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A comparative study on precision of pairwise comparison matrices

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Abstract
Pairwise comparisons have been a long-standing technique for comparing alternatives/criteria and their role has been pivotal in the development of modern decision-making methods such as the Analytic Hierarchy/Network Process (AHP/ANP), the Best-Worst method (BWM), PROMETHEE and many others. Pairwise comparisons can be performed within several frameworks such as multiplicative, additive and fuzzy representations of preferences, which are particular instances of a more general framework based on Abelian linearly ordered groups. Though multiplicative, additive and fuzzy representations of preferences are widely used in practice, it is unknown whether decision makers are equally precise in the three aforementioned representations when they measure objective data. Therefore, the aim of this paper is to design, carry out and analyse an experiment with over 200 respondents (undergraduate university students) from two countries, Czechia and Italy, to compare precision of the respondents in all three representations. In the experiment, respondents pairwise compared (by approximation) the areas of four geometric figures and then, the imprecision of their assessments was measured by computing the distance with the exact pairwise comparisons. We grouped the respondents in such a way that each participant was allowed to deal with a unique type of representation. The outcomes of the experiment indicate that the multiplicative approach is the most precise.

Keywords Multi-criteria decision making · Pairwise comparison matrix · Precision · Abelian linearly ordered group

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1 Introduction

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of decision elements, such as alternatives or criteria; the entry of a *Pairwise Comparisons Matrix* (PCM) $\mathbf{G} = [g_{ij}]_{n \times n}$ quantifies the preference intensity of x_i over x_j. The aggregation of the entries of a PCM allows us to obtain a weighted ranking on X; that is a mapping that assigns a real value to each x_i (Barzilai et al., 1987). Origins of the PCMs can be dated back to the early works of Catalan medieval scholar Llull (1274). The first notable work with an application in psychology on pairwise comparisons can be, however, attributed to Thurstone (1927) and his Law of Comparative Judgments. Currently, PCMs constitute the core of the Analytic Hierarchy Process (AHP) and the Analytic Network Process (ANP) (Saaty, 1977, 1980), and of popular theoretical frameworks for the Multiple Criteria Decision Making (MCDM). For further details about PCMs, the reader can refer to Ramík (2020).

The main advantage in using PCMs is that they allow a Decision Maker (DM) to compare two alternatives at a time, thus reducing the complexity of a decision-making problem; indeed, it is easier to perform pairwise comparisons than to provide directly a weighted ranking on all the alternatives, especially when the set X is large.

In the literature, several types of PCMs are proposed:

- if $g_{ij} \in]0, +\infty[$ represents a preference ratio, then $G = [g_{ij}]_{n \times n}$ is a *multiplicative* PCM (Barzilai and Golany, 1990). In a multiplicative PCM, $g_{ij} = 1$ if there is indifference between x_i and x_j, $g_{ij} > 1$ if x_i is strictly preferred to x_j, whereas $g_{ij} < 1$ expresses the reverse preference;
- if $g_{ij} \in \mathbb{R} =]-\infty, +\infty[$ represents a preference difference, then $G = [g_{ij}]_{n \times n}$ is an *additive* PCM (Barzilai, 1997). In an additive PCM, $g_{ij} = 0$ if there is indifference between x_i and x_j, $g_{ij} > 0$ if x_i is strictly preferred to x_j, whereas $g_{ij} < 0$ expresses the reverse preference;
- if $g_{ij} \in]0, 1[$ reflects a preference degree, then $G = [g_{ij}]_{n \times n}$ is a *fuzzy* PCM (Tanino, 1988). In a fuzzy PCM, $g_{ij} = 0.5$ if there is indifference between x_i and x_j, $g_{ij} > 0.5$ if x_i is strictly preferred to x_j, whereas $g_{ij} < 0.5$ expresses the reverse preference.

Multiplicative, additive and fuzzy PCMs share the same algebraic structure; that is, they are PCMs over Abelian linearly ordered (Alo)-groups (Cavallo and D'Apuzzo, 2009). Several authors follow this approach based on Alo-groups (e.g. Hou, 2016, Koczkodaj et al., 2016, Ramík, 2015 and Xia and Chen, 2015).

The foremost type of PCM, at least with respect to the number of real-world applications, is probably the multiplicative representation, used among others by Saaty (1980) in the theory of the Analytic Hierarchy Process (AHP) by using the finite semantic scale $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ instead of $]0, +\infty[$.

Though multiplicative, additive and fuzzy approaches are widely used in practice, to best of our knowledge, in the literature no study has been conducted in order to compare their reliability, with the exclusion of an experiment performed by Cavallo et al. (2019); the authors compare the DMs' coherence when they express subjective preferences by means of multiplicative, additive and fuzzy PCMs. However, no comparative study has been conducted in order to establish for which type of PCM, DMs are more precise when they measure objective data. This paper aims at filling this gap. We stress that if a DM measures objective data then (s)he can be perfectly coherent but not necessarily perfectly precise; on the other hand, if (s)he is perfectly precise then (s)he is perfectly coherent, too. In particular, we perform an experiment in order to compare the precision of these approaches when areas of four geometric figures have to be compared. In addition, we provide an appropriate methodology for the comparison of the approaches. The experiment is inspired by a validation study conducted by Saaty (2008); where, multiplicative pairwise comparisons have been applied to geometric areas in order to show the effectiveness of the AHP. In this paper, precision of additive and fuzzy pairwise comparisons, in addition to multiplicative ones, are analyzed.

The remainder of the paper is organized as follows: Sect. 2 provides preliminaries about Alo-groups and PCMs defined over Alo-groups. Section 3 describes the problem, the methodology and the results of the experiment. Section 4 provides conclusions and future work.

2 Preliminaries

In this section, we provide preliminaries about Alo-groups and PCMs defined over Alo-groups useful in the sequel; for further details the reader can refer to (Cavallo and D'Apuzzo, 2009).

Let G be a non-empty set, $\odot : G \times G \rightarrow G$ a binary operation on G, $\leq$ a weak order on G. Then, $\mathcal{G} = (G, \odot, \leq)$ is an *Abelian linearly ordered group*, *Alo-group* for short, if $(G, \odot)$ is an Abelian group and, for all $g_1, g_2, g_3 \in G$, it holds:

$$g_1 \leq g_2 \Rightarrow g_1 \odot g_3 \leq g_2 \odot g_3. \tag{1}$$

Let us denote with e the *identity element*, $g^{(-1)}$ the *inverse element* of $g \in G$ with respect to $\odot$, and $\div$ the *inverse operation* defined by $g_1 \div g_2 = g_1 \odot g_2^{(-1)}$ for all $g_1, g_2 \in G$.

Let $n \in \mathbb{N} \cup \{0\}$ and $g \in G$; then, the (n)-*natural- power* $g^{(n)}$ of $g \in G$ is:

$$g^{(n)} = \begin{cases} e, & \text{if} \quad n = 0 \\ g^{(n-1)} \odot g, & \text{if} \quad n \in \mathbb{N}. \end{cases} \tag{2}$$

By definition, an Alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *lattice ordered group* (Cavallo and D'Apuzzo, 2009), that is, for each $g_1, g_2 \in G$, there exists a maximal element $max\{g_1, g_2\}$. Cavallo and D'Apuzzo (2009) propose the following notions of $\mathcal{G}$-norm and $\mathcal{G}$-distance

$$\|.\|_{\mathcal{G}} : g \in G \rightarrow \|g\|_{\mathcal{G}} = \max\{g, g^{(-1)}\} \in G, \tag{3}$$

$$d_{\mathcal{G}} : (g_1, g_2) \in G \times G \to \|g_1 \div g_2\|_{\mathcal{G}} \in G; \tag{4}$$

the above notions are generalizations to $\mathcal{G}$ of the usual concepts of norm and distance.

