = (q\partial^\mu\chi + qA_\mu)(\partial_\mu S + qA_\mu + \hbar u_\mu),$$

where $u_\mu := \frac{i}{2} \frac{\bar{u}\partial_\mu u - (\partial_\mu \bar{u})u}{\bar{u}u}$, and we see that we recover the Hamilton–Jacobi equation in the $\hbar \rightarrow 0$ limit provided $q\chi$ is identified with the Hamilton–Jacobi function S and $k = -1$. Therefore in the semi-classical limit, constraints (16) and (17) are consistent with classical equations.

6 Coulomb Potential

Let us now consider the Coulomb potential for the hydrogen-like atom

$$A_0 = -\frac{Ze}{r}, \quad A_i = 0, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (20)$$

where e is the charge of the electron, $Z = 1, 2, 3, \dots$

The Hamiltonian for the Coulomb potential has a discrete spectrum hence a priori we would have to verify (16) and (17) for an infinite set of wave functions satisfying the corresponding spectral problem. For each such wave function there should exist a solution χ for (16) and (17). Let us concentrate on the ground state of the problem. When using the Dirac representation of gamma matrices, the corresponding wave function is [26]

$$\psi = g(r)e^{-iEt} \begin{pmatrix} 1 \\ 0 \\ iB \cos \theta \\ iB \sin \theta e^{i\phi} \end{pmatrix}, \quad (21)$$

$$g(r) = \mathcal{N}r^{\gamma-1}e^{-Zm_e\alpha r}, \quad B = \frac{(1-\gamma)}{\alpha}, \quad \gamma = \sqrt{1-Z^2\alpha^2},$$

where $\mathcal{N}$ is the normalization constant, m_e is the mass of the electron and $\alpha = e^2$ is the fine structure constant. Using these explicit expressions we find

$$\frac{i}{2}(\bar{\psi}\partial_0\psi - (\partial_0\bar{\psi})\psi) = E\bar{\psi}\psi,$$

$$\frac{i}{2}(\bar{\psi}\partial_i\psi - (\partial_i\bar{\psi})\psi) = \bar{\psi}\psi \frac{B^2}{(1-B^2)r^2}(x\delta_{i2} - y\delta_{i1}),$$

hence (17) for $q = -e$ becomes¹

$$-\frac{m_e^2}{k} = (e\partial_i\chi + eA_0)(-E + eA_0) - \frac{eB^2}{(1-B^2)r^2}\partial_\phi\chi. \quad (22)$$

Let us now consider an Ansatz

$$\chi = at + b\phi + c(r, \theta), \quad (23)$$

where a and b are constants. Substituting (23) to (22) we find that

$$\frac{m_e^2}{k} = \left(ea - \frac{\alpha}{r}\right)\left(E + \frac{\alpha}{r}\right) + \frac{eB^2}{(1-B^2)r^2}b.$$

The $1/r$ terms cancel only if we take $a = E/e$ which in turn fixes b and k as well, we have

$$a = E/e, \quad k = \frac{m_e^2}{E^2}, \quad b = \alpha^2 Z^2 \frac{1-B^2}{eB^2}.$$

Therefore, the value of k is fixed uniquely. Using the formula for the ground state energy $E = m_e\sqrt{1-Z^2\alpha^2}$ we find

$$k = \frac{1}{1-Z^2\alpha^2}. \quad (24)$$

In order to find $c(r, \theta)$ we will use the remaining continuity equation (16). The current components are

$$J_0 = q(A_0 + a)\bar{\psi}\psi, \quad J_i = q\partial_i\chi\bar{\psi}\psi,$$

hence $\partial^0 J_0 = 0$ and hence the continuity equation is $\partial^i J_i = 0$ or

¹ We use standard spherical co-ordinates $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

$$\bar{\psi}\psi\Delta\chi + \partial_i\chi\partial_i\bar{\psi}\psi = 0.$$

In spherical co-ordinates, using (23) and the fact that $\bar{\psi}\psi$ depends only on r , the continuity equation becomes

$$\partial_r^2 c + \left(\frac{2}{r} + \partial_r \ln \bar{\psi}\psi \right) \partial_r c + \frac{1}{r^2} \Delta_\theta c = 0, \quad (25)$$

where $\Delta_\theta = \frac{1}{\sin\theta} \partial_\theta(\sin\theta \partial_\theta)$. Many solutions exist to Eq. (25) e.g.

