Implicit max-stable extremal integrals

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Received: 19 August 2019 / Revised: 2 July 2020 / Accepted: 17 July 2020 /
Published online: 22 October 2020
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Abstract

Recently, the notion of implicit extreme value distributions has been established, which is based on a given loss function $f \geq 0$. From an application point of view, one is rather interested in extreme loss events that occur relative to f than in the corresponding extreme values itself. In this context, so-called f -implicit α -Fréchet max-stable distributions arise and have been used to construct independently scattered sup-measures that possess such margins. In this paper we solve an open problem in Goldbach (2016) by developing a stochastic integral of a deterministic function $g \geq 0$ with respect to implicit max-stable sup-measures. The resulting theory covers the construction of max-stable extremal integrals (see Stoev and Taqqu Extremes 8, 237–266 (2005)) and, at the same time, reveals striking parallels.

Keywords Implicit max-stable distributions · Independently scattered random sup-measures · Stochastic integrals · Implicit max-stable processes

AMS 2000 Subject Classifications 60G57 · 60G60 · 60G70

1 Introduction

The theory of implicit extreme values is highly motivated by application, such as hydrology (see the introductory example in Scheffler and Stoev (2017)), and tries to analyze the circumstances in which several impact factors lead to extreme loss or damage. Hence, different from classical extreme value theory (shortly: EVT), this perspective is less interested in the attained extreme values than in the study of complex systems that cause these extreme values. Particularly, the isolated impacts (components) of the system do not have to be extreme in any sense, but can still contribute to such *extreme loss events*.

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In this context it is reasonable to assume that the connection between the impact factors and the related loss is known. More precisely, throughout the paper we consider a fixed function $f : \mathbb{R}^d \rightarrow [0, \infty)$ that serves as some kind of *loss function*, depending on $d \geq 1$ impact factors. For technical reasons, we have to assume that f fulfills the following properties, which still appear natural for most examples:

- (i) f is continuous.
- (ii) $f(x) = 0$ if and only if $x = 0$.
- (iii) f is 1-homogeneous, i.e. we have $f(\lambda x) = \lambda f(x)$ for every $\lambda \geq 0$ and $x \in \mathbb{R}$.

Turning over to probability, we consider a random vector $X = (X^{(1)}, \dots, X^{(d)})$ modeling the joint behavior of the d impact factors. Then, for a sequence $(X_j)_{j \in \mathbb{N}}$ of identically distributed and independent (i.i.d.) random vectors, a major subject of classical multivariate EVT is to understand the asymptotic behavior (under possible normalization) of

$$M_k := \bigvee_{j=1}^k X_j := \max_{j=1, \dots, k} X_j \quad (1.1)$$

as $k \rightarrow \infty$, where the maximum is meant component-wise. Sometimes, one is also interested in the study of $\max_{j=1, \dots, k} f(X_j)$, which leads to an associated univariate problem.

In contrast and as motivated before already, we want to pursue an implicit approach instead. Thus, if we assume for a moment that the *observations* $X_1, X_2, \dots$ of the sample do not coincide, this suggests to consider

$$X_{j(k)} := \operatorname{argmax}_{j=1, \dots, k} f(X_j), \quad k \in \mathbb{N}. \quad (1.2)$$

Unfortunately, there will be ties in general. For this reason we replace the argmax function by the following $\vee_f$ -operation, which has been introduced in Goldbach (2016). Hence, let $\vee_f : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, defined by

$$\vee_f(x_1, x_2) := x_1 \vee_f x_2 := \begin{cases} x_1, & \text{if } f(x_1) \leq f(x_2) \\ x_2, & \text{if } f(x_1) < f(x_2). \end{cases} \quad (1.3)$$

Inductively, for $x_1, \dots, x_k \in \mathbb{R}^d$, we define

$$\bigvee_{j=1}^k x_j := \bigvee_{1 \leq j \leq k} x_j := x_1 \vee_f \dots \vee_f x_k := (x_1 \vee_f \dots x_{k-1}) \vee_f x_k.$$

Note that the resulting mapping is $\mathcal{B}((\mathbb{R}^d)^k) \rightarrow \mathcal{B}(\mathbb{R}^d)$ measurable (see Lemma 1.1.6 in Goldbach (2016)), where $\mathcal{B}(\mathbb{R}^d)$ denotes the collection of all Borel sets $A \subset \mathbb{R}^d$, and that the $\vee_f$ -operation is associative, but not commutative in general. However, it is a main feature of the $\vee_f$ -operation that the result is always part of the sample (in contrast to (1.1)).

Anyway, (1.2) can be rewritten as $X_{j(k)} = X_1 \vee_f \dots \vee_f X_k$ now. Also note that the study of $a_k^{-1} X_{j(k)}$, where $a_k > 0$ is suitable, and the characterization of possible limits in distribution (as $k \rightarrow \infty$) are the main topic of Scheffler and Stoev (2017). Scheffler and Stoev (2017) call these limits *implicit extreme value*

distributions. There it is also shown that, under mild assumptions, the class of implicit extreme value distributions coincides with the class of *implicit max-stable distributions*. Here, an $\mathbb{R}^d$ -valued random vector Y with distribution $\mu := \mathcal{L}(Y)$ is said to be implicit max-stable if, for every $k \in \mathbb{N}$, there exists some $b_k > 0$ such that

$$b_k^{-1} \bigvee_{j=1}^k Y_j = \bigvee_{j=1}^k b_k^{-1} Y_j \stackrel{d}{=} Y \quad (1.4)$$

holds true, where $Y_1, Y_2, \dots$ is an i.i.d. sequence with common distribution μ and where $\stackrel{d}{=}$ denotes equality in distribution. The importance of stability equalities like (1.4) is familiar, for example when using the “+”-operation or the “ $\vee$ ”-operation on $\mathbb{R}^d$ instead of $\vee_f$. Then, stochastic processes whose margins possess such distributions are of interest and are often constructed by using certain stochastic integrals with respect to *independently scattered random measures* (see Samoradnitsky and Taqqu (1994)) or corresponding *sup-measures* (see Stoev and Taqqu (2005)), respectively. Note that these random measures are related to a further measure, which is deterministic and which we will denote by m . More details can be found in Section 2 below. In any case the approach in Stoev and Taqqu (2005) leads to an *extremal integral*, which is well-defined for every function g that belongs to $L_+^\alpha(m)$, where

$$L^\alpha(m) := \left\{ g : E \rightarrow \mathbb{R} \mid g \text{ is measurable with } \|g\|_\alpha := \left(\int_E |g|^\alpha dm \right)^{1/\alpha} < \infty \right\} \quad (1.5)$$

and $L_+^\alpha(m) := \{g : E \rightarrow \mathbb{R}_+ \mid g \in L^\alpha(m)\}$. Here, $\alpha > 0$ is somehow connected to the underlying sup-measure. In particular, this diversity of possible *integrands* (or *kernel functions*) g allows in Stoev and Taqqu (2005) to study a deep relationship between extremal integral representations and max-stable stochastic processes that are well-known in literature.

Hence, in Goldbach (2016) the notion of a so-called *implicit sup-measure* is introduced (see Definition 2.2 below), which extends the sup-measures from Stoev and Taqqu (2005). In addition, Goldbach (2016) provides the existence of such implicit sup-measures, denoted by M in the sequel. Again the details will be discussed in Section 2. For the time being, we rather refer the reader to Example 3.1.15 in Goldbach (2016), where $X(t) := M([0, t])$ leads to an $\mathbb{R}^d$ -valued stochastic process $\mathbb{X} = \{X(t) : t \geq 0\}$ that is implicit max-stable in the sense of Definition 5.1 below. Also note that Goldbach (2016) proposes the definition of a certain stochastic integral with respect to M , which allows for the representation

$$X(t) = M([0, t]) = \int_{\mathbb{R}} \mathbb{1}_{[0, t]}(s) dM(s). \quad (1.6)$$

Here, $g(s) = \mathbb{1}_{[0, t]}(s)$ is a simple function in the sense of (2.3) below. Unfortunately, the definition of the stochastic integral in Goldbach (2016) is just valid for simple functions $g \geq 0$ and is therefore more or less limited to considerations as in (1.6). However, based on the yields in Stoev and Taqqu (2005), an extension of (1.6) could be very interesting accordingly.