2.1 Real continuous Alo-groups

An Alo-group $\mathcal{G} = (G, \odot, \leq)$ is called *continuous* if the operation $\odot$ is continuous, and *real* if G is a subset of the real line $\mathbb{R}$ and $\leq$ is the weak order on G inherited from the usual order on $\mathbb{R}$. From now on, we will assume that $\mathcal{G} = (G, \odot, \leq)$ is a real continuous Alo-group, with G being an open interval. Under these assumptions, the equation $x^{(n)} = g$ has a unique solution; thus, it is reasonable to consider the notions of (n)-*root* and $\mathcal{G}$-*mean* (Cavallo and D'Apuzzo, 2009).

For each $n \in \mathbb{N}$ and $g \in G$, the (n)-*root* of g, denoted by $g^{(\frac{1}{n})}$, is the unique solution of the equation $x^{(n)} = g$, that is:

$$\left(g^{(\frac{1}{n})}\right)^{(n)} = g,$$

and the $\mathcal{G}$-*mean* $m_{\mathcal{G}}(g_1, g_2, \dots, g_n)$ of the elements $g_1, g_2, \dots, g_n$ of G is expressed as follows:

$$m_{\mathcal{G}}(g_1, g_2, \dots, g_n) = \left(\bigodot_{i=1}^{n} g_i\right)^{(1/n)}. \tag{5}$$

Two real continuous Alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{H} = (H, \circ, \leq)$, with G and H being open intervals, are isomorphic (Cavallo and D'Apuzzo, 2009); thus, there exists a bijection $\phi : G \to H$ that is both a lattice isomorphism and group isomorphism, that is, for all $g_1, g_2 \in G$, it holds:

$$g_1 \leq g_2 \Leftrightarrow \phi(g_1) \leq \phi(g_2) \quad \textit{and} \quad \phi(g_1 \odot g_2) = \phi(g_1) \circ \phi(g_2). \tag{6}$$

Note that, the isomorphisms allows us to pass from an Alo-group to another isomorphic Alo-group; as an example, Cavallo and D'Apuzzo (2009) prove that:

$$d_{\mathcal{G}}(g_1, g_2) = \phi^{-1}(d_{\mathcal{H}}(\phi(g_1), \phi(g_2))), \tag{7}$$

$$m_{\mathcal{G}}(g_1, \dots, g_n) = \phi^{-1}(m_{\mathcal{H}}(\phi(g_1), \dots, \phi(g_n))), \tag{8}$$

with $g_1, \dots, g_n \in G$.

Examples of real continuous Alo-groups are the following ones:

Multiplicative Alo-group $\mathcal{M} = (\mathbb{R}^+, \cdot, \leq)$, where $\cdot$ is the usual multiplication on $\mathbb{R}$, $e = 1$ and the $\mathcal{M}$-mean of $m_1, m_2, \dots, m_n \in \mathbb{R}^+$ is the geometric mean:

$$m_{\mathcal{M}}(m_1, m_2, \dots, m_n) = \sqrt[n]{\prod_{i=1}^{n} m_i}. \tag{9}$$

$\mathcal{M}$-distance between m_1 and m_2 is $d_{\mathcal{M}}(m_1, m_2) = \max\{\frac{m_1}{m_2}, \frac{m_2}{m_1}\}$.

Additive Alo-group $\mathcal{A} = (\mathbb{R}, +, \leq)$, where $+$ is the usual addition on $\mathbb{R}$, $e = 0$ and the $\mathcal{A}$-mean of $a_1, a_2, \dots, a_n \in \mathbb{R}$ is the arithmetic mean:

$$m_{\mathcal{A}}(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^{n} a_i}{n}. \tag{10}$$

$\mathcal{A}$-distance between a_1 and a_2 is $d_{\mathcal{A}}(a_1, a_2) = \max\{a_1 - a_2, a_2 - a_1\} = |a_1 - a_2|$.

Fuzzy Alo-group $\mathcal{F} = (]0, 1[, \otimes, \leq)$, where $\otimes :]0, 1[^2 \to]0, 1[$ is the operation defined by

$$f_1 \otimes f_2 = \frac{f_1 f_2}{f_1 f_2 + (1 - f_1)(1 - f_2)}, \tag{11}$$

$e = 0.5$ and the $\mathcal{F}$-mean of $f_1, f_2, \dots, f_n \in]0, 1[$ is the following one:

$$m_{\mathcal{F}}(f_1, f_2, \dots, f_n) = \frac{\sqrt[n]{\prod_{i=1}^{n} f_i}}{\sqrt[n]{\prod_{i=1}^{n} f_i} + \sqrt[n]{\prod_{i=1}^{n} (1 - f_i)}}. \tag{12}$$

$\mathcal{F}$-distance between f_1 and f_2 is the following one:

$$d_{\mathcal{F}}(f_1, f_2) = max\left\{\frac{f_1(1 - f_2)}{f_1(1 - f_2) + (1 - f_1)f_2}, \frac{f_2(1 - f_1)}{f_2(1 - f_1) + (1 - f_2)f_1}\right\}. \tag{13}$$

Isomorphisms between multiplicative and additive Alo-groups are the following logarithmic and exponential functions:

$$h_a : x \in \mathbb{R}^+ \to \log_a x \in \mathbb{R} \quad h_a^{-1} : y \in \mathbb{R} \to a^y \in \mathbb{R}^+, \tag{14}$$

with a be a real number greater than 1.

Isomorphisms between multiplicative and fuzzy Alo-groups are the following ones:

$$g : x \in \mathbb{R}^+ \to \frac{x}{x + 1} \in]0, 1[, \quad g^{-1} : y \in]0, 1[\to \frac{y}{1 - y} \in \mathbb{R}^+. \tag{15}$$

For isomorphisms between further Alo-groups, the reader can refer to Cavallo and D'Apuzzo (2009) and Ramík (2015).

2.2 PCMs over an Alo-group

Let $\mathcal{G} = (G, \odot, \leq)$ be a real continuous Alo-group, with G an open interval. A PCM over $\mathcal{G} = (G, \odot, \leq)$ is defined as follows:

$$\mathbf{G} = [g_{ij}]_{n\times n} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}; \tag{16}$$

where $g_{ij} \in G$ quantifies the preference intensity of x_i over x_j (Cavallo and D'Apuzzo, 2009). We assume that each PCM $\mathbf{G} = [g_{ij}]_{n\times n}$ satisfies the following $\mathcal{G}$-reciprocity property:

$$g_{ji} = g_{ij}^{(-1)} \quad \forall\, i,j \in \{1,2,\dots,n\}.$$

Therefore, a multiplicative PCM is a PCM over the multiplicative Alo-group $\mathcal{M} = (\mathbb{R}^+, \cdot, \leq)$, an additive PCM is a PCM over the additive Alo-group $\mathcal{A} = (\mathbb{R}, +, \leq)$, and a fuzzy PCM is a PCM over the fuzzy Alo-group $\mathcal{F} = (]0,1[, \otimes, \leq)$.

