Extremes
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RESEARCH

Extremes of locally-homogenous vector-valued Gaussian processes

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Abstract
In this paper, we study the asymptotical behaviour of high exceedence probabilities for centered continuous $\mathbb{R}^n$-valued Gaussian random field $\boldsymbol{X}$ with covariance matrix satisfying $\Sigma - R(t+s,t) \sim \sum_{l=1}^{n} B_l(t)\,|s_l|^{\alpha_l}$ as $s \downarrow 0$. Such processes occur naturally as time transformations of homogenous random fields, and we present two asymptotic results of this nature as applications of our findings. The technical novelty of our proof consists in showing that the Slepian-Gordon inequality technique, essential in the univariate case, can also be successfully applied in the multivariate setup. This is noteworthy because this technique was previously believed to be inaccessible in this particular context.

Keywords Locally stationary · Gaussian random fields · Gaussian extremes · High exceedence probability

AMS 2000 Subject Classifications 60G15 · 60G70

1 Introduction

Despite the fact that the Gaussian extremes have been an active research area since at least the 60 s, up until recently little has been known about exact asymptotics of high exceedance probabilities of Gaussian processes *in the multivariate case*. A deep contribution Dębicki et al. (2020) has paved a way towards different problems of the following kind:

$$\mathbb{P}\{\exists\, t \in [0,T] \,:\, \boldsymbol{X}(t) > u\boldsymbol{b}\} \quad \text{as} \quad u \to \infty$$

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for $\boldsymbol{b} \in \mathbb{R}^d \backslash (-\infty, 0]^d$ and $\boldsymbol{X}$ being a continuous Gaussian process. Here ">" denotes the componentwise (Hadamard) comparison. As it turns out, these problems are much more challenging than the univariate ones due to the lack of several techniques which are crucial for the univariate case. The reader can find the detailed account of this shortage in the introduction to the aforementioned paper. Among these lacking techniques, the authors name the Slepian inequality and mention that its extension in the form of Gordon inequality is thought to be inapplicable if the compontents of $\boldsymbol{X}$ are not independent (see Dębicki et al. (2015) for the i.i.d. case).

In this contribution, we aim to achieve two goals. First, we extend (Dębicki et al. 2020, Theorem 2.1) on stationary processes to a certain class of homogenous Gaussian random fields defined on $[0, T]^n$, see Theorem 1. Second, we apply this result to the study of locally-homogenous Gaussian random fields. The corresponding result is presented in Theorem 2. The crucial step of the second part involves constructing two homogenous processes which stochastically dominate $\boldsymbol{X}$ on short intervals from above and from below. This is done by showing that a certain matrix-valued function is positive definite and subsequently applying the Gordon inequality.

As an application of our findings, we present asymptotic formulas for the time-transformed operator fractional Ornstein-Uhlenbeck process $\boldsymbol{Y}$ defined by the covariance matrix function

$$\mathbb{R}^2 \ni (t, s) \mapsto \exp\left(-|\varphi(t) - \varphi(s)|^H\right),$$

with H a symmetric matrix with eigenvalues from $(0, 1]$ and φ a strictly monotone continuously differentiable function. By Proposition 1,

$$\mathbb{P}\{\exists\, \boldsymbol{Y}(t) > u\boldsymbol{b}\} \sim c\, u^{1/h} \mathbb{P}\{\boldsymbol{Y}(0) > u\boldsymbol{b}\},$$

where h is the lowest eigenvalue of H and c is given in the form of an integral of Pickands-type constants over $[0, T]$. This result extends (Dębicki et al. 2020, Proposition 3.1). Another application concerns a class of continuous Gaussian processes associated to the following matrix-valued function:

$$\mathbb{R}^2 \ni (t, s) \mapsto \exp\left(-|t - s|^\alpha \left[B^+ + B^- \operatorname{sign}(t - s)\right]\right),$$

where $B^\pm = (B \pm B^\top)/2$ are symmetric and antisymmetric parts of a real $d \times d$ matrix B and $\alpha \in (0, 2]$. In Ievlev and Novikov (2023) we found the necessary and sufficient conditions on the pair (α, B) under which this function is positive definite (see Lemma 3) and thus generates a Gaussian process. Here we present an asymptotic result on the time-transformed version of this process, see Proposition 2.

The notion of locally stationary process was introduced by Berman in (1974) and its extremes were extensively studied afterwards in the papers by Hüsler (1990), Piterbarg (1996), Chan and Lai (2006) and many others. See also Piterbarg and Rodionov (2020), Qiao (2021) and Tan and Zheng (2020) for more recent contributions. Its multivariate counterpart, however, has not been considered so far due to the technical issues. The technique of Dębicki et al.

(2020) based on the uniform version of local Pickands lemma may in principle be applied to this class of processes, but it would require much stronger assumptions than those we impose in this contribution. Our result, presented in Theorem 2, should appear natural (if not obvious) for the specialist, but it still requires a rigorous proof, which involves imposing the right assumptions on the field $\boldsymbol{X}$.

The applicability of Gordon inequality in this context allows to significantly simplify the study of *classical* multivariate Gaussian extremes. In particular, the technical issue of uniformity in the single and double sums may be resolved by passing to a stationary dominating process. Therefore, besides the results here, we establish a simpler methodology compared to Dębicki et al. (2020) for dealing with non-stationary Gaussian random fields.

We want to point out that one possible direction in which our results can be extended is the family of $\alpha(t)$-locally stationary Gaussian random fields, see Hashorva and Ji (2016).

Brief organization of the paper Main results are presented in Section 2 with proofs relegated to Section 5. The applications are presented in the Section 3. Section 4 contains auxiliary results and technical lemmas. Appendix contains several known results taken from Dębicki et al. (2020) and reproduced here for reader's convenience in the adapted form.

2 Main results

Before proceeding to the theorems, let us introduce some relevant notation.

Vectors Throughout the paper points of $\mathbb{R}^d$ are written in bold letters (values of multivariate processes), while points of $[0,T]^n \subset \mathbb{R}^n$ (points of their domain) are written in the regular font. This does not lead to any confusion since their meaning can always be understood from the context, but allows to avoid visual clutter. All operations on vectors in both spaces, unless specified otherwise, are performed component-wise. For example, if t and s belong to $\mathbb{R}^n$, then ts denotes the vector $(t_i s_i)_{i=1,\dots,n}$. Similarly for t/s, e^t, $\lfloor t \rfloor$ and so on denoting vectors with components t_i/s_i, e^{t_i} and $\lfloor t_i \rfloor$ correspondingly. We write $t \geq s$ if $t_i \geq s_i$ for all their coordinates. By abuse of notation, we write $1 = (1,\dots,1) \in \mathbb{R}^n$ and $0 = (0,\dots,0) \in \mathbb{R}^n$. If $s > t$, then $[t,s]$ denotes the box $\{u : u_i \in [t_i, s_i]\}$.

Matrices If $A = (A_{ij})_{i,j=1,\dots,d}$ is a $d \times d$ matrix and $I, J \subset \{1,\dots,d\}$ are two index sets, we write A_{IJ} for the submatrix $(A_{ij})_{i \in I, j \in J}$. If $I = J$, we occasionally write A_I instead of A_{II}. $\|A\|$ denotes any fixed norm in the space of $d \times d$ matrices. Our formulas do not depend on the choice of the norm. For $\boldsymbol{w} \in \mathbb{R}^d$, $\mathrm{diag}\,(\boldsymbol{w})$ stands for the diagonal matrix with entries $w_1, w_2, \dots, w_d$ on the main diagonal. The notation $A \trianglerighteq 0$ means that A is positive definite and $A \triangleright 0$ means that A is strictly positive definite. If A is a real matrix, denote its symmetric and anti-symmetric parts by $A^{\pm} := (A \pm A^\top)/2$.

Quadratic programming problem Let Σ be a $d \times d$ real matrix with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \backslash (-\infty, 0]^d$, then by Lemma 7 the quadratic programming problem

$$\Pi_\Sigma(\boldsymbol{b}): \quad \text{minimize} \quad \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \quad \text{under the linear constraint} \quad \boldsymbol{x} \geq \boldsymbol{b}$$

has a unique solution $\widetilde{\boldsymbol{b}} \geq \boldsymbol{b}$ and there exists a unique non-empty index set $I \subset \{1, \dots, d\}$ such that

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I, \qquad \widetilde{\boldsymbol{b}}_J = \Sigma_{IJ}(\Sigma_{II})^{-1}\boldsymbol{b}_I \geq \boldsymbol{b}_J, \qquad \boldsymbol{w}_I = (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \qquad \boldsymbol{w}_J = \boldsymbol{0}_J,$$

where $\boldsymbol{w} := \Sigma^{-1}\widetilde{\boldsymbol{b}}$ and $J = \{1, \dots, d\} \backslash I$.

Other notation We use lower case constants $c_1, c_2, \dots$ to denote generic constants used in the proofs, whose exact values are not important and can be changed from line to line. The labeling of the constants starts anew in every proof. Let $f, g : [0, T]^n \to M$, where $M = \mathbb{R}^{d \times d}$, $\mathbb{R}^d$ or $\mathbb{R}$ be two matrix-valued, vector-valued or real-valued functions and $h : [0, T]^n \to \mathbb{R}$ be a real-valued function. We write "$f = g + o(h)$ as $t \to t_0$" if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $\|f(t) - g(t)\| \leq \varepsilon |h(t)|$. The next two subsections present our results on homogenous and locally homogenous fields.

2.1 Homogenous case

Let $\boldsymbol{X}(t)$, $t \in [0, T]^n$ be a centered homogenous and continuous Gaussian random field. Denote its covariance and variance matrices by

$$R(t, s) := \mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^\top(s)\right\} \quad \text{and} \quad \Sigma := R(0, 0).$$

Homogenity means that for each t and s in $[0, T]^n$

$$\mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^\top(s)\right\} = \mathbb{E}\left\{\boldsymbol{X}(t - s)\boldsymbol{X}^\top(0)\right\} = R(t - s, 0),$$

therefore we set in the following $R(t) := R(t, 0)$. It follows that $R(-t) = R^\top(t)$. The matrix $\Sigma - R(t)$ is positive definite, but not necessarily symmetric. Let $\boldsymbol{b} \in \mathbb{R}^d \backslash (-\infty, 0]^d$ and denote by $\widetilde{\boldsymbol{b}}$ and I the unique solution of $\Pi_\Sigma(\boldsymbol{b})$ and its I index set, see Lemma 7 for details. Set $\boldsymbol{w} := \Sigma^{-1}\widetilde{\boldsymbol{b}}$.