In effect, this paper mainly pursues two goals. On the one hand, we want to stimulate this new field of EVT (also see de Fondeville and Davison (2018) and Dombry and Ribatet (2015)), which often allows us to recover results from classical EVT as Example 4.6 will reveal. On the other hand, the subsequent findings could serve as a helpful tool to solve several problems that have already been discussed in literature. For instance, we think about the study of so-called *f-implicit max-infinitely-divisible distributions* (see Definition 2.1.1 in Goldbach (2016)). At the same time and based on the asymptotic theory in Scheffler and Stoev (2017), it might be tempting to construct *implicit max-stable processes* using the outcome in this paper.

It is also mentionable that we will obtain results that, at first glance, are similar to those in Stoev and Taqqu (2005), where a (classical) max-stable extremal integral has been constructed. Somehow this means that we have parallels to the notion of α -stable stochastic integrals, as proposed by Samoradnitsky and Taqqu (1994). However, our techniques are mostly different, since monotonicity arguments do not work any longer and since the $\vee_f$ -operation can be rough sometimes. Thus, we believe that these techniques are of independent interest.

The paper is organized as follows: In Section 2 we will provide a short review of the underlying concepts in order to understand (1.6) in more detail. In Section 3 we will expand (1.6) towards the notion of an (implicit) stochastic integral that, in the end, will be realized as a stochastic limit and that allows every function $g \in L_+^\alpha(m)$ to serve as kernel function. Then, Section 4 is devoted to present some useful properties, which emphasize our implicit approach and to analyze the dependence structures of the resulting stochastic integrals. Finally, as a main benefit of our theory, we will be able to construct interesting examples of so-called *f-implicit max-stable processes*. There, in Section 5, we will also give a short outlook on related problems that could be further work. Note that some rather technical proofs have been outsourced to the Appendix.

2 Preliminaries

In this section we briefly want to introduce some notation and, by the way, recall from Goldbach (2016) and Stoev and Taqqu (2005) what we know so far about (implicit) α -Fréchet distributions and their corresponding sup-measures. Throughout let $(\Omega, \mathcal{A}, \mathbb{P})$ be some underlying probability space on which all occurring random elements are defined.

As usual an $\mathbb{R}$ -valued random variable Z is said to be α -Fréchet distributed, where $\alpha > 0$, if

$$\mathbb{P}(Z \leq x) = \begin{cases} \exp(-\sigma^\alpha x^{-\alpha}), & x > 0 \\ 0, & x \leq 0. \end{cases} \quad (2.1)$$

Here, $\sigma \geq 0$ is referred to as the *scale* (coefficient) of Z and we write $Z \sim \Phi_\alpha(\sigma)$. Note that $\sigma = 0$ leads to the point measure in zero, i.e. $\Phi_\alpha(0) = \varepsilon_0$ for every $\alpha > 0$. In the case $\sigma = 1$ we say that Z is *standard* α -Fréchet distributed, abbreviated by $Z \sim \Phi_\alpha$.

Note that Proposition 3.19 and Theorem 4.2 in Scheffler and Stoev (2017) provide a characterization of all implicit max-stable distributions on $\mathbb{R}^d$. Then, in a very natural special case, the solutions of (1.4) lead to the following notion, which is due to Definition 3.1.2 in Goldbach (2016). Recall that the function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$, fulfilling the properties (i)-(iii) from Section 1, is fixed throughout the paper.

Definition 2.1 Fix $\alpha > 0$ and let κ be a probability measure on $\mathcal{B}(S)$, where $S := \{f = 1\}$ is a compact subset of $\mathbb{R}^d$. Then an $\mathbb{R}^d$ -valued random vector Y is said to have an *f-implicit α -Fréchet distribution with scale $\sigma \geq 0$ and angular part κ* if

$$Y \stackrel{d}{=} \sigma Z \Theta, \quad (2.2)$$

where the random variable $Z \sim \Phi_\alpha$ is independent of the S -valued random vector Θ , being κ -distributed. We write $Y \sim \Phi_{\alpha, \kappa}^f(\sigma)$ in this case. Moreover, let $\Phi_{\alpha, \kappa}^f := \Phi_{\alpha, \kappa}^f(1)$ again.

Observe that (2.2) is nothing else than $Y \stackrel{d}{=} Z_\sigma \Theta$ with $Z_\sigma \sim \Phi_\alpha(\sigma)$ and that $f(Y) \sim \Phi_\alpha(\sigma)$ in this case. Hence, the denomination *f-implicit α -Fréchet* becomes reasonable.

Now we can proceed with the consideration of independently scattered random measures and sup-measures, respectively. The first named ones have a long history, particularly in the context of α -stable or, more generally, infinitely-divisible stochastic integrals and processes. See Kremer and Scheffler (2019) and Rajput and Rosinski (1989) and Samoradnitsky and Taqqu (1994), just to mention a few. In this case, property (RM_2) of the following definition essentially has to be modified by using the “+”-operation. In contrast, when using the “ $\vee$ ”-operation instead (as established in Stoev and Taqqu (2005)), we are dealing with so-called (independently scattered) sup-measures. In the following we will refine this idea according to Definition 3.1.8 in Goldbach (2016), where the set $L_0^d := \{X : \Omega \rightarrow \mathbb{R}^d \mid X \text{ is random vector}\}$ ($d \geq 1$) is of interest.

Definition 2.2 Let $(E, \mathcal{E}, m)$ be a σ -finite measure space with $\mathcal{E}_0 := \{A \in \mathcal{E} : m(A) < \infty\}$. Then, for f as before, a mapping $M^f : \mathcal{E}_0 \rightarrow L_0^d$ is called an *f-implicit sup-measure* if the following conditions are fulfilled:

- (RM_1) For finitely many sets $A_1, \dots, A_k \in \mathcal{E}_0$ the corresponding random vectors $M^f(A_1), \dots, M^f(A_k)$ are independent.
- (RM_2) For any collection of disjoint sets $A_1, A_2, \dots \in \mathcal{E}_0$ such that $\bigcup_{j=1}^\infty A_j \in \mathcal{E}_0$ we have that

$$M^f\left(\bigcup_{j=1}^\infty A_j\right) = \bigvee_f M^f(A_j) = M^f(A_{j_0}) \quad \text{almost surely (a.s.),}$$

where j_0 is a random index.

In addition, if $M^f(A)$ has an *f-implicit α -Fréchet* distribution for every $A \in \mathcal{E}_0$ and some $\alpha > 0$, the *f-implicit sup-measure* $M^f = M_\alpha^f$ is said to be α -Fréchet.

Of course, the question arises whether non-trivial examples of such sup-measures do exist. A satisfying answer is given by the following statement, which is due to Theorem 3.1.12 in Goldbach (2016).

Proposition 2.3 Fix $\alpha > 0$ and an arbitrary probability measure κ on $\mathcal{B}(S)$. Then, for every σ -finite measure space $(E, \mathcal{E}, m)$, there exists an f -implicit α -Fréchet sup-measure $M^f : \mathcal{E}_0 \rightarrow L_0^d$ such that $M^f(A) \sim \Phi_{\alpha, \kappa}^f(m(A)^{1/\alpha})$ for every $A \in \mathcal{E}_0$.

From now on and by a little abuse of notation, we will neglect the fact that the sup-measure, which is provided by Proposition 2.3, depends on f, α, κ and m . Hence, we merely abbreviate this sup-measure by M . Then a function $g : E \rightarrow \mathbb{R}_+ = [0, \infty)$ is called *simple (with respect to $\mathcal{E}_0$)* if, for some $k \in \mathbb{N}$, there exist $\alpha_1, \dots, \alpha_k \geq 0$ (which should not be mixed up with the underlying *stability index* $\alpha > 0$) and disjoint sets $A_1, \dots, A_k \in \mathcal{E}_0$ such that the representation

$$g(s) = \sum_{j=1}^k \alpha_j \mathbb{1}_{A_j}(s), \quad s \in E \quad (2.3)$$

holds true. Certainly, such a representation is not unique. However, we get the following.