Under the assumption of $\mathcal{G}$-reciprocity, we set:

$$x_i \succ_{\mathbf{G}} x_j \Leftrightarrow g_{ij} > e, \qquad x_i \sim_{\mathbf{G}} x_j \Leftrightarrow g_{ij} = e, \tag{17}$$

where $x_i \succ_{\mathbf{G}} x_j$ and $x_i \sim_{\mathbf{G}} x_j$ stand for "x_i is strictly preferred to x_j" and "x_i is indifferent to x_j", respectively, and we denote:

$$x_i \succsim_{\mathbf{G}} x_j \Leftrightarrow (x_i \succ_{\mathbf{G}} x_j \; or \; x_i \sim_{\mathbf{G}} x_j) \Leftrightarrow g_{ij} \geq e, \tag{18}$$

where $x_i \succsim_{\mathbf{G}} x_j$ stands for "x_i is weakly preferred to x_j".

The minimal level of coherence that a $\mathcal{G}$-reciprocal PCM should satisfy is the $\mathcal{G}$-transitivity. $\mathbf{G} = [g_{ij}]_{n\times n}$ is *$\mathcal{G}$-transitive* if verifies the following condition:

$$g_{ij} \geq e, \; g_{jk} \geq e \Rightarrow g_{ik} \geq e. \tag{19}$$

$\mathcal{G}$-transitivity ensures a rearrangement $(i_1, i_2, \dots, i_n)$ of $\{1,2,\dots,n\}$ such that:

$$x_{i_1} \succsim_{\mathbf{G}} x_{i_2} \succsim_{\mathbf{G}} \cdots \succsim_{\mathbf{G}} x_{i_n}. \tag{20}$$

2.2.1 Distance between PCMs

Let $\mathbf{G}^1 = [g^1_{ij}]_{n\times n}$ and $\mathbf{G}^2 = [g^2_{ij}]_{n\times n}$ be PCMs over $\mathcal{G} = (G, \odot, \leq)$; then, Cavallo (2019) provides the following $\mathcal{G}$-distance between $\mathbf{G}^1$ and $\mathbf{G}^2$:

$$d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^2) = \left(\bigodot_{i=1}^{n-1} \bigodot_{j=i+1}^{n} d_{\mathcal{G}}(g^1_{ij}, g^2_{ij}) \right)^{\left(\frac{2}{n(n-1)}\right)}. \tag{21}$$

The $\mathcal{G}$-distance in (21) can be interpreted as a $\mathcal{G}$-mean (5) of $\mathcal{G}$-distances $d_{\mathcal{G}}(g^1_{ij}, g^2_{ij})$, with $i < j$, and it satisfies the following properties:

1. $d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^2) \geq e$;

2. $d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^2) = e \Leftrightarrow \mathbf{G}^1 = \mathbf{G}^2$;
3. $d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^2) = d_{\mathcal{G}}(\mathbf{G}^2, \mathbf{G}^1)$;
4. $d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^2) \leq d_{\mathcal{G}}(\mathbf{G}^1, \mathbf{G}^3) \odot d_{\mathcal{G}}(\mathbf{G}^3, \mathbf{G}^2)$.

Example 1 Let us consider the following multiplicative PCMs:

$$\mathbf{M}^1 = [m_{ij}^1]_{4\times4} = \begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{7} & \frac{1}{5} \\ 8 & 1 & \frac{1}{3} & \frac{1}{6} \\ 7 & 3 & 1 & 3 \\ 5 & 6 & \frac{1}{3} & 1 \end{bmatrix}, \quad \mathbf{M}^2 = [m_{ij}^2]_{4\times4} = \begin{bmatrix} 1 & \frac{1}{9} & 6 & \frac{1}{6} \\ 9 & 1 & \frac{1}{3} & 7 \\ \frac{1}{6} & 3 & 1 & \frac{1}{2} \\ 6 & \frac{1}{7} & 2 & 1 \end{bmatrix}.$$

Then, the distance between them is calculated as

$$\begin{aligned} d_{\mathcal{M}}(\mathbf{M}^1, \mathbf{M}^2) &= \sqrt[6]{\prod_{i=1}^{3} \prod_{j=i+1}^{4} d_{\mathcal{M}}(m_{ij}^1, m_{ij}^2)} \\ &= (d_{\mathcal{M}}(m_{12}^1, m_{12}^2) \cdot d_{\mathcal{M}}(m_{13}^1, m_{13}^2) \cdot d_{\mathcal{M}}(m_{14}^1, m_{14}^2) \cdot d_{\mathcal{M}}(m_{23}^1, m_{23}^2) \cdot \\ &\quad \cdot d_{\mathcal{M}}(m_{24}^1, m_{24}^2) \cdot d_{\mathcal{M}}(m_{34}^1, m_{34}^2))^{\frac{1}{6}} = 4.926, \end{aligned}$$

where $d_{\mathcal{M}}(m_{ij}^1, m_{ij}^2) = \max \left\{ \frac{m_{ij}^1}{m_{ij}^2}, \frac{m_{ij}^2}{m_{ij}^1} \right\}$.

Example 2 Let us consider the following additive PCMs:

$$\mathbf{A}^1 = [a_{ij}^1]_{4\times4} = \begin{bmatrix} 0 & -8 & -7 & -5 \\ 8 & 0 & -3 & -6 \\ 7 & 3 & 0 & 3 \\ 5 & 6 & -3 & 0 \end{bmatrix}, \quad \mathbf{A}^2 = [a_{ij}^2]_{4\times4} = \begin{bmatrix} 0 & -9 & 6 & -6 \\ 9 & 0 & -3 & 7 \\ -6 & 3 & 0 & -2 \\ 6 & -7 & 2 & 0 \end{bmatrix}.$$

Then, the distance between them is calculated as

$$\begin{aligned} d_{\mathcal{A}}(\mathbf{A}^1, \mathbf{A}^2) &= \frac{\sum_{i=1}^{3} \sum_{j=i+1}^{4} d_{\mathcal{A}}(a_{ij}^1, a_{ij}^2)}{6} \\ &= \frac{1}{6}(d_{\mathcal{A}}(a_{12}^1, a_{12}^2) + d_{\mathcal{A}}(a_{13}^1, a_{13}^2) + d_{\mathcal{A}}(a_{14}^1, a_{14}^2) + d_{\mathcal{A}}(a_{23}^1, a_{23}^2) \\ &\quad + d_{\mathcal{A}}(a_{24}^1, a_{24}^2) + d_{\mathcal{A}}(a_{34}^1, a_{34}^2)) = 5.5, \end{aligned}$$

where $d_{\mathcal{A}}(a_{ij}^1, a_{ij}^2) = |a_{ij}^1 - a_{ij}^2|$. Note that, the distance can also be computed by applying the isomorphism in (14) as follows:

$$d_{\mathcal{A}}(\mathbf{A}^1, \mathbf{A}^2) = h_a\left(d_{\mathcal{M}}(h_a^{-1}(\mathbf{A}^1), h_a^{-1}(\mathbf{A}^2)) \right),$$

with $h_a^{-1}(\mathbf{A}^1) = [h_a^{-1}(a_{ij}^1)]_{4\times4}$ and $h_a^{-1}(\mathbf{A}^2) = [h_a^{-1}(a_{ij}^2)]_{4\times4}$ be multiplicative PCMs.