In this section we impose the following assumptions:

A1 $\Sigma_{II} - R_{II}(t)$ is strictly positive definite for every $t \in (0, T]$

A2 There exist a collection $\mathbb{B} := (B_l)_{l=1,\dots,n}$ of real $d \times d$ matrices and a collection of numbers $\boldsymbol{\alpha} := (\alpha_l)_{l=1,\dots,n} \in (0, 2]^n$ such that

$$\Sigma - R(t) = \sum_{l=1}^{n} B_l\, |t_l|^{\alpha_l} + o\left(\sum_{l=1}^{n} |t_l|^{\alpha_l}\right) \quad \text{as} \quad t \downarrow 0, \tag{A2.1}$$

$$\boldsymbol{w}^\top B_l \boldsymbol{w} > 0 \quad \text{for all} \quad l = 1, \dots, n. \tag{A2.2}$$

Remark 1 It follows from (A2.1) that

$$\Sigma - R(t) \sim \sum_{l=1}^{n} \left[B_l \, |t_l|^{\alpha_l} \, \mathbb{1}_{t_l \geq 0} + B_l \, |t_l|^{\alpha_l} \, \mathbb{1}_{t_l < 0} \right]$$

as $t \to 0$ and B_l's satisfy

$$\widetilde{B}_l := B_l^+ \cos\left(\frac{\pi\alpha_l}{2}\right) - iB_l^- \sin\left(\frac{\pi\alpha_l}{2}\right) \trianglerighteq 0, \quad \text{where} \quad B^\pm := \frac{B \pm B^\top}{2}. \tag{1}$$

From this follows that $B_l^+ \trianglerighteq 0$.

Theorem 1 *If $\boldsymbol{X}$ is a centered homogenous and continuous Gaussian random field satisfying Assumptions A1 and A2, then*

$$\mathbb{P}\{\exists\, t \in [0, T]^n \, : \, \boldsymbol{X}(t) > u\boldsymbol{b}\} \sim T^n \, \mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}} \prod_{l=1}^{n} u^{2/\alpha_l} \, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\},$$

where the constant $\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}}$ is given by

$$\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}} := \lim_{\Lambda \to \infty} \frac{1}{\Lambda^n} \int_{\mathbb{R}^d} e^{\mathbf{1}^\top \boldsymbol{x}} \, \mathbb{P}\left\{ \exists\, t \in [0, \Lambda]^n \, : \, \sum_{l=1}^{n} \operatorname{diag}(\boldsymbol{w}) \left[\boldsymbol{Y}_l(t_l) - S_{\alpha_l, B_l}(t_l) \, \boldsymbol{w} \right] > \boldsymbol{x} \right\} d\boldsymbol{x} \in (0, \infty). \tag{2}$$

Here $\boldsymbol{Y}_l$ is a continuous Gaussian process associated to the covariance function

$$R_{\alpha_l, B_l}(t_l, s_l) := S_{\alpha_l, B_l}(t_l) + S_{\alpha_l, B_l}(-s_l) - S_{\alpha_l, B_l}(t_l - s_l), \qquad S_{\alpha_l, B_l}(t_l) := |t_l|^{\alpha_l} \left[B \mathbb{1}_{t_l \geq 0} + B^\top \mathbb{1}_{t_l < 0} \right].$$

2.2 Locally homogenous case

In this section $\boldsymbol{X}(t)$, $t \in [0, T]^n$ is a centered continuous Gaussian random field with covariance matrix

$$R(t, s) := \mathbb{E}\left\{ \boldsymbol{X}(t) \, \boldsymbol{X}^\top(s) \right\}$$

and variance matrix Σ satisfying $R(t, t) = R(0, 0) =: \Sigma$. We impose the following assumptions:

B1 $\Sigma_{II} - R_{II}(t)$ is strictly positive definite for every $t \in (0, T]$

B2 There exist a collection $\mathbb{B}(t) := (B_l(t))_{l=1,\dots,n}$ of continuous real $d \times d$ matrix-valued functions and a collection of numbers $\boldsymbol{\alpha} := (\alpha_l)_{l=1,\dots,n} \in (0, 2]^n$ such that

$$\Sigma - R(t+s,t) = \sum_{l=1}^{n} \left[B_l(t)\,|s_l|^{\alpha_l} \mathbb{1}_{s_l \geq 0} + B_l^\top(t)\,|s_l|^{\alpha_l}\,\mathbb{1}_{s_l<0} \right] + o\left(\sum_{l=1}^{n} |s_l|^{\alpha_l} \right) \quad \text{as} \quad t \to +0, \tag{B2.1}$$

where small-o is uniform in $t \in [0,T]^n$ and

$$\widetilde{B}_l(t) := B_l^+(t) \cos\left(\frac{\pi\alpha_l}{2}\right) - iB_l^-(t) \sin\left(\frac{\pi\alpha_l}{2}\right) \rhd 0 \quad \text{for all} \quad t \in [0,T]^n. \tag{B2.2}$$

Remark 2 From (B2.2) follows that $\boldsymbol{w}^\top B_l(t)\,\boldsymbol{w} > 0$ for all $t \in [0,T]^n$.

Theorem 2 *If $\boldsymbol{X}$ is a centered and continuous Gaussian random field satisfying Assumptions $B1$ and $B2$, then*

$$\mathbb{P}\{\exists\, t \in [0,T]^n \,:\, \boldsymbol{X}(t) > u\boldsymbol{b}\} \sim \int_{[0,T]^n} \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\, dt \prod_{l=1}^{n} u^{2/\alpha_l}\, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\},$$

where the constant $\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}}$ is given by (2).

3 Examples

3.1 Time-transformed operator fractional Ornstein-Ulhenbeck process

Let H be a symmetric matrix with all eigenvalues $h_1, \ldots, h_d$ belonging to $(0,1]$ and consider a stationary a.s. continuous $\mathbb{R}^d$-valued Gaussian process $\boldsymbol{X}(t)$, $t \geq 0$ with cmf

$$R(t,s) = \exp\left(-|t-s|^{2H}\right), \tag{3}$$

where $t^H = \exp(H \ln t)$ for $t > 0$. This process is known in the literature as the operator fractional Ornstein-Uhlenbeck process. In this section we consider its time-transformed version. Specifically, let φ be a continuously differentiable strictly monotone function. Define $\boldsymbol{Y}(t) := \boldsymbol{X}(\varphi(t))$. Let us show that this process is locally stationary in the sense defined above. Since H is symmetric, there exists an orthogonal matrix Q such that $H = Q\,\operatorname{diag}(h_1, \ldots, h_d)\,Q^\top$. Hence,

$$R(t+s,t) = I - Q\widetilde{I}Q^\top|\varphi(t+s) - \varphi(t)|^{2h} + O\left(|\varphi(t+s) - \varphi(t)|^2\right) \quad \text{as} \quad s \to 0,$$

with $h := \min_{i=1,\ldots,d} h_i$ and $[\widetilde{I}\,]_{ij} := \mathbb{1}_{i=j \text{ and } h=h_i}$. Since φ is differentiable, we have

$$R(t+s,t) = I - Q\widetilde{I}Q^\top|\varphi'(t)|^{2h}|s|^{2h} + O\left(|s|^{4h}\right) \quad \text{as} \quad s \to 0.$$

Then (B2) holds with $B(t) := Q\widetilde{I}Q^\top|\varphi'(t)|^{2h}$ and $\Sigma = I$. Note that $|\varphi'(t)| > 0$ since φ is strictly monotone. By Theorem 2 we have the following result:

Proposition 1 *Let $\boldsymbol{Y}(t) = \boldsymbol{X}(\varphi(t))$, $t \in [0,T]$, where φ is a continuously differentiable strictly monotone function and $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ is an operator fO-U process associated to the covariance (3) with a symmetric matrix H whose eigenvalues belong to $(0,1]$. Let $\widetilde{b}_j = \max\{b_j, 0\}$ for $j = 1, \dots, d$. If $\widetilde{\boldsymbol{b}}^\top Q\widetilde{I}Q^\top \widetilde{\boldsymbol{b}} > 0$, then*

$$\mathbb{P}\{\exists\, t \in [0,T] : \boldsymbol{Y}(t) > u\boldsymbol{b}\} \sim u^{1/h} \int_0^T \mathcal{H}_{2h, Q\widetilde{I}Q^\top |\varphi'(t)|^{2h}, \boldsymbol{w}}\, dt\ \mathbb{P}\{\boldsymbol{X}(\varphi(0)) > u\boldsymbol{b}\}.$$

3.2 A Gaussian process with α-homogenous log-covariance

In an upcoming paper Ievlev and Novikov (2023) we show the following result:

Theorem 3 *Let B be a real $d \times d$ matrix. If a matrix-valued function R defined by*

$$R(t,s) = \exp\left(-|t-s|^\alpha \left[B^+ + B^- \operatorname{sign}(t-s)\right]\right), \qquad t,\, s \in \mathbb{R}, \tag{4}$$

is positive-definite, then the condition (1) is satisfied. If, on the other hand, the condition (1) is satisfied. Then

- If $\alpha \in (0,1)$, then R is positive-definite if and only if B satisfies
$$B^{1/\alpha} + B^{1/\alpha, \top} \trianglerighteq 0. \tag{5}$$
- If $\alpha \in [1,2]$, then R is positive-definite.

Using the above result, define $\boldsymbol{X}(t)$, $t \in \mathbb{R}$ a stationary continuous Gaussian process associated to this covariance and let φ be a strictly increasing continuously differentiable function. Define $\boldsymbol{Y}(t) := \boldsymbol{X}(\varphi(t))$. The covariance of $\boldsymbol{Y}$ satisfies

$$R_{\boldsymbol{Y}}(t+s,t) \sim I - \left[B^+ + B^- \operatorname{sign}(s)\right] \left|\varphi'(t)\right|^\alpha |s|^\alpha + O\left(|s|^{2\alpha}\right) \quad \text{as} \quad s \to 0,$$

where we used the fact that $\operatorname{sign}(\varphi(t+s) - \varphi(t)) = \operatorname{sign}(s)$ since φ is increasing. Hence, the assumption B2.1 is satisfied with $B(t) = B|\varphi'(t)|^\alpha$. The validity of B2.2 follows from the fact that $|\varphi'(t)| > 0$ and our assumption on B. By Theorem 2, we have the following result:

Proposition 2 *Let $\boldsymbol{Y}(t) = \boldsymbol{X}(\varphi(t))$, $t \in [0,T]$, where φ is a strictly increasing continuously differentiable function and $\boldsymbol{X}$ is a process associated to the covariance (4), where B and α are such that this function is positive definite. Then*

$$\mathbb{P}\{\exists\, t \in [0,T] : \boldsymbol{Y}(t) > u\boldsymbol{b}\} \sim u^{2/\alpha} \int_0^T \mathcal{H}_{\alpha, B|\varphi'(t)|^\alpha, \boldsymbol{w}}\, dt\ \mathbb{P}\{\boldsymbol{X}(\varphi(0)) > u\boldsymbol{b}\}$$

as $u \to \infty$.

