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A New Contact Paradox

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Abstract
There is a well-known variety of contact paradoxes which are significantly linked to topology. The aim of this paper is to present a new paradox concerning contact with bodies composed of a denumerable infinity of parts. This paradox establishes the logical necessity, in a Newtonian context, of contact forces (herein called "phantom forces") that violate what is probably our most basic causal intuition, embodied in what I call the Principle of Influence: any force exerted on a body B induces (causes) change of movement of B or (inclusive disjunction) the emergence of internal forces in B. However, the above paradox can be made strictly compatible with a Newtonian framework by introducing phantom forces as ideal elements in the Hilbert sense, though it will be seen that this does not solve all the problems.

Keywords Contact · Infinity · Newtonian Forces · Paradox · Phantom Forces

1 Introduction. The Principle of Influence

A paradox is a particular type of problem, but there is no general consensus as to the exact nature of this particularity. Proposals range from the most demanding extreme to the most lax. The former includes the notion of paradox in the sense of logic (antinomy): contradiction between the conclusions of two inferences that seem equally sound. At the opposite extreme Sorensen (2003) could be cited, to whom a paradox is a kind of enigma, where none of its possible good responses (not all compatible with each other) need to be based on arguments. My own preferences tend toward the more lax sense, which also possesses the virtue of encompassing the other senses as particular cases. I understand a paradox as a problem arising from the conflict between two or more equally powerful intuitions (this is the meaning of the term adopted herein). So, for example, when both intuitions are based on apparently correct inferences from apparently true premises, a classical logical antinomy

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arises. At the opposite extreme, even optical illusions are examples of paradoxes (and I presume that Sorensen would agree with me on this). In these, at least one of the intuitions is purely visual ("visual paradoxes").

There is a well-known variety of contact paradoxes which are significantly linked to topology (Zimmerman, 1996). A second, less discussed, class of problems involves contact with bodies composed of a denumerable infinity of parts. Here the emphasis lies not so much in topology as in the possibility (or not) of coherently describing the physical interaction (forces involved) with such entities and/or the properties of such interaction (Alper & Bridger, 1998; Peijnenburg & Atkinson, 2010; Prosser, 2006). This will be my aim here. This is also where I shall seek to highlight the existence of a conflict between powerful intuitions that authorizes us to speak of paradox in the sense specified at the beginning. The order of presentation is as follows. This Sect.(1) introduces the principle of influence, which will be central to this paper up to Sect.5, and whose initial plausibility will also become evident. The following Sect.(2) introduces the infinite system, which (with several variations) will be involved in most of the analyses to be conducted. Having considered a non-paradoxical case in Sect.3 where the infinite system's properties are compatible with the principle of influence, the following Sect.(4) describes a variation of this system which is problematic. Its problematic nature is explained in Sect.5 as an incompatibility with the principle of influence (this is where a conflict between intuitions will be revealed; the source of the new contact paradox), which I propose to solve by renouncing this principle and by introducing the new concept of phantom force. Section6 explains that not all the paradoxical aspects of the infinite configuration analyzed can be accounted for by resorting to phantom forces, but the most important ones can (I hope). Section7 then presents a new, simpler and more elegant infinite configuration, whose analysis leads not only to the rejection of the principle of influence and the need for phantom forces, but to the actual need for phantom interactions (the relationship between the two concepts is explained here). Finally, Sect.8 puts forward (by way of conclusion) a way of understanding the role of phantom forces in the context of Newtonian mechanics that connects with the role of ideal elements in Hilbert's philosophy of mathematics. Furthermore, the essence of the new contact paradox is explained in detail.

1.1 The Principle of Influence

The notion of force is a theoretical construct that corresponds to a certain kind of physical experience. It must therefore be required that this construct has some kind of empirical relevance. Thus, a non-zero force acting on a body must have at least causal effects in the form of change of movement and/or non-zero internal forces induced within it. This is a plausible requirement for any reasonable theory on forces (even if it is not empirically correct, like Newtonian mechanics). The forces to which we are accustomed causally entail change of movement or (inclusive disjunction) the presence of some other forces (typically in the form of internal stresses in the bodies on which they act). This justifies the following intuitively evident principle, which I shall henceforth assume (unless expressly stated otherwise).

PRINCIPLE OF INFLUENCE: any force exerted on a body B induces (causes) change of movement of B or (inclusive disjunction) the emergence of internal forces in B.

Note that this principle cannot be summed up by stating that "force induces motion", i.e., basically that $F=m\cdot a$. Indeed, the term "F" in $F=m\cdot a$ designates the net force, which is the force causing acceleration (change of movement). But even when there is no acceleration and no net force, (balanced) forces may be present that cause the occurrence of internal stresses without causing change of movement. This is a possibility that is covered precisely by the principle of influence. The inclusive disjunction in its formulation makes it clear that, in addition to causing movement and internal forces, a force can cause movement without the emergence of internal forces (e.g. when acting upon a point particle) and can also cause the emergence of internal forces without movement (e.g. when a spring is compressed by the action of equal and opposite forces acting upon its ends). Note also that the term "induces" in the principle of influence formulation has the meaning of "causes" and is therefore sufficiently precise to be evaluated in the usual language of mathematical physics.

The role of principle of influence in infinite component systems is clear. Consider for example the following description of an infinite stack of slabs (Benardete, 1964). Lying upon the ground there is a slab of stone 1/2 thick, weighing 1/2. Resting squarely on this first slab is a second slab of stone 1/4 thick, weighing 1/4. Resting squarely on this second slab is a third slab of stone 1/8 thick, weighing 1/8, &c. *ad infinitum*. It follows that the i+1-th slab exerts an upward force of $1/2^{i+1}$ (on the i+2-th) and a downward force of $1/2^{i}$ (on the i-th). Now, suppose that a man of weight P climbs to the top of the stack of slabs. Obviously, he exerts a (downward) force of magnitude P on it. This force does not cause movement but it does cause new internal forces to emerge in the stack. The i+1-th slab now exerts an upward force of $P + (1/2^{i+1})$ (on the i+2-th) and a downward force of $P + (1/2^{i})$ (on the i-th).

Another illustrative example is as follows. Let us consider a set of infinite point particles p_{n+1} of identical unit mass at rest at points $x_n = 1/n$ (n=1,2,3,4….). Now let a particle p_1 also of unit mass and velocity v (moving to the left) approach them from the right (Laraudogoitia, 1996). Particle p_1 has been set in this state of motion by a given force F exerted on it to the left. Taking into account that in a binary collision between identical particles, the particles simply exchange their velocities, it is clear the final state resulting from the infinite sequence of successive binary collisions that takes place is as follows: an infinite set of point particles p_n of identical unit mass at rest at points $x_n = 1/n$ (n=1,2,3,4….). Indeed, velocity v is transferred first to p_2, then to p_3, and so on successively until no particle continues in movement. We shall use the term P for the system of infinite particles p_1, p_2, p_3, … p_n, …. Since it has been exerted on p_1, force F has also obviously been exerted on P. Even though it does not cause its movement [1], it does cause new internal forces to emerge in P. In effect, p_{i+1} experiences a force exerted on it by p_i and then exerts a force on p_{i+2}.