Example 3 Let us consider the following fuzzy PCMs:

$$\mathbf{F}^1 = [f^1_{ij}]_{4\times 4} = \begin{bmatrix} 0.5 & 0.1 & 0.2 & 0.2 \\ 0.9 & 0.5 & 0.3 & 0.1 \\ 0.8 & 0.7 & 0.5 & 0.8 \\ 0.8 & 0.9 & 0.2 & 0.5 \end{bmatrix}, \quad \mathbf{F}^2 = [f^2_{ij}]_{4\times 4} = \begin{bmatrix} 0.5 & 0.1 & 0.9 & 0.2 \\ 0.9 & 0.5 & 0.2 & 0.8 \\ 0.1 & 0.8 & 0.5 & 0.3 \\ 0.8 & 0.2 & 0.7 & 0.5 \end{bmatrix}.$$

Then, the following equality holds:

$$d_{\mathcal{F}}(\mathbf{F}^1, \mathbf{F}^2) = \frac{\sqrt[6]{\prod_{i=1}^{3}\prod_{j=i+1}^{4} d_{\mathcal{F}}(f^1_{ij}, f^2_{ij})}}{\sqrt[6]{\prod_{i=1}^{3}\prod_{j=i+1}^{4} d_{\mathcal{F}}(f^1_{ij}, f^2_{ij})} + \sqrt[6]{\prod_{i=1}^{3}\prod_{j=i+1}^{4} (1 - d_{\mathcal{F}}(f^1_{ij}, f^2_{ij}))}} = 0.84,$$

where $d_{\mathcal{F}}(f^1_{ij}, f^2_{ij}) = \max\left\{ \frac{f^1_{ij}(1-f^2_{ij})}{f^1_{ij}(1-f^2_{ij})+(1-f^1_{ij})f^2_{ij}}; \frac{f^2_{ij}(1-f^1_{ij})}{f^2_{ij}(1-f^1_{ij})+(1-f^2_{ij})f^1_{ij}} \right\}$. Note that, the distance can also be computed by applying the isomorphism in (15) as follows:

$$d_{\mathcal{F}}(\mathbf{F}^1, \mathbf{F}^2) = g\left(d_{\mathcal{M}}(g^{-1}(\mathbf{F}^1), g^{-1}(\mathbf{F}^2)) \right),$$

with $g^{-1}(\mathbf{F}^1) = [g^{-1}(f^1_{ij})]_{4\times 4}$ and $g^{-1}(\mathbf{F}^2) = [g^{-1}(f^2_{ij})]_{4\times 4}$ be multiplicative PCMs.

Applications of the $\mathcal{G}$-distance in (21) are provided by Brunelli and Cavallo (2020) for measuring incoherence of PCMs and by Cavallo and Ishizaka (2023) for computing matching errors in PCMs.

3 A comparative study

As introduced in Sect. 1, the study has been inspired by a validation study conducted by Saaty (2008). The goal is to compare the imprecision of multiplicative, additive and fuzzy approaches when the areas of four geometric figures have to be compared. It means that the geometric figures have to be pairwise compared by means of area ratios on the interval $]0, +\infty[$, area differences on the interval $]-\infty, +\infty[$ and relatives intensities of the areas on the interval $]0, 1[$. The study has been conducted by designing an experiment, where a set of respondents pairwise compared (by approximation) the areas of the geometric figures and then, the imprecision of their assessments was measured by computing the distance with the exact pairwise comparisons. We grouped the respondents in such a way that each participant was allowed to deal with a unique type of approach.

3.1 Geometric figures and exact PCMs for the experiment

In this section, we provide the geometric figures used in the experiment and the exact multiplicative, additive and fuzzy PCMs. Let us consider the circle, the rectangle,

Fig. 1 Geometric figures used in the experiment for measuring the imprecision of the multiplicative and fuzzy approaches

the triangle and the square in Figure 1, which areas are $A_1 = 9.126$, $A_2 = 2.553$, $A_3 = 4.503$ and $A_4 = 1.3$, respectively.

By pairwise comparing the areas, we obtain the following multiplicative and fuzzy PCMs:

$$\mathbf{M} = [m_{ij}]_{4\times4} = \left[\frac{A_i}{A_j}\right]_{4\times4} = \begin{bmatrix} 1 & 3.574 & 2.027 & 7.020 \\ 0.280 & 1 & 0.567 & 1.964 \\ 0.493 & 1.764 & 1 & 3.464 \\ 0.142 & 0.509 & 0.289 & 1 \end{bmatrix}; \tag{22}$$

$$\mathbf{F} = [f_{ij}]_{4\times4} = \left[\frac{A_i}{A_i + A_j}\right]_{4\times4} = \begin{bmatrix} 0.5 & 0.781 & 0.670 & 0.875 \\ 0.219 & 0.5 & 0.362 & 0.663 \\ 0.330 & 0.638 & 0.5 & 0.776 \\ 0.125 & 0.337 & 0.224 & 0.5 \end{bmatrix}. \tag{23}$$

$\mathbf{M} = [m_{ij}]_{4\times4}$ and $\mathbf{F} = [f_{ij}]_{4\times4}$ represent the area ratios on the interval $]0, +\infty[$ and the relative intensities of the areas on the interval $]0, 1[$, respectively. As an example, the fourth columns of the PCMs in (22) and (23) represent the pairwise comparisons between the area of each geometric figure in Fig. 1 and the area of the square.

Note that for all real positive numbers c, figures with areas equal to $c \cdot A_1$, $c \cdot A_2$, $c \cdot A_3$ and $c \cdot A_4$ generate the PCMs in (22) and (23).

The multiplicative PCM $\mathbf{M} = [m_{ij}]_{4\times4}$ and the fuzzy PCM $\mathbf{F} = [f_{ij}]_{4\times4}$ are isomorphic; indeed, for each $i, j \in \{1, \dots, 4\}$, we have:

$$f_{ij} = g(m_{ij}) = \frac{m_{ij}}{m_{ij} + 1}, \tag{24}$$

where g is the isomorphism in (15).

Let a be a real number greater than 1. Then, the following additive PCM:

$$\mathbf{A} = [a_{ij}]_{4\times4} = [\log_a A_i - \log_a A_j]_{4\times4}$$

is isomorphic to both $\mathbf{M}$ and $\mathbf{F}$; indeed, for each $i, j \in \{1, \dots, 4\}$, we have:

$$a_{ij} = h_a(m_{ij}) = \log_a(m_{ij}), \tag{25}$$

and

$$a_{ij} = h_a(g^{-1}(f_{ij})) = \log_a\left(\frac{f_{ij}}{1-f_{ij}}\right), \tag{26}$$

where h_a and g are the isomorphisms in (14) and (15), respectively. Note that, the additive PCM with entries equal to $A_i - A_j$ is isomorphic neither to $\mathbf{M}$ nor to $\mathbf{F}$.