4 Auxiliary results

4.1 Lemma on positive definiteness

Lemma 1 *Let B be a real $d \times d$ matrix satisfying*

$$\widetilde{B} = B_+ \sin\left(\frac{\pi\alpha}{2}\right) - iB_- \cos\left(\frac{\pi\alpha}{2}\right) \rhd 0. \tag{6}$$

Then there exists a collection of complex numbers $\{\lambda_k\}_{k=1,\ldots,d}$ satisfying

$$\operatorname{Re} \lambda_k = 1, \qquad |\operatorname{Im} \lambda_k| < \left|\tan\left(\frac{\pi\alpha}{2}\right)\right| \tag{7}$$

and a collection of strictly positive definite Hermitian matrices $\{V_k\}_{k=1,\ldots,d}$ of rank one such that

$$B = \sum_{k=1}^{d} \lambda_k V_k. \tag{8}$$

Proof Note that B can be represented as follows:

$$B = B_+ + iB'_-, \qquad B'_- := -iB_-, \qquad B_\pm := \frac{B \pm B^\top}{2}.$$

Here B_+ is symmetric and strictly positive definite by (6) and B'_- is Hermitian. Hence, there exists an invertible real matrix A such that $B_+ = AA^\top$. Note that for each unitaty matrix Q holds

$$QA^{-1}B_+A^{-\top}Q^* = QQ^* = I.$$

Since B'_- is Hermitian, so is $A^{-1}B'_-A^{-\top}$ and therefore there exists a unitary matrix Q and a real diagonal matrix D such that

$$A^{-1}B'_-A^{-\top} = Q^*DQ.$$

Denote $V := AQ^*$. Therefore, we have the following representations of B_+

$$VV^* = AQ^*QA^\top = AA^\top = B_+ \tag{9}$$

and B'_-

$$VDV^* = AQ^*DQA^\top = AA^{-1}B'_-A^{-\top}A^\top = B'_-. \tag{10}$$

Hence, for B we have

$$B = B_+ + iB'_- = VV^* + iVDV^* = V\Big[I + iD\Big]V^*.$$

Set next

$$\lambda_k := 1 + iD_{kk}, \qquad V_k := V\mathcal{D}_k V^*, \tag{11}$$

where $[\mathcal{D}_k]_{ml} = \delta_{km}\delta_{kl}$ is the diagonal matrix with 1 at k-th place. Clearly, V_k's are Hermitian, positive definite, of rank one and (8) is satisfied. It remains to show that the inequality (7) is also satisfied. To this end, use (9) and (10) to rewrite $\widetilde{B}$ as

$$\widetilde{B} = V\Big[I\cos\Big(\frac{\pi\alpha}{2}\Big) - iD\sin\Big(\frac{\pi\alpha}{2}\Big)\Big]V^* \rhd 0.$$

Therefore, we have

$$I\cos\Big(\frac{\pi\alpha}{2}\Big) - iD\sin\Big(\frac{\pi\alpha}{2}\Big) \rhd 0,$$

which implies (7). $\square$

Lemma 2 *Under the conditions of Lemma* 1, *the functions given by*

$$\mathcal{E}_{\alpha,B,k}(t) := \exp\left(-d\lambda_k V_k|t|^\alpha\right)\mathbb{1}_{t\geq 0} + \exp\left(-d\,\overline{\lambda}_k\, V_k|t|^\alpha\right)\mathbb{1}_{t<0}$$

with λ_k, V_k *and* α *from Lemma* 1 *are all positive definite complex matrix-valued functions. Let* $\Sigma = AA^\top$ *be a strictly positive definite matrix and define*

$$\mathcal{E}_{\alpha,B}(t) := \frac{1}{2d}A\sum_{k=1}^{d}\Big[\mathcal{E}_{\alpha,A^{-1}BA^{-\top},k}(t) + \overline{\mathcal{E}_{\alpha,A^{-1}BA^{-\top},k}(t)}\Big]A^\top.$$

Then $\mathcal{E}_{\alpha,B}(t)$ *is positive definite real matrix-valued function satisfying*

$$\mathcal{E}_{\alpha,B}(t) = \Sigma - B|t|^\alpha\mathbb{1}_{t\geq 0} - B^\top|t|^\alpha\mathbb{1}_{t<0} + o(|t|^\alpha) \quad \text{as} \quad t\to 0.$$

Proof Since $V_k = V^*\mathcal{D}_k V$ by (11), there exists $\mu_k > 0$ and a unitary matrix U such that $V_k = \mu_k U^*\mathcal{D}_k U$. Hence,

$$\exp\left(-d\big[1 + i\,\mathrm{Im}\,\lambda_k \mathrm{sign}(t)\big]V_k|t|^\alpha\right) = U^*\exp\left(-d\mu_k\Big[1 + i\,\mathrm{Im}\,\lambda_k\mathrm{sign}(t)]\mathcal{D}_k\,|t|^\alpha\right)U.$$

Positive definiteness of this function is therefore equivalent to that of a scalar-valued function

$$\exp\left(-d\mu_k\Big[1 + i\,\mathrm{Im}\,\lambda_k\,\mathrm{sign}(t)\Big]|t|^\alpha\right),$$

which follows from (7). The second claim follows from (8) and the fact that

$$\widetilde{B} \rhd 0 \implies \widetilde{A^{-1}BA^{-\top}} = A^{-1}\widetilde{B}A^{-\top} \rhd 0$$

by a direct computation. $\square$

4.2 Double sum bound

Define for $k \in \mathbb{Z}^d \backslash \{0\}$ and $\Lambda > 0$ the double events' probabilities by

$$P_{\boldsymbol{b}}(k, \Lambda) := \mathbb{P}\left\{ \begin{array}{ll} \exists\, t \in \Lambda[0,1]^n : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda[k, k+1] : & \boldsymbol{X}(s) > u\boldsymbol{b} \end{array} \right\}.$$

Lemma 3 (Double sum bound). *If $\boldsymbol{X}(t)$, $t \in [0,T]^n$ is a centered continuous Gaussian field satifying Assumption A2, then there exist positive constants C and ε such that for every $k \in \mathbb{Z}^d \backslash \{0\}$ with $1 < |k_l| \le N_u(\varepsilon)$ for all l and $\Lambda > 0$ holds*

$$\frac{P_{\boldsymbol{b}}(k, \Lambda)}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \le C \Lambda^{\#\{l : k_l = 0\}} \prod_{l : k_l \neq 0} \left(|k_l| - 1\right)^{-2} \exp\left(-\frac{1}{4} \boldsymbol{w}^\top B_l \boldsymbol{w}\, \Lambda^{\alpha_l} \left(|k_l| - 1\right)^{\alpha_l} \right)$$

Remark 3 Note that the conditions of the lemma demand that there be no l's such that $k_l = \pm 1$. This is not a coincidence: the adjacent double events are to be estimated differently. See the proof of Theorem 1 for details.

Proof Without loss of generality assume that $I = \{1, \dots, n\}$. Then

$$\begin{aligned} P_{\boldsymbol{b}}(k, \Lambda) &\le \mathbb{P}\left\{ \exists\, (t,s) \in \Lambda u^{-2/\alpha}[k, k+1] \times [0,1] : \frac{1}{2}\Big[\boldsymbol{X}(t) + \boldsymbol{X}(s) \Big] > u\boldsymbol{b} \right\} \\ &= u^{-d} \int_{\mathbb{R}^d} \mathbb{P}\left\{ \exists\, (t,s) \in [0, \Lambda]^{2n} : \chi_{u,k,\boldsymbol{x}}(t,s) > \boldsymbol{x} \right\} \varphi_{u,k}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u} \right) d\boldsymbol{x}, \end{aligned} \tag{12}$$

where

$$\chi_{u,k,\boldsymbol{x}}(t,s) := u\left(\boldsymbol{X}_{u,k}(t,s) - u\boldsymbol{b} \,\middle|\, \boldsymbol{X}_{u,k}(0,0) = u\boldsymbol{b} - \frac{\boldsymbol{x}}{u} \right) + \boldsymbol{x}$$

with

$$\boldsymbol{X}_{u,k}(t,s) := \frac{1}{2}\Big[\boldsymbol{X}\left(\Lambda u^{-2/\alpha} k + u^{-2/\alpha} t\right) + \boldsymbol{X}\left(u^{-2/\alpha} s\right) \Big]$$

and $\varphi_{u,k}$ is the pdf of $\boldsymbol{X}_{u,k}(0,0) \overset{d}{=} N(0, \Sigma_{u,k})$, where

$$\begin{aligned} \Sigma_{u,k} &:= \mathbb{E}\left\{ \boldsymbol{X}_{u,k}(0,0) \boldsymbol{X}_{u,k}^\top(0,0) \right\} = \frac{1}{4}\Big[2\Sigma + R\left(\Lambda u^{-2/\alpha} k\right) + R\left(-\Lambda u^{-2/\alpha} k\right) \Big] \\ &= \Sigma - u^{-2} \sum_{l=1}^{n} \Big[B_l + B_l^\top \Big] \Lambda^{\alpha_l} k_l^{\alpha_l} + o\left(u^{-2/\alpha} \Lambda k\right). \end{aligned} \tag{13}$$

First, bound $\varphi_{u,k}$ as follows:

$$\varphi_{u,k}\left(u\boldsymbol{b} - \frac{\boldsymbol{x}}{u} \right) \le \varphi(u\boldsymbol{b}) \exp\left(\frac{u^2}{2} \boldsymbol{b}^\top \Big[\Sigma^{-1} - \Sigma_{u,k}^{-1} \Big] \boldsymbol{b} \right) \exp\left(\boldsymbol{b}^\top \Sigma_{u,k}^{-1} \boldsymbol{x} \right),$$

where φ is the pdf of $N(0,\Sigma)$. Plugging this into (12) and noting that $u^{-d}\,\varphi(u\boldsymbol{b}) = \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}$, we obtain the following bound:

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \le \exp\left(\frac{u^2}{2}\boldsymbol{b}^\top\left[\Sigma^{-1} - \Sigma_{u,k}^{-1}\right]\boldsymbol{b}\right)\int_{\mathbb{R}^d}\exp\left(\boldsymbol{b}^\top\Sigma_{u,k}^{-1}\boldsymbol{x}\right) \mathbb{P}\left\{\exists\,(t,s)\in[0,\Lambda]^{2n}\,:\,\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > \boldsymbol{x}\right\}d\boldsymbol{x}\,. \tag{14}$$

At this point we split the proof into three parts: estimation of the integral, estimation of the exponent in front of it and their comparison.

The exponent in front of the integral By (13), we have

$$\Sigma^{-1} - \Sigma_{u,k}^{-1} = -u^{-2}\sum_{l=1}^{n}\Sigma^{-1}\left[B_l + B_l^\top\right]\Sigma^{-1}\Lambda^{\alpha_l}|k_l|^{\alpha_l} + o\left(u^{-2/\alpha}\Lambda k\right). \tag{15}$$

Therefore,

$$\frac{u^2}{2}\boldsymbol{b}^\top\left[\Sigma^{-1} - \Sigma_{u,k}^{-1}\right]\boldsymbol{b} = -\sum_{l=1}^{n}\boldsymbol{w}^\top B_l\,\boldsymbol{w}\,\Lambda^{\alpha_l}|k_l|^{\alpha_l} + u^2 o\left(u^{-2/\alpha}\Lambda k\right). \tag{16}$$

By our assumptions,

$$\sup_{-N_u(\varepsilon)\le k\le N_u(\varepsilon)} u^2\left|o\left(u^{-2/\alpha}\Lambda k\right)\right|\xrightarrow[u\to\infty]{} 0.$$

The integral First note that

$$\exp\left(\boldsymbol{b}^\top\Sigma_{u,k}^{-1}\boldsymbol{x}\right) = \exp\left((\boldsymbol{w} + o(u^{-2/\alpha}\Lambda k))^\top\boldsymbol{x}\right)$$

where the small-o term tends to zero uniformly in k. We will drop this term from now on to simplify the notation. To bound the remaining integral we will use Lemma 8, which gives

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\exists\,(t,s)\in[0,\Lambda]^{2n}\,:\,\boldsymbol{\chi}_{u,k,\boldsymbol{x}}(t,s) > u\boldsymbol{b}\right\}d\boldsymbol{x} \le c_1\exp\left(c_2(G+\sigma^2)\right) \tag{17}$$

with some positive constants c_1 and c_2. Here $G\in\mathbb{R}$ and $\sigma^2 > 0$ are numbers (depending on k and u) such that

$$\sup_{F\subset\{1,\dots,d\}}\ \sup_{(t,s)\in[0,\Lambda]^{2n}} \boldsymbol{w}_F^\top\,\mathbb{E}\left\{\boldsymbol{\chi}_{u,k,\boldsymbol{x},F}(t,s)\right\} \le G + \varepsilon\sum_{j=1}^{d}|x_j| \tag{18}$$

and

$$\sup_{F\subset\{1,\dots,d\}}\ \sup_{(t,s)\in[0,\Lambda]^{2n}} \mathrm{Var}\left\{\boldsymbol{w}_F^\top\,\boldsymbol{\chi}_{u,k,\boldsymbol{x},F}(t,s)\right\} \le \sigma^2.$$

To apply this lemma we need to find such numbers.