Definition 2.4 Let $g : E \rightarrow \mathbb{R}_+$ be simple and assume that a corresponding representation for g is given by (2.3). Then the $\mathbb{R}^d$ -valued random vector

$$I_M^{\vee f}(g) := I(g) := \int_E^{\vee f} g(s) dM(s) := \bigvee_{j=1}^k \alpha_j M(A_j) \quad (2.4)$$

is uniquely determined by g a.s. (see Proposition 3.2.3 in Goldbach (2016)), which means that $I(g)$ does not depend on a particular representation for g . We call $I(g)$ the (f -implicit extremal) integral of g (with respect to M).

Several properties of the (f -implicit extremal) integral $I(g)$ for simple functions g can be found in Proposition 3.2.4 of Goldbach (2016). Later, in the context of Theorem 4.2, we will study them into more detail. For the moment, it suffices just to mention the following one:

$$I(g) \sim \Phi_{\alpha, \kappa}^f(\|g\|_\alpha), \quad \text{where} \quad \|g\|_\alpha = \left(\int_E g(s)^\alpha dm(s) \right)^{1/\alpha} \quad (2.5)$$

according to (1.5). Finally, Corollary 3.2.5 in Goldbach (2016) shows that $I(g) = I(\tilde{g})$ a.s., provided that g and $\tilde{g}$ coincide m -almost everywhere (a.e.). This is one of the reasons why m is often referred to as the *control measure* of M .

Remark 2.5 Stoev and Taqqu (2005) introduce an extremal integral for deterministic functions $g \geq 0$ with respect to certain α -Fréchet sup-measures, which leads to $\mathbb{R}_+$ -

valued random variables. We omit the details. However, using Corollary 3.1.16 in Goldbach (2016) and the fact that

$$f(x_1 \vee_f x_2) = f(x_1) \vee f(x_2) \quad \text{for any } x_1, x_2 \in \mathbb{R}^d, \quad (2.6)$$

it is easy to check that the random variable $f(I(g))$ equals the corresponding extremal integral in Stoev and Taqqu (2005), provided that the integrand $g \geq 0$ is a simple function.

3 Extension of the integral

Recall that the sup-measure M as well as the underlying ingredients f, α, κ and m are fixed throughout. We want to start with a definition that appears not only general, but also natural in our context. In addition, it follows former examples in literature, which also deal with stochastic integrals (see Kremer and Scheffler (2019), Rajput and Rosinski (1989), Samoradnitsky and Taqqu (1994) and Stoev and Taqqu (2005) again). However, we will restrict ourselves to the consideration of $\mathbb{R}_+$ -valued functions as integrands.

Definition 3.1 A measurable function $g : E \rightarrow \mathbb{R}_+$ is called *integrable with respect to M* (shortly: *M -integrable*) if there exists a sequence of simple functions $(g_n)_n$ fulfilling $g_n \uparrow g$ such that the sequence $(I(g_n))_n$ converges in probability (on $\mathbb{R}^d$). Here, $g_n \uparrow g$ means that $g_n(s) \leq g_{n+1}(s)$ for every $n \in \mathbb{N}$ and $s \in E$ together with $\sup_{n \in \mathbb{N}} g_n(s) = \lim_{n \rightarrow \infty} g_n(s) = g(s)$. Finally, let $\mathcal{I}(M)$ denote the set of all functions $g : E \rightarrow \mathbb{R}_+$ that are M -integrable.

The main aim of this section is to answer two questions that immediately arise from the previous definition:

- (1) Which functions belong to $\mathcal{I}(M)$?
- (2) Given a function $g \in \mathcal{I}(M)$. Does the stochastic limit $I(g) := \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g_n)$ depend on the choice of (g_n) ? And if not, what are the properties of $I(g)$? (This will be mostly the subject of Section 4.)

We have seen in Definition 2.4 that the integral for simple functions essentially uses the $\vee_f$ -operation. However, the pursued extension will also benefit from an operation that is quite related and that we will introduce now.

Definition 3.2 For $k \geq 2$ and $x_1, \dots, x_k \in \mathbb{R}^d$ arbitrary let $j_0 \in \{1, \dots, k\}$ be the index such that $x_1 \vee_f \dots \vee_f x_k = x_{j_0}$. Then we define

$$\bigvee_{f, j=1}^k x_j := \bigvee_{f, 1 \leq j \neq j_0 \leq k} x_j.$$

The following observation combines both operations from a probabilistic point of view and, at the same time, reveals aspects from classical EVT. Therefore, recall (2.1).

Lemma 3.3 Assume that $X_1, \dots, X_k$ are $\mathbb{R}^d$ -valued and independent random vectors, where $f(X_j) \sim \Phi_\alpha(\sigma_j)$ with scale $\sigma_j \geq 0$ for $j = 1, \dots, k$. Then we have the following:

$$\forall \gamma > 0: \quad \mathbb{P} \left(f \left(\bigvee_{j=1}^k X_j \right) \leq (1 + \gamma) f \left(\bigvee_{j=1}^k X_j \right) \right) \leq 1 - (1 + \gamma)^{-\alpha}.$$

Proof Without loss of generality we can assume that $\sigma_j > 0$ for $j = 1, \dots, k$. To start with a general observation, let Y_1 and Y_2 be independent random variables, where $Y_i \sim \Phi_\alpha(\rho_i)$ with scale $\rho_i > 0$ for $i = 1, 2$. Fix $\gamma > 0$. Then a standard calculation, using the substitution $1 - \hat{\gamma} = (1 + \gamma)^{-\alpha}$, shows that

$$\begin{aligned} \mathbb{P}(Y_1 \leq Y_2 \leq (1 + \gamma)Y_1) &= \int_0^\infty [e^{-\rho_2^\alpha (1+\gamma)^{-\alpha} x^{-\alpha}} - e^{-\rho_2^\alpha x^{-\alpha}}] \alpha \rho_1^\alpha x^{-\alpha-1} e^{-\rho_1^\alpha x^{-\alpha}} dx \\ &= \frac{\rho_1^\alpha}{\rho_1^\alpha + (1 + \gamma)^{-\alpha} \rho_2^\alpha} - \frac{\rho_1^\alpha}{\rho_1^\alpha + \rho_2^\alpha} \end{aligned} \quad (3.1)$$

$$\begin{aligned} &= \frac{\hat{\gamma} \rho_2^\alpha}{\rho_1^\alpha + \rho_2^\alpha} \times \frac{\rho_1^\alpha}{\rho_1^\alpha + (1 - \hat{\gamma}) \rho_2^\alpha} \\ &\leq \frac{\hat{\gamma} \rho_2^\alpha}{\rho_1^\alpha + \rho_2^\alpha}. \end{aligned} \quad (3.2)$$

Now let $Z^{(i)} = \max_{1 \leq j \neq i \leq k} f(X_j)$ for $1 \leq i \leq k$. Then we see on the one hand that $Z^{(i)}$ is independent from $f(X_i)$. On the other hand, $Z^{(i)}$ is also α -Fréchet distributed, namely with scale $(\sum_{1 \leq j \neq i \leq k} \sigma_j^\alpha)^{1/\alpha}$. Moreover, note that

$$\left\{ f \left(\bigvee_{j=1}^k X_j \right) \leq (1 + \gamma) f \left(\bigvee_{j=1}^k X_j \right) \right\} = \bigcup_{i=1}^k \{Z^{(i)} \leq f(X_i) \leq (1 + \gamma)Z^{(i)}\} \quad (3.3)$$

holds true. Then, for every $i \in \{1, \dots, k\}$, we can apply (3.2) to $Y_1 = Z^{(i)}$ and $Y_2 = f(X_i)$, respectively. Using (3.3), this gives the assertion, since

$$\sum_{i=1}^k \frac{\hat{\gamma} \sigma_i^\alpha}{\sum_{1 \leq j \neq i \leq k} \sigma_j^\alpha + \sigma_i^\alpha} = \hat{\gamma} = 1 - (1 + \gamma)^{-\alpha}.$$

□

Remember that (2.4) does not depend on a particular representation for g . However, a similar approach using the operation from Definition 3.2 would lead to serious problems. In order to overcome them, we need some additional notation.