$$c(r) = \text{const.} \int \frac{dr}{r^2 \bar{\psi}\psi}, \quad \text{or} \quad c(\theta) = \text{const.} \int \frac{d\theta}{\ln \sin \theta}.$$

One can also look for a general solution for $c(r, \theta)$ by separating variables $c(r, \theta) = c_1(r)c_2(\theta)$ —although this is not needed here as we are only interested in the existence of the solution to Eqs. (16) and (17). However, general dependence on t and ϕ is important for our considerations since it will be crucial in the next section. Below we find general form of χ satisfying conditions (16) and (17).

6.1 General Solution

Without any loss of generality we first write χ as

$$\chi = \frac{E}{e}t + \alpha^2 Z^2 \frac{1-B^2}{eB^2} \phi + c(r, \theta) + d\bar{\chi}(t, \phi, r, \theta), \quad (26)$$

where $\bar{\chi}$ is arbitrary and d is a convenient constant. Substituting (26) to (22) we obtain

$$C_1 \partial_t \bar{\chi} + C_2 \partial_\phi \bar{\chi} = C_3, \\ C_1 = e \left(\frac{Z\alpha}{r} + E \right), \quad C_2 = \frac{eB^2}{(1-B^2)r^2}, \quad C_3 = \frac{m_e^2}{k} - E. \quad (27)$$

A general solution of (27) can be obtained using the method of characteristics, we find that χ must be of the following functional form

$$\bar{\chi} = f(C_1 \phi - C_2 t) + \frac{C_3}{C_1} t, \quad (28)$$

where $f(\cdot)$ is an arbitrary differentiable function. Therefore we have obtained a general solution of (17) however there is still a crucial constraint coming from the continuity equation (16). Substituting (28) to (15) we obtain

$$J_0 = -e \left(A_0 - C_2 f' + \frac{C_3}{C_1} \right) \bar{\psi}\psi, \quad J_i = -e \partial_i \chi \bar{\psi}\psi,$$

where we use the abbreviation $f' = f'(\xi)|_{\xi=C_1\phi-C_2t}$. With the above current components we have

$$\partial_0 J_0 = -eC_2^2 \bar{\psi} \psi f'',$$

and

$$\begin{aligned} \partial_i J_i &= \bar{\psi} \psi \partial_i \partial_i \bar{\chi} + \partial_i \bar{\chi} \partial_i \bar{\psi} \psi \\ &= \bar{\psi} \psi \left[\partial_r^2 \bar{\chi} + \left(\frac{2}{r} + \partial_r \ln \bar{\psi} \psi \right) \partial_r \bar{\chi} + \frac{1}{r^2} \Delta_{\theta\phi} \bar{\chi} \right], \end{aligned}$$

where $\Delta_{\theta\phi}$ is standard $\Delta_{\theta\phi} = \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \partial_\phi^2$. Using

$$\begin{aligned} \partial_r \bar{\chi} &= (\phi \partial_r C_1 - t \partial_r C_2) f' + t C_3 \partial_r \frac{1}{C_1}, \\ \partial_r^2 \bar{\chi} &= (\phi \partial_r C_1 - t \partial_r C_2)^2 f'' + (\phi \partial_r^2 C_1 - t \partial_r^2 C_2) f' + t C_3 \partial_r^2 \frac{1}{C_1}, \end{aligned}$$

(and similar for ∂_θ derivatives) and $\partial_\phi^2 \bar{\chi} = C_1^2 f''$ we observe that the continuity equation $\partial^\mu J_\mu = 0$ will contain terms of order t^2 and ϕ^2 . The only way to cancel them is by a condition $f'' = 0$ i.e.