Finding G By the formulas on conditional Gaussian distribution, we have

$$\mathbb{E}\left\{\chi_{u,k,\boldsymbol{x}}(t,s)\right\} = u\left[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\right]\Sigma_{u,k}^{-1}\left[u\boldsymbol{b} - \frac{\boldsymbol{x}}{u}\right], \tag{19}$$

where $R_{u,k}(t,s,t',s')$ is the covariance of $\chi_{u,k,\boldsymbol{x}}(t,s)$. Note that this covariance does not depend on $\boldsymbol{x}$. The $\boldsymbol{x}$-term can clearly be bounded by

$$\left\|\left[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\right]\Sigma_{u,k}^{-1}\boldsymbol{x}\right\| \le \varepsilon \sum_{j=1}^{d} |x_j|.$$

Let us bound the $\boldsymbol{b}$-contribution. A direct computation gives

$$\begin{aligned}&\Sigma_{u,k} - R_{u,k}(t,s,0,0)\\ &\sim \frac{1}{4u^2}\sum_{l=1}^{n}\left[S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(\Lambda k_l + t_l) + S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l) - S_{\alpha_l,B_l}(\Lambda k_l)\right]\end{aligned} \tag{20}$$

uniformly in $k \in [-N_u(\varepsilon), N_u(\varepsilon)]$. By (15)

$$\begin{aligned}u^2\,\boldsymbol{w}_F^\top\left[\left[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\right]\Sigma_{u,k}^{-1}\boldsymbol{b}\right]_F &\sim u^2\,\boldsymbol{w}_F^\top\left[\left[\Sigma_{u,k} - R_{u,k}(t,s,0,0)\right]\boldsymbol{w}\right]_F\\ &\sim \frac{1}{4}\sum_{l=1}^{n}[A_{1,l} + A_{2,l} + A_{3,l}]\end{aligned}$$

uniformly in $k \in [-N_u(\varepsilon), N_u(\varepsilon)]$, where

$$\begin{aligned}A_{1,l} &:= \boldsymbol{w}_F^\top\left[\left[S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(t_l)\right]\boldsymbol{w}\right]_F,\\ A_{2,l} &:= \boldsymbol{w}_F^\top\left[\left[S_{\alpha_l,B_l}(\Lambda k_l + t_l) - S_{\alpha_l,B_l}(\Lambda k_l)\right]\boldsymbol{w}\right]_F,\\ A_{3,l} &:= \boldsymbol{w}_F^\top\left[\left[S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l)\right]\boldsymbol{w}\right]_F.\end{aligned}$$

The first can be bounded as follows:

$$\left|A_{1,l}\right| \le |\boldsymbol{w}|^2\left[\left\|S_{\alpha_l,B_l}(s_l)\right\| + \left\|S_{\alpha_l,B_l}(t_l)\right\|\right] \le 2\Lambda^{\alpha_l}|\boldsymbol{w}|^2\left\|B_l\right\|.$$

$A_{2,l}$ and $A_{3,l}$ can be bounded for $k_l \ne 0$ similarly as follows:

$$\left|A_{2,l}\right| \le |\boldsymbol{w}|^2\|B\|\left[\left|\Lambda k_l + t_l\right|^{\alpha_l} - \left|\Lambda k_l\right|^{\alpha_l}\right] \le c_2\,\Lambda^{\alpha_l}|k_l|^{\alpha_l - 1}.$$

Therefore, the inequality (18) is satisfied with

$$G = c_2\sum_{l=1}^{n}\Lambda^{\alpha_l}(1 + |k_l|^{\alpha_l - 1}\mathbb{1}_{k_l \ne 0}). \tag{21}$$

Finding σ^2 We have

$$\operatorname{Var}\left\{\boldsymbol{w}_F^\top \chi_{u,k,\boldsymbol{x},F}(t,s)\right\} = \sum_{j',j\in F} w_j w_{j'} \operatorname{cov}\left(\chi_{u,k,\boldsymbol{x},j}(t,s), \chi_{u,k,\boldsymbol{x},j'}(t,s)\right) \le c_3 \sum_{j,j'} \left[\mathcal{R}_{u,k,\boldsymbol{x}}(t,s,t,s)\right]_{j,j'},$$

where

$$\mathcal{R}_{u,k,\boldsymbol{x}}(t,s,t',s') := \mathbb{E}\left\{\chi_{u,k,\boldsymbol{x}}(t,s)\,\chi_{u,k,\boldsymbol{x}}^\top(t',s')\right\} = R_{u,k}(t,s,t',s') - R_{u,k}(t,s,0,0)\Sigma_{u,k}^{-1}R_{u,k}(0,0,t',s')$$
$$\sim \frac{1}{4}\sum_{l=1}^{n}\left[A_{1,l}+A_{2,l}+A_{3,l}+A_{4,l}+A_{5,l}+A_{6,l}\right],$$

where

$$\begin{aligned}
A_{1,l} &:= S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(s_l) + S_{\alpha_l,B_l}(-t_l') + S_{\alpha_l,B_l}(-s_l'),\\
A_{2,l} &:= S_{\alpha_l,B_l}(s_l - \Lambda k_l) - S_{\alpha_l,B_l}(-\Lambda k_l),\\
A_{3,l} &:= S_{\alpha_l,B_l}(t_l + \Lambda k_l) - S_{\alpha_l,B_l}(\Lambda k_l),\\
A_{4,l} &:= -S_{\alpha_l,B_l}(s - s') - S_{\alpha_l,B_l}(t - t'),\\
A_{5,l} &:= S_{\alpha_l,B_l}(-\Lambda k_l - t_l') - S_{\alpha_l,B_l}(-\Lambda k_l - t_l' + s_l),\\
A_{6,l} &:= S_{\alpha_l,B_l}(\Lambda k_l - s_l') - S_{\alpha_l,B_l}(\Lambda k_l - s_l' + t_l).
\end{aligned}$$

Similarly to how we bounded differences of this form above, we obtain

$$\|A_{1,l}\|,\ \|A_{4,l}\| \le c_4\Lambda^{\alpha_l}, \qquad \|A_{2,l}\|,\ \|A_{3,l}\|,\ \|A_{5,l}\|,\ \|A_{6,l}\| \le c_5\Lambda^{\alpha_l}|k_l|^{\alpha_l-1}.$$

Hence, the inequality (18) is satisfied with

$$\sigma^2 = c_6\sum_{l=1}^{n}\Lambda^{\alpha_l}\left(1+|k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right) \tag{22}$$

as $u\to\infty$.

Proceeding with the integral Combining (21) and (22) with (17), we find

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top\boldsymbol{x}}\,\mathbb{P}\left\{\exists\,(t,s)\in[0,\Lambda]^{2n} : \chi_{u,k,\boldsymbol{x}}(t,s) > u\boldsymbol{b}\right\}d\boldsymbol{x} \le c_6\exp\left(c_7\sum_{l=1}^{n}\Lambda^{\alpha_l}\left(1+|k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right)\right).$$

By (16) and (14), we have

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}} \le c_8\exp\left(-\sum_{l=1}^{n}\Lambda^{\alpha_l}\left[\frac{\boldsymbol{w}^\top B_l\boldsymbol{w}}{2}|k_l|^{\alpha_l} - c_7\left(1+|k_l|^{\alpha_l-1}\mathbb{1}_{k_l\neq 0}\right)\right]\right). \tag{23}$$

If $|k_l|$ is large enough, we have

$$\frac{\boldsymbol{w}^\top B_l\boldsymbol{w}}{2}|k_l|^{\alpha_l} - c_7\left(1+|k_l|^{\alpha_l}\right) \ge \frac{\boldsymbol{w}^\top B_l\boldsymbol{w}}{4}.$$

Lifting the assumption that $|k_l|$ is large Let K be such that for $|k_l|\ge K$ holds

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}} \le c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l \boldsymbol{w}\, \Lambda^{\alpha_l} |k_l|^{\alpha_l}\right).$$

It suffices to consider the case when some of k_l's satisfy $1 < |k_l| < K$. Assume for simplicity that there is exactly one such l that $|k_l| < K$, take $\Lambda' > 0$ such that $\Lambda' < \Lambda$ and bound $P_{\boldsymbol{b}}$ as follows:

$$\begin{aligned} P_{\boldsymbol{b}}(k,\Lambda) &\le \sum_{0\le p_l, q_l \le \lceil \Lambda/\Lambda' \rceil} \mathbb{P}\left\{ \begin{matrix} \exists\, t \in \Lambda' u^{-2/\alpha}[\Lambda k/\Lambda' + q_l 1_l, \Lambda k/\Lambda' + q_l 1_l + 1]: & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda' u^{-2/\alpha}[p_l 1_l, p_l 1_l + 1]: & \boldsymbol{X}(s) > u\boldsymbol{b} \end{matrix} \right\} \\ &= \sum_{0\le p_l, q_l \le \lceil \Lambda/\Lambda' \rceil} P_{\boldsymbol{b}}(\Lambda k/\Lambda' + (q_l - p_l) 1_l, \Lambda'). \end{aligned} \tag{24}$$

Here $1_l \in \mathbb{Z}^d$ such that $[1_l]_{l'} = \delta_{l,l'}$. Choose $\Lambda' := \Lambda(|k_l| - 1)/K$. Then

$$k_l' := \Lambda k_l/\Lambda' + q_l - p_l \ge \Lambda k_l/\Lambda' - \Lambda/\Lambda' = \Lambda(k_l - 1)/\Lambda' \ge K$$

and therefore

$$\begin{aligned} \frac{P_{\boldsymbol{b}}(k',\Lambda')}{\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}} &\le c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l \boldsymbol{w}\, \Lambda'^{\alpha_l} |k_l'|^{\alpha_l}\right) \\ &= c_8 \exp\left(-\frac{1}{4}\sum_{l=1}^{n} \boldsymbol{w}^\top B_l \boldsymbol{w}\, \Lambda^{\alpha_l} (|k_l| - 1)^{\alpha_l}\right). \end{aligned} \tag{25}$$

It remains to note that the number of terms in the sum (24) is at most $\lceil \Lambda/\Lambda' \rceil^2 \le 2K^2/(|k_l| - 1)^2$.

