Plimpton 322: A Study of Rectangles

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Abstract

Plimpton 322 is one of the most sophisticated and interesting mathematical objects from antiquity. It is often regarded as teacher's list of school problems, however new analysis suggests that it relates to a particular geometric problem in contemporary surveying.

Keywords Plimpton 322 · Pythagorean triples · Regular numbers · Surveying · proto-Trigonometry · Mesopotamia

1 Introduction

Plimpton 322 is one of the most remarkable mathematical objects from antiquity (Neugebauer, 1969, p. 40). This broken clay tablet dates from the Old Babylonian (OB) period (1900–1600 BCE) and contains a table of “Pythagorean triples” over a millennium before Pythagoras was born.

The tablet has been the subject of intense study. We reconsider some of the usual assumptions about Plimpton 322 and conclude that it was a mathematical study of the individual *sides* of Pythagorean triples.

This article is organized into three main parts: an introductory section that consolidates recent scholarship on Mesopotamian mathematics, a section that analyses Plimpton 322 itself, and a final section where we consider our new analysis in the context of contemporary surveying.

2 Background

Mesopotamian mathematics is fundamentally about lists (Proust, 2015, p. 215). Numerical calculations in administration, engineering and surveying were performed by trained professionals, called **scribes**, whose understanding of mathematics was founded in lists. In this section, we redefine several familiar mathematical concepts based on the author's understanding of source materials and secondary literature. The purpose of these definitions is to

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highlight the unique aspects of the Mesopotamian list-based approach to mathematics. We begin with a somewhat novel definition of number.

2.1 Numbers

Scribes used a sexagesimal (base 60) number system. Their digits were written by accumulating symbols that represent 10 and 1, in that order. For instance, 16 was written as one 10 followed by six 1s. Similarly, 5 was written as no 10s followed by five 1s. We shall write these digits as 00, 01, 02, 03, ..., 58, and 59. The null digit 00 warrants special attention because it was written as no 10s and no 1s. In other words, the null digit appears as a blank space.¹

A number is a finite sequence of digits. Because scribes wrote the null digit as blank space, intermediate null digits appear as gaps within the number. More importantly, any leading or trailing null digits are lost in the vacant space surrounding the number.

Definition 1 A **number**, often called a **Sexagesimal Place-Value Notation (SPVN) number**, is a finite sequence of sexagesimal digits 00, 01, ..., 58, and 59 where both the first and last digits are non-null. We separate the digits of a number using a colon. The digit to the left of a colon represents units that are sixty times greater than the digit to the right.²

The loss of surrounding null digits means that SPVN numbers are only determined up to a multiple of 60. For example, the number 16 : 00 : 05 itself is ambiguous and can mean any rational number of the form

$$16 \times 60^n + 0 \times 60^{n-1} + 5 \times 60^{n-2}, n \in \mathbb{Z}.$$

This ambiguity is sometimes resolved by context.

Decimal numbers are fundamentally different from SPVN numbers since the former can have leading zeros, trailing zeros, and a radix point while the latter can not. Several modern mathematical concepts must be adjusted to accommodate the different and more ancient SPVN number system. For example, in SPVN arithmetic $42 \times 10 = 7$ which means that 7 is a multiple of 10 with the usual understanding of the term “multiple”. This is clearly unsatisfactory, and the problem is that our usual understanding of the term “multiple” is too modern.^{3 4}

¹ A separate symbol was occasionally used to represent the consecutive absence of 10s and 1s (Friberg & Al-Rawi, 2016, p. 254), (Friberg & Al-Rawi, 2016, p. 3). The use of an explicit symbol to represent null became more common over time. See also (Høyrup, 2002, p. 294).

² The use of a colon to separate sexagesimal digits is familiar from modern time-keeping, and following Proust (2020) we use the same symbol for SPVN numbers. We also omit any initial zero from the first digit of a number, so we write 6 : 40 instead of 06 : 40.

³ See (Proust, 2016, p. 13) for a discussion of this issue.

⁴ It is worth mentioning that there is no SPVN expression for the number zero. This is hardly a flaw since the number zero is a relatively modern invention that the SPVN system was never intended to deal with. See also (Høyrup, 2002, p. 185).

Table 1 The standard table of reciprocals which lists pairs of numbers a and $\bar{a}$ such that $a \times \bar{a} = 1$. Some texts omit duplicate entries marked with * that appear elsewhere on the table

2	30	9	6 : 40	20	3	36	1 : 40	1	1
3	20	10	6	24	2 : 30	40	1 : 30	1 : 04	56 : 15
4	15	12	5	25	2 : 24	45	1 : 20	1 : 12*	50*
5	12	15	4	27	2 : 13 : 20	48	1 : 15	1 : 15*	48*
6	10	16	3 : 45	30	2	50	1 : 12	1 : 20*	45*
8	7 : 30	18	3 : 20	32	1 : 52 : 30	54	1 : 06 : 40	1 : 21	44 : 26 : 40

2.2 Regular Numbers

Before we can address the concept of multiple we must first recognize that many SPVN numbers are actually units, called regular numbers.

Definition 2 The SPVN number a is **regular** if there exists some SPVN number $\bar{a}$ such that $a \times \bar{a} = 1$, the number $\bar{a}$ is the **reciprocal** of a , and any number which is not regular is **irregular**.⁵

The standard table of reciprocals (Table 1) is a canonical piece of scribal equipment found throughout the OB period. It lists every one and two-digit regular number from 2 to 1 : 21 together with their corresponding reciprocals, and was memorized by scribes during their training. This meant that scribes could instantly recognize all regular and irregular numbers within this range. These numbers play a fundamental role in Mesopotamian mathematics and, following (Proust, 2012, p. 391), we reflect their importance with a further definition.

Definition 3 An **elementary regular number** is any number a or $\bar{a}$ from the standard table of reciprocals (Table 1). An **elementary irregular number** is any one or two-digit number between 2 and 1 : 21 that is not regular.

In other words, we regard the elementary regular numbers as obviously regular and the elementary irregular numbers as obviously irregular.

Scribes knew that certain numbers became smaller when multiplied by certain reciprocals. In fact, they knew precisely which numbers could be reduced and how to reduce them. For example, they knew 5 : 55 : 57 : 25 : 18 : 45 could be reduced with multiplication by the reciprocal of 3 : 45, and that numbers like 7 or 5 : 09 : 01 could not be reduced at all. Their ability to reduce numbers was based on an understanding of “factors” and “multiples” that is slightly different from the usual understanding of these terms.

⁵ Irrational numbers are different again to irregular numbers. While irrational numbers do not feature in this particular discussion, the interested reader is referred to (Høyrup, 2002, p.297-8).

2.3 Factors and Multiples

We return to decimal for a moment. For decimal numbers, it is well known that any integer with final digit 2, 4, 6, 8, or 0 is a multiple of two, and any integer with final digit 5 or 0 is a multiple of five. Other multiples are possible if we look at more digits. For instance, any decimal integer whose final three digits are

$$125, 250, 375, 500, 625, 750, 875 \text{ or } 000$$

is a multiple of 5^3 . Generally, if a is a proper factor of 10^k , $k \in \mathbb{Z}^+$ then a decimal integer is a multiple of a exactly when its final k digits match one of the prescribed endings

$$1 \times a, 2 \times a, \dots, \left(\frac{10^k}{a} - 1 \right) \times a$$

or are all zero. Our definition of multiple for SPVN numbers is similar, except that we limit the number of final digits to at most two, omit trailing null digits, and have many proper factors to consider because 60 is a superior highly composite number. This definition is one of the novel aspects of the paper.

Let A_1 be the set of proper factors of 60^1 in SPVN form

$$A_1 = \{30, 20, 15, 12, 10, 6, 5, 4, 3, 2\}.$$

Similarly let A_2 be the set of proper factors of 60^2 that are not factors of 60^1 , in SPVN form

$$A_2 = \{50, 48, 45, 40, 36, 25, 24, 18, 16, 9, 8, \\ 7 : 30, 6 : 40, 3 : 45, 3 : 20, 2 : 30, 2 : 24, 1 : 40, 1 : 30, 1 : 20, 1 : 15, 1 : 12\}.$$

Definition 4 The SPVN number n is a **multiple** of $a \in A_k$, $k \in \{1, 2\}$ exactly when the final k digits of n match one of the prescribed endings

$$1 \times a, 2 \times a, \dots, (\bar{a} - 1) \times a.$$

Equivalently we say that a is a **factor** of n .