$$f = E(r, \theta)(C_1 \phi - C_2 t) + F(r, \theta),$$

by using which the continuity equation becomes

$$\begin{aligned} \frac{2E}{r} + \frac{\cot\theta}{r^2} \partial_\theta F + \left[(1 + \partial_r \ln \bar{\psi} \psi) E \cdot C_1 + \frac{C_1}{r^2} \cot\theta \right] \phi \\ + \left[\left(1 + \frac{2}{r} \right) C_3 \partial_r^2 \frac{1}{C_1} - (1 + \partial_r \ln \bar{\psi} \psi) E \cdot C_2 - \frac{C_2 \cot\theta}{r^2} \right] t = 0. \end{aligned}$$

The above equation needs to be satisfied for all t and ϕ therefore we have in fact three independent equations. Solving the first two we find

$$E = -\frac{\cot\theta}{r^2(1 + \partial_r \ln \bar{\psi} \psi)}, \quad F = \frac{2}{r(1 + \partial_r \ln \bar{\psi} \psi)} \theta + g(r),$$

substituting to the third we obtain

$$\left(1 + \frac{2}{r} \right) C_3 \partial_r^2 \frac{1}{C_1} = 0.$$

There is still the remaining k parameter in C_3 and so we can set $C_3 = 0$ as in (24). Therefore we obtain a general solution for $\bar{\chi}$ (and hence χ) which is

$$\bar{\chi} = -\frac{\cot\theta}{r^2} (C_1 \phi - C_2 t) + \frac{2}{r} \theta + g(r). \quad (29)$$

6.2 Consistency Check

From (16) we deduce that $\int dV J_0$ is conserved. That conserved quantity is quadratic in the fermion field. On the other hand, in the Dirac theory we already have conserved quantities which involve quadratic fermion terms—the Dirac current j_μ^D and the phase part of the Dirac current j_μ^P (and the spin current $j_\mu^S = m j_\mu^D - j_\mu^P$). It is therefore reasonable to ask if they are dependent i.e. if there exist constants C and D such that

$$-e(A^\mu + \partial^\mu \chi) \bar{\psi} \psi = C j_\mu^D + D j_\mu^P. \quad (30)$$

To verify if the above identity takes place we calculate the RHS of (30) for the ground state (21). For $\mu = 1, 2, 3$ we have

$$C j_1^D + D j_1^P = y f(r) \bar{\psi} \psi, \quad C j_2^D + D j_2^P = -x f(r) \bar{\psi} \psi, \quad C j_3^D + D j_3^P = 0,$$

where

$$f(r) = -C \frac{2Bm}{(1-B^2)r} + D \frac{B^2}{(1-B^2)r^2}.$$

Because for the Coulomb potential we have $A_i = 0$, Eq. (30) implies the following set of equations

$$-e \partial_1 \chi = y f(r), \quad -e \partial_2 \chi = -x f(r), \quad -e \partial_3 \chi = 0.$$

These equations have no solutions, which can be seen as follows: the third equation implies that χ does not depend on the z coordinate. Then taking the z derivative of the first two equations we find that $\partial_3 f = 0$ —which is not the case. Therefore we have shown that the identity (30) cannot hold for all μ .

On the other hand, in order to argue that the conserved quantity $\int dV J_0$ is dependent on the conserved charges of the Dirac theory, it is sufficient to show that (30) is true for $\mu = 0$. That turns out to be achievable. We have

$$C j_0^D + D j_0^P = C m \frac{1+B^2}{1-B^2} \bar{\psi} \psi + D(E - eA_0) \bar{\psi} \psi,$$

therefore (30) for $\mu = 0$ becomes

$$-e(A_0 + \partial_0 \chi) = C m \frac{1+B^2}{1-B^2} + D(E - eA_0),$$

which is solved by

$$\chi = \left(-C m \frac{1+B^2}{e(1-B^2)} - D \frac{E}{e} \right) t + (D-1)A_0 t + g(r, \theta, \phi).$$

The above solution needs to be equal to the one obtained previously i.e. (26) with (29). However this can be achieved only if $D = 1$, $C = 0$ and $d = 0$ which gives $J_0 = j_0^P$. Only then is the conserved charge coming from $\partial_\mu J^\mu = 0$ equal to the

conserved charge of the Dirac theory. Note that in this case the general solution (26) simplifies into our original Ansatz (23)

$$\chi = \frac{E}{e}t + \alpha^2 Z^2 \frac{1-B^2}{eB^2} \phi + c(r, \theta). \quad (31)$$