What makes the principle of influence eminently plausible is its weak character. It speaks of the causal effects of a force without going into much detail. For example, it does not specify exactly where the force that produces such effects acts (beyond generically indicating that it does so "on a body B"). This lack of specification averts numerous problems when dealing with material systems with an infinite number of component parts. For example, in the previous case of the man on the stack of slabs, what this principle says is quite clear and blatantly true. This is precisely because it makes no commitment to any specific statement concerning which part (if any) of the stack of slabs is in contact with the man and therefore feels his "direct and immediate" pressure. Also note that the principle of influence DOES NOT DEFINE the way in which a force acts, what it does is DESCRIBE (very succinctly)

[1] F causes the movement of p_1, but not of P, whose center of masses remains fixed at x = 0. By definition, the movement of P is the movement of its center of masses. Broadly speaking, as is well known, the movement of a material body is the movement of its center of masses.

the way in which a force acts. There is therefore no danger of circularity due to the fact that forces appear in it as possible effects of forces.

Neither is what I call the Principle of Influence (POI) a dubious eventual intuition, but rather something deeply rooted in the very nature of Newtonian mechanics. And it is easy to see why this is so. Newtonian mechanics is traditionally divided into three parts: statics, kinematics and dynamics. Only the first and last parts explicitly consider the role of forces. The first considers them as the cause of internal stresses in bodies in equilibrium and the third as the cause of their changes of movement. It is precisely these two aspects, internal stresses and change of movement, which are reflected in the POI. Its meaning is therefore clear and is very precisely grounded in the content provided by statics and dynamics as parts of classical mechanics. No one in the philosophy of physics community can be found who has postulated such a principle as the POI. The reason for this is not, however, its mysterious or controversial nature but rather its trivial nature: no one has as yet considered it worthwhile to make something so seemingly self-evident explicit. Of course (as previously stated) both aspects mentioned above (internal stresses and change of movement) do not always need to manifest themselves simultaneously. Indeed, if force F on a body is not balanced, it causes a change of movement (acceleration) and internal stresses, which would deform it were it not rigid (unless it is a point particle); and if F is balanced, it causes internal stresses (and not acceleration), which would also deform it were it not rigid. For example, two equal and opposite forces acting on an uncompressed spring cause only internal stresses within it, but no movement of its center of masses and, therefore, according to note (1), no movement of the spring itself (even though there is movement of the spring's two halves, which move towards each other during the compression process as a result of the non-balanced forces acting upon each one). Similarly, a couple of forces generates a torque that spins a homogeneous disc which also causes internal stresses therein, but its center of masses does not move (and, in this sense, the disc does not move, as stated in note (1), even though it rotates). Finally, an increase in the pressure of an ideal gas in volume V alters the internal stresses inside (its internal pressure) also without changing the position of its center of masses. Thus, it is clear that the POI is a very general principle in Newtonian mechanics, but by no means a vague principle. Its concrete, quantitative manifestations depend on the specific application of the laws of statics and/or dynamics, whose most general features are described by the POI. Finally, since forces are par excellence the most fundamental causal agents in a Newtonian conception of the world, the POI reflects what is probably our most fundamental causal intuition, given that it is a Newtonian conception that underlies most of our physical intuitions (now far from Aristotelian "common sense physics").

2 Starting Point

As shall be seen, the new contact paradox that I propose violates the principle of influence and is based on a variant of the configuration introduced for other purposes by Laraudogoitia 2020. This mechanical system is in turn a variation of previous similar configurations. The important point, however, (as will be seen in detail below) is that what I have to say drawing on my version is entirely new and different from what has been stated thus far in the context of literature on infinite systems. I shall describe the initial configuration as follows.

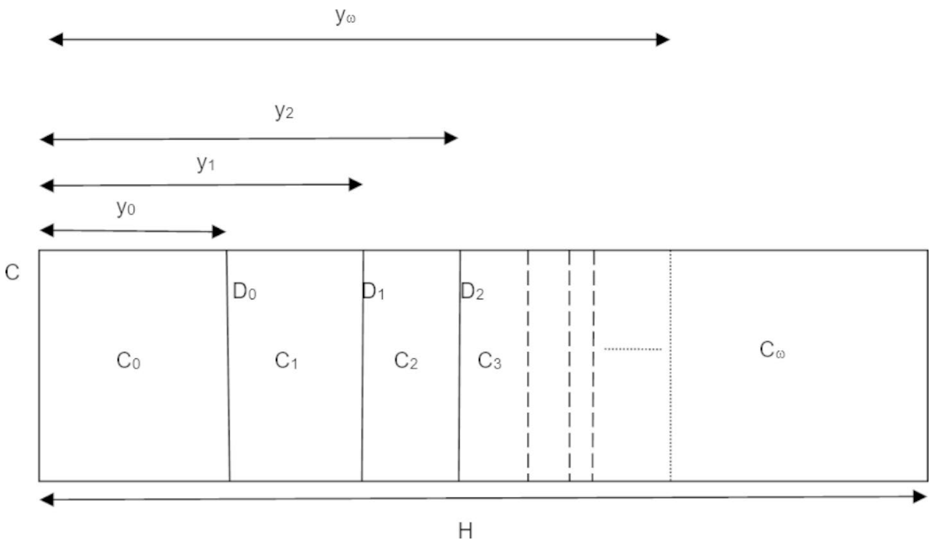

Figure1A hollow cylinder with an infinity of separate compartments

A rigid, hollow cylinder C of unit section and length H contains an infinite number of rigid circular disks D_0, D_1, D_2, D_3, …, D_n, … (order type ω) that fit perfectly inside C and divide its interior into an infinity of separate compartments C_0, C_1, C_2, C_3, …, C_n, … C_ω (order type ω + 1). The location of each D_i in the interior of C may be identified, for instance, by measuring its horizontal distance y_i from the left end of C, as shown in Fig. 1. Note that y_ω is the limit of the D_i locations (namely, $\lim_{i\to\infty} y_i = y_\omega$) but it is not the location of any disks. There is no D_ω. I assume that the masses of the D_i decrease sufficiently for the total mass of the configuration to be finite, and that the thickness of the D_i is decreasing sufficiently. I also assume that the C_i compartments ($0 < i < \omega$) are full of gas at different pressures. As we already know, where there is no friction between the D_i and C's inner surface, the evolution of the whole will direct us to a final location of the former (generally changing their y_i coordinates) that guarantees pressures will be equal in all the C_i compartments. In order to make things more interesting and highlight the new contact paradox presented herein, I will use only a marginally different context: suppose there is friction between every D_i and C to the extent that the net horizontal force required to move any D_i inside C must be greater than a certain finite value $k^* > 2$. This will ensure that no D_i slides inside C in any of the cases discussed below. It must also be assumed that C_0 and C_ω are initially empty, so their internal pressure is $p(C_0) = p(C_\omega) = 0$. I will use the following abbreviations to refer to forces exerted on material bodies by material bodies:

F[A/B] = Force that body A exerts on body B.