- Definition 3.4** (a) Let g be a simple function with representation $g = \sum_{j \in J} \alpha_j \mathbb{1}_{A_j}$ for some finite index set J , where $A_j \in \mathcal{E}_0$ are disjoint and where $\alpha_j \geq 0$ for every $j \in J$. Although we do not claim that $\cup_{j \in J} A_j = E$, we call $\{A_j : j \in J\}$ a *partition* (of g) in this case and write $g \sim (A_j, \alpha_j)_{j \in J}$.
- (b) Assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions. Then we write $\mathcal{P}_1 \leq \mathcal{P}_2$ if the following holds true: Any set from $\mathcal{P}_1$ can be represented by an appropriate union over sets belonging to $\mathcal{P}_2$.
- (c) Consider a sequence (g_n) of simple functions. Then a corresponding sequence of representations

$$g_n(s) = \sum_{j=1}^{k_n} \alpha_j^{(n)} \mathbb{1}_{A_j^{(n)}}(s), \quad s \in E \quad (3.4)$$

is called *consistent* if $\mathcal{P}_n \leq \mathcal{P}_{n+1}$, where $\mathcal{P}_n := \{A_1^{(n)}, \dots, A_{k_n}^{(n)}\}$ for every $n \in \mathbb{N}$.

Note that the following remark, in particular part (b), fixes the problem that we addressed above Definition 3.4. Anyway, its proofs are easy and therefore left to the reader.

Remark 3.5 (a) Suppose that g_1, g_2 are simple functions that can be represented by $g_i \sim (A_j^{(i)}, \alpha_j^{(i)})_{j=1, \dots, k_i}$ with $\mathcal{P}_i = \{A_1^{(i)}, \dots, A_{k_i}^{(i)}\}$ for $i = 1, 2$. Then we can always find a *common partition* $\mathcal{P}$, which fulfills $\mathcal{P}_1, \mathcal{P}_2 \leq \mathcal{P}$. More precisely, define

$$A_0^{(1)} := \bigcup_{j=1}^{k_2} A_j^{(2)} \setminus \bigcup_{j=1}^{k_1} A_j^{(1)}, \quad A_0^{(2)} := \bigcup_{j=1}^{k_1} A_j^{(1)} \setminus \bigcup_{j=1}^{k_2} A_j^{(2)}$$

and let $\alpha_0^{(1)} = \alpha_0^{(2)} = 0$. Hence, we observe that

$$\mathcal{P} = \{A_{j_1}^{(1)} \cap A_{j_2}^{(2)} : 0 \leq j_1 \leq k_1, 0 \leq j_2 \leq k_2\}$$

has the desired properties, since we can write

$$g_i(s) = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \alpha_{j_i}^{(i)} \mathbb{1}_{A_{j_1}^{(1)} \cap A_{j_2}^{(2)}}(s) \quad \text{for every } s \in E \text{ and } i = 1, 2.$$

- (b) Consider a sequence (g_n) of simple functions and assume that a consistent sequence of representations is given by (3.4), which is always possible due to part (a). Moreover, assume that $g_n \uparrow$, which means that the sequence (g_n) itself is increasing. Then, using (RM2), it follows for every $n \in \mathbb{N}$ that

$$f \left(\bigvee_{j=1}^{k_n^*} \alpha_j^{(n)} M(A_j^{(n)}) \right) \leq f \left(\bigvee_{j=1}^{k_{n+1}^*} \alpha_j^{(n+1)} M(A_j^{(n+1)}) \right) \quad \text{a.s.}$$

Equipped with the previous observations, we are now able to enhance the idea of Lemma 3.3. Note that, for real-valued functions $g, g_1, g_2, \dots$ on E , we shortly write

$g_n \leq g$, provided that $g_n(s) \leq g(s)$ holds true for every $n \in \mathbb{N}$ and $s \in E$. Also recall (1.5).

Proposition 3.6 *Let (g_n) be a sequence of simple functions fulfilling $g_n \leq g$ for some $g \in L_+^\alpha(m)$ and assume that a consistent sequence of representations is given by (3.4). Define*

$$X_n := I(g_n) = \bigvee_{j=1}^{k_n} \alpha_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad X_n^* := \bigvee_{j=1}^{k_n} \alpha_j^{(n)*} M(A_j^{(n)}).$$

Moreover, assume that there exist further sequences $(h_{1,n}), (h_{2,n})$ of simple functions such that $h_{1,n} \leq g_n \leq h_{2,n}$ and $h_{i,n} \uparrow g$ for $i = 1, 2$ as $n \rightarrow \infty$. Then, for any $\varepsilon > 0$, there exist a set $A \in \mathcal{A}$ with $\mathbb{P}(A) \geq 1 - \varepsilon$ as well as some $\delta > 0$ and $N \in \mathbb{N}$ such that we have

$$f(X_n)(\omega) \geq (1 + \delta)f(X_n^*)(\omega) \quad \text{for every } n \geq N \text{ and } \omega \in A. \quad (3.5)$$

The proof of this result is given in the [Appendix](#). Anyway, a reformulation of Proposition 3.6 essentially states that we observe *gaps* behind the attained maxima, which appear with a demanded probability and where the size of these gaps depends on the given probability. We will now try to benefit from these gaps and, therefore, handle some of the troubles that can be caused by the $\vee_f$ -operation. Recall the set S from Definition 2.1.

Lemma 3.7 *Consider $\alpha_j, \beta_j > 0$ and $x_j \in \mathbb{R}^d$ for $j = 1, \dots, k$ and some $k \in \mathbb{N}$, where $\gamma := \min\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k\}$ and $\rho := \max\{|\alpha_j - \beta_j| : j = 1, \dots, k\}$. Let*

$$\zeta := \bigvee_{j=1}^k \alpha_j x_j \quad \text{and} \quad \zeta^* := \bigvee_{j=1}^k \alpha_j^* x_j.$$

Furthermore, assume that there exists some $\delta > 0$ such that we have $f(\zeta) \geq (1 + \delta)f(\zeta^)$. Then, provided that $\rho < \gamma(\sqrt{1 + \delta} - 1)$, the following relation holds true:*

$$\left\| \bigvee_{j=1}^k \alpha_j x_j - \bigvee_{j=1}^k \beta_j x_j \right\| \leq C \rho \max_{j=1, \dots, k} f(x_j),$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$ and where $C := \max\{\|x\| : x \in S\}$.

Proof The case $\zeta = 0$ is equivalent to $x_1 = \dots = x_k = 0$ and therefore obvious. Else let j_0 be the index fulfilling $\zeta = \alpha_{j_0} x_{j_0} > 0$, which particularly implies that $f(x_{j_0}) > 0$. We generally note that $\|x\| = f(x)\|x/f(x)\|$ for every $x \in \mathbb{R}^d \setminus \{0\}$. This means that we have $\|x\| \leq C f(x)$ for every $x \in \mathbb{R}^d$. Thus, since $\|\cdot\|$ is symmetric and since f is 1-homogeneous, the assertion would follow if we could

show that $f(\beta_1 x_1) \vee_f \cdots \vee_f f(\beta_k x_k) = \beta_{j_0} x_{j_0}$. For this purpose, we first observe that

$$\frac{\alpha_j}{\beta_j} = 1 + \frac{\alpha_j - \beta_j}{\beta_j} \leq 1 + \frac{\rho}{\gamma} \quad \text{and} \quad \frac{\beta_j}{\alpha_j} \leq 1 + \frac{\rho}{\gamma} \quad (j = 1, \dots, k)$$

hold true. Fix $j \neq j_0$. Then, using the given assumptions, we obtain that

$$\begin{aligned} f(\beta_j x_j) &\leq \left(1 + \frac{\rho}{\gamma}\right) f(\alpha_j x_j) \leq \left(1 + \frac{\rho}{\gamma}\right) (1 + \delta)^{-1} f(\alpha_{j_0} x_{j_0}) \\ &\leq \left(1 + \frac{\rho}{\gamma}\right)^2 (1 + \delta)^{-1} f(\beta_{j_0} x_{j_0}) < f(\beta_{j_0} x_{j_0}), \end{aligned}$$

which completes the proof. Note that we benefited from $f(x_{j_0}) > 0$ in the last step. $\square$

The next result is the main step in order to extend the definition of the f -implicit extremal integral. Its proof uses an auxiliary result (namely Lemma 6.1) and both can be found in the [Appendix](#)

Proposition 3.8 *Assume that $g \in L_+^\alpha(m)$ and that (g_n) is a sequence of simple functions fulfilling $g_n \leq g$ together with $g_n(s) \rightarrow g(s)$ for m -a.e. $s \in E$. Then there exists a sequence of increasing sets $E_1, E_2, \dots \in \mathcal{E}_0$ with $m(E \setminus \bigcup_{l=1}^\infty E_l) = 0$ and such that, for any $l \in \mathbb{N}$, the sequence $(I(g_n \mathbb{1}_{E_l}))$ converges in probability as $n \rightarrow \infty$.*

Before putting things together, we have to deal with the remainder $g_n \mathbb{1}_{E_l^c} := g_n \mathbb{1}_{E \setminus E_l}$. Note that a proof of the following result is also given in the [Appendix](#).