For instance, the number $5 : 55 : 57 : 25 : 18 : 45$ is a multiple of $3 : 45$ because it ends with $18 : 45 = 5 \times 3 : 45$. On the other hand, the possible endings for a multiple of 10 are 10, 20, 30, 40, and 50. Since 7 does not match one of these endings we conclude that 7 is not a multiple of 10, despite the fact that $7 = 42 \times 10$ in SPVN arithmetic. Defining the concept of multiple in terms of the final digits is not an entirely new idea, indeed Neugebauer's triaxial index grid is just a special case of this definition (Neugebauer, 1934, p. 9–15). See also (Friberg & Al-Rawi, 2016, p. 63–73) and (Proust, 2007, p. 170 footnote 313).⁶

Scribes would only consider factors that were obvious from the final one or two digits of a number. Because they were not willing to look beyond these digits, our definition

⁶ Proust suggested that a scribe would know that a number ending in $18 : 45$ has a factor of $3 : 45$ because $8 : 45$ can be split into $5 + 3 : 45$ (Proust, 2012, p.403), (Proust, 2016, p. 13). Unfortunately the same can be said for $28 : 45$ and $38 : 45$, which are not multiples of $3 : 45$.

Table 2 The 3 : 45-multiplication table

1	3 : 45	6	22 : 30	11	41 : 15	16	1	30	1 : 52 : 30
2	7 : 30	7	26 : 15	12	45	17	1 : 03 : 45	40	2 : 30
3	11 : 15	8	30	13	48 : 45	18	1 : 07 : 30	50	3 : 07 : 30
4	15	9	33 : 45	14	52 : 30	19	1 : 11 : 15		
5	18 : 45	10	37 : 30	15	56 : 15	20	1 : 15		

explicitly excludes the elementary regular numbers 27, 32, 54, 1 : 04, 1 : 21 and their reciprocals from consideration.

Scribes were very familiar with the multiples of $a \in A_1 \cup A_2$ and learned them by rote in scribal school (Robson, 2008, p. 97–106). To be precise, scribes would memorize lists with the following standard format

$$1 \times a, 2 \times a, \dots, 19 \times a, 20 \times a, 30 \times a, 40 \times a, 50 \times a.$$

Such lists are called ***a*-multiplication tables** and were memorized to facilitate multiplication. Although it seems that some tables served an additional purpose related to the determination of factors. For example, CBS 6095 (Neugebauer & Sachs, 1945, p.23) (Table 2) is a 3 : 45-multiplication table. Memorization of this table enabled scribes to easily multiply by 3 : 45, but it also allowed them to immediately recognize any multiple of 3 : 45 by matching the final two digits with this table.

How does this definition of factor fit with the Mesopotamian list-based approach to mathematics? The **combined table** is another standard piece of scribal equipment. It consists of the standard table of reciprocals followed by a homogeneous selection of about forty *a*-multiplication tables. Instances of the combined table are found throughout OB times across multiple archaeological sites and, aside from isolated variations, are basically uniform in structure (Sachs, 1947, p. 221). While the combined table was almost certainly used for multiplication, the inclusion of the standard table of reciprocals and the curious selection of multiplication tables suggests more is true.

Friberg argued that the combined table could be used to facilitate division (Friberg, 2007, p. 92). Perhaps it is more accurate to say that the combined table served as a reference table for all things related to multiplication, including the identification and removal of factors. The selection of *a*-multiplication tables includes every $a \in A_1 \cup A_2$, and this would explain how scribes were able to identify factors based on the final one or two digits alone. The standard table of reciprocals was also included because it showed how to remove these factors. In other words, the combined table relates to the identification and removal of factors in the following sense: the multiplication tables showed which numbers have factors, and the standard table of reciprocals showed how to remove those factors.^{7,8,9}

⁷ The combined table also contained multiplication tables related to practical quantities such as brick weights, see (Friberg, 2007, p. 89).

⁸ The lexicographic order of the combined table and the restriction of multiplication tables to $A_1 \cup A_2$ (or certain practical quantities) meant that the 50-multiplication table was always first in the combined table. See (Friberg, 2007, p. 91–92) for an alternative hypothesis.

⁹ The purpose of the 22 : 30-multiplication table appears to be related to the factorization of regular numbers. The purpose of the 44 : 26 : 40-multiplication table remains unclear.

Table 3 CBS 1215 #21:
computation of a many-place
reciprocal

Line		
9.	5 : 55 : 57 : 25 : 18 : 45	16
10.	1 : 34 : 55 : 18 : 45	16
11.	25 : 18 : 45	16
12.	6 : 45	1 : 20
13.	9	6 : 40
14.	8 : 53 : 20	
15.	2 : 22 : 13 : 20	
16.	37 : 55 : 33 : 20	
17.	10 : 06 : 48 : 53 : 20	

2.4 Factorization

Scribes used their understanding of factors to reduce large numerical problems to smaller problems that could be solved by reference to standard tables. For example, a scribe could reduce the problem $6 : 14 : 24 \times x = 2 : 36$ by removing a factor of 12 from $6 : 14 : 24$. This simplification could be repeated, eventually reducing the problem to $13 \times x = 5 : 25$ which is easily solved by reference to the combined table.

Here we define **factorization** as the process of simplifying a large problem through repeated identification and removal of factors, which halts once the problem is simple enough to be solved directly. The process was first recognized by Sachs in 1947, who called it “the Technique” for finding reciprocals (Sachs, 1947, p. 223). But this is just one application, it has since become apparent that scribes also used factorization to solve linear equations and find square roots. Examples are discussed below.

When does factorization stop? Most scholars agree that a scribe would have “pushed the simplification of the numbers as far as possible by dividing out all regular factors” (Bruins, 1957, p. 28) or until “no regular factors remained” (Britton et al., 2011, p. 534). Here we emphasize that factorization continues at the pleasure of the scribe. It is a tool that helps them solve large problems, and ceases once the problem is small enough to be solved directly. This usually means that all factors are removed, but not always. This subtle distinction is important for our later analysis.

This discussion focuses on factorization in general. It is different to the specific analysis of factorization for regular numbers, such as Sachs (1947); Proust (2012). It is much easier to determine the presence of factors for regular numbers because regular numbers have fewer possible one and two-digit endings. For example, any regular number ending in $2 : 40$ must be a multiple of $2 : 40$ (Proust 2007, p. 174–175). This level of simplicity does not extend to numbers in general. Indeed, the irregular number $12 : 40$ ends in $2 : 40$ but it is not a multiple of $2 : 40$.

Our first example (Table 3) shows how a scribe computes the reciprocal of $5 : 55 : 57 : 25 : 18 : 45$ and is taken from lines 9 to 17 of CBS 1215 #21 (Friberg & Al-Rawi, 2016, p. 74–75). The “computation of (reciprocal pairs of) regular many-place sexagesimal numbers that are not listed in the Standard table of reciprocals” is often referred to as the trailing-part algorithm (Friberg, 1990, p. 550). However, we regard it more generally as a specific application of factorization.

Table 4 3N-T 362+366:
computation of a many-place
reciprocal

Line			
4.	17 : 46 : 40		9
5.	2 : 40	2	22 : 30
6.		3 : 22 : 30	

Here and throughout we used bold-font to emphasize those digits used to determine factors during factorization.

Table 3 shows that the scribe computed the reciprocal of $5 : 55 : 57 : 25 : 18 : 45$ by iteratively breaking off elementary regular factors. The reciprocal of each factor was recorded on the right, and then the reciprocal of the whole computed from these parts

$$\overline{5 : 55 : 57 : 25 : 18 : 45} = 6 : 40 \times 1 : 20 \times 16^3 = 10 : 06 : 48 : 53 : 20.$$

To be precise, the task is to compute the reciprocal of $5 : 55 : 57 : 25 : 18 : 45$. The answer is not obvious and so the scribe seeks to reduce the problem by removing an elementary regular factor. The final digits $18 : 45$ show this number contains a factor of $3 : 45$, which can be removed with multiplication by the reciprocal $\overline{3 : 45} = 16$. This reciprocal is recorded on the right and the problem is reduced to finding the reciprocal of

$$5 : 55 : 57 : 25 : 18 : 45 \times 16 = 1 : 34 : 55 : 18 : 45.$$

Line ten begins with $1 : 34 : 55 : 18 : 45$, from which another factor of $3 : 45$ is found. The reciprocal 16 is recorded on the right, and the problem is reduced further

$$1 : 34 : 55 : 18 : 45 \times 16 = 25 : 18 : 45$$

Line eleven begins with $25 : 18 : 45$, from which still another factor of $3 : 45$ is found. The reciprocal 16 recorded, and the problem is reduced again

$$25 : 18 : 45 \times 16 = 6 : 45.$$

Line twelve begins with $6 : 45$. The reciprocal of this number is still not obvious, so the scribe reduces the problem by removing an elementary regular factor of 45 (the final two digits $6 : 45 = 9 \times 45$ show the factor of 45 is present). The reciprocal $\overline{45} = 1 : 20$ is recorded on the right, and the problem is reduced to finding the reciprocal of

$$6 : 45 \times 1 : 20 = 9.$$

Line thirteen begins with 9, the reciprocal $\overline{9} = 6 : 40$ appears on the right and factorization stops because there is nothing left to remove. The product of these reciprocals is calculated in lines fourteen to seventeen:

$$\begin{aligned} \overline{5 : 55 : 57 : 25 : 18 : 45} &= 6 : 40 \times 1 : 20 \times 16^3 \\ &= (8 : 53 : 20) \times 16^3 \\ &= (2 : 22 : 13 : 20) \times 16^2 \\ &= (37 : 55 : 33 : 20) \times 16 \\ &= 10 : 06 : 48 : 53 : 20. \end{aligned}$$

Table 5 UET 6/2 222: the square root of a many-place number

Line			
1.		1 : 03 : 45	
2.		1 : 03 : 45	
3.	15	1: 07: 44: 03: 45	16
4.	15	18: 03: 45	16
5.	17	4 : 49	
6.		3 : 45	
7.		1 : 03 : 45	

In this example, the scribe repeated factorization until no factors remained. This is not the case in 3N-T 362+366 (Robson, 2000, p. 22), which shows the calculation of another many-place reciprocal $17 : 46 : 40 = 3 : 22 : 30$. Lines 4 to 6 are given in Table 4.