Let us set $a = A_4$; then, we obtain the following additive PCM:

$$\mathbf{A} = [a_{ij}]_{4\times 4} = [\log_{A_4} A_i - \log_{A_4} A_j]_{4\times 4} = \begin{bmatrix} 0 & 4.855 & 2.692 & 7.428 \\ -4.855 & 0 & -2.163 & 2.573 \\ -2.692 & 2.163 & 0 & 4.735 \\ -7.428 & -2.573 & -4.735 & 0 \end{bmatrix}. \tag{27}$$

$\mathbf{A} = [a_{ij}]_{4\times 4}$ represents the area differences on the interval $]-\infty, +\infty[$ of the geometric figures in Fig. 2, which areas are $\log_{A_4} A_1 = 8.428$, $\log_{A_4} A_2 = 3.573$, $\log_{A_4} A_3 = 5.735$ and $\log_{A_4} A_4 = 1$. As an example, the fourth column of the PCM in (27) represents the pairwise comparisons between the area of each geometric figure in Fig. 2 and the area of the square.

3.2 The methodology

We applied an opinion survey with a sample of 108 students of master courses in economics and management at the School of Business Administration in Karvina of the Silesian University, Czech Republic, and 120 students of the Department of Architecture of University of Naples Federico II, Italy. The students completed the survey during a class period and received instructions to fill it out. On average, the experiment lasted twenty minutes.

As done by Cavallo et al. (2019), in the design of the experiment, we created four surveys and we grouped the respondents by survey, in such a way that each participant was allowed to answer only one. Thus, each survey was filled out by $N = 57$ participants (i.e., 27 Czech students and 30 Italian students). This "between-subject" designed experiment was more appropriate for our purpose than a "within-subject" designed experiment, where participants were exposed to more than one survey (i.e., all the four surveys). It has been long observed that within-subject cannot be used when independence of multiple exposure is not warrant; in fact, the respondents

Fig. 2 Geometric figures used in the experiment for measuring the imprecision of the additive approach

have a reference point when they are responding the second survey. The reader can refer to the literature mentioned by Cavallo et al. (2019).

The survey forms are the following ones:

Q_M The respondent has to perform 6 pairwise comparisons; that is, for each pair of figures in Fig. 1, (s)he has to select the bigger one and to establish the ratio between the biggest area and the smallest (i.e., a real number in $[1, +\infty[$). As an example, if two areas are equal to 5 and 3 then the ratio is equal to $\frac{5}{3} = 1.667$;

Q_A The respondent has to perform 6 pairwise comparisons; that is, for each pair of figures in Figure 2, (s)he has to select the bigger one and to establish the difference between the biggest area and the smallest (i.e., a real number in $[0, +\infty[$), by assuming that area of the square, which represents the measure unit, is equal to 1. As an example, if two areas are equal to 5 and 3 then the difference is equal to $5 - 3 = 2$;

Q_F The respondent has to perform 6 pairwise comparisons; that is, for each pair of figures in Fig. 1, (s)he has to select the bigger one and to establish the relative intensity of the biggest area on the interval $[0.5,\ 1[$ in such a way that the sum of the intensities of the two areas is equal to 1. As an example, if two areas are equal to 5 and 3 then the biggest intensity is equal to $\frac{5}{5+3} = 0.625$ (the smallest is equal to 0.375);

Q_S The respondent has to perform 6 pairwise comparisons; that is, for each pair of figures in Fig. 1, (s)he has to select the bigger one and to establish how much the biggest figure is bigger than the other one by using a positive integer value in Table 1. As an example, if two areas are equal to 5 and 3 then the closest value in the Saaty's scale to the ratio $\frac{5}{3}$ is equal to 2.

Note that, although there is not an Alo-group defined on Saaty's scale, since multiplicative PCMs over Saaty's scale are used in several real world applications,

Table 1 Saaty's scale

Value	Semantic meaning	Inverse value
1	the two figures have the same area	1
3	a figure is weakly bigger than the other one	$\frac{1}{3}$
5	a figure is strongly bigger than the other one	$\frac{1}{5}$
7	a figure is very strongly bigger than the other one	$\frac{1}{7}$
9	a figure is absolutely bigger than the other one	$\frac{1}{9}$
2, 4, 6, 8	intermediate values	$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$

they will be also considered in our experiment; thus, we assume that the exact pairwise comparisons are provided by the multiplicative PCM in (22) and we set $\mathbf{S} = [s_{ij}]_{4\times4} = \mathbf{M} = [m_{ij}]_{4\times4}$.

The survey forms have been developed by means of Google Forms (https://www.google.it/intl/en/forms/about/) in such a way that each question is mandatory (each PCM is complete); moreover, we added constraints that allows us to obtain a suitable range for the entries of the PCMs. As an example, Fig. 3 provides a question in the multiplicative survey Q_M.

In order to measure and compare the imprecision of the four types of PCMs (i.e., multiplicative, additive, fuzzy, Saaty), we perform the following steps:

STEP 1. By the pairwise comparisons of the areas of circle, rectangle, triangle and square in Fig. 1 and in Fig. 2 performed by the students by filling out the assigned survey, we obtain the following four sets:

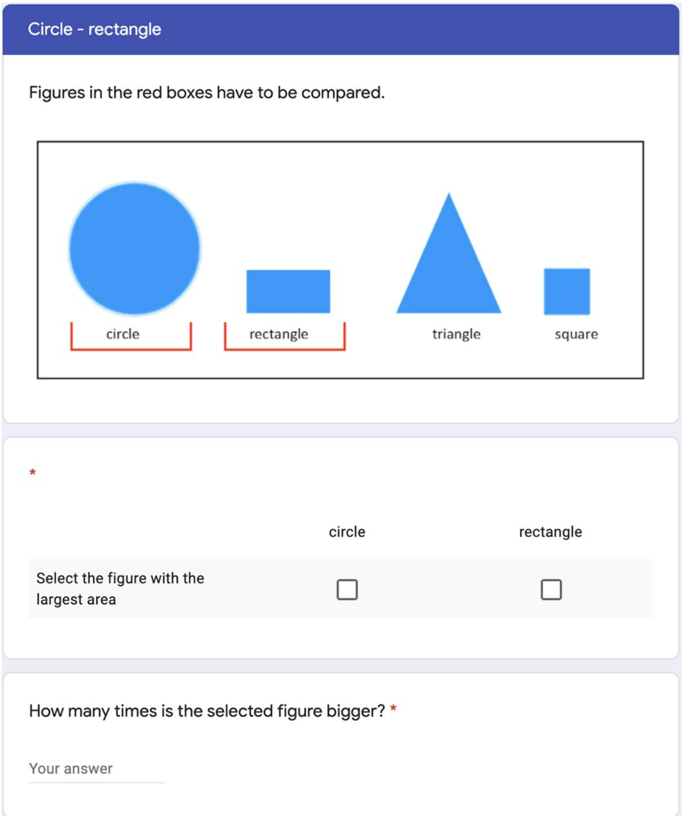

Fig. 3 Example of multiplicative question

- $M = \{\mathbf{M}^k = [m_{ij}^k]_{4\times4}\ |k \in \{1, \ldots, N\}\}$, with $\mathbf{M}^k = [m_{ij}^k]_{4\times4}$ be a multiplicative PCM, approximation of $\mathbf{M}$ in (22), obtained by the k-th student who filled out Q_M;
- $A = \{\mathbf{A}^k = [a_{ij}^k]_{4\times4}\ |k \in \{1, \ldots, N\}\}$, with $\mathbf{A}^k = [a_{ij}^k]_{4\times4}$ be an additive PCM, approximation of $\mathbf{A}$ in (27), obtained by the k-th student who filled out Q_A;
- $F = \{\mathbf{F}^k = [f_{ij}^k]_{4\times4}\ |k \in \{1, \ldots, N\}\}$, with $\mathbf{F}^k = [f_{ij}^k]_{4\times4}$ be a fuzzy PCM, approximation of $\mathbf{F}$ in (23), obtained by the k-th student who filled out Q_F;
- $S = \{\mathbf{S}^k = [s_{ij}^k]_{4\times4}\ |k \in \{1, \ldots, N\}\}$, with $\mathbf{S}^k = [s_{ij}^k]_{4\times4}$ be a multiplicative PCM, approximation of $\mathbf{M}$ in (22) with entries on the Saaty's scale, obtained by the k-th student who filled out Q_S.