Lemma 3.9 *In the situation of Proposition 3.8 we have the following: For any $\varepsilon > 0$ there exists some $L = L(\varepsilon)$ such that*

$$\forall n, l \geq L : \quad \mathbb{P}(I(g_n) \neq I(g_n \mathbb{1}_{E_l})) \leq \varepsilon. \quad (3.6)$$

Finally, we are able to answer the questions from the beginning of this section. In this context, recall Definition 3.1 and notice that (3.7) below will respect the definition of $I(g)$ so far (see (2.4)). Also note that the proof of part (b) of the following result benefits from the fact that we stated Proposition 3.8 in an extensive way. That is we did not demand the convergence $g_n \rightarrow g$ to be monotone in the first place.

Theorem 3.10 *We have the following:*

- (a) $\mathcal{I}(M) = L_+^\alpha(m)$, which is independent of f .
- (b) Assume that $g \in L_+^\alpha(m)$ and that (g_n) is a sequence of simple functions fulfilling $g_n \uparrow g$. Then the sequence $(I(g_n))$ converges in probability and this limit does a.s. not depend on the particular choice of simple functions (g_n) with $g_n \uparrow g$.

We call this limit the (f -implicit extremal) integral of g (with respect to M) and write

$$I_M^{\vee_f}(g) := I(g) := \int_E^{\vee_f} g(s) dM(s) := \int_E^{\vee_f} g(s) M(ds) := \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g_n). \quad (3.7)$$

Proof We first prove part (b): Fix $\varepsilon > 0$ and consider increasing sets $E_1, E_2, \dots \in \mathcal{E}_0$ as provided by Proposition 3.8. According to Lemma 3.9 there exists some L such that we have $\mathbb{P}(I(g_n) \neq I(g_n \mathbb{1}_{E_l})) \leq \varepsilon/3$ for every $n, l \geq L$. At the same time, Proposition 3.8 states that $(I(g_n \mathbb{1}_{E_l}))_n$ is Cauchy (in probability), i.e. we can find some N fulfilling

$$\mathbb{P}(\|I(g_m \mathbb{1}_{E_L}) - I(g_n \mathbb{1}_{E_L})\| \geq \varepsilon/3) \leq \varepsilon/3 \quad \text{for every } m, n \geq N.$$

Note that the event $\{\|I(g_m) - I(g_n)\| \geq \varepsilon\}$ is a subset of

$$\{\|I(g_m \mathbb{1}_{E_L}) - I(g_n \mathbb{1}_{E_L})\| \geq \varepsilon/3\} \cup \bigcup_{i \in \{m, n\}} \{\|I(g_i) - I(g_i \mathbb{1}_{E_L})\| \geq \varepsilon/3\}. \quad (3.8)$$

Hence, for every $m, n \geq \max\{L, N\}$, we easily conclude that $\mathbb{P}(\|I(g_m) - I(g_n)\| \geq \varepsilon) \leq \varepsilon$, which, as in the proof of Proposition 3.8 (see the [Appendix](#)), shows that $(I(g_n))$ converges in probability. Denote this limit by X and consider a different sequence of simple functions (g'_n) , still fulfilling $g'_n \uparrow g$. Then, repeating the previous ideas, we obtain that $(I(g'_n))$ converges in probability, say with limit X' . Define a further sequence of simple functions $(h_v)_{v \in \mathbb{N}}$ by $h_{2n-1} := g_n$ and $h_{2n} := g'_n$ for every $n \in \mathbb{N}$, respectively. In particular, we observe that Proposition 3.8 as well as Lemma 3.9 still apply to (h_v) such that $(I(h_v))$ converges, too. However, by regarding suitable subsequences, it follows that X and X' coincide a.s.

Concerning part (a), consider $g \in L_+^\alpha(m)$ and choose a sequence (g_n) of simple functions with $g_n \uparrow g$ (see section 2.3 in Stoev and Taqqu (2005) for example to verify that such a sequence always exists). By what we have just proved, it follows that $g \in \mathcal{I}(M)$ and therefore that $L_+^\alpha(m) \subset \mathcal{I}(M)$. Conversely, fix $g \in \mathcal{I}(M)$ and let (g_n) be a proper sequence of simple functions in the sense of Definition 3.1. Denote the associated stochastic limit of $(I(g_n))$ by Y and observe that we have $\|g_n\|_\alpha \uparrow \|g\|_\alpha \in [0, \infty]$ by the monotone convergence theorem. At the same time,

$$\forall x > 0: \quad \mathbb{P}(f(I(g_n)) \leq x) = \exp(-\|g_n\|_\alpha^\alpha x^{-\alpha}) \quad (3.9)$$

holds true, while Proposition 3.2.4 in Goldbach (2016) implies that $(f(I(g_n)))$ is increasing a.s. However, by the continuous mapping theorem, the corresponding limit coincides with $f(Y)$ a.s. In view of (3.9) and since $f(Y)$ is $[0, \infty)$ -valued, it is easy to check that $\|g\|_\alpha < \infty$, i.e. we have that $g \in L_+^\alpha(m)$ and therefore that $L_+^\alpha(m) \supset \mathcal{I}(M)$. $\square$

4 Fundamental properties

Based on Theorem 3.10, it appears natural to study properties of the mapping $\mathcal{I}(M) \ni g \mapsto I(g)$ in the sequel. Actually, we already encountered some of them, for example in the proof of Lemma 3.9 (see (5.23)). A closer look on (5.23) reveals that, at least for simple functions g , the stochastic integral manages to overcome some of the problems that occur in the context of the $\vee_f$ -operation. It will be crucial to gain a corresponding insight for functions $g \in L_+^\alpha(m)$. Therefore, we start with the following preparation, whose proof is given in the [Appendix](#).

Lemma 4.1 *Let $g_1, g_2 \in L_+^\alpha(m)$ such that $g_1 \leq g_2$. Then $I(g_1) \vee_f I(g_2) = I(g_2) \vee_f I(g_1)$ holds true a.s.*

As announced before already, we want to proceed with some illuminating properties of the f -implicit extremal integral that mostly extend from the consideration of simple functions. In this context, recall from Goldbach (2016) the *partial order* $\leq_f$ on $\mathbb{R}^d$, being defined by

$$x \leq_f y \quad :\Leftrightarrow \quad f(x) < f(y) \text{ or } x = y. \quad (4.1)$$

See Proposition 1.3.2 and Lemma 1.3.3 in Goldbach (2016) for several properties concerning this binary relation. Also note that the proof of Lemma 4.1, in particular the set A from (5.26), already anticipated this relation to some extent. This means that we have $x \leq_f y$ or $y \leq_f x$ if and only if $x \vee_f y = y \vee_f x$.