In the following only horizontal forces will be of interest, therefore F[A/B] will be positive or negative depending on whether the force of A on B is directed to the right or to the left respectively. Occasionally, in order to be clear, I shall also add the subscript notation:

F_x[A/B] = X-type force that body A exerts on body B.

So the case may arise where F[A/B] = F_x[A/B] + F_y[A/B] + ….

In particular, if the force that body A exerts on body B is type X, then F[A/B] = F_x [A/B]. To ease notation, when it is clear from the context that F[A/B] = F_x [A/B], I will not make this explicit and, in which case, I will use F_x [A/B] or F[A/B] indistinctly to designate exactly the same thing.

3 Case I. An Unproblematic Case

The following unproblematic case is first considered (which we will call Case I): $p(C_i)=1/i$ $(i>0)$. Clearly, $p(C_i)<k^*$, hence no D_i slides inside C. Given that $p(C_i)>p(C_{i+1})$ $(i>0)$, it is evident that every D_i $(i>0)$ is given a net push to the right E_i by the gases that it is in contact with. However, D_i is at rest, so a force $-E_i$ acts on it to the left, evidently exerted by C. So, each D_i $(i>0)$ pushes C to the right, in turn, with force E_i. Moreover, $F_{friction}[D_0/C]=-F_{friction}[C/D_0]$. Since the net force on D_0 must also be zero (considering that D_0 does not slide inside C either) $F_{friction}[C/D_0]+F[\text{the gas in } C_1/D_0]=0$. I shall use the abbreviation GC_α to refer to the material system formed by the gas in compartment C_α. It follows that $F_{friction}[C/D_0]=-F[GC_1/D_0]=1$. For $i>0$, $F_{friction}[D_i/C]=E_i=-F_{friction}[C/D_i]$. As the net force on D_i $(i>0)$ must be null (considering D_i does not slide inside C, which is at rest) and, therefore $F_{friction}[C/D_i]+F[\text{the gases in } C_i \text{ and } C_{i+1}/D_i]=0$ $(i>0)$, it follows that $-F_{friction}[C/D_i]=F[\text{the gases in } C_i \text{ and } C_{i+1}/D_i]=(1/i)-[1/(i+1)]=1/[i(i+1)]$ $(i>0)$. I will also use the abbreviation S_1+S_2 to denote the material system made up of subsystems S_1 and S_2 (S_1 and S_2 being exclusive and exhaustive parts of S_1+S_2). In general, with $S_1+S_2+S_3+\ldots$, I will designate the material system composed of the denumerable infinity of subsystems $S_1, S_2, S_3, \ldots$ ($S_1, S_2, S_3, \ldots$ being exclusive and exhaustive parts of $S_1+S_2+S_3+\ldots$). Therefore, $F_{friction}[D_i/C]=-F_{friction}[C/D_i]=F[GC_i+GC_{i+1}/D_i]=1/[i(i+1)]$ $(i>0)$ and hence $F_{friction}[D_1+D_2+D_3+\ldots/C]=\sum F_{friction}[D_i/C]=\sum E_i=\sum 1/[i(i+1)]=1$. This force is annulled by the force D_0 exerts on $C=F_{friction}[D_0/C]=-1$, thus we can deduce the existence of equilibrium.

4 Case II. A Problematic Case

Case II will differ from Case I in just one respect: compartment C_ω will also contain gas and, furthermore, at pressure $p(C_\omega)=1$ (therefore identical to $p(C_1)$). However this new pressure affects (obviously, at best, indirectly) the pressures of the gases in the other compartments, in no event would value 2 be exceeded (as $1+(1/i)\leq 2$). Since $2<k^*$, it follows that none of the D_i $(i\geq 0)$ pistons will alter their position inside cylinder C. Consequently, neither will any of the gas pressures. It will continue to be $p(C_i)=1/i$ $(i>0)$, $p(C_0)=0$ y $p(C_\omega)=1$. It should be acknowledged that this new configuration with non-empty compartment C_ω exemplifies the typical contact problem in the case of a series of material bodies of ordinal type $\omega+1$. This problem is by no means new in the literature. Nonetheless, as mentioned earlier, it is what I am about to say in relation it that is novel.

As previously stated, when I mention forces, I will always be referring to horizontal forces (understood as forces parallel to the cylinder axis, the X axis, so to speak). The reason is that the forces in a perpendicular direction to the X axis (which exist, and are exerted by the gases on the side walls of cylinder C) play no part in the problem to be discussed in this paper. Given the known pressures, it is still true that $F_{friction}[D_1+D_2+D_3+\ldots/C]=1$ y $F_{friction}[D_0/C]=-1$. But now too $F[GC_\omega/C]=1$. Since $F_{friction}[D_0/C]+F_{friction}[D_1+D_2+D_3+$

… /C] + F[the gas in C_ω/C] = 1 ≠ 0, apparently the net force on C is not null, so C would seem to cease to be in equilibrium simply because it has filled compartment C_ω. This semblance will turn out to be deceptive but is directly linked to the contact paradox discussed in this paper.

The gas in C_ω is at unit pressure. The force exerted on C by this gas is F[GC_ω/C] = 1. [2] The existence of such pressure also implies (again, according to Newton's third law of action and reaction, see note (2)) that GC_ω must exert a force of unit magnitude directed to the left. In a sufficiently generic way, we could say that this force acts on the material system made up of the set of D_i (i ≥ 0) plus the set of gases enclosed between them. In other words, it acts on system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … although (clearly) it does not do so on any of the component parts D_0, GC_1, D_1, GC_2, D_2, GC_3, …. [3] Hence, F[GC_ω/$D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + …] = − 1.