Lifting the assumption that all k_l's are non-zero By (23) and (25)

$$\frac{P_{\boldsymbol{b}}(k,\Lambda)}{\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}} \le c_8 \prod_{l:k_l\ne 0} \exp\left(-\frac{1}{4}\boldsymbol{w}^\top B_l \boldsymbol{w}\, \Lambda^{\alpha_l}\left(|k_l| - 1\right)^{\alpha_l}\right) \prod_{l:k_l=0} \exp\left(c_7 \Lambda^{\alpha_l}\right). \tag{26}$$

Similarly to the previous point of the proof, take $\Lambda' \in (0,\Lambda)$ and assume for simplicity that there is only one l such that $k_l = 0$. Note that

$$P_{\boldsymbol{b}}(k,\Lambda) \le \sum_{0\le p \le \lceil \Lambda/\Lambda' \rceil} \mathbb{P}\left\{ \begin{matrix} \exists\, t_j \in \Lambda u^{-2/\alpha_j}[k_j, k_j+1],\ j \ne l & : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, t_l \in \Lambda' u^{-2/\alpha_l}[p, p+1] & & \\ \exists\, s_j \in \Lambda u^{-2/\alpha_j}[0, 0+1],\ j \ne l & : & \boldsymbol{X}(s) > u\boldsymbol{b} \\ \exists\, s_l \in \Lambda' u^{-2/\alpha_l}[p, p+1] & & \end{matrix} \right\}. \tag{27}$$

A similar proof to what we used above shows that each term of this sum is at most

$$c_8 \prod_{l' \neq l} \exp\left(-\frac{1}{4} \boldsymbol{w}^\top B_l \boldsymbol{w} \, \Lambda^{\alpha_l} \left(|k_l| - 1\right)^{\alpha_l}\right) \exp\left(c_7 \Lambda'^{\alpha_l}\right) \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}.$$

The number of terms in the sum (27) is at most $\lceil \Lambda/\Lambda' \rceil$, hence

$$\frac{P_{\boldsymbol{b}}(k, \Lambda)}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \leq c_9 \Lambda \prod_{l \neq l'} \exp\left(-\frac{1}{4} \boldsymbol{w}^\top B_l \boldsymbol{w} \, \Lambda^{\alpha_l} \left(|k_l|^{\alpha_l} - 1\right)^{\alpha_l}\right),$$

where $c_9 = 2c_8 \exp(c_7 \Lambda'^{\alpha'})/\Lambda'$. The general case when there is several l's such that $k_l = 0$ can be addressed similarly. □

5 Proofs of the main results

5.1 Proof of theorem 1

Proof We begin the proof by splitting $[0, T]^n$ into pieces of Pickands scale

$$[0, T]^n = \Lambda u^{-2/\alpha} \bigcup_{k \leq N_u} [k, k+1], \quad \text{where} \quad N_u(T) := \left\lceil \frac{T}{\Lambda u^{-2/\alpha}} \right\rceil$$

and using Bonferroni inequality to obtain

$$\Sigma_1' - \Sigma_2 \leq \mathbb{P}\{\exists\, t \in [0, T]^n \, : \, \boldsymbol{X}(t) > u\boldsymbol{b}\} \leq \Sigma_1,$$

where

$$\Sigma_1 := \sum_{0 \leq k \leq N_u(T)} \mathbb{P}\left\{\exists\, t \in \Lambda u^{-2/\alpha}[k, k+1] \, : \, \boldsymbol{X}(t) > u\boldsymbol{b}\right\},$$

$$\Sigma_2 := \sum_{\substack{0 \,\leq\, k, j \,\leq\, N_u(T) \\ k \,\neq\, j}} \mathbb{P}\left\{\begin{array}{l} \exists\, t \in \Lambda u^{-2/\alpha}[k, k+1] \, : \, \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[j, j+1] \, : \, \boldsymbol{X}(s) > u\boldsymbol{b} \end{array}\right\}.$$

and Σ_1' is defined by the same formula as Σ_1 but with $N - 1$ instead of N in the upper summation limit. At this point we split the proof into two parts. First, we will focus on finding the exact asymptotics of the single sum $\Sigma_1 \sim \Sigma_1'$, and then demonstrate that the double sum Σ_2 is negligible with respect to Σ_1.

Since $\boldsymbol{X}$ is homogenous, we can easily compute the single sum

$$\Sigma_1 = \left\lceil \prod_{l=1}^{n} N_{u,l}(T) \right\rceil \mathbb{P}\left\{\exists\, t \in \Lambda u^{-2/\alpha}[0, 1]^n \, : \, \boldsymbol{X}(t) > u\boldsymbol{b}\right\}.$$

Applying local Pickands Lemma 5, we obtain

$$\Sigma_1' \sim \Sigma_1 \sim T^n \left[\prod_{l=1}^{n} u^{-2/\alpha_l} \right] \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)}{\Lambda^n} \, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}.$$

Since $E \mapsto \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}(E)$ is subadditive, we have that the limit

$$\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}} := \lim_{\Lambda\to\infty} \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)}{\Lambda^n}$$

exists and is finite. We will show that it is also positive after dealing with the double sum.

Double sum By stationarity we have that

$$\Sigma_2 = \sum_{\substack{0 \le k, j \le N_u(T) \\ k \ne j}} \mathbb{P}\left\{ \begin{array}{ll} \exists\, t \in \Lambda u^{-2/\alpha}[k-j, k-j+1] : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] : & \boldsymbol{X}(s) > u\boldsymbol{b} \end{array} \right\}.$$

Reindexing the sum by $q = k - j$, we obtain

$$\Sigma_2 = \prod_{l=1}^{n} N_{u,l}(T) \sum_{\substack{-N_u(T) \le q \le N_u(T) \\ q \ne 0}} \mathbb{P}\left\{ \begin{array}{ll} \exists\, t \in \Lambda u^{-2/\alpha}[q, q+1] : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] : & \boldsymbol{X}(s) > u\boldsymbol{b} \end{array} \right\}.$$

Denote the double events' probabilities by

$$P_{\boldsymbol{b}}(q,\Lambda) := \mathbb{P}\left\{ \begin{array}{ll} \exists\, t \in \Lambda u^{-2/\alpha}[q, q+1] : & \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] : & \boldsymbol{X}(s) > u\boldsymbol{b} \end{array} \right\}.$$

Take some $\varepsilon \in (0, T)$ and divide the sum in two parts:

$$\sum_{\substack{0 \le q \le N_u \\ q \ne 0}} P_{\boldsymbol{b}}(q,\Lambda) = \sum_{\exists\, l :\, |q_l| > N_{u,l}(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda) + \sum_{-N_u(\varepsilon) \le q \le N_u(\varepsilon)} P_{\boldsymbol{b}}(q,\Lambda). \tag{28}$$

Terms of the first sum can be bounded as follows:

$$\begin{aligned} P_{\boldsymbol{b}}(q,\Lambda) &\le \mathbb{P}\left\{ \exists\, (t,s) \in \Lambda u^{-2/\alpha}([q,q+1] \times [0,1]) : \frac{1}{2}[\boldsymbol{X}(t) + \boldsymbol{X}(s)] > u\boldsymbol{b} \right\} \\ &\le \mathbb{P}\left\{ \exists\, (t,s) \in \Lambda u^{-2/\alpha}([q,q+1] \times [0,1]) : \frac{1}{2}\left[\boldsymbol{X}_I(t) + \boldsymbol{X}_I(s)\right] > u\boldsymbol{b}_I \right\}. \end{aligned}$$

Let $\Sigma(t,s)$ denote the variance matrix of $(\boldsymbol{X}(t) + \boldsymbol{X}(s))/2$:

$$\Sigma(t,s) = \frac{1}{4}\Big[2\Sigma + R(t-s) + R(s-t)\Big].$$

In view of Assumption A1, the matrix $(\Sigma_{II}(t,s))^{-1}) - (\Sigma_{II})^{-1}$ is strictly positive definite for $t \neq s$, which implies

$$\tau := \inf\left\{ \inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II}(t,s))^{-1} \boldsymbol{x}_I \;\middle|\; (t,s) \in \Lambda u^{-2/\alpha}([q, q+1] \times [0,1)) \right\}$$
$$\ge \tau_1 := \inf\left\{ \inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II}(t,s))^{-1} \boldsymbol{x}_I \;\middle|\; (t,s) \in [0,T]^n \,:\, |t_l - s_l| > \varepsilon \right\} > \tau_0 := \inf_{\boldsymbol{x}_I \ge \boldsymbol{b}_I} \boldsymbol{x}_I^\top (\Sigma_{II})^{-1} \boldsymbol{x}_I > 0.$$

Note that the condition $\exists\, l \,:\, |q_l| > N_u(\varepsilon)$ allows us to separate $\delta(u,\varepsilon) := \tau - \tau_0$ from 0 by $\delta(\varepsilon) := \tau_1 - \tau_0 > 0$, which depends on ε, but does not depend on u. Since $\tau_0 = \boldsymbol{b}_I^\top (\Sigma_{II})^{-1} \boldsymbol{b}_I = \boldsymbol{b}^\top \Sigma^{-1} \boldsymbol{b}$, we obtain by using the Piterbarg inequality (34) the following upper bound:

$$\mathbb{P}\left\{ \exists\,(t,s) \in \Lambda u^{-2/\alpha}([q, q+1] \times [0,1)) \,:\, \frac{1}{2}\left[\boldsymbol{X}(t) + \boldsymbol{X}(s)\right] > u\boldsymbol{b} \right\}$$
$$\le c_1\, u^{2n/\gamma - 1} \operatorname{mes}\left(\Lambda u^{-2/\alpha}([q, q+1] \times [0,1])\right) \exp\left(-\frac{u^2 \tau}{2}\right) \le c_2\, \Lambda^{2n}\, u^M \exp\left(-\frac{u^2}{2}\left[\boldsymbol{b}^\top \Sigma^{-1} \boldsymbol{b} + \delta(\varepsilon)\right]\right),$$

which is negligible with respect to $\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}$ as $u \to \infty$. Summing these bounds, we obtain

$$\limsup_{u \to \infty} \frac{\sum\limits_{\exists\, l:|q_l| > N_u(\varepsilon)} P_{\boldsymbol{b}}(q, \Lambda)}{\prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} = 0.$$

To bound the second sum in (28), we divide it further into

$$\sum_{-N_u(\varepsilon) \le q \le N_u(\varepsilon)} P_{\boldsymbol{b}}(q, \Lambda) = \sum_{\substack{-N_u(\varepsilon) \,\le\, q \,\le\, N_u(\varepsilon) \\ \exists\, l \,:\, |q_l| = 1}} P_{\boldsymbol{b}}(q, \Lambda) + \sum_{\substack{-N_u(\varepsilon) \,\le\, q \,\le\, N_u(\varepsilon) \\ \forall\, l \,:\, |q_l| \neq 1}} P_{\boldsymbol{b}}(q, \Lambda) =: A_1 + A_2. \tag{29}$$

The probabilities of the second sum can be estimated by Lemma 3 as follows:

$$\frac{P_{\boldsymbol{b}}(q, \Lambda)}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \le c\Lambda^{\#\{l:k_l = 0\}} \prod_{l:k_l \neq 0} \left(|k_l| - 1\right)^{-2} \exp\left(-\frac{1}{4}\, \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} \left(|k_l| - 1\right)^{\alpha_l}\right)$$

and therefore

$$\lim_{\Lambda \to \infty} \limsup_{u \to \infty} \frac{A_2}{\mathcal{H}_{\boldsymbol{\alpha}, \mathbb{B}, \boldsymbol{w}}([0, \Lambda]^n) \prod\limits_{l=1}^{n} u^{-2/\alpha_l} \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}}$$
$$\le c_1 \lim_{\Lambda \to \infty} \sum_{l} \Lambda^{\#\{l:k_l = 0\} - n} \exp\left(-\frac{1}{8}\, \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l}\right) = 0.$$