Theorem 4.2 *Let $g_1, g_2 \in L_+^\alpha(m)$.*

- (i) (*f -implicit α -Fréchet*) *The random vector $I(g_1)$ is f -implicit α -Fréchet distributed in the sense of Definition 2.1. More precisely, $I(g_1) \sim \Phi_{\alpha, \kappa}^f(\|g_1\|_\alpha)$.*
- (ii) (*f -implicit max-linearity*) *For $a, b \geq 0$ we have that*

$$I(ag_1 \vee bg_2) = aI(g_1) \vee_f bI(g_2) \quad \text{a.s.}, \quad (4.2)$$

which particularly means that $I(g_1)$ and $I(g_2)$ commute under the $\vee_f$ -operation.

- (iii) (*f -implicit independence*) *The random vectors $I(g_1)$ and $I(g_2)$ are independent if and only if $g_1 g_2 = 0$ m-a.e.*
- (iv) (*f -implicit monotonicity*) *We have: $g_1 \leq g_2$ m-a.e. if and only if $I(g_1) \leq_f I(g_2)$ a.s. In addition, $g_1 = g_2$ m-a.e. is equivalent to $I(g_1) = I(g_2)$ a.s.*

Proof For simple functions g_1 and g_2 , the whole statement follows from Proposition 3.2.4 and Corollary 3.2.5 in Goldbach (2016), respectively. We will use this fact without explicit reference in the sequel. Moreover and without loss of generality, we can assume that $\|g_i\|_\alpha > 0$ for $i = 1, 2$. Throughout let $(g_{1,n})$ and $(g_{2,n})$ be sequences of simple functions such that $g_{i,n} \uparrow g_i$ for $i = 1, 2$ and $n \rightarrow \infty$. In particular, Theorem 3.10 states that

$$I(g_{i,n}) \rightarrow I(g_i) \quad \text{in probability} \quad (\text{for } i = 1, 2 \text{ as } n \rightarrow \infty). \quad (4.3)$$

- (i) Since we have that $I(g_{1,n}) \sim \Phi_{\alpha, \kappa}^f(\|g_{1,n}\|_\alpha)$, while $\|g_{1,n}\|_\alpha \uparrow \|g_1\|_\alpha$ by the monotone convergence theorem, the assertion follows from (4.3) by passing through the limit.
- (ii) Obviously, the homogeneity property $I(ag_1) = aI(g_1)$ extends from the consideration of simple functions to the present case. Therefore it suffices to consider the case $a = b = 1$ in the following. Then, since $g_{1,n} \vee g_{2,n} \uparrow g_1 \vee g_2 \in L_+^\alpha(m)$, we derive from Theorem 3.10 together with the accuracy of (4.2) for simple functions that

$$I(g_1 \vee g_2) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g_{1,n} \vee g_{2,n}) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} (I(g_{1,n}) \vee_f I(g_{2,n})).$$

At this point, recall (4.3) and note that $I(g_{1,n}) \vee_f I(g_{2,n}) \in \{I(g_{1,n}), I(g_{2,n})\}$ for every $n \in \mathbb{N}$. Hence, we need that the $\vee_f$ -operation is continuous, which is not true in general (see Example 1.1.4 in Goldbach (2016)). However, in order to ensure continuity in our situation (and therefore to obtain the assertion), we merely need that

$$I(g_1) \vee_f I(g_2) = I(g_2) \vee_f I(g_1) \quad \text{a.s.} \quad (4.4)$$

To prove (4.4), we first consider the case $g_1 g_2 = 0$ m -a.e. Then, without loss of generality, we can also assume that $g_{1,n} g_{2,n} = 0$ holds true m -a.e., which means that $I(g_{1,n})$ and $I(g_{2,n})$ are independent for every $n \in \mathbb{N}$. On the one hand, it is clear that the corresponding stochastic limits, namely $I(g_1)$ and $I(g_2)$, preserve this property. Hence, $f(I(g_1))$ and $f(I(g_2))$ are independent, too. At the same time, we have that $f(I(g_i)) \sim \Phi_\alpha(\|g_i\|_\alpha)$ due to part (i), which means that $f(I(g_1)) \neq f(I(g_2))$ a.s. (essentially use (3.2) for $\gamma = 0$). In particular, (4.4) is fulfilled, provided that $g_1 g_2 = 0$ m -a.e. Moreover, Lemma 4.1 states that (4.4) is still true as long as we have $g_1 \leq g_2$. Finally, writing $g_i = g_i \mathbb{1}_{g_1 \leq g_2} \vee \mathbb{1}_{g_1 > g_2}$ for general $g_i \in L_+^\alpha(m)$ and using the associativity of the $\vee_f$ -operation, we can combine both observations to derive that

$$\begin{aligned} I(g_1) \vee_f I(g_2) &= I(g_1 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 > g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 > g_2}) \\ &= I(g_1 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 > g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 > g_2}) \\ &= I(g_2 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 > g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 > g_2}) \\ &= I(g_2 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_2 \mathbb{1}_{g_1 > g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 \leq g_2}) \vee_f I(g_1 \mathbb{1}_{g_1 > g_2}) \\ &= I(g_2) \vee_f I(g_1) \end{aligned}$$

holds true a.s., which shows (4.4).

- (iii) The *if*-part turns out to be a by-product of the proof of part (ii) before. Conversely, assume that $I(g_1)$ and $I(g_2)$ are independent. Then, although g_1 and g_2 are not necessarily simple functions, properties (i) and (ii) allow to imitate the according part in the proof of Proposition 3.2.4 (iv) in Goldbach (2016) to conclude that $g_1 g_2 = 0$ m -a.e.
- (iv) Let us first prove the *only if*-part, where we can assume that $g_{1,n} \leq g_{2,n}$ again. It follows that $f(I(g_{1,n})) \leq f(I(g_{2,n}))$ a.s. and hence that $f(I(g_1)) \leq f(I(g_2))$ a.s. In view of (4.1) this would already imply that $I(g_1) \leq_f I(g_2)$ a.s., provided that $\mathbb{P}(A) = 0$ holds true, where the set A is defined as in (5.26). Actually, this was just the outcome of the proof of Lemma 4.1.

Conversely, if $I(g_1) \leq_f I(g_2)$ a.s., we can exactly use the idea that has been presented in the proof of Proposition 3.2.4 (iii) in Goldbach (2016) to obtain that $g_1 \leq g_2$ m -a.e., since this only uses the properties (i) and (ii) again. For the additional statement of part (iv), merely note the following observation:

$$\forall x, y \in \mathbb{R}^d : \quad x = y \quad \Leftrightarrow \quad x \leq_f y \quad \text{and} \quad y \leq_f x.$$

□

The next result characterizes the convergence in probability of the occurring stochastic integrals, namely in terms of the corresponding kernel functions that belong to $L_+^\alpha(m)$. As a by-product, it also shows that any sequence of approximating functions (g_n) can be used in (3.7) to reach $I(g)$ as long as one of the conditions in (4.5) below holds true. More precisely, we are neither restricted to simple functions nor to monotone sequences anymore.

Theorem 4.3 *Consider $g, g_1, g_2, \dots \in L_+^\alpha(m)$. Then, as $n \rightarrow \infty$, we have:*

$$I(g_n) \xrightarrow{\mathbb{P}} I(g) \Leftrightarrow \int_E |g_n^\alpha - g^\alpha| dm \rightarrow 0 \Leftrightarrow \int_E |g_n - g|^\alpha dm \rightarrow 0. \quad (4.5)$$

Before proving Theorem 4.3, we need two auxiliary results, where the first one deals with the deliverance from monotone sequences that we announced already before. Anyway, both proofs can be found in the [Appendix](#).

Lemma 4.4 *Consider $g \in L_+^\alpha(m)$ and assume that (g_n) is a sequence of simple functions fulfilling $g_n \leq g$ together with $g_n(s) \rightarrow g(s)$ for m -a.e. $s \in E$. Then we obtain that $I(g_n) \rightarrow I(g)$ in probability as $n \rightarrow \infty$.*

Note that the following observation could be stated in a more general framework. However, by doing it this way, it will allow an easy application within the proof of Theorem 4.3 below.