STEP 2. For each $k \in \{1, \ldots, N\}$, we compute the following multiplicative PCMs:

$$\begin{aligned} h_{A_4}^{-1}(\mathbf{A}^k) =& [h_{A_4}^{-1}(a_{ij}^k)]_{4\times4}; \\ g^{-1}(\mathbf{F}^k) =& [g^{-1}(f_{ij}^k)]_{4\times4}, \end{aligned}$$

that are isomorphic to $\mathbf{A}^k$ and $\mathbf{F}^k$, respectively, and they are approximations of $\mathbf{M}$ in (22).

STEP 3. For each $k \in \{1, \ldots, N\}$, by applying the distance (21) between two PCMs over the multiplicative Alo-group, we compute the following imprecisions of the approximations:

$$d_{\mathcal{M}}(\mathbf{M}, \mathbf{M}^k) = \sqrt[\frac{n(n-1)}{2}]{\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} d_{\mathcal{M}}(m_{ij}, m_{ij}^k)};$$

$$d_{\mathcal{M}}(\mathbf{M}, h_{A_4}^{-1}(\mathbf{A}^k)) = \sqrt[\frac{n(n-1)}{2}]{\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} d_{\mathcal{M}}\left(m_{ij}, h_{A_4}^{-1}(a_{ij}^k)\right)};$$

$$d_{\mathcal{M}}(\mathbf{M}, g^{-1}(\mathbf{F}^k)) = \sqrt[\frac{n(n-1)}{2}]{\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} d_{\mathcal{M}}\left(m_{ij}, g^{-1}(f_{ij}^k)\right)};$$

$$d_{\mathcal{M}}(\mathbf{M}, \mathbf{S}^k) = \sqrt[\frac{n(n-1)}{2}]{\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} d_{\mathcal{M}}(m_{ij}, s_{ij}^k)}.$$

STEP 4. For each approach, we compute the geometric mean of the imprecisions:

$$m_{\mathcal{M}}\left(d_{\mathcal{M}}(\mathbf{M},\mathbf{M}^1),\ldots,d_{\mathcal{M}}(\mathbf{M},\mathbf{M}^N)\right) = \sqrt[N]{\prod_{k=1}^{N} d_{\mathcal{M}}(\mathbf{M},\mathbf{M}^k)};$$

$$m_{\mathcal{M}}\left(d_{\mathcal{M}}\left(\mathbf{M},h_{A_4}^{-1}(\mathbf{A}^1)\right),\ldots,d_{\mathcal{M}}\left(\mathbf{M},h_{A_4}^{-1}(\mathbf{A}^N)\right)\right) = \sqrt[N]{\prod_{k=1}^{N} d_{\mathcal{M}}\left(\mathbf{M},h_{A_4}^{-1}(\mathbf{A}^k)\right)};$$

$$m_{\mathcal{M}}\left(d_{\mathcal{M}}\left(\mathbf{M},g^{-1}(\mathbf{F}^1)\right),\ldots,d_{\mathcal{M}}\left(\mathbf{M},g^{-1}(\mathbf{F}^N)\right)\right) = \sqrt[N]{\prod_{k=1}^{N} d_{\mathcal{M}}\left(\mathbf{M},g^{-1}(\mathbf{F}^k)\right)};$$

$$m_{\mathcal{M}}\left(d_{\mathcal{M}}(\mathbf{M},\mathbf{S}^1),\ldots,d_{\mathcal{M}}(\mathbf{M},\mathbf{S}^N)\right) = \sqrt[N]{\prod_{k=1}^{N} d_{\mathcal{M}}(\mathbf{M},\mathbf{S}^k)}.$$

They represent the averages (i.e., geometric means) of the imprecisions, computed on the multiplicative Alo-group, obtained by the N surveys Q_M, the N surveys Q_A, the N surveys Q_F and the N surveys Q_S, respectively.

Remark 1 The choices done in STEP 2, STEP 3 and STEP 4 to choose the multiplicative Alo-group for computing the approximations, the imprecisions and the averages of the imprecisions, do not affect the results. Indeed, if, instead to compare the geometric means of the imprecisions in STEP 4, we compare the isomorphic arithmetic means on the additive Alo-group or the isomorphic fuzzy means on the fuzzy Alo-group, then, we obtain equivalent results, because by properties of $\mathcal{G}$-distance and $\mathcal{G}$-mean in (7) and (8), by relationships among $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{F}$ in (24), (25) and (26), and isomorphisms in (14) and (15), we have that:

$$\begin{aligned}
&m_{\mathcal{M}}\left(d_{\mathcal{M}}(\mathbf{M},\mathbf{M}^1),\ldots,d_{\mathcal{M}}(\mathbf{M},\mathbf{M}^N)\right) = h_{A_4}^{-1}\left(m_{\mathcal{A}}\left(d_{\mathcal{A}}(\mathbf{A},h_{A_4}(\mathbf{M}^1)),\ldots,d_{\mathcal{A}}(\mathbf{A},h_{A_4}(\mathbf{M}^N))\right)\right)\\
&\quad = g^{-1}\left(m_{\mathcal{F}}\left(d_{\mathcal{F}}(\mathbf{F},g(\mathbf{M}^1)),\ldots,d_{\mathcal{F}}(\mathbf{F},g(\mathbf{M}^N))\right)\right);\\
&m_{\mathcal{M}}\left(d_{\mathcal{M}}\left(\mathbf{M},h_{A_4}^{-1}(\mathbf{A}^1)\right),\ldots,d_{\mathcal{M}}\left(\mathbf{M},h_{A_4}^{-1}(\mathbf{A}^N)\right)\right) = h_{A_4}^{-1}\left(m_{\mathcal{A}}\left(d_{\mathcal{A}}(\mathbf{A},\mathbf{A}^1),\ldots,d_{\mathcal{A}}(\mathbf{A},\mathbf{A}^N)\right)\right)\\
&\quad = g^{-1}\left(m_{\mathcal{F}}\left(d_{\mathcal{F}}\left(\mathbf{F},g(h_{A_4}^{-1}(\mathbf{A}^1))\right),\ldots,d_{\mathcal{F}}\left(\mathbf{F},g(h_{A_4}^{-1}(\mathbf{A}^N))\right)\right)\right);\\
&m_{\mathcal{M}}\left(d_{\mathcal{M}}\left(\mathbf{M},g^{-1}(\mathbf{F}^1)\right),\ldots,d_{\mathcal{M}}\left(\mathbf{M},g^{-1}(\mathbf{F}^N)\right)\right) = h_{A_4}^{-1}\left(m_{\mathcal{A}}\left(d_{\mathcal{A}}\left(\mathbf{A},h_{A_4}(g^{-1}(\mathbf{F}^1))\right),\ldots,d_{\mathcal{A}}\left(\mathbf{A},h_{A_4}(g^{-1}(\mathbf{F}^N))\right)\right)\right)\\
&\quad = g^{-1}\left(m_{\mathcal{F}}\left(d_{\mathcal{F}}(\mathbf{F},\mathbf{F}^1),\ldots,d_{\mathcal{F}}(\mathbf{F},\mathbf{F}^N)\right)\right);\\
&m_{\mathcal{M}}\left(d_{\mathcal{M}}(\mathbf{M},\mathbf{S}^1),\ldots,d_{\mathcal{M}}(\mathbf{M},\mathbf{S}^N)\right) = h_{A_4}^{-1}\left(m_{\mathcal{A}}\left(d_{\mathcal{A}}(\mathbf{A},h_{A_4}(\mathbf{S}^1),\ldots,d_{\mathcal{A}}(\mathbf{A},h_{A_4}(\mathbf{S}^N))\right)\right)\\
&\quad = g^{-1}\left(m_{\mathcal{F}}\left(d_{\mathcal{F}}(\mathbf{F},g(\mathbf{S}^1),\ldots,d_{\mathcal{F}}(\mathbf{F},g(\mathbf{S}^N))\right)\right).
\end{aligned}$$