Line four follows directly from factorization: the scribe does not know the reciprocal of $17 : 46 : 40$ and seeks to simplify the problem by breaking off an elementary regular factor. The scribe can immediately see that $6 : 40$ is a factor. This is because, having memorized the $6 : 40$ -multiplication table, a scribe would have recognized the final digits as $46 : 40 = 7 \times 6 : 40$. The scribe records the reciprocal of this factor $6 : 40 = 9$ on the right, and then removes it with $17 : 46 : 40 \times 9 = 2 : 40$.

Line five begins with $2 : 40$. This number could have been broken down into the elementary regular factors $2 : 40 = 2 \times 1 : 20$ but instead factorization halts. This is important because factorization has stopped before all the elementary regular factors were removed. The scribe may have noticed that $2 : 40 = 2 \times 1 : 20$ and computed the reciprocal directly as half of $1 : 20 = 45$

$$\overline{2 : 40} = \overline{2} \times \overline{1 : 20} = \overline{2} \times 45 = 22 : 30.$$

This is perhaps why the superfluous 2 appears in line five of the text.

As before, the final line contains the reciprocal $17 : 46 : 40$ which was calculated as the product of the individual reciprocals

$$\overline{17 : 46 : 40} = 9 \times 22 : 30 = 3 : 22 : 30.$$

This example is important because it emphasizes the idiosyncratic nature of Mesopotamian factorization. Scribes do not necessarily seek to remove each and every factor as with modern factorization. Instead, an individual scribe will cease removing factors once they can solve the problem directly.¹⁰

Our next example comes from VAT 7532 (Friberg, 2007, p. 23) where, in modern terms, a scribe solves the linear equation

$$6 : 14 : 24 \times x = 2 : 36.$$

The scribe simply states that $x = 25$ so we can do no more than guess how they arrived at this answer. Factorization can reduce this problem to

$$13 \times x = 5 : 25.$$

¹⁰ For an alternative geometric interpretation of this text, see (Robson, 2008, p. 107–109).

This cannot be reduced further, however a scribe would have already recognized the answer because $13 \times 25 = 5 : 25$ is familiar from the 25-multiplication table. This is typical, almost all known exercises can be solved with factorization and the combined table alone (Sachs, 1952, p. 151).

Our final example shows how factorization was used to calculate square roots such as $\sqrt{1 : 07 : 44 : 03 : 45}$ and is taken from the reverse side of UET 6/2 222 (Friberg, 2000, p. 108) (Table 5). The text begins by computing $1 : 03 : 45^2 = 1 : 07 : 44 : 03 : 45$ in lines one to three. This is followed by a variation on factorization where only *square* elementary regular factors are removed. As usual the reciprocal of each factor is recorded on the right, but additionally the square root of each factor is recorded on the left.

The product of these individual square roots yields the expected result $1 : 03 : 45$ in lines six and seven. In modern language, the scribe made two symmetrical calculations

$$\begin{aligned} 1 : 03 : 45^2 &= 1 : 07 : 44 : 03 : 45 \\ \sqrt{1 : 07 : 44 : 03 : 45} &= 15 \times \sqrt{18 : 03 : 45} \\ &= 15^2 \times \sqrt{4 : 49} \\ &= 15^2 \times 17 = 1 : 03 : 45. \end{aligned}$$

The scribe computes the square root of this many-place number by successively removing square factors until the problem is small enough to be solved directly. In line three, a square factor of $3 : 45$ (i.e. $3 : 45 = 15^2$) is apparent from the final two digits. The scribe records the reciprocal $3 : 45 = 16$ on the right and the square root $\sqrt{3 : 45} = 15$ on the left. Then the factor is removed

$$1 : 07 : 44 : 03 : 45 \times 16 = 18 : 03 : 45.$$

In line four the procedure repeats. Another square factor of $3 : 45$ is identified, its reciprocal and square root recorded, and then the factor is removed

$$18 : 03 : 45 \times 16 = 4 : 49.$$

The problem has now been reduced to finding the square root of $4 : 49$. Factorization halts because the answer, 17, can be found directly from a table of square roots (Friberg, 2007, p. 50).

In summary, scribal mathematics is fundamentally about lists and procedures. Large problems are reduced to small problems by factorization, and small problems are solved directly from standard tables.

2.5 Diagonal Triples

A Pythagorean triple is a right triangle whose three sides are all integers where the square of the hypotenuse equals the sum of the squares of the other two sides. The equivalent Mesopotamian understanding of this fundamental object is slightly different.

Table 6 MS 3971 §3d: generation of a diagonal triple from the regular number 1 : 20

Line	
1.	The 4th. 1 : 20 is the regular number, 45 its reciprocal. 1 : 20 and 45
2.	heap, 2 : 05 it gives. Half of 2 : 05 break, 1 : 02 : 30 it gives.
3.	1 : 02 : 30 square, 1 : 05 : 06 : 15.
4.	1 from the length tear off, 5 : 06 : 15 it gives.
5.	5 : 06 : 15 makes 17 : 30 square-root.
6.	17 : 30, the short side of the 4th diagonal.

Table 7 Generation parameters, intermediate values, and results corresponding to the OB school exercise MS 3971 §3. The text specifies that we are to examine the diagonals, and then in part §4 we are told to find a rectangle with diagonal 7. Of the five given rectangles, only §3d can be rescaled to have diagonal 7

Subsection	x	δ	δ^2	β
§3a	1 : 04	1 : 00 : 07 : 30	1 : 00 : 15 : 00 : 56 : 15	3 : 52 : 30
§3b	1 : 40	1 : 08	1 : 17 : 04	32
§3c	1 : 30	1 : 05	1 : 10 : 25	25
§3d	1 : 20	1 : 02 : 30	1 : 05 : 06 : 15	17 : 30
§3e	1 : 12	1 : 01	1 : 02 : 01	11

Definition 5 A **diagonal triple** is a triple of SPVN numbers (b, ℓ, d) corresponding to the sides and diagonal of a rectangle where $b^2 + \ell^2 = d^2$ and with $b < \ell$. If $\ell = 1$ then the triple is said to be **normalized** and the special notation $(\beta, 1, \delta)$ is used.

A diagonal triple is a rectangle whose sides and diagonal are SPVN numbers, as distinct from a Pythagorean triple which is a right triangle over the positive integers. We use the term diagonal triple instead of Pythagorean triple from now on, and refer to its measurements as the short side b , the long side ℓ and diagonal d .¹¹

For a period spanning 1500 years, from OB times to Seleucid times, it was known that a regular number x generates the normalized diagonal triple $(\beta, 1, \delta)$ where

$$\delta = \overline{2}(x + \bar{x}) \text{ and } \beta = \sqrt{\delta^2 - 1} = \overline{2}(x - \bar{x}).$$

For example, MS 3971 §3 (Friberg, 2007, p. 252–253), also see below) is an exercise where five different regular numbers are used to generate five normalized diagonal triples. The fourth in this series is given in Table 6.

The procedure begins with the elementary regular number $x = 1 : 20$. In lines one and two this is used to compute the diagonal as the average of x and its reciprocal $\delta = \overline{2}(x + \bar{x}) = 1 : 02 : 30$. In line three the scribe computes the square of the diagonal $\delta^2 = 1 : 05 : 06 : 15$. In line four the scribe computes the square of the short side according to the “Pythagorean” relation $\beta^2 = \delta^2 - 1 = 5 : 06 : 15$. Factorization was almost certainly used during the square root calculation $\beta = \sqrt{5 : 06 : 15} = 17 : 30$ in line five but

¹¹ A scribe would have understood a diagonal triple as a rectangle, with the equivalent statement about right triangles a secondary interpretation as per Britton et al. (2011); Friberg (2007); Høyrup (2002).

all details of this calculation are omitted. In any case, this example demonstrates a key characteristic of Mesopotamian mathematics: the questions were designed so they could be solved using only standard procedures and tables. It is because of this tradition, not lucky coincidence, that this square root can be found using only the techniques we have discussed thus far¹²

$$\sqrt{5 : 06 : 15} = 30 \times \sqrt{20 : 25} = 30 \times 5 \times \sqrt{49} = 30 \times 5 \times 7 = 17 : 30.$$

2.6 MS 3971 §3 and §4, a Small Plimpton 322

It is instructive to prefix our main discussion of Plimpton 322 with a discussion of MS 3971, a sequence of OB mathematical exercises published by Friberg in 2007. The similarities between Plimpton 322 and MS 3971 part §3 are “striking”, as discussed by (Britton et al., 2011, p. 588), (Proust, 2011, p. 664) and (Friberg, 2007, p. 252–254), although the connection between Plimpton 322 and MS 3971 parts §3 and §4 together has not been previously considered.