5 Case II. The New Contact Paradox. Phantom Forces

The force GC_ω exerts on system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … violates the principle of influence. We shall see why. The presence of gas at unit pressure in C_ω leads to a force acting on itself by system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … (obviously F[$D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … /GC_ω] = − F[C/GC_ω] = F[GC_ω/C] = 1). But there is no contact interaction (internal force) which has been induced by the gas at unit pressure in C_ω between the D_i (i ≥ 0) and the gases enclosed between them. Moreover, no matter how system $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … is broken down (even purely formally) into exclusive and exhaustive parts, there is no contact interaction (internal force) between the parts which has been induced by the gas at unit pressure in C_ω. That is, GC_ω at unit pressure does not cause the emergence of internal forces in $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … that did not previously exist, when C_ω was empty. In fact, this unit pressure, added to the pressure of any of the gases enclosed between two contiguous pistons (which is never higher than 1), in no event exceeds the k* > 2 value required for the displacement of at least some D_i inside C. However, if the D_i do not move, the initial values of the gas pressures in the C_i compartments will not change either. And if these pressures do not change, neither will the friction forces that kept the different D_i pistons in their initial positions (i.e. before compartment C_ω was filled with gas at unit pressure). This means that there is no contact interaction (internal force) which has been induced by the gas at unit pressure in C_ω between the D_i (i ≥ 0) and the enclosed gases between them, as stated above. And, as also stated earlier, GC_ω does not cause the emergence of internal forces in $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + … that did not previously exist, when C_ω was empty. We therefore observe an instance of contact interaction that violates the principle of influence. This is a truly unique type of interaction one is tempted to call "phantom". The gas in C_ω exerts a unit force on $D_0 + GC_1 + D_1 + GC_2 + D_2 + GC_3$ + …,

[2] The fact that a gas is subjected to pressure p implies that its environment exerts certain forces on it. The fact that it also acts on its environment is a consequence of the law of action and reaction.

[3] This is an example of what has been called "global" interaction in the literature. See, for example, Laraudogoitia (2005). Similarly, in Sect.1 we saw that the man of weight P exerts a force of magnitude P on the stack of slabs although (clearly) he does not do so on any of the component slabs.

the set consisting of the D_i ($i \geq 0$) and the GC_i ($1 \leq i < \omega$) gases, [4] without inducing any internal force between them or, indeed, causing any kind of movement (nor any internal stress in the D_i that did not previously exist, when C_ω was empty). Its sole function is purely formal: to ensure compliance with the law of action and reaction (since, according to the definition of pressure, it has been seen earlier that $F[D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots /GC_\omega]=1$). We will describe this phantom-formal character by writing "ph" in subscript to denote this force: $F_{ph}[GC_\omega/ D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots] = -1$. Note that the phantom force is the force that GC_ω exerts on $D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots$, but not the force that $D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots$ exerts on GC_ω. This latter force, unsurprisingly, causes the emergence of internal forces in GC_ω in the form of gaseous pressure.

Comparison with Benardete's infinite stack of slabs may be enlightening here. Before the man climbs the stack, the i+1-th slab exerts an upward force of $1/2^{i+1}$ on the i+2-th. Once he has climbed up, this force has a value of $P+(1/2^{i+1})$. In other words, the man's weight has caused the emergence of internal forces in the stack, altering those existing previously. In our case, prior to filling C_ω with gas, the internal forces in system $D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots$ had certain defined values. However, these values are not altered in any way when C_ω is filled with gas at unit pressure! The man's weight has a causal influence on the stack of slabs on which it acts (altering the internal stresses within it). However, the pressure of the gas in C_ω has no causal influence on system $D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots$ on which it acts (it does not alter the internal stresses within it). The contrast can be seen clearly intuitively in this way. Suppose that one and only one of the slabs in Benardete's infinite stack of slabs (let us say, i+1-th) is an elastic but non-rigid solid. The weight of the other slabs will have deformed (compressed) it to some extent. However, when the man climbs to the top of the stack, the compression deformation of slab i+1-th will be greater, displaying an increase in the stress to which it was subjected (an increase caused by the man's weight). Alternatively, now suppose that in our system in Fig. 1, one and only one of the circular disks (let us say, i+1-th, D_{i+1}) is an elastic but non-rigid solid. The gas pressure in C_{i+1} and C_{i+2} will have deformed (compressed) it to some extent. However, when compartment C_ω is filled with gas at unit pressure, deformation of D_{i+1} is not be altered in the slightest (and neither are pressures $p(C_{i+1})$ and $p(C_{i+2})$), proving that the stress to which it was subjected has not changed. In short, the paradox of Benardete's slabs does not violate the Principle of Influence, but rather confirms it. As confirmed (as far as I know) by all the examples considered thus far in the relevant literature. The new contact paradox presented above is radically different on this point, because the configuration in Fig. 1 in Case II does in fact violate this principle. It constitutes the first counterexample to the POI, that is to say, to our most fundamental intuitions concerning the causal role of forces in the Newtonian conception of classical mechanics. It can also be seen in this way: the Case II configuration presents a conflict between the powerful intuition based on Newton's third law of action and reaction (a law requiring the existence of force $F[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots] = -1$) and the no less powerful intuition based on the Principle of Influence (a Principle requiring causal efficacy from $F[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots]$ which $F[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3 + \ldots]$ does not have). The new contact paradox arises from this conflict. Hence, in infinite physical systems as in Fig. 1, it is not only a question that the constraints make the contact with a sequence of physical objects of ordinal type ω difficult to implement into the dynam-

[4] Some philosophers would prefer to speak here of the fusion of material bodies D_i ($i \geq 0$) and gases GC_i ($1 \leq i < \omega$). The difference between the two terms is irrelevant for the purposes of this paper.

ics, or that it is implemented using forces that play a purely formal role. It is a question that, in such systems, one of our most fundamental "mechanical intuitions" must be abandoned: it could be the principle of influence, but it could also be Newton's third law (although this is not the solution I will advocate here). Both intuitions are deeply rooted in the usual interpretation of Newtonian mechanics, which thus warrants that the new contact paradox be treated as a problem for philosophers of physics.

Again the comparison with the paradox of Benardete's slabs is illuminating. Here the difficulty that justifies reference to "paradox" (a difficulty yet to be discussed) is of a completely different nature (in particular, as already seen, there is no issue with Newton's laws or the POI). In Benardete's words (1964): "Let us plant our foot firmly upon this infinite pile of slabs. But what is there to plant our foot upon? *There simply is no top surface to the whole pile.* For the pile is constituted exclusively by the slabs *ex hypothesi*: absolutely nothing else has been added." (p. 237). It is easy to extract the opposing intuitions underlying the paradox of Benardete's slabs from this statement. As is evident, the core of the issue here relates to the question of the topologically open or closed (or neither) character of material objects, that is, to the question of whether or not the topological boundary of any object O belongs to O. This is at the heart of a much debated (and subtle) metaphysical question that overlaps with what is sometimes referred to as the mathematical philosophy of contact. However, it is, in any event, far removed from the kind of questions that are central to the new contact paradox.

5.1 The New Contact Paradox in a Simple Form

Herein is the new contact paradox in a simple form. From the analysis conducted above, it can be seen that, ultimately, the gases in compartments C_i $(0<i<\omega)$ do not play any essential role in the new contact paradox. It follows that this will arise in a simpler form when all the C_i $(0 \leq i<\omega)$ are empty, leaving only gas in C_ω (at the usual pressure $p(C_\omega)=1$). In this case, this gas will not have any causal influence on system $D_0+D_1+D_2+\ldots$, on which it nevertheless exerts force $F_{ph}[GC_\omega/\ D_0+D_1+D_2+\ldots\ ]=-1$. That is, given that prior to filling C_ω, the internal forces in $D_0+D_1+D_2+\ldots$ are now null, when $p(C_\omega)=1$ continue to be null. In order to see this formally, one needs only to reproduce all the developments of the previous sections (cases I and II) by simply suppressing the occurrences of GC_α $(1 \leq \alpha<\omega)$ and performing $p(C_\alpha)=0$ $(0 \leq \alpha<\omega)$. Nevertheless, analysis of these sections is important in order to appreciate the meaning of the new contact paradox in the more overall context of the paradoxical non-phantom forces seen in Sect.6 below.