6.3 Final Phase Constraints

The existence of the scalar function χ is a necessary condition for the equivalence principle to hold at quantum level. As previously mentioned, that principle fixes (albeit not completely) the phase of the wave function. If (A_μ, ψ) is an arbitrary pair satisfying the Dirac equation then, in order to make the condition (9) hold, there must exist a scalar χ such that a pair $(A_\mu + \partial_\mu \chi, e^{iq\chi} \psi)$ satisfies (9) on top of the Dirac equation. One immediately notices that there is something disturbing with the phase (26): the coefficient next to ϕ is a function of r , θ and α . For spinors the transformation $\phi \rightarrow \phi + 4\pi$ should leave the wave function invariant, however, that can be achieved only if $d = 0$ and

$$\frac{1-B^2}{B^2} \alpha^2 Z^2 \in \mathbf{Z}/2. \quad (32)$$

The condition $d = 0$ is consistent with the previous discussion where we demanded that the conserved charge $\int J_0 dV$ is precisely the Dirac charge. However the additional relation (32) is new. It can be written as

$$-\frac{4}{Z^2} \left(1 - \frac{Z^2(Z^2 + 1)}{2} \alpha^2 + \sqrt{1 - Z^2 \alpha^2} \right) = n, \quad (33)$$

where $n \in \mathbf{Z}$ and where we substituted B . The only Z independent solution of (33) is $\alpha = 0$ (with $n = 0$). Other solutions of the above equation are Z dependent with

$$\alpha_\pm^2 = \frac{4 + n + nZ^2 \pm 2\sqrt{4 - 2n(1 + 1/Z^2)}}{2(1 + Z^2)^2},$$

where for each Z the range of n is such that $\alpha_\pm$ is positive.

This result clearly contradicts the existence of the physically acceptable phase χ . While the phase exists, i.e. the solution of Eqs. (16), (17) exists, it is true only for $\alpha = 0$ or $\alpha_\pm$ that depend on Z . Therefore we conclude that in this sense Einstein's Equivalence Principle is inconsistent with constraints coming from relativistic quantum mechanics.

6.4 Sufficient Condition

As a final check, just like we did in the case of the free particle, we now also verify the original condition (12). Because the only non-zero component of A_μ is A_0 we may write (12) as

$$\left(-iqA_0\partial_0 - q^2A_0^2 - \frac{m^2}{k}\right)\psi = q^2A_0\partial_0\chi\psi + \left(iq\partial_\mu\chi\partial^\mu + \frac{i}{2}q\partial^2\chi\right)\psi.$$

Using the explicit formula for ψ (21) and χ (23) we can write the above condition as

$$\left(-qEA_0 - q^2A_0^2 - \frac{m^2}{k}\right)\psi = (q^2aA_0 + qaE)\psi - \frac{iq}{2}\left(2\frac{g'}{g}\partial_r c + \Delta\right)\psi + \frac{1}{r}q\partial_\theta c\psi_1 - iqb\psi_2, \quad (34)$$

where a , b and c are as in (23), g is as in (21), Δ is the Laplace operator, $q = -e$, spinors ψ_1, ψ_2 are

$$\psi_1 = ge^{-iEt} \begin{pmatrix} 0 \\ 0 \\ -B \sin \theta \\ B \cos \theta e^{i\phi} \end{pmatrix}, \quad \psi_2 = ge^{-iEt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -B \sin \theta e^{i\phi} \end{pmatrix}.$$

Spinors ψ , ψ_1 and ψ_2 are linearly independent therefore coefficients standing next to them in (34), should be 0. For ψ_1 and ψ_2 we obtain $\partial_\theta c = 0$ and $b = 0$. For ψ the coefficient is a complex number hence we have two conditions in this case. The one corresponding to the imaginary part can be made 0 by a suitable choice of $c(r)$. However, the real part [i.e. the first line in (34)] can never be 0 for a Coulomb potential due to the A_0^2 term unless $q = 0$ hence $\alpha = 0$. This result is consistent with the conclusion made in the previous section based on the phase constraints.