Part §3 begins with instructions that we should inspect the diagonals of five rectangles *aššum 5 šilpatum amari-ka*, literally “when/because your seeing the 5 diagonals”. A less literal translation would be “Carefully examine the 5 diagonals”. This is followed by the generation of five diagonal triples using the standard method shown above in Table 6. Of interest are the initial generation parameters x , diagonals δ , intermediate values δ^2 , and short sides β which have been summarized in Table 7.

What can be said about these diagonals? The numbers 1 : 01, 1 : 05, and 1 : 08 can be immediately recognized as elementary irregular numbers (recall that the student knows all the one and two-digit regular and irregular numbers between 2 and 1 : 21 by rote). A small amount of simplification is required to deduce that 1 : 00 : 07 : 30 is irregular. This is because once a single factor is removed, what remains is a number without any elementary regular factors

$$1 : 00 : 07 : 30 \times \overline{7 : 30} = 8 : 01.$$

Only 1 : 02 : 30 is regular, which is apparent once a single factor is removed

$$1 : 02 : 30 \times \overline{2 : 30} = 25.$$

These calculations present no difficulty for a student familiar with the combined table and versed in factorization. Part §3 concludes by repeating “5 diagonals”, so evidently we were supposed to realize something important about these numbers.

Part §3 is immediately followed by part §4 where the student is asked to find a rectangle with diagonal 7 or, in other words, to find a rectangle (b, ℓ, d) which can be rescaled by a factor of $\overline{7d}$ into

$$\overline{7d}(b, \ell, d) = (7\overline{bd}, 7\overline{\ell d}, 7).$$

¹² Note that 15 is regarded as a square number because $30^2 = 15$.

Fig. 1 Plimpton 322. Photograph courtesy of the Rare Books and Manuscripts Library, Columbia University; photograph by Andrew Kelly

This might seem trivial, but it is not. The question requires the student to use a rectangle with a regular diagonal (i.e. a rectangle where $\bar{d}$ exists). There are infinitely many diagonal triples, but only two have this property: the rectangle from §3d whose diagonal we were told to examine, and the simple (3, 4, 5) rectangle. It is not surprising that the student elects to answer this question with the numerically simple rectangle

$$7 \times \bar{5} \times (3, 4, 5) = (4 : 12, 5 : 36, 7).$$

But it cannot be a coincidence that the other possible rectangle appeared in section §3, especially considering the explicit and repeated emphasis on its diagonal. The alternative answer, using the rectangle from §3d, would have been

$$7 \times \overline{1 : 02 : 30} \times (17 : 30, 1, 1 : 02 : 30) = (1 : 57 : 36, 6 : 43 : 12, 7).$$

The lesson from MS 3971 §3 and §4 is that rectangles with regular sides are important because only the regular sides can be rescaled to arbitrary lengths such as 7. Naturally, (3, 4, 5) is the preferred triple on account of its three regular sides. Although diagonal triples with two regular sides are also useful in certain circumstances, such as (17 : 30, 1, 1 : 02 : 30) in this example.¹³

3 Plimpton 322

The object known as Plimpton 322 (Fig. 1) is a fragment of a clay tablet from ancient Mesopotamia. It was sold by the archaeologist, adventurer, academic, and antiquities dealer Edgar Banks to the famous publisher and collector George Plimpton in about 1922, who

¹³ The (17 : 30, 1, 1 : 02 : 30) triple is perhaps more familiar when it is rescaled to $24 \times (17 : 30, 1, 1 : 02 : 30) = (7, 24, 25)$.

Table 8 Translation of Plimpton 322, with errors underlined

	Column <i>I</i> Square of the diagonal. Subtract 1 and then the square of the short side comes up	<i>II</i> fb-si _g of the of short side	<i>III</i> fb-si _g of the diagonal	<i>IV</i> Row
Position of the break	1 : 59 : 00 : 15	1 : 59	2 : 49	1
	1 : 56 : 56 : 58 : 14 : <u>56</u> : 15	56 : 07	3 : <u>12</u> : 01	2
	1 : 55 : 07 : 41 : 15 : 33 : 45	1 : 16 : 41	1 : 50 : 49	3
	1 : 53 : 10 : 29 : 32 : 52 : 16	3 : 31 : 49	5 : 09 : 01	4
	1 : 48 : 54 : 01 : 40	1 : 05	1 : 37	5
	1 : 47 : 06 : 41 : 40	5 : 19	8 : 01	6
	1 : 43 : 11 : 56 : 28 : 26 : 40	38 : 11	59 : 01	7
	1 : 41 : 33 : <u>59</u> : 03 : 45	13 : 19	20 : 49	8
	1 : 38 : 33 : 36 : 36	<u>9</u> : 01	12 : 49	9
	1 : 35 : 10 : 02 : 28 : 27 : 24 : 26 : 40	1 : 22 : 41	2 : 16 : 01	10
	1 : 33 : 45	45	1 : 15	11
	1 : 29 : 21 : 54 : 02 : 15	27 : 59	48 : 49	12
	1 : 27 : 00 : 03 : 45	<u>7</u> : <u>12</u> : <u>01</u>	4 : 49	13
	1 : 25 : 48 : 51 : 35 : 06 : 40	29 : 31	53 : 49	14
	1 : 23 : 13 : 46 : 40	56	53	15

then bequeathed it along with the rest of his collection to Columbia University in 1936 where it resides today. Neugebauer and Sachs showed that it contains diagonal triples in 1945 (Neugebauer & Sachs, 1945, p. 38–41), and subsequently it has become one of the most interesting and well studied mathematical objects of the ancient world.

The tablet is broken on the left, and the position of the break suggests this is only the latter part of a larger original.

We summarise some of the significant work regarding Plimpton 322 and present a new theory based on the above understanding of SPVN numbers and factorization. This is followed by a discussion of some of the many theories regarding its purpose.

3.1 Translation

The extant fragment of Plimpton 322 contains a table with four columns and fifteen rows. The column headings have been well studied and can be translated as “The square of the diagonal. Subtract 1 and then the square of the short side comes up”, “fb-si_g of the short side”, “fb-si_g of the diagonal” and “Row”.

The heading of column *I* indicates that the text concerns the diagonals δ and short sides β of rectangles satisfying $\delta^2 - 1 = \beta^2$, which is to say the text is about normalized diagonal triples $(\beta, 1, \delta)$. This is confirmed by the numerical values of δ^2 found in this column, starting with a rectangle that is almost a square and gradually becoming flatter row by row.

The suffix *ma* on the verb for subtraction, which indicates logical or temporal flow, makes it clear that actual subtraction is intended and we are not at liberty to reinterpret this operation in equivalent ways, such as $\delta^2 = \beta^2 + 1$.¹⁴

The headings of columns *II* and *III* contain the word *ib-si₈* which is usually translated as “square-side” or “equalside” meaning square root (Robson, 2001, p. 174), (Friberg, 2007, p. 434–435), (Britton et al., 2011, p. 526). However, there are exceptions: the term is used in MLC 2078 in reference to the result of the exponential and logarithmic operations (Neugebauer & Sachs, 1945, p. 35), and in MS 3048 in reference to the result of an exotic arithmetical operation (Friberg, 2007, p. 62–63). Neugebauer, suspicious that Plimpton 322 might be another exception, gave an intentionally vague translation of *ib-si₈* as “solving number” which indicates “the number which is the result of some operation” (Neugebauer & Sachs, 1945, p. 36). We follow Neugebauer in taking *ib-si₈* to mean the result of some operation and otherwise leave this ambiguous term untranslated.

The meaning of the fourth column is quite clear, as this is simply a row number (Table 8).

3.2 Generation Parameters

Neugebauer and Sachs proposed that the diagonal triples in Plimpton 322 were generated from two elementary regular numbers p and q chosen from the standard table of reciprocals with $p > q$. For simplicity of exposition, but without loss of generality, we assume p and q are relatively prime. These numbers determine a pair of reciprocals $p\bar{q}$ and $q\bar{p}$ which generate the diagonal triple $(\beta, 1, \delta)$ as

$$\begin{aligned}\delta &= \overline{2(p\bar{q} + q\bar{p})} = \overline{2pq(p^2 + q^2)}, \\ \beta &= \overline{2(p\bar{q} - q\bar{p})} = \overline{2pq(p^2 - q^2)}.\end{aligned}$$

A full list of the parameter values, and a discussion of how they might have been chosen, is given in (Proust, 2011, p. 663–664).¹⁵

In 1964, de Solla Price observed that if we restrict $p\bar{q} < 2 : 24 \approx 1 + \sqrt{2}$ (de Solla Price, 1964, p. 222) then 38 diagonal triples are produced, and ordered by “flatness” the first 15 of these triples match the first 15 entries of Plimpton 322 exactly. The remaining 23 fill the blank space on the tablet which had been ruled as if the author expected to place additional entries there. Moreover, we might reasonably expect a scribe to make this restriction because it is equivalent to saying that the short side β is literally shorter than the long side 1.