It should not be forgotten that the new contact paradox "in a simple form" has the advantage of showing, in a much clearer way, the broad spectrum of circumstances in which its central ingredient, phantom forces, may appear. Contact "at the limit" with some ordinal type ω series of bodies (which themselves are not always in contact with each other) seems to be necessary, although this is obviously not enough (as shown in Benardete's example of the man standing on the stack of slabs). Pinning things down is an open task here; a task transcending what has been done thus far in the philosophy of contact and analysis of its paradoxes.

5.2 More Phantom Forces

After the brief hiatus in 5.1, let us return to the overall situation regarding Case II. The consequences do not end here and lead to the need for new forces with the same phantom characteristics. Since system D_i ($i \geq 0$) plus gases GC_i ($1 \leq i < \omega$) was in equilibrium with C_ω empty, it will cease to be so if the only thing that has changed in this respect is the presence of the new force on said system $F_{ph}[GC_\omega/ D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots] = -1$ as deduced above. Therefore, (according to Newton's first law of equilibrium) something must counteract it. The only option is cylinder C, so there must be a new "phantom" force $F_{ph}[C/ D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots]=1$, which neither induces movement or any internal force in the set made up of the D_i ($i \geq 0$) and gases GC_i ($1 \leq i<\omega$) (i.e. no internal force that did not previously exist, when C_ω was empty). Again, this phantom force has a purely formal role to play, in this case ensuring compliance with the law of equilibrium. Finally, the principle of action and reaction requires a final additional force $F[D_0+GC_1+D_1+GC_2+D_2 +GC_3 + \dots /C] = -1$. As $F[GC_\omega/C]=1$, equilibrium of C is finally reached. [5]

By abstracting the specific details, what characterizes a phantom force becomes clear. The force F exerted on an extended body B is a phantom force if and only if it does not induce (cause) change of movement of B or the emergence of internal forces in B. Consequently, a phantom force violates the principle of influence. One might believe that such entities are idle. One could populate the world with a multitude of entities that (apparently at least) are of this kind, all of which can be eliminated using Occam's razor. In reality, the gods of Olympus, elves or demons play no role in the causal structure of the world. However, phantom forces are not exactly of the same "empty" nature. Although they violate the principle of influence, they play (at least indirectly) a role in the causal description of the world through the Newtonian laws of equilibrium, and action and reaction. The reason is that, as we have seen, they are required by these laws. $F_{ph}[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots] = -1$ was required by Newton's third law (the law of action and reaction) and $F_{ph}[C/ D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots]=1$ by Newton's first law (law of equilibrium).

6 Paradoxical Non-Phantom Forces

6.1 Reactions to Phantom Forces

The role of "phantom" forces in the contact paradox is more subtle than the above comments suggest. We have seen that, by virtue of $F[GC_\omega/C]=1$, $F_{ph}[GC_\omega/ D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots] = -1$, and that friction is the cause of forces $F_{friction}[D_1+D_2+D_3 + \dots /C]=1$ and $F_{friction}[D_0/C]=-1$ (both frictional forces were previously obtained in the unproblematic case in Sect.3). As the gases in the compartments do not interact by friction, it follows from the penultimate equality that $F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4 + \dots\dots /C]=1$, a force also caused by friction (evidently, the friction of D_i, $i \geq 1$, with C). Furthermore, as there are no interactions at a distance in our model, $F[GC_\omega/D_0]=F[GC_\omega/GC_1]=0$. The principle of superposition of forces therefore leads to $F_{ph}[GC_\omega/ D_1+GC_2+D_2+GC_3+D_3+GC_4+ \dots] =$

[5] Note that we have written $F[D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots /C]$ and not $F_{ph}[D_0+GC_1+D_1+GC_2+D_2+GC_3 + \dots /C]$; the reason being that this is a force with obvious causal power. It induces internal stresses in cylinder C by opposing $F[GC_\omega/C]$, thus reaching equilibrium.

– 1.[6] Since system $D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots$ was already in equilibrium with C_ω empty, it will cease to be so if the only thing that has changed in its respect is the presence of the new force $F_{ph}[GC_\omega/\ D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]$. Therefore, (according to the law of equilibrium) something must counteract it. It cannot be the gas in compartment C_1 (GC_1) because this gas has already annulled the force that C exerts by friction on system $D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots$ from the beginning. Indeed, it follows directly from what was seen in unproblematic Case I (Sect.3) that $F[GC_1/D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]=F[GC_1/D_1]=1=F_{friction}[D_1+D_2+D_3+\dots/C]=$ (as the gases in the compartments do not interact by friction)$=F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots/C]=-F_{friction}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]$. As before, the only option is cylinder C, so there must be a new phantom force $F_{ph}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]=1$, which neither induces movement nor any internal force in the set of D_i ($i\geq 1$) and gases GC_i ($2\leq i<\omega$). Also, as before, this phantom force has a purely formal role to play, ensuring compliance with the law of equilibrium. Lastly, the principle of action and reaction requires a final, additional force: $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots/C]=-1$. As $F[GC_\omega/C]=1$, equilibrium of C is finally reached (note that C was already in equilibrium with C_ω empty). The interesting point here is that there are two non-zero forces of a very different nature acting between the same material bodies. As seen above, $F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots\dots/C]=1$ is a force caused by friction and, as we have just seen, $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots/C]=-1$ is a "non-phantom" force[7] originating in phantom force $F_{ph}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]=1$ (which, in turn, originated in the phantom force $F_{ph}[GC_\omega/\ D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]$). In contrast, non-phantom force $F[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]=-1$ (from Sect.5.2 and note (5)) should be taken into account, which coexists only in a trivial sense with zero force $F_{friction}[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]=$ (as the gases in the compartments do not interact by friction)$=F_{friction}[D_0+D_1+D_2+\dots/C]=F_{friction}[D_0/C]+F_{friction}[D_1+D_2+D_3+\dots/C]=$ (as seen in Case I in Sect.3) $=-1+1=0$.