In particular, since the isomorphisms h_{A_4} and g are strictly increasing functions, the order is preserved by passing from multiplicative approach to arithmetic or fuzzy one. Thus, the choice of the approach, among multiplicative, additive and fuzzy, to compare the imprecisions could be a measurement option in our methodology; indeed, for each choice of approach, we reach the same result on the imprecision. However, we stress that our choice (i.e., multiplicative approach) is the most efficient because, in STEP 1, both the PCMs M_k and the PCMs S_k are multiplicative PCMs; thus, the multiplicative choice allows us to compute the smallest number of isomorphic PCMs in STEP 2.

The following example shows how the above steps are performed in the experimentation.

Example 4 Let us suppose that eight students fill out the questionnaires Q_M, Q_A, Q_F and Q_S.

STEP 1. By the questionnaires, we obtain the following multiplicative, additive, fuzzy and Saaty PCMs:

$$\mathbf{M^1} = \begin{bmatrix} 1 & 4 & 1.9 & 6.5 \\ 0.25 & 1 & 0.5 & 2 \\ 0.526 & 2 & 1 & 3 \\ 0.154 & 0.5 & 0.333 & 1 \end{bmatrix}, \quad \mathbf{M^2} = \begin{bmatrix} 1 & 3.8 & 2 & 6 \\ 0.263 & 1 & 0.333 & 1.5 \\ 0.5 & 3 & 1 & 3.2 \\ 0.167 & 0.667 & 0.313 & 1 \end{bmatrix};$$

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 4 & 3.5 & 8.3 \\ -4 & 0 & -2.5 & 2 \\ -3.5 & 2.5 & 0 & 6 \\ -8.3 & -2 & -6 & 0 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 3.2 & 1 & 7 \\ -3.2 & 0 & -3 & 3.5 \\ -1 & 3 & 0 & 4 \\ -7 & -3.5 & -4 & 0 \end{bmatrix};$$

$$\mathbf{F}^1 = \begin{bmatrix} 0.5 & 0.8 & 0.6 & 0.9 \\ 0.2 & 0.5 & 0.4 & 0.6 \\ 0.4 & 0.6 & 0.5 & 0.8 \\ 0.1 & 0.4 & 0.2 & 0.5 \end{bmatrix}, \quad \mathbf{F}^2 = \begin{bmatrix} 0.5 & 0.75 & 0.6 & 0.85 \\ 0.25 & 0.5 & 0.35 & 0.65 \\ 0.4 & 0.65 & 0.5 & 0.7 \\ 0.15 & 0.35 & 0.3 & 0.5 \end{bmatrix};$$

$$\mathbf{S^1} = \begin{bmatrix} 1 & 3 & 2 & 7 \\ \frac{1}{3} & 1 & \frac{1}{3} & 2 \\ \frac{1}{2} & 3 & 1 & 3 \\ \frac{1}{7} & \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}, \quad \mathbf{S^2} = \begin{bmatrix} 1 & 5 & 2 & 6 \\ \frac{1}{5} & 1 & \frac{1}{3} & 3 \\ \frac{1}{2} & 3 & 1 & 4 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}.$$

STEP 2. We compute the following multiplicative PCMs:

$$h_{A_4}^{-1}(\mathbf{A}^1) = \begin{bmatrix} 1 & 2.856 & 2.505 & 8.825 \\ 0.350 & 1 & 0.519 & 1.690 \\ 0.399 & 1.927 & 1 & 4.827 \\ 0.113 & 0.592 & 0.207 & 1 \end{bmatrix}, \quad h_{A_4}^{-1}(\mathbf{A}^2) = \begin{bmatrix} 1 & 2.315 & 1.3 & 6.275 \\ 0.432 & 1 & 0.455 & 2.505 \\ 0.769 & 2.197 & 1 & 2.856 \\ 0.159 & 0.399 & 0.350 & 1 \end{bmatrix};$$

$$g^{-1}(\mathbf{F}^1) = \begin{bmatrix} 1 & 4 & 1.5 & 9 \\ 0.25 & 1 & 0.667 & 1.5 \\ 0.667 & 1.5 & 1 & 4 \\ 0.111 & 0.667 & 0.25 & 1 \end{bmatrix}, \quad g^{-1}(\mathbf{F}^2) = \begin{bmatrix} 1 & 3 & 1.5 & 5.667 \\ 0.333 & 1 & 0.538 & 1.857 \\ 0.667 & 1.857 & 1 & 2.333 \\ 0.176 & 0.538 & 0.429 & 1 \end{bmatrix}.$$

STEP 3. We compute the following multiplicative imprecisions:

$$d_{\mathcal{M}}(\mathbf{M}, \mathbf{M}^1) = 1.094, \quad d_{\mathcal{M}}(\mathbf{M}, \mathbf{M}^2) = 1.203;$$
$$d_{\mathcal{M}}(\mathbf{M}, h_{A_4}^{-1}(\mathbf{A}^1)) = 1.229, \quad d_{\mathcal{M}}(\mathbf{M}, h_{A_4}^{-1}(\mathbf{A}^2)) = 1.316;$$
$$d_{\mathcal{M}}(\mathbf{M}, g^{-1}(\mathbf{F}^1)) = 1.229, \quad d_{\mathcal{M}}(\mathbf{M}, g^{-1}(\mathbf{F}^2)) = 1.220;$$
$$d_{\mathcal{M}}(\mathbf{M}, \mathbf{S}^1) = 1.159, \quad d_{\mathcal{M}}(\mathbf{M}, \mathbf{S}^2) = 1.307.$$

STEP 4. We compute the following geometric means $m_{\mathcal{M}}$ of the multiplicative imprecisions $d_{\mathcal{M}}$:

$$m_{\mathcal{M}}(1.094,\ 1.203) = 1.148;$$
$$m_{\mathcal{M}}(1.229,\ 1.316) = 1.271;$$
$$m_{\mathcal{M}}(1.229,\ 1.220) = 1.224;$$
$$m_{\mathcal{M}}(1.159,\ 1.307) = 1.230;$$

obtained by surveys Q_M, Q_A, Q_F and Q_S, respectively.