Alternatively, the starting point may not have been two parameters p and q chosen from the standard table of reciprocals but a single parameter x chosen from a hypothetical larger table of reciprocals (Bruins, 1949, p. 630), (Bruins, 1957, p. 26), (Britton et al., 2011, p. 534). The rows of Plimpton 322 could have been generated by simply proceeding through such a table and computing

¹⁴ Bruins (Bruins, 1957, p. 26) maintains there is no initial 1 in this column, which would mean the values are that of β^2 instead of δ^2 . But there is wide agreement that the values of δ^2 are visible in several places, see (Britton et al., 2011, p. 524).

¹⁵ It is more correct to say that the values of p and q were *almost* all chosen from the standard table of reciprocals because one additional reciprocal is used (Neugebauer and Sachs 1945, p. 41).

Table 9 Plimpton 322: the five errors

Row	Column	Error type	Correction
2	<i>I</i>	Type 1 or copy	<u>56</u> should be 50 : 06.
2	<i>III</i>	Type 2	<u>12 : 01</u> should be 13.
8	<i>I</i>	Type 1	<u>59</u> should be 45 : 14.
9	<i>II</i>	Copy	<u>9</u> should be 8.
13	<i>II</i>	Copy	<u>7 : 12 : 01</u> should be 2 : 41.

Table 10 If we believe that Plimpton 322 contains diagonal triples with common factors removed then there are an additional four errors. Two corrections are possible in row 15, either 56 is changed to 28 in column *II* or alternatively, 53 is changed to 1 : 46 in column *III*

Row	Column	Actual entry	Expected entry
2	<i>III</i>	3 : <u>12 : 01</u>	1 : 20 : 25
11	<i>II</i>	45	3
11	<i>III</i>	1 : 15	5
15	<i>II (III)</i>	56 (53)	28 (1 : 46)

$$\delta = \bar{2}(x + \bar{x}),$$

$$\beta = \bar{2}(x - \bar{x}).$$

So Bruins advocates a standard method of generation with an advanced table of reciprocals; while Neugebauer advocates an advanced method of generation with the standard table of reciprocals. Neither the advanced method nor the advanced table of reciprocals are otherwise known to have existed at the time.

There has been much discussion about the method of generation and what it can tell us about the possible contents of the missing columns. It seems reasonable that the missing columns contained at least the diagonal δ and short side β as argued by (Britton et al., 2011, p. 542) and supported by the three column headings which speak of operations performed to these numbers. However, Robson believes that the missing fragment of the tablet contained only the generation parameters (Robson, 2001, p. 201), and Friberg believes the missing fragment contained the generation parameters in addition to the δ and β values (Friberg, 2007, p. 448).

3.3 Errors

Many scholars believe that Mesopotamian scribes performed SPVN calculations using some kind of computational device similar to an abacus or counting board Woods (2017), Proust (2016), Proust (2000), Middeke-Conlin (2020). The device could potentially be in the form of an auxiliary clay tablet or wax writing board (Robson, 2008, p. 78). This device, whatever it may be, is susceptible to two kinds of error. For emphasis, we underline those digits that are considered erroneous. The first kind is **copy error** which occurs when SPVN numbers are carelessly transferred between the device and tablet (Proust, 2000, p.298). For instance, in Plimpton 322 row 9 it appears that the number 8 : 01 was carelessly copied as 9 : 01. Such errors can occur whenever numbers are copied, either

Table 11 A possible reconstruction of the factorization for row 2: (left) $\beta = 58 : 27 : 17 : 30$ is reduced to $56 : 07$ (right) a misplaced intermediate null in $16 : \underline{00} : 05$ caused the erroneous reduction of $\delta = 1 : 23 : 46 : 02 : 30$ to $3 : \underline{12} : \underline{01}$

$58 : 27 : 17 : \mathbf{30}$	2	$1 : 23 : 46 : \mathbf{02} : \mathbf{30}$	24
$1 : 56 : 54 : \mathbf{35}$	12	$33 : 30 : \mathbf{25}$	12
$23 : 22 : \mathbf{55}$	12	$6 : 42 : \mathbf{05}$	12
$4 : 40 : \mathbf{35}$	12	$1 : 20 : \mathbf{25}$	12
$56 : 07$		$16 : \underline{00} : \mathbf{05}$	12
		$3 : \underline{12} : \underline{01}$	

as they enter the device before computation or as they are copied from the device after computation.

Friberg calls the second category of error **telescoping error** (Friberg, 2007, p. 21). Telescoping errors occur during computation, as opposed to copy errors that occur before or after. According to Proust there are three possible types of computational error: the merging of two consecutive digits, the insertion of an extraneous null digit, and the omission of a digit. These are known as types 1, 2 and 3 respectively (Proust, 2000, p.298).

For example, the correct value for δ^2 in row 8 is

$$1 : 41 : 33 : 45 : 14 : 03 : 45.$$

However, what the scribe actually wrote was

$$1 : 41 : 33 : \underline{59} : 03 : 45.$$

The digits $45 : 15$ were accidentally combined into the single digit 59, and so we say this is a type 1 error. Similarly, the correct value for δ^2 in row 2 is

$$1 : 56 : 56 : 58 : 14 : 50 : 06 : 15.$$

However, what the scribe actually wrote was

$$1 : 56 : 56 : 58 : 14 : \underline{56} : 15$$

The digits $50 : 06$ were accidentally combined into the single digit 56. This appears to be a type 1 error, although it could also be considered a copy error.

An example of a type 2 error would be if the value $16 : \underline{00} : 05$ appeared instead of the correct value $16 : 05$. Finally, the loss of the middle digit 36 from the number $19 : 36 : 15$ would be considered a type 3 error. See Table 9 for a list of the errors in Plimpton 322 and their types. We argue that the error in row 2 resulted from a type 2 computational error during factorization and that the error in row 13 resulted from a copy error.

With a single exception (Friberg (2007)) it is universally believed that columns *II* and *III* correspond to the sides of a diagonal triple after common regular factors were removed, see for instance (Neugebauer & Sachs, 1945, p. 40), (Friberg, 1981, p. 291), (Robson, 2001, p.192–193), (Britton et al., 2011, p. 560) and (Mansfield & Wildberger, 2017, p.399). However, the four entries listed in Table 10 are inconsistent with this hypothesis. The single entries in rows 2 and 15 are usually dismissed as uncategorized computational errors. But the two entries in row 11 are much harder to dismiss this way. This row would almost certainly be correct because it contains the simplest and most familiar diagonal triple, yet it must be treated as an error (Neugebauer & Sachs, 1945, p. 40) or ignored Robson (2001).

Table 12 A possible reconstruction of the factorization steps for row 15: (left) $\beta = 37 : 20$ is factorized to 56 (right) $\delta = 1 : 10 : 40$ is factorized to 53

37 : 20	3	1 : 10 : 40	1 : 30
1 : 52	30	1 : 46	30
56			53

Table 13 A possible reconstruction of the factorization steps for row 5: (left) $\beta = 54 : 10$ is factorized to 1 : 05 (right) $\delta = 1 : 20 : 50$ is factorized to 1 : 37

54 : 10	6	1 : 20 : 50	6
5 : 25	12	8 : 05	12
1 : 05		1 : 37	

Table 14 A possible reconstruction of the factorization steps for row 13: (left) $\beta^2 = 27 : 00 : 03 : 45$ is mistaken for β and factorized to 7 : 12 : 01 (right) $\delta = 1 : 12 : 15$ is factorized to 4 : 49

<u>27 : 00 : 03 : 45</u>	16	1 : 12 : 15	4
<u>7 : 12 : 01</u>		4 : 49	

Britton et. al. do both and label this row a “discrepancy without significance” (Britton et al., 2011, p. 535). Friberg believes this is not an error at all, and instead suggests this row was already sufficiently reduced for the purpose of computing squares (Friberg, 1981, p. 295–6) or square roots (Friberg, 2007, p. 436). We extend Friberg’s idea and argue that all the rows were sufficiently reduced, although for a different purpose.

3.4 Solving Numbers

The heading for column *II* tells us that it contains the result of some operation performed on the short side β . Similarly, the heading for column *III* tells us that it contains the result of some operation performed on the diagonal δ . It is unlikely that the operation was performed on the squared values β^2 and δ^2 because the heading tells us so and because the type 1 computational error in row 8 column *I* was not transmitted to the latter columns (Friberg, 2007, p. 449).