Next, we show how to bound the first sum. Assume for simplicity that q is such that $|q_l| = 1$ and $|q_{l'}| \neq 1$ for all $l' \neq l$. We have

$$
\begin{aligned}
P_{\boldsymbol{b}}(q,\Lambda) &= \mathbb{P}\left\{\begin{array}{ll} \forall j \neq l\ \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1] & : \quad \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, t_l \in \Lambda u^{-2/\alpha_l}[1,2] & \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] \quad : & \qquad\quad \boldsymbol{X}(s) > u\boldsymbol{b} \end{array}\right\} \\
&\leq \mathbb{P}\left\{\begin{array}{ll} \forall j \neq l\ \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1] & \\ \exists\, t_l \in u^{-2/\alpha_l}\left[\Lambda+\sqrt{\Lambda}, 2\Lambda+\sqrt{\Lambda}\right] & : \quad \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \Lambda u^{-2/\alpha}[0,1] \quad : & \qquad\quad \boldsymbol{X}(s) > u\boldsymbol{b} \end{array}\right\} \\
&\quad + \mathbb{P}\left\{\begin{array}{ll} \exists\, t_j \in \Lambda u^{-2/\alpha_j}[q_j, q_j+1]\ \forall j \neq l & \\ \exists\, t_l \in u^{-2/\alpha_l}\left[\Lambda, \Lambda+\sqrt{\Lambda}\right] & : \boldsymbol{X}(t) > u\boldsymbol{b} \end{array}\right\} =: A_3 + A_4.
\end{aligned}
$$

The first probability on the right satisfies the conditions of Lemma 3, and therefore

$$
\begin{aligned}
\frac{A_3}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \leq c_1 \Lambda^{\#\{l : k_l = 0\}} \prod_{l' \neq l,\, k_l \neq 0} \exp\left(-\frac{1}{4}\, \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l} \left(|q_l| - 1\right)^{\alpha_l}\right) \\
\exp\left(-\frac{1}{4}\, \boldsymbol{w}^\top B_l\, \boldsymbol{w}\, \Lambda^{\alpha_l/2}\right)
\end{aligned}
$$

Therefore, we obtain

$$
\lim_{\Lambda\to\infty} \limsup_{u\to\infty} \frac{\sum_l A_3}{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n) \prod_{l=1}^n u^{-2/\alpha_l}\, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} = 0.
$$

For A_4, we have by Lemma 5

$$
\frac{A_4}{\mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \sim \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\Big([0,\Lambda] \times \ldots \times \left[0, \sqrt{\Lambda}\right] \times \ldots [0,\Lambda]\Big).
$$

Consequently, we have

$$
\begin{aligned}
&\lim_{\Lambda\to\infty} \limsup_{u\to\infty} \frac{\sum_l A_4}{T^n \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n) \prod_{l=1}^n u^{-2/\alpha_l}\, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}} \\
&\quad = \lim_{\Lambda\to\infty} \frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}\Big([0,\Lambda] \times \ldots \times \left[0, \sqrt{\Lambda}\right] \times \ldots [0,\Lambda]\Big)}{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)} \leq \lim_{\Lambda\to\infty} \Lambda^{-1/2} = 0.
\end{aligned}
$$

The general case of $q_{\mathcal{I}} \in \{\pm 1\}$ for $\mathcal{I} \subset \{1, \ldots, n\}$ can be addressed similarly.

Positivity of the Pickands constant To show that the constant is positive we can use the following lower bound:

$$\begin{aligned}\limsup_{u\to\infty}\frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda]^n)}{\Lambda^n} &\geq \liminf_{u\to\infty}\frac{\mathbb{P}\{\exists\, t\in[0,T]^n \boldsymbol{X}(t)>u\boldsymbol{b}\}}{T^n\prod_{l=1}^{n}u^{-2/\alpha_l}\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}}\\ &\geq \liminf_{u\to\infty}\frac{\mathbb{P}\{\exists\, t\in[0,\varepsilon]^n \boldsymbol{X}(t)>u\boldsymbol{b}\}}{T^n\prod_{l=1}^{n}u^{-2/\alpha_l}\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}}\\ &\geq \liminf_{u\to\infty}\frac{\widetilde{\Sigma}_1-\widetilde{\Sigma}_2}{T^n\prod_{l=1}^{n}u^{-2/\alpha_l}\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}},\end{aligned} \tag{30}$$

where $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are the single and double sum with some Λ' instead of Λ and without odd (in all coordinates) intervals:

$$\widetilde{\Sigma}_1 := \sum_{0\leq k\leq \widetilde{N}_u(\varepsilon)}\mathbb{P}\left\{\exists\, t\in\Lambda' u^{-2/\alpha}[2k,2k+1]\,:\,\boldsymbol{X}(t)>u\boldsymbol{b}\right\},$$

$$\widetilde{\Sigma}_2 := \sum_{\substack{0\,\leq\, k,\,j\,\leq\,\widetilde{N}_u(\varepsilon)\\ k\,\neq\, j}}\mathbb{P}\left\{\begin{array}{l}\exists\, t\in\Lambda' u^{-2/\alpha}[2k,2k+1]\,:\,\boldsymbol{X}(t)>u\boldsymbol{b}\\ \exists\, s\in\Lambda u^{-2/\alpha}[2j,2j+1]\,:\,\boldsymbol{X}(s)>u\boldsymbol{b}\end{array}\right\}$$

and $\widetilde{N}_u(\varepsilon)=\lfloor\varepsilon/2\Lambda' u^{-2/\alpha}\rfloor$. By the same reasoning as above,

$$\liminf_{u\to\infty}\frac{\widetilde{\Sigma}_1}{\prod_{l=1}^{n}u^{-2/\alpha_l}\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}}=\left(\frac{\varepsilon}{2}\right)^n\frac{\mathcal{H}_{\boldsymbol{\alpha},\mathbb{B},\boldsymbol{w}}([0,\Lambda'])}{\Lambda'^n},$$

and

$$\limsup_{u\to\infty}\frac{\widetilde{\Sigma}_2}{\prod_{l=1}^{n}u^{-2/\alpha_l}\mathbb{P}\{\boldsymbol{X}(0)>u\boldsymbol{b}\}}\leq c\left(\frac{\varepsilon}{2}\right)^n\sum_{l}\Lambda'^{\#\{l:k_l=0\}-n}\prod_{k_l\neq 0}\exp\left(-\frac{1}{4}\boldsymbol{w}^\top B_l\boldsymbol{w}\,\Lambda'^{\alpha_l}\right)$$

Taking Λ' to be large enough, we find that the difference in (30) is separated from zero. Hence, its limit is positive. $\square$

5.2 Proof of theorem 2

Proof We begin the proof by splitting $[0, T]$ into intervals of some small enough $\delta > 0$

$$[0,T]^n = \delta \bigcup_{k \le N_\delta} [k, k+1], \qquad N_\delta := \left\lceil \frac{T}{\delta} \right\rceil,$$

and applying the Bonferroni inequality, which yields

$$\Sigma_1' - \Sigma_2 \le \mathbb{P}\{\exists\, t \in [0,T] : \boldsymbol{X}(t) > u\boldsymbol{b}\} \le \Sigma_1,$$

where

$$\Sigma_1 := \sum_{k \le N_\delta} \mathbb{P}\{\exists\, t \in \delta[k,k+1] : \boldsymbol{X}(t) > u\boldsymbol{b}\},$$

$$\Sigma_2 := \sum_{\substack{k,j \le N_\delta \\ k \ne j}} \mathbb{P}\left\{ \begin{array}{l} \exists\, t \in \delta[k,k+1] : \boldsymbol{X}(t) > u\boldsymbol{b} \\ \exists\, s \in \delta[j,j+1] : \boldsymbol{X}(s) > u\boldsymbol{b} \end{array} \right\}$$

and Σ_1' is defined by the same formula as Σ_1, but with $(N-1)$ instead of N in the upper limit of summation. At this point we split the proof into two parts. First, we will focus on finding the exact asymptotics of the single sum $\Sigma_1 \sim \Sigma_1'$, and then demonstrate that the double sum Σ_2 is negligible with respect to Σ_1.

Single sum Let min and max applied to a matrix denote component-wise minimum and maximum and let J denote a $d \times d$ matrix of all ones: $J_{kj} = 1$. Take $\varepsilon > 0$ and for each l define two matrices, which bound $B_l(t)$ on $\delta[k, k+1]$ component-wise from below and from above by

$$B_{l,k,\varepsilon,+} := \min_{t \in \delta[k,k+1]} B_t - \varepsilon J, \qquad B_{l,k,\varepsilon,-} := \max_{t \in \delta[k,k+1]} B_t + \varepsilon J.$$

Since for all $t \in [0, T]$ we have $\widetilde{B}_t \rhd 0$ strictly, it follows that $\widetilde{B_{k,\varepsilon,\pm}} \rhd 0$ if ε is small enough. Denote

$$\mathbb{B}_{k,\varepsilon,\pm} := (B_{l,k,\varepsilon,\pm})_{l=1,\dots,n}.$$

By Lemma 2 the real matrix-valued functions $\mathcal{E}_{\alpha_l, B_{l,k,\varepsilon,\pm}}(s_l)$ are positive definite and give rise to the following bounds on the covariance of $\boldsymbol{X}$:

$$\sum_{l=1}^n \mathcal{E}_{\alpha_l, B_{l,k,\varepsilon,-}}(s_l) \le R(t+s, t) \le \sum_{l=1}^n \mathcal{E}_{\alpha_l, B_{l,k,\varepsilon,+}}(s_l)$$

for small enough s. These functions generate two stationary Gaussian processes $\boldsymbol{Y}_{l,k,\varepsilon,\pm}(s)$, $s \in \mathbb{R}$, which by Lemma 4 provide us with bounds on the high excursion probabilities on $\delta[k, k+1]$:

$$\mathbb{P}\{\exists\, t \in \delta[k,k+1] : \boldsymbol{X}(t) > u\boldsymbol{b}\} \le \mathbb{P}\left\{ \exists\, t \in \delta[k,k+1] : \sum_{l=1}^{n} \boldsymbol{Y}_{l,k,\varepsilon,-}(t) > u\boldsymbol{b} \right\}$$

$$\ge \mathbb{P}\left\{ \exists\, t \in \delta[k,k+1] : \sum_{l=1}^{n} \boldsymbol{Y}_{l,k,\varepsilon,+}(t) > u\boldsymbol{b} \right\}$$

Note that the sign plus is on the left and minus is on the right.
Applying Theorem 1, we find that

$$\mathbb{P}\Big\{\exists\, t \in \delta[k,k+1] : \boldsymbol{Y}_{k,\varepsilon,\pm}(t) > u\boldsymbol{b}\Big\} \sim \delta^n\, \mathcal{H}_{\boldsymbol{\alpha},\, \mathbb{B}_{k,\varepsilon,\pm},\, \boldsymbol{w}} \prod_{l=1}^{n} u^{-2/\alpha_l}\, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}.$$

By adding together all the terms, we obtain

$$\left[\sum_{k=1}^{N_u-1} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,+},\boldsymbol{w}}\, \delta^n\right] u^{2/\alpha}\, \mathbb{P}\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\} \le \Sigma_1' \le \Sigma_1 \le \left[\sum_{k=1}^{N_u} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,-},\boldsymbol{w}}\, \delta^n\right] u^{2/\alpha}\, \mathbb{P}\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\}.$$

By continuity of $B \mapsto \mathcal{H}_{\alpha,B,\boldsymbol{b}}$, we have that

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \sum_{k=1}^{N_\delta} \mathcal{H}_{\alpha,\mathbb{B}_{k,\varepsilon,\pm},\boldsymbol{w}}\, \delta^n \xrightarrow[\delta\to 0]{} \int_0^T \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\, dt\,.$$

Hence, as $u \to \infty$,

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0} \Sigma_1' \sim \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \Sigma_1 \sim \left[\int_0^T \mathcal{H}_{\boldsymbol{\alpha},\mathbb{B}(t),\boldsymbol{w}}\, dt\right] u^{2/\alpha}\, \mathbb{P}\{\boldsymbol{X}(0) > u\,\boldsymbol{b}\}.$$

Double sum The double sum can be estimated by the same argument as in the proof of Theorem 1. □

Appendix

Gordon inequality

The following Slepian-type lemma is stated in Dębicki et al. (2015) for the case where $T \subset \mathbb{R}$, but it can be extended to the following version by standard techniques. Due to its complexity we present it here without proof.