By 1.148 < 1.224 < 1.230 < 1.271, we have that the multiplicative PCMs are on average more precise than fuzzy PCMs that are on average more precise than Saaty PCMs that are on average more precise than additive PCMs.

As stressed in Remark 1, we would have obtained the same order if we had computed the isomorphic additive imprecisions $d_{\mathcal{A}}$ or fuzzy imprecisions $d_{\mathcal{F}}$ and then, compared the isomorphic arithmetic means $m_{\mathcal{A}}$ or the fuzzy means $m_{\mathcal{F}}$.

3.3 Results and discussion

In each survey, by selecting the bigger figure for each pair of figures (e.g., see Fig. 3), we gave students the possibility to provide an ordinal ranking of the areas of the figures even when wrong areas (obtained by wrong entries of the PCMs) were provided. Thus, we discarded about 10% of the surveys, i.e., surveys where the students were not transitive or provided an incorrect ranking of the figures (see (19) and (20)); indeed, the selection of the wrong figure shows incoherence or an error that does not depend by the method (multiplicative, additive, fuzzy or Saaty). The averages of the imprecisions obtained from the surveys Q_M, Q_A, Q_F and Q_S, by performing STEP 4 of the our methodology, are shown in Table 2.

Table 2 shows that the multiplicative approach is always the most precise because the average of the imprecisions obtained from Q_M is the smallest for both Italy and Czechia. Note that, for Italian respondents, the biggest average of the imprecisions is provided by Saaty's scale and, for Czech respondents, it is provided by the fuzzy approach. Of course, as stressed in Remark 1, by applying the isomorphisms in (14), with $a = 1.3$, and (15), we obtain the same rankings of the averages of the imprecisions (see Table 3 and Table 4).

To analyze if the differences among the means are significant, we performed ANalysis Of VAriance (ANOVA) tests at a significance level $\alpha = 0.05$. A significance level of 0.05 indicates a 5% risk of concluding that a difference exists when there is no actual difference. An ANOVA test (King, 2010) is a way to find out if survey or experiment results are significant. It is a type of statistical test used to determine if there is a statistically significant difference between categorical groups by testing for differences of means using variance. We recall that if ANOVA test provides a p-value less than the significance level then the null hypothesis is rejected, and we can conclude that not all of the group means are equal. The F value is used along with the p-value in deciding whether the results are significant enough to reject the null hypothesis. If ANOVA test provides a large F value (i.e. one that is bigger than the critical value F_{crit}), it means the difference is significant. Statistical analyses were conducted using StatPlus (https://www.analystsoft.com/en/).

ANOVA results are shown in Table 5; thus, we have that:

- for Italy, the null hypothesis is rejected and there are highly significant differences among all the means provided by Q_M, Q_A, Q_F and Q_S because p-value is equal to 0 and F value is higher than the critical value F_{crit};
- for Czechia, there are not significant differences among the means provided by Q_M, Q_A, Q_F and Q_S because p-value is greater than the significance level α and F value is smaller than the critical value F_{crit}.

Since, for Italian respondents, the null hypothesis is rejected, we performed all pairwise comparisons in order to determine which differences are statistically significant. The results are shown in Table 6. In particular, we stress that the difference between the mean provided by Q_M and each other mean is significant.

Thus, we can conclude that statistically, while for Czech respondents all the approaches provide the same imprecision, for Italian respondents the multiplicative approach provides the least imprecision. A possible reason for the different results

Table 2 Averages (i.e., geometric means) of imprecisions obtained in STEP 4 for each University

University	$m_{\mathcal{M}}$			
	Q_M	Q_A	Q_F	Q_S
Italy	1.321	1.485	1.598	1.720
Czechia	1.382	1.416	1.501	1.493

Table 3 Averages (i.e., arithmetic means) of imprecisions for each University

University	$m_{\mathcal{A}}$			
	Q_M	Q_A	Q_F	Q_S
Italy	1.061	1.506	1.786	2.066
Czechia	1.233	1.326	1.547	1.528

Table 4 Averages (i.e., fuzzy means) of imprecisions for each University

University	$m_{\mathcal{F}}$			
	Q_M	Q_A	Q_F	Q_S
Italy	0.569	0.598	0.615	0.632
Czechia	0.580	0.586	0.6	0.599

Table 5 ANOVA results

University	$F_{crit} = 2.694$	
	p-value	F
Italy	0	19.099
Czechia	0.092	2.256

obtained for the two groups of respondents may be found in different background knowledge and former experience concerning pairwise comparisons method. Indeed, the students from the Silesian University had some former experience with AHP, and so with Saaty's scale, therefore, they could be more experienced when using pairwise comparisons. Saaty's approach should to be avoided for measuring objective data both whenever the additional notion of ratio is not given and whenever the exact values of the ratios are not close to the values in the Saaty's scale.

4 Conclusions and future work

To best of our knowledge, this paper represents the first comparative study conducted in order to establish for which type of PCM (i.e., multiplicative, additive, fuzzy), DMs are more precise when they compare objective data. By following the idea proposed by Cavallo et al. (2019), where the authors compare the coherence of multiplicative, additive and fuzzy approaches for subjective preferences, and inspired by a validation study conducted by Saaty (1980), where the author shows the effectiveness of the multiplicative PCMs for the prioritization of geometric figures, we performed an experiment in order to compare the precision of these types of PCMs when areas of geometric figures have to be pairwise measured.

The experiment involves students from Silesian University and University of Naples Federico II, and it shows that the multiplicative approach is the most precise; the difference is statistically significant for Italian respondents.

Table 6 Italian respondents. Pairwise multiple comparisons between means. Y= there is significant difference. N= there is not significant difference

	Q_M	Q_A	Q_F	Q_S
Q_M	–	Y	Y	Y
Q_A	–	–	N	Y
Q_F	–	–	–	N
Q_S	–	–	–	–

It should be noted that in an experiment setting the order of comparisons and the size of the problem cannot be neglected, see (Bozóki et al., 2013), hence our results cannot be simply generalized and further research is needed.

We stress that our experimental results are valid in comparisons of areas of several geometric figures but no experiment has been performed for establishing which type of comparison (i.e. multiplicative, additive, fuzzy or linguistic) is the most precise in comparison of further objective data, e.g. sizes of objects and weights of objects. The application of a specific approach can also depend on the DM's problem, his/her personal preferences and views; thus, our future work will be the application of our comparative methodology to further real decision making problems concerning the measure of objective data, e.g., weights of objects and electricity consumption of household appliances (Whitaker, 2007), proposed in validation examples of the AHP. Finally, in usual decision making problems, the comparisons between alternatives is often different from the comparisons of areas; indeed, in these problems there are not exact values to approximate because the decision maker has to express subjective preferences between alternatives; indeed, experimental results obtained by Cavallo et al. (2019) show that the additive comparisons provide the worst results when decision maker expresses subjective preferences.

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Data availability The data that support the findings of this study are available from the corresponding authors upon request.

Code availability Our research did not involve any code.

Declarations

Conflict of interest We have no conflict of interest to disclose.

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