Neugebauer (Neugebauer & Sachs, 1945, p. 40) and Bruins (Bruins, 1957, p. 26–28) originally supposed that the operation was to remove common factors, and this assumption was adopted by almost every subsequent study. Here we argue that the operation is simply factorization; not common factorization as previously supposed. Rows 11 and 15 are correct under this hypothesis and yield valuable information, and row 2 contains a single type 2 computational error. Moreover, the rows which retain regular factors show that the scribe deliberately chose to cease factorization early, and this gives us a clue as to what they were looking for. Those numbers which retain regular factors were *sufficient* for something.

We begin with the generation procedure for row 1. Starting with the parameters $p = 12$ and $q = 5$ (or $x = p\bar{q} = 2 : 24$) the scribe computes the short side $\beta = 2pq(p^2 - q^2) = 59 : 30$ and diagonal $\delta = 2pq(p^2 + q^2) = 1 : 24 : 30$. The value in

column *I* is obtained by computing the square of the diagonal $\delta^2 = 1 : 59 : 00 : 15$. The value in column *II* is obtained by removing factors from the short side, and the value in column *III* is obtained by removing factors from the diagonal. From this row, it is impossible to tell if the two factorizations were synchronous (as usually supposed) or independent (as argued here and by Friberg (2007)) because both numbers have only a single regular factor $2pq = 30$. In fact, all rows share a factor of at least $2pq$ by construction. We can only distinguish synchronous from independent factorization through analysis of the rows where additional regular factors are present. This occurs in rows 2, 5, 11, and 15, which warrant careful attention.

Row 2 makes it clear that factorization is independent, simply because more factors are removed from the diagonal than from the short side (Friberg, 2007, p. 436). The steps of the factorization procedure are given in Table 11. An error occurred during factorization of the diagonal since $3 : 12 : 01$ appears instead of the correct value $3 : 13$. This error can be easily explained by the insertion of an intermediate null during the penultimate step of factorization, i.e. a type 2 computational error. The penultimate step of factorization should have been $16 : 05 \times 12 = 3 : 13$. However, if we introduce an intermediate null then $16 : 05$ becomes $16 : \underline{00} : 05$ and when this error is carried forward it produces the number found on the tablet

$$16 : \underline{00} : 05 \times 12 = 3 : \underline{12} : 01.$$

The nature of this error adds weight to our hypothesis that the operation in these columns was indeed factorization.

In row 15 the values $\delta = 1 : 10 : 40$ and $\beta = 37 : 20$ share a common factor of $1 : 20$, which is removed from δ but not from β (see Table 12). This reinforces the idea that factorization was independent. Moreover, the scribe would have been acutely aware that the reduced value of β (56) still has regular factors but halts factorization anyway. Whatever they were looking for, it was already apparent from 56.

Similarly in row 11 the values $\beta = 45$ and $\delta = 1 : 15$ are not factorized at all, despite their shared factor 15. Again, whatever the scribe was looking for, it must be apparent from these numbers without any reduction.

In row 5 the value $\beta = 54 : 10$ is reduced to $1 : 05$ (see Table 13) and this is quite interesting. Clearly, $1 : 05$ has a regular factor but the scribe does not remove it. The scribe seems to stop removing factors once none are left, or once the numbers are sufficiently small.

The error in row 13 can be explained as a copy error as follows. The scribe should have copied the value of β into the computational device and removed elementary regular factors through factorization. However, it seems that the scribe copied the wrong number and put β^2 there instead. Factorization of β^2 results in the number we see on the tablet (see Table 14). This was suggested by Britton et. al. (Britton et al., 2011, p. 538), but only works if we assume the columns were factorized independently of one another. For an alternative hypothesis, see (Bruins, 1957, p. 28).

So the numbers in columns *II* and *III* appear to be the result of independent factorization of β and δ . In row 11 (columns *II* and *III*) and rows 5 and 15 (column *II*), factorization terminates because the numbers 45, $1 : 15$, $1 : 05$, and 56 were sufficient in some sense, and in all other cases factorization terminates when no factors remain. See Table 15 for a restoration of the missing parts of Plimpton 322 based on this understanding of its contents.

This analysis answers some questions about Plimpton 322, but raises others. Why is the long side absent? Why would a scribe factorize a number but not record any of the factors?

What is sufficient about the numbers 45, 1 : 15, 1 : 05, and 56 that allowed the scribe to stop factorization? No reason is specified in Plimpton 322, but one answer resolves all these questions: the sides were factorized to determine if they are regular or not. The long side is regular by construction so it is not considered, the removed factors were discarded because they are irrelevant, and the numbers 45, 1 : 15, 1 : 05, and 56 are sufficient in the sense that a scribe would immediately recognize them as regular or irregular.

In summary, MS 3971 and Plimpton 322 are very similar indeed. One tablet invites us to investigate the regularity of the diagonal for five rectangles, and the other invites us to investigate the regularity of the short side and diagonal for fifteen (or more likely 38) rectangles.

3.5 Past Hypotheses

We have presented a new interpretation of Plimpton 322 as a table of rectangles that shows which sides are regular and which are not. What was the purpose of the text? Answers to this question are both speculative and necessary. Speculative because the text itself does not provide an answer, and necessary because any interpretation must fit within the wider context of Mesopotamian mathematics. The remainder of this section summarizes the views of some major studies and is followed by a final section where we view Plimpton 322 in relation to recent discoveries concerning contemporary land measurement.

3.5.1 A Theoretical Document

Neugebauer believed that Plimpton 322 was a theoretical document motivated by a need to understand diagonal triples.

In other words it was known during the whole duration of Babylonian mathematics that the sum of the squares of the lengths of the sides of a right triangle equals the square of the length of the hypotenuse. This geometrical fact having once been discovered, it is quite natural to assume that all triples of numbers ℓ , b and d which satisfy the relation $\ell^2 + b^2 = d^2$ can be used as sides of a right triangle. It is furthermore a normal step to ask the question: When do numbers ℓ , b , d satisfy the above relation? Consequently it is not too surprising that we find the Babylonian mathematicians investigating the number-theoretical problem of producing “Pythagorean numbers”. (Neugebauer, 1969, p. 36)

Britton et. al. agree that Plimpton 322 is a theoretical investigation into diagonal triples. In light of MS 3971 part §3 they suggest that Plimpton 322 is an elucidation of the relationship between diagonal triple property $\delta^2 - 1 = \beta^2$ and the procedure for completing the square (Britton et al., 2011, p.561). Here we have considered MS 3971 parts §3 and §4 together, and argued that both this tablet and Plimpton 322 are investigations into diagonal triples with regular sides.

Friberg observed there is a problem with this line of thinking. If we assume that Plimpton 322 was a theoretical investigation into diagonal triples with common factors removed, as both Neugebauer and Britton et. al. did, then it should contain 3 and 5 in row 11 and not 45 and 1 : 15. In 1981 Friberg used this as “very close to a proof that no application of ‘number theory’, in the proper sense of the word, was involved in the construction

of the table on Plimpton 322” (Friberg, 1981, p. 296). This view was recently advanced again in (Friberg, 2007, p. 434).

Friberg is right to object: Plimpton 322 cannot be a theoretical study of diagonal triples with common factors removed. However, this objection may be safely set aside because it now seems clear that Plimpton 322 has nothing to do with the concept of common factors. Plimpton 322 may well have a theoretical character and we shall return to this idea later.

3.5.2 A Teacher’s List

The idea that Plimpton 322 might be related to practical land measurement was conjectured by Price, but this conjecture was quickly abandoned in favor of its possible use as a teacher’s list of geometric problems with exact solutions.

Thus, on a purely arithmetical basis there is erected a “trigonometric” corpus that could be used for practical mensuration, or more probably for the setting out of series of practice problems in mensuration, all of which would be capable of exact numerical solution. (de Solla Price, 1964, p. 13)

Recall that the standard method for generating diagonal triples begins with a regular number x that is used to calculate $\delta = \overline{2}(x + \overline{x})$ and then $\beta = \sqrt{\delta^2 - 1}$. Since the value δ^2 occurs in column *I*, many believe that Plimpton 322 is a list of parameters related to the generation of diagonal triples. While column *I* could have enabled a teacher to easily check the intermediate δ^2 value in this procedure, the likelihood of this possibility has been significantly overstated.

A Mesopotamian teacher’s list is a collection of problems that students can answer using standard techniques and tables, and they carefully avoided any problem that was not easily solved by these methods. Consider for instance the following excerpt from the teacher’s list BM 80209 (Robson, 2001, p. 117).

If the area is 8:20 what is the circumference?
If the area is 2:13:20 what is the circumference?
If the area is 3:28:20 what is the circumference?
If the area is 5 what is the circumference?

Using the standard scribal approximation $\pi \approx 3$, a circle with area A has circumference $2\sqrt{\pi \times A} \approx 2\sqrt{3 \times A}$. This produces very simple answers to the questions posed above: 10, 40, 50 and 1 respectively. But these answers are more than just “exact numerical solution”s. They were carefully chosen to ensure the problems were tractable. In other words, the answers could be found using standard tables.