6.2 How Paradoxical Can the Reaction to a Phantom Force Be?

This previous point highlights an additional paradoxical dimension of the philosophy of contact in the presence of phantom forces. Even if the reaction force to a phantom force is not in itself phantom, it can also be of an enigmatic nature. This is exactly what happens in the case of force $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots/C]$ (and, to a somewhat lesser extent, in the case of $F[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]$). As seen, $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots/C]$ (just as $F[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]$) is neither a frictional force nor a usual contact force originating in the impenetrability of matter (such as that which typically arises when two material bodies press against each other). However (in contrast to $F[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]$), it can coexist with them, as exem-

[6] The principle of superposition of forces applies here to a finite number of forces, namely: $F[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots]=F[GC_\omega/\ D_0]+F[GC_\omega/GC_1]+F[GC_\omega/\ D_1+GC_2+D_2+GC_3+D_3+GC_4+\dots]$. When the number of forces considered is infinite, it is not always satisfied. As seen at the end of Sect.4, $F[GC_\omega/D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots]=-1$ even though $\forall i\geq 0 F[GC_\omega/D_i]=0$ and $\forall i$ ($1\leq i<\omega$) $F[GC_\omega/GC_i]=0$. This is characteristic of many standard examples in the literature of systems with infinite components (such as the slab stack seen in Sect.1).

[7] This is exactly the same as $F[D_0+GC_1+D_1+GC_2+D_2+GC_3+\dots/C]$, whose non-phantom character was substantiated in note 5.

plified by the case of the non-zero friction force that $D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots$. exerts on C. Even if the reaction force to a phantom force does not violate the principle of influence, it may in itself be a mysterious entity. We lack an adequate causal mechanism to explain its genesis in the physics of contact. This is the reason for subscript "?" in $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$.

Note that $F[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$, the total force that $D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots$ exerts on C, has a value of $F[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]=F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots\ldots/C]+F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]=1-1=0$. Also, (as should be the case, given C's state of equilibrium) $F[D_0+GC_1/C]=F[D_0/C]=F_{friction}[D_0/C]=F_{friction}[D_0+GC_1/C]=-F[GC_\omega/C]=-1$, so that $F[D_0+GC_1/C]+F[GC_\omega/C]=-1+1=0$. We have seen that friction is the physical cause of term $F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots\ldots/C]$, but that the physical cause of $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$ remains an enigma (if it seems shocking to discuss physical causes when considering a system such as in Fig.1, which clearly cannot exist in reality, please refer to my discussion of the concept of physical system in Sect.8). In particular, it is an enigma how $D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots$ can exert equal and opposite forces on C. Likewise, when GC_ω was empty, $F_{friction}[D_0+GC_1/C]$ and $F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots\ldots/C]$ were already creating internal forces of stress in C. However, on filling GC_ω with gas, new internal stresses in C are generated by $F[GC_\omega/C]$ and enigmatic force $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$, confirming the latter's non-phantom character (in accordance with notes (7) and (5)).

7 Some Final Variations

7.1 Trivial

There is nothing special regarding value $p(C_\omega)=1$ in the contact paradox. Many other values will serve equally well to illustrate it. Most certainly, any that do not entail the possibility of pressures equal to or greater than k*. Let us take for example $p(C_\omega)=1/3$. The gas in C_ω now exerts a 1/3 force on system $D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots$ ($F_{ph}[GC_\omega/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]=-1/3$) but without exerting it directly on any of the D_i, GC_{i+1} ($i\geq 1$) or inducing any force between them (or inside them). The forces involving only C, D_i and GC_{i+1} ($i\geq 1$) remain unaltered (taking $p(C_\omega)=1/3$ changes nothing to this effect, like taking for example $p(C_\omega)=1$ or $p(C_\omega)=0$). The single role of $F_{ph}[GC_\omega/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]$ is purely formal: to ensure compliance with the law of action and reaction (given that $p(C_\omega)=1/3$). Moreover, as $F[GC_1/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]=-F_{friction}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]=1$ (as seen in Sect.6.1), it follows that $F_{ph}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]=1/3$ (this, which is a consequence of the law of equilibrium, is analogous to the result $F_{ph}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]=1$ seen earlier for the case of $p(C_\omega)=1$). Thus, $F[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$ (the sum of a force caused by friction and another that is not)$=F_{friction}[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]+F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]=1-1/3$, which again ensures equilibrium of C, because $F[D_0+GC_1/C]=-1$ and $F[GC_\omega/C]=1/3$. As in the case of $p(C_\omega)=1$, now $F_?[D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots/C]$ is also required by the principle of action and reaction applied to $F_{ph}[C/D_1+GC_2+D_2+GC_3+D_3+GC_4+\ldots]$. And the diagnosis is clearly the same. The phantom forces that violate the principle of influence have a purely formal role: to ensure compliance with the law of action and reaction and law of equilibrium. And the paradoxical non-phantom forces are reaction forces to phantom forces.

7.2 Non-Trivial. The New Contact Paradox in Its Simplest Form. Phantom Interactions

Herein is the new contact paradox in its simplest form. In addition to its ultimate simplicity, (I believe) it has the added interest of showing how (unlike the examples thus far discussed) the reaction force to a phantom force can itself be phantom, which leads to the concept of phantom interaction. S_n shall denote an elastic spherical body of radius $R_n=(n+1)/n$ ($n \in Z^+$), when unstressed, and zero thickness (for simplicity). S_{n+1} will have a smaller radius $R_{n+1}=(n+2)/(n+1)<R_n$. Let us assume that the elasticities are such that, when S_{n+1}'s interior is filled with gas at pressure p, its volume grows until the new radius takes value $R^*_{n+1}=(n+2)/(n+1)+(p/(p+1))[((n+1)/n)-((n+2)/(n+1))]=(n+2)/(n+1)+(p/(p+1))[1/(n(n+1))]$ [8]. Note that $R^*_{n+1}<R_n$. The physical system formed by the infinite concentric S_n spheres (embedded in one another) will be designated by Σ. It is assumed that the S_n masses decrease sufficiently for the configuration's total mass to be finite. As is evident, the common interior to all the Σ spheres is a given ball B. We shall now assume that ball B is filled with gas G at pressure p. The resulting configuration can be mentally divided by symmetry plane Π, as in Fig.2 (marked by a thick dashed line). Gas G is split into left and right parts G_{left}, G_{right}. Analogously, Σ is split into parts Σ_{left} and Σ_{right}.