Lemma 4 (Gordon inequality). *Let $\boldsymbol{X}(t)$, $t \in T$ and $\boldsymbol{Y}(t)$, $t \in T$ be two centered separable vector-valued Gaussian processes with values in $\mathbb{R}^d$ defined on a separable metric space T. If for all $t, s \in T$ holds*

$$R_{\boldsymbol{X}}(t,t) = R_{\boldsymbol{Y}}(t,t), \qquad R_{\boldsymbol{X}}(t,s) \ge R_{\boldsymbol{Y}}(t,s),$$

then for $\boldsymbol{u} \in \mathbb{R}^d$ holds

$$\mathbb{P}\{\exists\, t \in T\, :\, \boldsymbol{X}(t) > \boldsymbol{u}\} \le \mathbb{P}\{\exists\, t \in T\, :\, \boldsymbol{Y}(t) > \boldsymbol{u}\}.$$

Local Pickands lemma

The reader may find the uniform multivariate version of the local Pickands lemma in Dębicki et al. (2020). However, for the needs of this paper this strong result is not necessary, since we obtain uniformity using Gordon's inequality (Lemma 4). This is why we present here a simplified version of the local Pickands lemma.

Lemma 5 *Let $\boldsymbol{X}(t)$, $t \in [0,T]^n$ be a centered Hölder continuous homogenous Gaussian random field with values in $\mathbb{R}^d$ and covariance R satisfying*

$$\Sigma - R(t) = \sum_{l=1}^{n} \left[B_l |t_l|^{\alpha_l} \mathbb{1}_{t_l \ge 0} + B_l^\top |t_l|^{\alpha_l} \mathbb{1}_{t_l < 0} \right] + \epsilon(t), \qquad \epsilon(t) = o\left(\sum_{l=1}^{n} |t_l|^{\alpha_l} \right) \quad \text{as} \quad t \to 0,$$

where B_l's are some $d \times d$ real matrices and $\alpha_l \in (0,2]$. Denote $\alpha := (\alpha_l)_{l=1,\dots,n}$, $B := (B_l)_{l=1,\dots,n}$ and $\boldsymbol{w} := \Sigma^{-1} \widetilde{\boldsymbol{b}}$, where $\widetilde{\boldsymbol{b}}$ is the unique solution of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$. Then the matrix-valued functions $\mathcal{R}_{\alpha_l,B_l} : \mathbb{R} \to \mathbb{R}^{d\times d}$ defined by

$$\mathcal{R}_{\alpha_l,B_l}(t_l, s_l) := S_{\alpha_l,B_l}(t_l) + S_{\alpha_l,B_l}(-s_l) - S_{\alpha_l,B_l}(t_l - s_l), \qquad S_{\alpha_l,B_l}(t_l) = |t_l|^{\alpha_l} \left[B_l \mathbb{1}_{t_l \ge 0} + B_l^\top \mathbb{1}_{t_l < 0} \right]$$

are positive definite and for any closed $E \subset [0,T]$ containing 0 and closed holds

$$\mathbb{P}\left\{\exists\, t \in u^{-2/\alpha} E\, :\, \boldsymbol{X}(t) > u\boldsymbol{b}\right\} \sim \mathcal{H}_{\alpha,B,\boldsymbol{w}}(E)\, \mathbb{P}\{\boldsymbol{X}(0) > u\boldsymbol{b}\}$$

with

$$\mathcal{H}_{\alpha,B,\boldsymbol{w}}(E) = \int_{\mathbb{R}^d} e^{\boldsymbol{1}^\top \boldsymbol{x}}\, \mathbb{P}\left\{ \exists\, t \in E\, :\, \sum_{l=1}^{n} \operatorname{diag}(\boldsymbol{w}) \left[\boldsymbol{Y}_l(t_l) - S_{\alpha_l,B_l}(t_l)\, \boldsymbol{w} \right] > \boldsymbol{x} \right\} d\boldsymbol{x} \in (0,\infty),$$

where $\boldsymbol{Y}_l$ is a continuous zero mean Gaussian process associated to the covariance function $\mathcal{R}_{\alpha_l,B_l}(t,s)$.

Borell-TIS and Piterbarg inequalities

Lemma 6 *Let $(\boldsymbol{Z}(t))_{t\in E}$, $E \subset \mathbb{R}^k$ be a separable centered d-dimensional vector-valued Gaussian random field having components with a.s. continuous paths. Assume that $\Sigma(t) = \mathbb{E}\left\{\boldsymbol{Z}(t)\boldsymbol{Z}(t)^\top\right\}$ is non-singular for all $t \in E$. Let $\boldsymbol{b} \in \mathbb{R}^d \backslash (-\infty, 0]^d$ and define $\sigma_{\boldsymbol{b}}^2(t)$ by*

$$\sigma_{\boldsymbol{b}}^{-2}(t) := \min_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1}(t)\, \boldsymbol{x}.$$

If $\sigma_{\boldsymbol{b}}^2 := \sup_{t\in E} \sigma_{\boldsymbol{b}}^2(t) \in (0,\infty)$, then there exists some positive constant μ such that for all $u > \mu$

$$\mathbb{P}\{\exists\, t \in E : \boldsymbol{Z}(t) > u\boldsymbol{b}\} \le \exp\left(-\frac{(u-\mu)^2}{2\sigma_{\boldsymbol{b}}^2}\right). \tag{31}$$

If further for some $C \in (0,\infty)$ and $\boldsymbol{\gamma} \in (0,2]^k$

$$\sum_{1\le i\le k} \mathbb{E}\left\{ \left(Z_i(t) - Z_i(s)\right)^2 \right\} \le C \sum_{m=1}^{k} |t_m - s_m|^{\gamma_m} \tag{32}$$

and

$$\left\| \Sigma^{-1}(t) - \Sigma^{-1}(s) \right\|_F \le C \sum_{m=1}^{k} |t_m - s_m|^{\gamma_m} \tag{33}$$

hold for all $t, s \in E$, then for all u positive

$$\mathbb{P}\{\exists\, t \in E : \boldsymbol{Z}(t) > u\boldsymbol{b}\} \le C_* \mathrm{mes}(E)\, u^{2d/\gamma-1} \exp\left(-\frac{u^2}{2\sigma_{\boldsymbol{b}}^2}\right), \tag{34}$$

where C_* is some positive constant not depending on u. In particular, if $\sigma_{\boldsymbol{b}}^2(t)$ is continuous and achieves its unique maximum at some fixed point $t_* \in E$, then (34) is still valid if (32) and (33) are assumed to hold only for all $t, s \in E$ in an open neighborhood of t_*.

Quadratic programming problem

For a given non-singular $d \times d$ real matrix Σ we consider the quadratic programming problem

$$\Pi_\Sigma(\boldsymbol{b}) : \text{minimize } \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \text{ under the linear constraint } \boldsymbol{x} \ge \boldsymbol{b}. \tag{35}$$

Below $J = \{1,\dots,d\}\backslash I$ can be empty; the claim in (37) is formulated under the assumption that J is non-empty.

Lemma 7 *Let $d \ge 2$ and Σ a $d \times d$ symmetric positive definite matrix with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \backslash (-\infty,]^d$, then $\Pi_\Sigma(\boldsymbol{b})$ has a unique solution $\widetilde{\boldsymbol{b}}$ and there exists a unique non-empty index set $I \subset \{1,\dots,d\}$ with $m \le d$ elements such that*

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I \ne \boldsymbol{0}_I \tag{36}$$

$$\widetilde{\boldsymbol{b}}_J = \Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_J, \qquad (\Sigma_{II})^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \tag{37}$$

$$\min_{\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} = \widetilde{\boldsymbol{b}}^\top \Sigma^{-1} \widetilde{\boldsymbol{b}} = \boldsymbol{b}_I^\top (\Sigma_{II})^{-1} \boldsymbol{b}_I > 0, \tag{38}$$

$$\max_{z \in [0,\infty)^d : z^\top \boldsymbol{b} > 0} \frac{(z^\top \boldsymbol{b})^2}{z^\top \Sigma z} = \frac{(\boldsymbol{w}^\top \boldsymbol{b})^2}{\boldsymbol{w}^\top \Sigma \boldsymbol{w}} = \min_{\boldsymbol{x} \geq \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}, \tag{39}$$

with $\boldsymbol{w} = \Sigma^{-1} \widetilde{\boldsymbol{b}}$ *satisfying* $\boldsymbol{w}_I = (\Sigma_{II})^{-1} \boldsymbol{b}_I > \boldsymbol{0}_I$, $\boldsymbol{w}_J = \boldsymbol{0}_J$.