Plimpton 322 contains many large prime numbers in columns *II* and *III*, such as 5 : 09 : 01. These large primes may be interpreted as exact solutions to some unspecified problem, but they are quite different from the “very simple solutions” usually found in school exercises (Høyrup, 2002, p. 386, footnote 475) and standard tables.

Nevertheless, Friberg concludes

... that the purpose of the author of Plimpton 322 was to write a “teacher’s aid” for setting up and solving problems involving right triangles. (Friberg, 1981, p. 302)

Robson agrees that

It would have enabled a teacher to set his students repeated exercises on the same mathematical problem, and to check their intermediate and final answers without repeating the calculations himself. (Robson, 2002, p. 118)

This is a very bold conclusion. If Plimpton 322 were a teacher's list then it should contain carefully chosen problems. Presumably, the task involves computing the square root of the intermediate value in column *I* to obtain the results (ib-si_8) in columns *II* and *III*. However, the difficulty of this square root step ranges from trivial (row 11), to very difficult (row 10) and perhaps even impossible with available techniques (row 4) (Mansfield & Wildberger, 2017, p. 404). A modern analogy would be to say that it contains a mix of elementary school problems alongside the unsolved conjectures of mathematics and hence does not serve as a list of problems for any given audience. On this point we agree with Britton et al. (2011) that Plimpton 322 is a complete list of *all* diagonal triples produced by the generation procedure and, unlike known teacher's lists or student exercises, has not been curated for students.

Speculation that Plimpton 322 is a teacher's list is largely based on a lack of viable alternatives, with Friberg concluding Plimpton 322 is not related to number theory and hence must be a teacher's list (Friberg, 2007, p. 434) (Friberg, 1981, p. 296) and Robson similarly concluding it is a teacher's list because no other hypothesis seemed likely (Robson, 2001, p. 113).

3.5.3 No Trig as We Know It

The word trigonometry is universally used to mean the study of the ratios of the sides of right triangles as functions of angle (sin, cos, tan and the reciprocal ratios csc, sec, cot), and it is quite clear that these functions are not part of Mesopotamian mathematics. Any interpretation of Plimpton 322 as a table of trigonometric functions is rightly dismissed by Robson as anachronistic, who is far from alone on this point. There is vast consensus that Plimpton 322 is not about trigonometry as we know it (Van Brummelen, 2009, p. 14). However, it is misleading to say that

... there could be no notion of measurable angle in the Old Babylonian period. Without a well-defined centre or radius there could be no mechanism for conceptualising or measuring angles, and therefore the popular interpretation of Plimpton 322 as some sort of trigonometric table becomes meaningless. (Robson, 2001, p.182–183)

Scribes measured and understood just one angle: the right angle. The Akkadian word *mutarrittum*, meaning “direction of the plumb line” was used as a metaphor for the perpendicular side of a shape (Høyrup, 2002, p. 228–229) and an interest in perpendicularity is apparent from early sketches in education and surveying (Robson, 2001, p. 182 footnote 18). In particular, diagonal triples were used by surveyors to construct shapes with perpendicular sides by least the OB period (see below, also Mansfield (2020)). So while it is meaningless to consider Plimpton 322 as a table of trigonometric functions, it is *not* meaningless to emphasize the practical importance of right angles in Mesopotamian surveying and consider Plimpton 322 as a table of rectangles.

Price conjectured that Plimpton 322 is “trigonometric” in the sense that it relates to practical mensuration. While Plimpton 322 cannot be a trigonometric table in the usual sense, it may still relate to practical mensuration as proposed by Mansfield & Wildberger (2017).

We are now in a position to refine these conjectures into a new hypothesis: that Plimpton 322 was a theoretical investigation into a certain problem in contemporary land measurement.

4 Contemporary Land Measurement

We have presented a new analysis of Plimpton 322 as a table of rectangles with information about which sides are regular and which sides are not. This section anticipates the question “why?”, as in what problem could have motivated such a list? Unfortunately, the answer to this question was lost thousands of years ago and we do not hope for definitive answers. However, it is important to establish that our interpretation fits within the wider context of Mesopotamian mathematics and that there was at least some contemporary interest in rectangles with regular sides.

Tables of rectangles are arguably the earliest mathematical texts (Proust, 2020, p. 346). For example, VAT 12593 (Deimel, 1922, No. 82), MS 3047 (Friberg, 2007, p. 160) and Feliu (2012) are tables of rectangles and squares that date from the Early Dynastic period (ca. 2,600–2,350 BCE). These very early tables of rectangles show the simple relationship between perimeter and surface area.

These tables were used by **surveyors**, mathematically trained scribes specialized in land measurement. Surveying was a highly respected profession in ancient Mesopotamia. The 1-rod reed and measuring rope are not only the surveyor’s tools but also public symbols of fairness often found in the hands of goddesses and kings (Robson, 2008, p. 117–122). These tools and their function in surveying are mentioned in *Enki and the World Order* where the Sumerian god Enki determines the destiny of Nisaba, the patron goddess of scribes (Robson, 2008, p. 188):

My illustrious sister, holy Nisaba,
is to receive the 1-rod reed.
The lapis lazuli rope is to hang from her arm.
She is to proclaim all the great divine powers.
She is to fix boundaries and mark borders. She is to be the scribe of the Land.
The gods’ eating and drinking are to be in her hands.

A **field plan** is a sketch of a field and its measurements made by a surveyor. Field plans are found from the very earliest times (Lecompte, 2020, p. 285–287). Most known examples date from the Ur III period (2000–1900 BCE) and concern large agricultural fields belonging to institutions such as palaces or temples (Nemet-Nejat, 1982, p. 19). Surveyors would measure these fields to estimate the expected size of the harvest, so essentially these early field plans are just agricultural estimates (Liverani, 1990, p. 155). To facilitate its measurement, a field was subdivided into shapes that are approximately rectangles, right trapezoids, and right triangles. However these shapes are just approximations and the rectangles “seem never to have had equal sides or truly right angles” (Dunham, 1986, p. 33).

Surveyors would have been aware that their perpendicular lines were not entirely accurate, however this method of surveying was canonical and any imperfection would have been an “acceptable discrepancy” at the time (Middeke-Conlin, 2020, p. 123).

The margins of acceptable discrepancy narrowed during the OB period as land ownership began to shift away from institutions and towards private individuals. Cadastral accuracy became increasingly important to avoid private disputes over boundaries

Fig. 2 Si.427 Obverse. Photograph by and courtesy of the İstanbul Arkeoloji Müzeleri

(Nemet-Nejat, 1982, p. 19). This is apparent from an OB poem about two quarreling scribes where the senior scribe admonishes the junior scribe's ability to survey land properly (Vanstiphout, 2003, p. 589). The senior scribe says

Go to divide a plot, and you are not able to divide the plot;
go to apportion a field, and you cannot even hold the tape and rod properly.
The field pegs you are unable to place; you cannot figure out its shape,
so that when wronged men have a quarrel you are not able to bring peace,
but you allow brother to attack brother.
Among the scribes, you (alone) are unfit for the clay.

To which the junior scribe retorts that his surveying brings peace to the hearts of wronged men

When I go to apportion a field, I can apportion the pieces,
so that when wronged men have a quarrel, I soothe their hearts and [...].
Brother will be at peace with brother,

This new phase in surveying is best illustrated by the remarkably accurate field plan Si.427 (Fig. 2). Fr. Vincent Scheil discovered and cataloged Si.427 along with many other tablets from the 1894 French archaeological expedition at Sippar (Scheil, 1902, p. 134). A partial edition was published in 1895 (Scheil 1895, p. 33) and a complete edition in 2020

Table 15 A reconstruction of the full table from Plimpton 322. Possible variations depend upon which factors the scribe chose to remove. For example, in line 35 the scribe is free to factorize $6 : 20$ as $6 : 20 \times 10 = 38$ or $6 : 20 \times 20 = 19$