This is evidently an equilibrium configuration in which the radius of S_n continues to be R_n, i.e., as if gas G were not present. This is so because: (a) no elastic sphere S_{n+1} filled with gas inside at finite pressure p would dilate so much as to come into contact with elastic sphere S_n (note that $R^*_{n+1}<R_n$); (b) no elastic sphere S_n is FULL of gas inside (it is only partially filled). Using the notation in Sect.2, it is clear that $F[\Sigma/G]=0$, but this is a net force, and of little interest. More revealing is what occurs regarding Σ_{left}, Σ_{right}, G_{left} and G_{right}. Fluid statics guarantees, unremarkably, that $F[G_{left}/G_{right}]=-F[G_{right}/G_{left}]=F[\Sigma_{left}/G_{left}]=-F[\Sigma_{right}/G_{right}]=p\pi$. However, the reaction forces of G_{left} on Σ_{left} and G_{right} on Σ_{right} are phantom forces: $F_{ph}[G_{left}/\Sigma_{left}]=-p\pi=-F_{ph}[G_{right}/\Sigma_{right}]$. This is evident since, regardless of whether or not there is gas under pressure in the common interior of the elastic spheres (previously called ball B), these spheres will neither move nor be subjected to any internal stress (each sphere's surface is always separated from the other surfaces by a finite distance), which violates the principle of influence. At this point, Σ_{left} and Σ_{right} resemble the sequence of material bodies $D_0+D_1+D_2+\ldots$ in Sect.5.1. There is, however, a highly significant difference. Σ_{left} is in equilibrium and $F_{ph}[G_{left}/\Sigma_{left}]=-p\pi$. The force balancing this must evidently be a phantom force exerted by Σ_{right}: $F_{ph}[\Sigma_{right}/\Sigma_{left}]=p\pi$. And likewise, since $F_{ph}[G_{right}/\Sigma_{right}]=p\pi$, the force which balances it must be the phantom force exerted by Σ_{left}: $F_{ph}[\Sigma_{left}/\Sigma_{right}]=-p\pi$. The important point is that $F_{ph}[\Sigma_{right}/\Sigma_{left}]$ and $F_{ph}[\Sigma_{left}/\Sigma_{right}]$ are a pair of phantom forces of action and reaction. This shows that, as stated at the beginning, the reaction force to a phantom force can itself be phantom. When, as in this case, a force and its reaction are both phantom, then it is appropriate to use the term phantom interaction. The situation can be summarized by saying that the configuration in Fig.2 not only shows

[8] This shows that elastic spheres S_n are increasingly "less elastic" as n grows, though none of them is rigid.

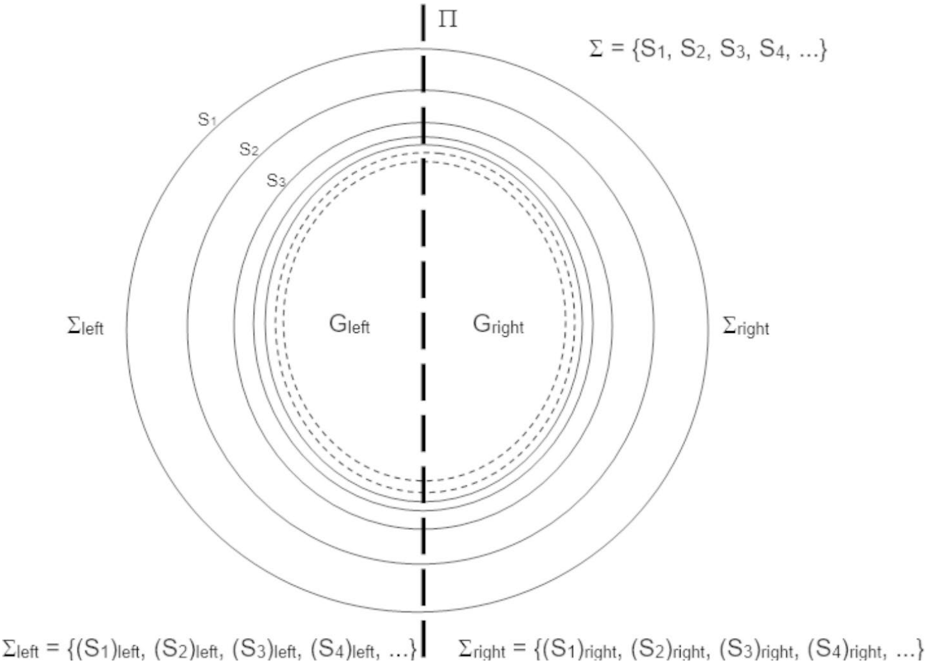

Figure2A hollow sphere with an infinity of inner concentric hollow spheres

the existence of phantom forces but also of phantom interactions: this is the case of the interaction between Σ_{left} and Σ_{right}. This was not possible with the system in Fig.1, and shows the extremes to which the purely formal role (in order to ensure compliance with the law of action and reaction and law of equilibrium) of phantom forces that violate the principle of influence can go.

8 Tentative Conclusion. Comments on Physical Systems, Infinitism and the New Contact Paradox

I would like to mention a positive interpretation of the new contact paradox. Perhaps this is a paradox that we should accept in our theoretical treatment of mechanical forces. By unambiguously allowing what I have called phantom forces (which violate the principle of influence), we are allowing the Newtonian laws involved (the first and third) to be formulated with total generality (as they usually are). Otherwise, by rejecting such forces we would be forced to complicate the highly intuitive and simple axiomatic format of Newton's laws (their extreme generality) in view of configurations (such as in Case II and Sect.7.2), where some of these laws would fail. The situation is suggestively similar to what can be found in other theoretical domains. The closest probably being that of projective geometry, where "ideal elements" are accepted without any visual or "empirical" correlate (ideal points and ideal lines as opposed to ordinary points and ordinary lines) in order to enable axioms of the projective plane to be formulated simply and generally. Even though the theory of ideal elements has its origin in Hilbert's philosophy of mathematics, he himself drew a clear analogy between the use of ideal elements in mathematics (which we add to make a smoother theory) and the use of hypotheses involving theoretical terms in physics (in his words: "All of physical thought …. appears for this view as nothing other than an implicit application of the method of the extension of a system through the addition of ideal elements" (Hilbert, 1919, p. 159)). As Hallett (1990) points out when interpreting Hilbert's theory of geometry: "… there is no absolute difference in status between theoretical terms like 'force' in physics, 'point' in geometry and 'irrational' in analysis" (1990, p. 223) and "Once new concepts or 'ideal elements' or new theoretical terms have been accepted, then they exist in the sense in which *any* theoretical entities exist" (1990, p. 239). Admitting phantom forces as ideal elements in a simple and general formulation of Newtonian mechanics avoids the main difficulties seen in the previous points, albeit not all of them, as evidenced by the paradoxical non-phantom forces in Sect.6. The concept of force has indeed come under suspicion since at least the 18th century (see, for example, D'Alembert's *Traité de dynamique*). In the 19th century, three distinct strategies had already been clearly defined (Capecchi, 2014) that persist to this day (with varying degrees of support and nuances): to consider the concept of force as a derived concept (e.g. Kirchhoff), a primitive concept (Euler) or an obscure and therefore rejectable concept (Hertz). I do not intend to take sides in this controversy here, but I believe that the new contact paradox may be relevant to this effect. If one considers phantom forces as ideal elements, then one seems to be closer to the viewpoint that regards forces as primitive elements of mechanics. If, on the other hand, one considers phantom forces to be an unacceptable construct and incapable of properly accounting for the paradox, then one may be favoring conceptions such as Hertz's, which advocate an essentially kinematic formulation of mechanics.