Lemma 4.5 *Let $(h_{1,n}) \subset L_+^\alpha(m)$ fulfill $X = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(h_{1,n})$ for some random vector X . Assume that $(h_{2,n})$ is a further sequence of functions such that we have $0 \leq h_{2,n} \leq \gamma_n \mathbb{1}_A$ for every $n \in \mathbb{N}$, where $A \in \mathcal{E}_0$ and where $(\gamma_n) \subset \mathbb{R}_+$ converges to zero. Then we have that $I(h_{1,n} \vee h_{2,n}) \rightarrow X$ in probability, too (as $n \rightarrow \infty$).*

Proof of Theorem 4.3 Note that the last-mentioned equivalence in (4.5) corresponds to Lemma 2.3 in Stoev and Taqqu (2005). Hence, if we recall (1.5), the present proof reduces to the following:

$$I(g_n) \xrightarrow{\mathbb{P}} I(g) \Leftrightarrow \|g_n^\alpha - g^\alpha\|_1 \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \quad (4.6)$$

Observe that, in the case $\|g\|_\alpha = 0$, we have $I(g) = 0$ a.s. together with $\|g_n^\alpha - g^\alpha\|_1 = \|g_n^\alpha\|_1$. Since Theorem 4.3 implies that $I(g_n) \sim \Phi_{\alpha, \kappa}^f(\|g_n\|_\alpha)$ (see (2.2)), it is easy to verify that (4.6) holds true in this case. Thus, let us assume that $0 < \|g\|_\alpha < \infty$ in the sequel.

Then, in order to prove (4.6), we first suppose that $I(g_n) \rightarrow I(g)$ in probability, which also implies that $f(I(g_n)) \rightarrow f(I(g))$ by the continuous mapping theorem. Recall that $f(I(g_n)) \sim \Phi_\alpha(\|g_n\|_\alpha)$ and $f(I(g)) \sim \Phi_\alpha(\|g\|_\alpha)$, respectively. Hence, a combination of (2.6) and (4.2) shows that

$$f(I(g_n)) \vee f(I(g)) = f(I(g_n) \vee_f I(g)) = f(I(g_n \vee g)) \quad \text{a.s.}$$

Accordingly, it follows that $f(I(g_n) \vee f(I(g))) \sim \Phi_\alpha(\|g_n \vee g\|_\alpha)$. Based on this, we can mostly follow the proof of Theorem 2.1 in Stoev and Taqqu (2005) to obtain that $\|g_n^\alpha - g^\alpha\|_1 \rightarrow 0$. The details are left to the reader.

Conversely, assume that $\|g_n^\alpha - g^\alpha\|_1 \rightarrow 0$ holds true. Unfortunately, this merely implies that there exists a suitable subsequence along which g_n converges to g (m -a.e.). At the same time, it only remains to prove that $I(g_n) \rightarrow I(g)$ in probability, which can be characterized in the following way (use Theorem 20.5 in Billingsley (2008) for instance): Each subsequence of $(I(g_n))$ contains a further subsequence that converges to $I(g)$ in probability. Keeping this in mind, the following steps will reveal that, without loss of generality, we can already assume that $g_n \rightarrow g$ m -a.e as $n \rightarrow \infty$. Now fix $\varepsilon > 0$. Then we can use Egorov's Theorem again (see the proof of Proposition 3.8 in the Appendix) to obtain a sequence of increasing sets $E'_1, E'_2, \dots \in \mathcal{E}$ with $m(E \setminus \bigcup_{l=1}^\infty E'_l) = 0$ and such that, for every $l \in \mathbb{N}$, the convergence $g_n \mathbb{1}_{E'_l} \rightarrow g \mathbb{1}_{E'_l}$ holds uniformly. Clearly, the previous observation remains true for $E_1, E_2, \dots$ (instead of $E'_1, E'_2, \dots$), defined by

$$E_l := E'_l \cap (\{1/l \leq g \leq l\} \cup \{g = 0\}), \quad l \in \mathbb{N}.$$

Note that $\|g_n\|_\alpha^\alpha \rightarrow \|g\|_\alpha^\alpha > 0$ by assumption. Using this together with Theorem 4.2 and the fact that, for any $n, l \in \mathbb{N}$, the estimation

$$\|g_n \mathbb{1}_{E_l^\complement}\|_\alpha^\alpha \leq \|g_n^\alpha - g^\alpha\|_1^\alpha + \|g \mathbb{1}_{E_l^\complement}\|_\alpha^\alpha$$

is valid, we can argue as in the proof of Lemma 3.9 (see the Appendix) to verify that (3.6) is fulfilled accordingly. Obviously, the argument includes the consideration of the function g . More precisely, if we let $g_0 := g$ for the moment, there exists some $L = L(\varepsilon)$ such that

$$\forall n \in \{0, L, L+1, \dots\} \quad \forall l \geq L : \quad \mathbb{P}(I(g_n) \neq I(g_n \mathbb{1}_{E_l})) \leq \varepsilon/3.$$

If we use the triangular inequality (compare (3.8)), it follows for any $n \geq L$ that

$$\mathbb{P}(\|I(g_n) - I(g)\| \geq \varepsilon) \leq \frac{2\varepsilon}{3} + \mathbb{P}(\|I(g_n \mathbb{1}_{E_L}) - I(g \mathbb{1}_{E_L})\| \geq \varepsilon/3).$$

Hence, as already argued elsewhere, it suffices to show that the following relation holds true:

$$\mathbb{P}(\|I(g_n \mathbb{1}_{E_L}) - I(g \mathbb{1}_{E_L})\| \geq \varepsilon/3) \leq \frac{\varepsilon}{3} \quad \text{for almost all } n. \quad (4.7)$$

For every $n \in \mathbb{N}$, let $(g_{n,v})_v$ be a sequence of simple functions with $g_{n,v} \uparrow g_n$ and such that the convergence $g_{n,v} \mathbb{1}_{E_L} \rightarrow g_n \mathbb{1}_{E_L}$ holds uniformly (as $v \rightarrow \infty$). Note that this is possible, since $g \mathbb{1}_{E_L} \leq L$, which means that $g_n \mathbb{1}_{E_L}$ is also bounded (at least for almost all n). Recall the notation $\|\cdot\|_\infty$ from (5.19). Then, according to Theorem 3.10, we can even find a strictly increasing sequence $(v(n))_n$ of naturals such that, for those $n \in \mathbb{N}$,

$$\|g_n \mathbb{1}_{E_L} - g_{n,v(n)} \mathbb{1}_{E_L}\|_\infty \leq 1/n \quad \text{and} \quad \mathbb{P}(\|I(g_n \mathbb{1}_{E_L}) - I(g_{n,v(n)} \mathbb{1}_{E_L})\| \geq 1/n) \leq 1/n$$

hold true. In other words, as $n \rightarrow \infty$, we have that

$$\|g_n \mathbb{1}_{E_L} - g_{n,v(n)} \mathbb{1}_{E_L}\|_\infty \rightarrow 0 \quad \text{and} \quad I(g_n \mathbb{1}_{E_L}) - I(g_{n,v(n)} \mathbb{1}_{E_L}) \xrightarrow{\mathbb{P}} 0. \quad (4.8)$$

In addition, we define the sequence (η_n) by

$$\eta_n := \frac{L^{-1}}{L^{-1} + \|g_n \mathbb{1}_{E_L} - g \mathbb{1}_{E_L}\|_\infty} \in (0, 1]$$

and observe that, for every $n \in \mathbb{N}$ and $s \in E_L \setminus \{g = 0\}$, the following calculation is valid:

$$g_n(s) \leq \frac{g(s) + \|g_n \mathbb{1}_{E_L} - g \mathbb{1}_{E_L}\|_\infty}{g(s)} g(s) \leq \left(1 + \frac{\|g_n \mathbb{1}_{E_L} - g \mathbb{1}_{E_L}\|_\infty}{L^{-1}}\right) g(s) = \eta_n^{-1} g(s).$$