Short side	Diagonal	Square of the diagonal. Subtract 1 and then the square of the short side comes up	fb-si_8 of the short side	fb-si_8 of the diagonal	Row
59:30	1:24:30	1:59:00:15	1:59	2:49	1
58:27:17:30	1:23:46:02:30	1:56:56:58:14:50:06:15	56:07	3:13	2
57:30:45	1:23:06:45	1:55:07:41:15:33:45	1:16:41	1:50:49	3
56:29:04	1:22:24:16	1:53:10:29:32:52:16	3:31:49	5:09:01	4
54:10	1:20:50	1:48:54:01:40	1:05	1:37	5
53:10	1:20:10	1:47:06:41:40	5:19	8:01	6
50:54:40	1:18:41:20	1:43:11:56:28:26:40	38:11	59:01	7
49:56:15	1:18:03:45	1:41:33:45:14:03:45	13:19	20:49	8
48:06	1:16:54	1:38:33:36:36	8:01	12:49	9
45:56:06:40	1:15:33:53:20	1:35:10:02:28:27:24:26:40	1:22:41	2:16:01	10
45	1:15	1:33:45	45	1:15	11
41:58:30	1:13:13:30	1:29:21:54:02:15	27:59	48:49	12
40:15	1:12:15	1:27:00:03:45	2:41	4:49	13
39:21:20	1:11:45:20	1:25:48:51:35:06:40	29:31	53:49	14
37:20	1:10:40	1:23:13:46:40	56	53	15
36:27:30	1:10:12:30	1:22:09:12:36:15	35	5:37	16
32:50:50	1:08:24:10	1:17:58:56:24:01:40	7:53	3:17	17
32	1:08	1:17:04	32	1:08	18
30:04:53:20	1:07:07:06:40	1:15:04:53:43:54:04:26:40	1:07:41	2:31:01	19
29:15	1:06:45	1:14:15:33:45	39	1:29	20
27:40:30	1:06:04:30	1:12:45:54:20:15	41	14:41	21
25	1:05	1:10:25	25	1:05	22
24:11:40	1:04:41:40	1:09:45:22:16:06:40	14:31	38:49	23
22:22	1:04:02	1:08:20:16:04	11:11	32:01	24
21:34:22:30	1:03:45:37:30	1:07:45:23:26:38:26:15	34:31	1:42:01	25
20:51:15	1:03:31:15	1:07:14:53:46:33:45	16:41	50:49	26
20:04	1:03:16	1:06:42:40:16	5:01	15:49	27
18:16:40	1:02:43:20	1:05:34:04:37:46:40	5:29	18:49	28
17:30	1:02:30	1:05:06:15	35	25	29
14:57:45	1:01:50:15	1:03:43:52:35:03:45	2:13	27:29	30
13:30	1:01:30	1:03:02:15	27	41	31
11	1:01	1:02:01	11	1:01	32
10:14:35	1:00:52:05	1:01:44:55:12:40:25	59	29:13	33
07:05	1:00:25	1:00:50:10:25	17	29	34
06:20	1:00:20	1:00:40:06:40	19	3:01	35
04:37:20	1:00:10:40	1:00:21:21:53:46:40	52	11:17	36
03:52:30	1:00:07:30	1:00:15:00:56:15	31	8:01	37
02:27	1:00:03	1:00:06:00:09	49	20:01	38

Mansfield (2020). Si.427 is one of the most complete examples of applied geometry from the ancient world and can be found on display at the İstanbul Arkeoloji Müzeleri.

Like earlier field plans, Si.427 retains the subdivision into rectangles, right trapezoids, and right triangles. But unlike earlier field plans it concerns the sale of private land and the measurements have been made with unusually high precision. The rectangles themselves are most remarkable because they actually have opposite sides of equal length, which is unique and suggests that OB surveyors had devised a way to create perpendicular lines more accurately than before.

Establishing perpendicular lines is a delicate task that usually requires specialized equipment. How could a surveyor create accurate perpendicular lines with just a measuring rod, rope, and pegs? The answer lies at the boundaries where we find three shapes (two rectangles and a right triangle) with the dimensions of diagonal triples. The perpendicular sides of these shapes were likely extended by sight to form the lines found in the subdivision. Diagonal triples were used to create rectangular altars in ancient India (Datta, 1932, p. 64–66), so it is unsurprising to find that they were also used in ancient Mesopotamia. This confirms Adams' conjecture that the Mesopotamian interest in diagonal triples was driven by the "increasing need for cadastral accuracy in resolving disputes over private land sales and tenure" (Adams, 2009, p. 6), but raises new questions. The surveyor who wrote Si.427 used the $(7 : 30, 18, 19 : 30) = 1 : 30 \times (5, 12, 13)$ and $(4, 7 : 30, 8 : 30) = 30 \times (8, 15, 17)$ diagonal triples. Why were $(5, 12, 13)$ and $(8, 15, 17)$ chosen instead of the simple $(3, 4, 5)$ triple? What determined the scaling factors $1 : 30$ and 30 ?

We conjecture that the actual shape of the region would influence which diagonal triple was used—with narrow triples chosen to match narrow regions. Indeed, in Si.427 the regions are too narrow to accommodate the width of the $(3, 4, 5)$ triple, which is probably why other triples were chosen.

The Roman surveyor Balbus used the $(3, 4, 5)$ triple as an auxiliary rectangle that can be discarded after its sides were used to create perpendicular lines (Bohlin 2013, p. 23). This method is still popular today, which might lead us to expect that the Mesopotamian use of diagonal triples should be similar and there is no need for scaling or other triples.

However, the Mesopotamian use of diagonal triples appears to be genuinely different. Instead of creating a small auxiliary shape, OB surveyors would create a whole region with the dimensions of a diagonal triple. It seems that one side of the region was fixed and used to determine the scaling factor. This is suggested by the land measurement exercise YBC 8633 (Neugebauer & Sachs, 1945, p. 53–55), (Høyrup, 2002, p. 254–257) where a boundary of length $1 : 40$ is selected, and the $(3, 4, 5)$ triple is enlarged by a "bundling" factor of $1 : 40 \times 5 = 20$ so that its diagonal matches this boundary.

The field plan Si.427 shows that a variety of diagonal triples were used in surveying, and the issue of determining the scaling, or "bundling", factor for these triples is especially relevant. The exercise MS 3971 §4 showed that only the regular side of a rectangle can be scaled to an arbitrary length, and it now seems this is not just a toy exercise but related to a real problem in cadastral surveying. The diagonal triples $(5, 12, 13)$ and $(8, 15, 17)$ from Si.427 were probably chosen because their regular short and long sides are amenable to arbitrary rescaling.

In conclusion, OB surveyors created accurate perpendicular lines from a variety of diagonal triples and those with regular sides were particularly useful. Was Plimpton 322 inspired by this cadastral interest in rectangles with regular sides? Perhaps, but we cannot hope for definitive answers to such questions. Instead, through MS 3971 and Si.427 we have established that scribes and surveyors had a theoretical and practical interest in

rectangles with regular sides, and this gives contextual support to our new interpretation of Plimpton 322.

5 A Study of Rectangles

As Britton et. al. observed, previous analysis of Plimpton 322 was hampered by the assumption that it “was created to serve a restricted pedagogical purpose” (Britton et al., 2011, p. 543). They drop this assumption and conclude it is mathematics of some kind, as originally suggested by Neugebauer (Britton et al., 2011, p. 561). Our analysis agrees with that of Britton et. al. and goes further. Here we have dropped the assumption that Plimpton 322 was about diagonal triples with common factors removed, and this has allowed us to reach a new conclusion.

Our conclusion is that Plimpton 322 is a study of rectangles. Its author systematically generated as many as 38 rectangles of the form $(\beta, 1, \delta)$ and recorded values for δ^2 , the factorization of β , and the factorization of δ . The values β and δ are referenced in the column headings and were probably present on the missing fragment.

We have proposed that the factorizations for β and δ were intended to demonstrate the regularity of these sides. Four key observations support this interpretation

1. it explains why Plimpton 322 omits analysis of the long side, there was no need to investigate the long side because it is regular by construction,
2. it explains why only the result of factorization is recorded and individual factors discarded,
3. it explains why some entries retain regular factors, and the most compelling reason is
4. the entries in Table 10 are not actually errors but the result of factorization which halts soon as the regularity of the side becomes apparent.

Through this interpretation, we have reduced the number of errors in Plimpton 322 down to the five listed in Table 9. This makes our new interpretation more consistent with the text than any previous hypothesis.

What is the purpose of the square numbers of column *I*? We conjecture, with (Britton et al., 2011, p. 559), that this column was intended to give meaning to the β and δ values by verifying that they satisfy the diagonal triple relation $\delta^2 - 1 = \beta^2$. This is one of the few ways that a scribe could specify that the table concerns genuine diagonal triples and not simply rectangles.

6 Final Remarks

The understanding of Plimpton 322 was a long and arduous road, each author having made their humble little contributions. This article is no different. Here we have attempted to make sense of columns *II* and *III* in light of an improved understanding of factorization and without the assumption that these columns were reduced synchronously.

Neugebauer originally suggested that Plimpton 322 was theoretical in nature and Price later suggested it was related to practical mensuration. Paradoxically, it now seems both were correct and that Plimpton 322 was a theoretical investigation into rectangles with

regular sides that was motivated, directly or indirectly, by the use of these objects in contemporary surveying.

We conclude that Plimpton 322 is an investigation into rectangles with regular sides. This does seem related to practical mensuration as briefly suggested by de Solla Price (1964) and more fully by Mansfield & Wildberger (2017), however the precise extent remains unknown. It could have been motivated by a particular practical need, or by a purely theoretical interest in geometry. Although it is more likely that the answer lies somewhere between these two extremes. In any case, Plimpton 322 has nothing to do with the modern study of trigonometry developed by Greek astronomers measuring the sky. Instead, this “proto-trigonometric” study of rectangles seems to have originated from the problems faced by Mesopotamian surveyors measuring the ground.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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