The systems in Figs.1 and 2, analyzed in this paper, are examples of physical systems. By physical system I mean a system whose essential characteristics can be studied with the aid of a standard physical theory (in our case classical Newtonian mechanics). Here the expression "physical system" can be considered synonymous to "physical model", although I tend to prefer the first expression since the general concept of system seems to me to be more manageable and interesting than the general concept of model (at least for non-formal disciplines). Note that a physical system need not be physically realizable (analogous to how a model need not model anything real), and the systems in Figs.1 and 2 are most certainly not (basically because of their infinite character). They are, however, analyzable in the context of Newtonian mechanics, so the causal vocabulary used in certain parts (consider, for example, reference to physical causes in Sect.6.2) holds its own. And this is so, not because the systems referred to are physically realizable (there is agreement that they are not), but because the causal content of the Newtonian theory within the framework they are studied is (precisely as a consequence of said study) directly transferable to such systems. The fact that the systems in Figs.1 and 2 are non-physically realizable physical systems also explains why their analysis in the context of Newtonian mechanics has consequences for this theory under which they are analyzed but no consequences for reality. Lastly, having consequences for the theory under which it is analyzed but not for reality is a feature that the new contact paradox shares with many other theoretical results concerning infinite physical systems (be it in the philosophical literature or in the more technical scientific literature) such as the beautiful supertask (Laraudogoitia, 1996). In this sense, the new contact paradox therefore belongs to a well-established "tradition" in science and in its philosophical reflection.

The new contact paradox arises in some physical systems with an infinite denumerable number of component parts. Faced with this, there is always the drastic option of avoiding difficulties by rejecting the actual infinitude in physical models (like the Newtonian models considered here). There are at least three overriding objections to this course of action. First, the fact that it ignores (and condemns) from the outset the existence of a multitude of infinite physical systems studied in the philosophical, physical and mathematical literature (not even always in the form of supertasks). In many of these cases interesting results are obtained that help frame the scope of validity and interpretation of the physical theories involved. Second, the fact that it even rejects cases of uninteresting infinite physical systems that pose no problem of inconsistency in the physical theories admitting them. And third, the fact that (as a result) it fails to discriminate between both types of systems. In particular, it puts on an equal footing (unchecked, which is scarcely compatible with philosophical practice):

a. cases of interesting infinite systems from which something can be learned;
b. other cases that are quite simply incompatible with the physical theory in which they are studied;
c. cases that are perfectly consistent, albeit trivial.

My personal intuition is that the new contact paradox falls within the framework of type (a) cases. Moreover, it is conceptually different from all other cases of this type seen in the literature. In addition, the differentiation criterion is clear: none of the other infinite Newtonian systems studied to date (and, of course, neither non-infinite systems) contradicts the principle of influence (POI). Only the new contact paradox does so. Given this contradiction, one might consider including the new contact paradox within type (b) cases. In such a circumstance, it would simply be argued that the infinite material configurations of the new contact paradox go against Newtonian physics. There are, however, two immediate objections to this:

1) The paradox does not arise if (everything else remaining the same) compartment C_ω in Fig. 1 is simply left empty (with no gas) in such a configuration. This already seems to imply that it is not the presence of the actual infinite as such that is the problem; rather that the root of the paradox lies elsewhere. Furthermore, it would be necessary to explain how it is possible that the mere absence or not of a single material body (in this case the gas in C_ω) is what makes the difference between an infinite material system that does not go against Newtonian physics and one that does. [9]

2) This incompatibility does not occur, strictly speaking, with Newtonian physics (summarily condensed into Newton's three laws) but with the principle of influence (POI). I have in fact proposed something along these lines here. Continuing to uphold Newton's laws (Newtonian mechanics) and rejecting the universal validity of POI.

[9] The analogy (yet to be explored) is interesting between this and what Sainsbury (2009) calls the "principle of tolerance" in the analysis of paradoxes of vagueness ("a gram cannot make the difference between not enough wood to make a table and enough wood", p.48). This analogy is more pertinent if one takes into account that, even where there is gas in C_ω, the paradox will not arise if its pressure $p(C_\omega)$ is high enough to move some final segment of discs D_i despite friction (for this reason a value of $p(C_\omega)$ was chosen in Sect. 4, which would ensure that $p(C_\omega)+p(C_i)<k^*$). And even where $p(C_\omega)$ is small, no paradox will arise in this case if friction is low enough (for the same reason).

Note also that, in the new contact paradox, the ordinal type of the set of compartments filled with gas is $\omega+1$. In the absence of gas in compartment C_{ω}, the ordinal type is ω. Yet the contrast between ω and $\omega+1$ cannot be what makes the difference between an infinite material system that does not go against Newtonian physics and one that does. To see this, one only needs to go back to Benardete's example (1964) considered in the introduction, where a man of weight P climbs to the top of the infinite stack of slabs. Here, the set of relevant material bodies' ordinal type (man and slabs) is $\omega+1$. And, as made clear from my analysis, this material system not only does it not go against Newtonian physics, it also does not go against the POI. We conclude that the contrast between ω and $\omega+1$ can in no way be what makes the difference between an infinite material system that does not go against the POI and one that does [10].

I would like to conclude with a consideration that summarizes the essence of this paper in a very succinct and expressive way. At first sight it may seem impossible for two physical systems X and Y to interact in such a way that X's state is independent of Y's presence or absence. It has been shown, however, that in principle there is no such impossibility if X has a natural order of type ω and the joint system X+Y one of type $\omega+1$ (a condition which, albeit not sufficient, appears to be necessary) [11]. This is precisely the essence of the new contact paradox.

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Declarations

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[10] Another interesting example shows that the ordinal type alone is not sufficient to account for the paradox. Assuming that in the configuration in Fig.2 we add one more concentric sphere, S_{∞}, of unit radius. The new system continues to have a natural ordinal type $\omega+1$ (to be considered as consisting of S_1, S_2, S_3, … S_{∞}+G). If S_{∞} is also an elastic sphere, then nothing substantial changes by its presence and we face the same contact paradox and its phantom forces. However, if S_{∞} is a rigid sphere, the paradox disappears because S_{∞}+G does not interact in any way with the elastic spheres system S_1, S_2, S_3, …. If, on the contrary, it is spheres S_1, S_2, S_3, … that are rigid, while S_{∞} is elastic (or even where S_{∞} does not exist), the paradox reappears.

[11] In Case II, X is formed, for example, by D_0, GC_1, D_1, GC_2, D_2, GC_3, … and X+Y by D_0, GC_1, D_1, GC_2, D_2, GC_3, … GC_{ω}. In Sect.7.2, X is formed, for example, by $(S_1)_{left}$, $(S_2)_{left}$, $(S_3)_{left}$, $(S_4)_{left}$, … and X+Y by $(S_1)_{left}$, $(S_2)_{left}$, $(S_3)_{left}$, $(S_4)_{left}$, … G_{left}.

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