Integral estimate

Lemma 8 *If a family of Hölder continuous random fields* $\chi_{\boldsymbol{x}}(t)$, $t \in [0, \Lambda]^n$ *indexed by* $\boldsymbol{x} \in \mathbb{R}^d$ *and jointly measurable with respect to* $(t, \boldsymbol{x})$ *satisfies*

$$\sup_{F \subset \{1,\dots,d\}} \sup_{t \in [0,\Lambda]^n} \boldsymbol{w}_F^\top \mathbb{E}\left\{\chi_{\boldsymbol{x},F}(t)\right\} \leq G + \varepsilon \sum_{j=1}^d |x_j|$$

and

$$\sup_{F \subset \{1,\dots,d\}} \sup_{t \in [0,\Lambda]^n} \operatorname{Var}\left\{\boldsymbol{w}_F^\top \chi_{\boldsymbol{x},F}(t)\right\} \leq \sigma^2,$$

with some constants $\boldsymbol{w} > \boldsymbol{0}$, $\sigma^2 > 0$, $G \in \mathbb{R}$ *and small enough* $\varepsilon > 0$, *then there exist constants* C, $c > 0$ *independent of* Λ, *such that the following inequality holds*:

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \chi_{\boldsymbol{x}}(t) > \boldsymbol{x}\right\} d\boldsymbol{x} \leq C e^{c(G + \sigma^2)}.$$

Proof of Lemma 8 Define a collection of sets $\Omega_F = \left\{\boldsymbol{x} \in \mathbb{R}^d \,:\, \boldsymbol{x}_F > \boldsymbol{0},\ \boldsymbol{x}_{F^c} < \boldsymbol{0}\right\}$ indexed by $F \subset \{1, \dots, d\}$ and split the integral:

$$\int_{\mathbb{R}^d} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \chi_{\boldsymbol{x}}(t) > \boldsymbol{x}\right\} d\boldsymbol{x} = \sum_{F \in 2^d} \int_{\Omega_F} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \chi_{\boldsymbol{x}}(t) > \boldsymbol{x}\right\} d\boldsymbol{x}\,.$$

For $\boldsymbol{x} \in \Omega_F$ the probability under the integral may be bounded as follows:

$$\begin{aligned}
\mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \chi_{\boldsymbol{x}}(t) > \boldsymbol{x}\right\} &\leq \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \boldsymbol{w}_F^\top\left(\chi_{\boldsymbol{x},F}(t) - \mathbb{E}\left\{\chi_{\boldsymbol{x},F}(t)\right\}\right) > \boldsymbol{w}_F^\top \boldsymbol{x}_F - \boldsymbol{w}_F^\top \mathbb{E}\left\{\chi_{\boldsymbol{x},F}(t)\right\}\right\} \\
&\leq \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \boldsymbol{w}_F^\top\left(\chi_{\boldsymbol{x}} - \mathbb{E}\left\{\chi_{\boldsymbol{x}}(t)\right\}\right) > \boldsymbol{w}_F^\top \boldsymbol{x}_F - G - \varepsilon \sum_{j=1}^d |x_j|\right\} \\
&= \mathbb{P}\left\{\exists\, t \in [0, \Lambda]^n \,:\, \eta_{\boldsymbol{x},F}(t) > r_{F,\varepsilon}(\boldsymbol{x}) - G\right\},
\end{aligned}$$

where

$$r_{F,\varepsilon}(\boldsymbol{x}) = \boldsymbol{w}_F^\top \boldsymbol{x}_F - \varepsilon \sum_{j=1}^d |x_j| \qquad \text{and} \qquad \eta_{\boldsymbol{x},F}(t) = \boldsymbol{w}_F^\top\left(\chi_{\boldsymbol{x},F}(t) - \mathbb{E}\left\{\boldsymbol{x}_{\boldsymbol{x},F}(t)\right\}\right).$$

Let us split the domain Ω_F into two parts

$$\Omega_{F,+} = \left\{ \boldsymbol{x} \in \Omega_F \, : \, r_{F,\varepsilon}(\boldsymbol{x}) > G \right\} \qquad \text{and} \qquad \Omega_{F,-} = \Omega_F \backslash \Omega_{F,+}.$$

Let us first deal with the integral over $\Omega_{F,-}$. It follows from $\boldsymbol{w}_F^\top \boldsymbol{x}_F - \varepsilon \sum_{j=1}^d |x_j| < G$ that

$$\sum_{j \in F} (w_i - \varepsilon)|x_j| - \varepsilon \sum_{j \in F^c} |x_j| < G$$

or, with $w_* = \min_{j \in F} w_j > 0$ and $\varepsilon < w_*$,

$$\varepsilon \sum_{j \in F} |x_j| \leq \frac{\varepsilon G}{w_* - \varepsilon} + \frac{\varepsilon^2}{w_* - \varepsilon} \sum_{j \in F^c} |x_j|.$$

Therefore, with $r = r_{F,\varepsilon}(\boldsymbol{x})$, we have

$$\begin{aligned} \boldsymbol{w}^\top \boldsymbol{x} &= r + \boldsymbol{w}_{F^c}^\top \boldsymbol{x}_{F^c} + \varepsilon \sum_{j=1}^d |x_j| = r + \varepsilon \sum_{j \in F} |x_j| - \sum_{j \in F^c} (w_j - \varepsilon)|x_j| \\ &\leq r + \frac{\varepsilon G}{w_* - \varepsilon} - \left(w_* - \frac{\varepsilon^2}{w_* - \varepsilon} - \varepsilon \right) \sum_{j \in F^c} |x_j| \leq r + \frac{\varepsilon G}{w_* - \varepsilon}, \end{aligned}$$

provided that ε is small enough. Bounding the probability under the integral by 1 and changing the variables, we obtain

$$\begin{aligned} &\int_{\Omega_{F,-}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{ \exists\, t \in [0, \Lambda]^n \, : \, \boldsymbol{\chi}_{\boldsymbol{x}}(t) > \boldsymbol{x} \right\} d\boldsymbol{x} \leq \int_{\Omega_{F,-}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, d\boldsymbol{x} = \int_{-\infty}^G dr \int dS \, e^{\boldsymbol{w}^\top \boldsymbol{x}} \, r^{d-1} \\ &\leq \int_{-\infty}^G \int dS \, e^{r + \varepsilon G/(w_* - \varepsilon)} r^{d-1} \, dr \, dS \leq c_1 e^{\varepsilon G/(w_* - \varepsilon)} \int_{-\infty}^G e^{(1+\varepsilon) r} \, dr = c_1 \, e^{c_2 G}. \end{aligned}$$

Next, we concentrate on the intergral over $\Omega_{F,+}$. By Piterbarg inequality (34), we have the following uniform in $\boldsymbol{x} \in \Omega_{F,+}$ upper bound:

$$\mathbb{P}\left\{ \exists\, t \in [0, \Lambda]^n \, : \, \eta_{\boldsymbol{x},F}(t) > \boldsymbol{x} \right\} \leq c_3 \left(\frac{r(\boldsymbol{x}) - G}{\sigma} \right)^{2/\gamma} \exp\left(-\frac{1}{2} \left(\frac{r(\boldsymbol{x}) - G}{\sigma} \right)^2 \right).$$

Plugging this bound into the integral and changing the variables, we obtain

$$\begin{aligned} &\int_{\Omega_{F,+}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \, \mathbb{P}\left\{ \exists\, t \in [0, \Lambda]^n \, : \, \boldsymbol{\chi}_{\boldsymbol{x}}(t) > \boldsymbol{x} \right\} d\boldsymbol{x} \\ &\leq c_3 \int_{\Omega_{F,+}} e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r(\boldsymbol{x}) - G}{\sigma} \right)^{2/\gamma} \exp\left(-\frac{1}{2} \left(\frac{r(\boldsymbol{x}) - G}{\sigma} \right)^2 \right) d\boldsymbol{x} \\ &= c_3 \int_G^\infty dr \int dS \, e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r - G}{\sigma} \right)^{2/\gamma + d - 1} \exp\left(-\frac{1}{2} \left(\frac{r - G}{\sigma} \right)^2 \right). \end{aligned}$$

Note that with $w^* = \max_{i=1,\dots,d} w_i$ we have

$$r = \sum_{i\in F}(w_i - \varepsilon)|x_i| - \varepsilon \sum_{i\in F^c} |x_i| \geq (w^* - \varepsilon) \sum_{i\in F} |x_i| - \varepsilon \sum_{i\in F^c} |x_i|$$

and it follows that for all $\varepsilon < w^*$ the following bound holds:

$$\varepsilon \sum_{i\in F} |x_i| \leq \frac{\varepsilon r}{w^* - \varepsilon} + \frac{\varepsilon^2}{w^* - \varepsilon} \sum_{i\in F^c} |x_i|.$$

This bound yields

$$(\boldsymbol{w}, \boldsymbol{x}) = r + \varepsilon \sum_{i\in F}^{d} |x_i| - \sum_{i\in F^c}(w_i - \varepsilon)|x_i| \leq \left(1 + \frac{\varepsilon}{w^* - \varepsilon}\right) r - \left(w_* - \varepsilon - \frac{\varepsilon^2}{w^* - \varepsilon}\right) \sum_{i\in F^c} |x_i|,$$

from which for small enough ε follows that $(\boldsymbol{w}, \boldsymbol{x}) \leq (1 + \varepsilon')r$, with $\varepsilon' = \varepsilon/(w^* - \varepsilon)$. Hence,

$$\begin{aligned} c_3 \int_G^\infty dr \int dS\, e^{\boldsymbol{w}^\top \boldsymbol{x}} \left(\frac{r-G}{\sigma}\right)^{2/\gamma+d-1} \exp\left(-\frac{1}{2}\left(\frac{r-G}{\sigma}\right)^2\right) \\ \leq c_4 \int_{-\infty}^\infty e^{(1+\varepsilon')r} \exp\left(-\frac{1}{2}\left(\frac{r-G}{\sigma}\right)^2\right) dr \leq c_5\, e^{c_6(G+\sigma^2)}, \end{aligned}$$

where in the last step we used the Gaussian mgf formula $\mathbb{E}\left\{e^{t\mathcal{N}(\mu,\sigma^2)}\right\} = e^{t\mu + t^2\sigma^2/2}$ with $t = 1 + \varepsilon'$. □

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Declarations

Ethical approval This study is a purely mathematical research paper and does not involve human subjects, animal subjects, or ethical considerations. Therefore, no ethical approval was required for this research.

Conflict of interest The author declares that there are no conflicts of interest regarding this research study.

References

Berman, S.M.: Sojourns and extremes of Gaussian processes. Ann. Probab. 999–1026 MR 0372976 (1974)

Chan, H.P., Lai, T.L.: Maxima of asymptotically Gaussian random fields and moderate deviation approximations to boundary crossing probabilities of sums of random variables with multidimensional indices. Ann. Probab. **34**(1)80–121 MR 2206343 (2006)

Dębicki, K., Hashorva, E., Ji, L., Tabiś, K.: Extremes of vector-valued Gaussian processes: exact asymptotics. Stochastic Process. Appl. **125**(11)4039–4065 MR 3385594 (2015)

Dębicki, K., Hashorva, E., Wang, L.: Extremes of vector-valued Gaussian processes. Stochastic Process. Appl. **130**(9)5802–5837 MR 4127347 (2020)

Hashorva, E., Ji, L.: Extremes of $\alpha(t)$-locally stationary Gaussian random fields. Trans. Am. Math. Soc. **368**(1)1–26 MR 3413855 (2016)

Hüsler, J.: Extreme values and high boundary crossings of locally stationary Gaussian processes. Ann. Probab. 1141–1158 MR 1062062 (1990)

Ievlev, P., Novikov, S.: A matrix-valued Schoenberg's problem with applications to Gaussian processes. Electron. Commun. Probab. **28**, 1–12 (2023). https://doi.org/10.1214/23-ECP562

Piterbarg, V.I.: Asymptotic methods in the theory of Gaussian processes and fields. American Mathematical Soc. **148** MR 1361884 (1996)

Piterbarg, V.I., Rodionov, I.V.: High excursions of Bessel and related random processes. Stochastic Processes and their Applications **130**(8)4859–4872 MR 4108474 (2020)

Qiao, W.: Extremes of locally stationary Gaussian and chi fields on manifolds. Stochastic Processes and their Applications **133**, 166–192 MR 4192351 (2021)

Tan, Z., Zheng, S.: Extremes of a type of locally stationary Gaussian random fields with applications to Shepp statistics. J. Theor. Probab. **33**(4)2258–2279 MR 4166199 (2020)