The above calculation uncovers also the main reason for the non-existence of χ satisfying (12). A crucial step in the above proof is related to the linear independence of ψ , ψ_1 and ψ_2 which results in three independent conditions that in the end cannot be met. Independence of ψ , ψ_1 and ψ_2 is possible thanks to the spin-1/2 degrees of freedom—this would not have been possible if ψ was a spin-0 complex valued function. Therefore it seems very plausible that non-existence of χ satisfying (12) is a rather generic feature due to spin-1/2 degrees of freedom of the wave function.

7 Discussion

Generalization of Einstein's elevator thought experiment to the case of charged objects implies a constraint involving electromagnetic potential and momentum (2). Quantization of (2) results in equations which need to be satisfied by the wavefunction and the electromagnetic potential. These differential equations in fact fix the phase of the wave function strongly (but not completely). For binding potentials, the phase will in general depend on the azimuthal angle, therefore the coefficient next to that angle needs to be a half-integer in order to make the wave function unambiguous. This procedure applied to the Coulomb potential results in certain

algebraic equations that need to be satisfied by the fine structure constant. We have verified that all the solutions of those equations are different from the experimental value of α implying that the Equivalence Principle is inconstant with those quantum mechanical systems (Coulomb interaction). In our view such inconsistencies will always appear in the case of binding potentials, they certainly appear for all Hydrogen-like potentials.

We have concluded above that in the simplest theoretical treatment of the hydrogen-like potentials, the condition (32) coming from the phase constraints of the wave-function, cannot be satisfied. However there are several approximations that we are using to arrive at this result. To be more specific we are neglecting the magnetic moment of the nucleus, radiative corrections, nuclear effects (finite charge-radius of the nucleon) and two body relativistic effects. In principle we should be considering the above corrections and in fact any other, albeit very small, contributions. Therefore in a more complete framework we would obtain a generalization of condition (32) which could be written as

$$\frac{1 - B^2}{B^2} \alpha^2 Z^2 + \Delta \in \mathbf{Z}/2, \quad (35)$$

where Δ represents all possible corrections and therefore may depend on many other dimensionless constants such as m_e/m_n , α , corrections coming from quantum electrodynamics etc. In our view it is unlikely that these corrections would help to meet the condition (35), mainly because the left hand side of (35) depends explicitly on Z while the right hand side is a half-integer. With no loss we can conclude that (35), if true, would imply that

$$\frac{1 - B^2}{B^2} \alpha^2 Z^2 + \Delta = \frac{n}{2} + f(\alpha, Z, m_e/m_N, \dots),$$

where f is some function such that for all Z s, $f = 0$ if α is numerically equal to the fine structure constant. One cannot exclude the existence of such f based on the considerations set forth in the paper however we find such possibility unlikely. Therefore, if only phase constraints are considered, a more careful conclusion would be to say that either Einstein's Equivalence Principle is inconstant with the quantum theory already at atomic level, or there exists some quite unexpected relation involving α and all other dimensionless constants that appear in Δ .

In our view a more likely explanation of the apparent disagreement lies in the notion of spin for fermions, as it does not have a classical realization, whereas the equivalence principle is a statement from classical physics. We have made this statement more explicit in the last section. Indeed, the quantum constraint (9) that we are considering, does not contain any spin operators but has to be satisfied by all spinor components of ψ separately. In the end, this is the consequence of the fact that the formulation of the Einstein's Equivalence Principle makes no mention at all of the spin of the particle.

Finally, we would like to comment on potential relevance of the result presented in this paper in modern approaches to unify Quantum Mechanics and General Relativity. It is a general belief that since gravitational interaction is much weaker

than all the Standard Model interactions, there is no point in including gravity into the picture as long as we are considering distances much bigger than the Planck scale. That approach, where the dynamics at smaller scales is ignored as long as only larger scales are considered, is in fact common to all branches of Physics and it seems very reasonable. However, given the main result advocated in this paper, we may ask what is the purpose of trying to unify gravity and quantum mechanics at Planck scale when it seems that they disagree already at atomic scale? The current paper certainly does not give an answer to this question. However, as mentioned above, we would like to indicate that a possible direction to resolve the above disagreement may require an extension of the equivalence principle to the case of particles with half-integer spin. That does not simply mean adding new fermion fields to the theory but would rather require a certain extension of the classical geometry in such a way that spin variables appear there naturally.

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