In view of $g_{n,v(n)} \leq g_n$ this shows that $h_{1,n} := \eta_n g_{n,v(n)} \mathbb{1}_{E_L \setminus \{g=0\}}$ defines a simple function fulfilling $h_{1,n} \leq g \mathbb{1}_{E_L}$ for every $n \in \mathbb{N}$. Combine (4.8) with $\|g_n \mathbb{1}_{E_L} - g \mathbb{1}_{E_L}\|_\infty \rightarrow 0$ (see above) and note that $\eta_n \rightarrow 1$ to verify that $\|h_{1,n} - g \mathbb{1}_{E_L}\|_\infty \rightarrow 0$. In particular, Lemma 4.4 implies that $I(h_{1,n}) \rightarrow I(g \mathbb{1}_{E_L})$ in probability. Finally, let $h_{2,n} := \eta_n g_{n,v(n)} \mathbb{1}_{E_L \cap \{g=0\}}$ and observe that

$$h_n := h_{1,n} \vee h_{2,n} = \eta_n g_{n,v(n)} \mathbb{1}_{E_L}, \quad n \in \mathbb{N}. \quad (4.9)$$

Using $g_{n,v(n)} \leq g_n$ again, we also conclude that $h_{2,n} \leq \|g_n \mathbb{1}_{E_L} - g \mathbb{1}_{E_L}\|_\infty \mathbb{1}_{E_L \cap \{g=0\}}$ holds true. Hence, the assumptions of Lemma 4.5 are fulfilled and we obtain that $I(h_n) \rightarrow I(g \mathbb{1}_{E_L})$ in probability. Finally, we benefit from the intimate relation between $g_{n,v(n)}$ and $h(n)$. More precisely, (4.9) and the homogeneity of the stochastic integral lead to the fact that

$$I(g_{n,v(n)} \mathbb{1}_{E_L}) - I(h_n) = (\eta_n^{-1} - 1)I(h_n) \rightarrow 0 \cdot I(g \mathbb{1}_{E_L}) = 0$$

in probability as $n \rightarrow \infty$. If we write

$$I(g_n \mathbb{1}_{E_L}) - I(g \mathbb{1}_{E_L}) = I(g_n \mathbb{1}_{E_L}) - I(g_{n,v(n)} \mathbb{1}_{E_L}) + I(g_{n,v(n)} \mathbb{1}_{E_L}) - I(h_n) + I(h_n) - I(g \mathbb{1}_{E_L})$$

and use the previous outcome, it follows that $I(g_n \mathbb{1}_{E_L}) - I(g \mathbb{1}_{E_L}) \rightarrow 0$ in probability, which particularly implies the accuracy of (4.7). This completes the proof. $\square$

Let us remark that there is no intuitive counterpart to Proposition 2.8 in Stoev and Taqqu (2005) (even if we use the $\leq_f$ order). However, we now illustrate that we immediately retrieve the max-stable extremal integral that has been constructed in Stoev and Taqqu (2005), leading to univariate random variables. For this purpose and as already indicated in the proof of Proposition 3.6 (see the Appendix), we merely have to manipulate our approach by considering $L_+^\alpha(m) \ni g \mapsto f(I(g))$ instead. On the other hand, there is also a straight possibility to do so, which means that we should finally talk about concrete choices of the loss function f .

Example 4.6 Obviously, every norm on $\mathbb{R}^d$ can serve as loss function. However, for the rest of this example, let us consider the special case $d = 1$ with

$$f_0 = |\cdot| \quad \text{and} \quad S_0 = \{f_0 = 1\} = \{-1, 1\}.$$

Then, for $x_1, x_2 \geq 0$ (which is the typical setting in the context of classical EVT), we see that

$$x_1 \vee_{f_0} x_2 = x_1 \vee x_2 \quad \text{as well as} \quad x_1 \leq_{f_0} x_2 \Leftrightarrow x_1 \leq x_2. \quad (4.10)$$

At the same time, letting $\kappa = \varepsilon_1$ within Proposition 2.3, we obtain that $M^{f_0}(A) \sim \Phi_\alpha(m(A)^{1/\alpha})$. In particular, we have that $M^{f_0}(A) \geq 0$ a.s. for every $A \in \mathcal{E}_0$. It follows that the observation (4.10) remains accordingly true for the occurring stochastic integrals, at least a.s. For instance note that (4.2) becomes

$$I(ag_1 \vee bg_2) = aI(g_1) \vee bI(g_2) \quad \text{a.s.}$$

in this case, which is just the so-called *max-linearity* in the sense of Stoev and Taqqu (2005). In general, it turns out that Theorem 3.10, Theorem 4.2, and Theorem 4.3 are natural extensions of the corresponding results in Stoev and Taqqu (2005).

The rest of this section will be devoted to a different aspect. Actually, one important benefit of stochastic integral representations is that we can implement various dependence structures and that we can control them by an appropriate choice of kernel functions. In this context, properties (ii)-(iv) of Theorem 4.2 gave a first insight. Now fix $l \in \mathbb{N}$ and $g_1, \dots, g_l \in L_+^\alpha(m)$. Then we want to understand the joint distribution of the $\mathbb{R}^{l \cdot d}$ -valued random vector $(I(g_1), \dots, I(g_l))$. Hence, for $r > 0$ and $F \in \mathcal{B}(S)$, recall from Goldbach (2016) the notation

$$D(r, F) := \{x = \tau\theta \in \mathbb{R}^d \setminus \{0\} : \tau \leq r, \theta \in F\} = \{x \in \mathbb{R}^d \setminus \{0\} : f(x) \leq r, x/f(x) \in F\}.$$

By Theorem 4.2 (i) it follows for every g_j with $\|g_j\|_\alpha > 0$ that

$$\mathbb{P}(I(g_j) \in D(r, F)) = \kappa(F) \exp(-\|g_j\|_\alpha^\alpha r^{-\alpha}).$$

Conversely, if $\|g_j\|_\alpha = 0$, we have that $I(g_j) = 0$ a.s., which is rather uninteresting. Therefore let us assume that $\|g_j\|_\alpha > 0$ for $j = 1, \dots, l$, which leads to $\mathbb{P}(I(g_j) \in \mathbb{R}^d \setminus \{0\}) = 1$. Then, Lemma 3.1.4 (a) in Goldbach (2016) shows the distribution of $(I(g_1), \dots, I(g_l))$ is entirely determined by the probabilities

$$\mathbb{P}(I(g_j) \in D(r_j, F_j) : j = 1, \dots, l), \quad r_1, \dots, r_l > 0 \text{ and } F_1, \dots, F_l \in \mathcal{B}(S). \quad (4.11)$$

Let us first consider the radial behavior of $(I(g_1), \dots, I(g_l))$, which corresponds to the setting in Stoev and Taqqu (2005) again.

Lemma 4.7 *For $r_1, \dots, r_l > 0$ we have that*

$$\begin{aligned} \mathbb{P}(f(I(g_j)) \leq r_j : j = 1, \dots, l) &= \mathbb{P}(I(g_j) \in D(r_j, S) : j = 1, \dots, l) \\ &= \exp\left(-\int_E \left(\bigvee_{j=1}^l g_j(s)/r_j\right)^\alpha m(ds)\right) \\ &= \exp\left(-\left\|\bigvee_{j=1}^l g_j/r_j\right\|_\alpha^\alpha\right). \end{aligned}$$

Proof By homogeneity of f and the f -implicit max-linearity, we obtain that

$$\mathbb{P}(f(I(g_j)) \leq r_j : j = 1, \dots, l) = \mathbb{P}(\bigvee_{j=1}^l f(r_j^{-1} I(g_j)) \leq 1) = \mathbb{P}(f(\bigvee_{j=1}^l g_j/r_j) \leq 1).$$

In view of $f(I(g)) \sim \Phi_\alpha(\|g\|_\alpha)$ this gives the assertion. $\square$

As we have just seen, (4.11) can be rewritten as $\mathbb{P}(I(g_j/r_j) \in D(1, F_j) : j = 1, \dots, l)$. Hence, it suffices to assume that $r_j = 1$ (for $j = 1, \dots, l$) in the sequel.

Moreover, $a \wedge b$ denotes the minimum of a and b (accordingly for finitely many real numbers).

Theorem 4.8 *In the case $l = 2$ the following identity holds true for all $F_1, F_2 \in \mathcal{B}(S)$:*

$$\begin{aligned} \mathbb{P}(I(g_1) \in D(1, F_1), I(g_2) \in D(1, F_2)) \\ = \kappa(F_1 \cap F_2) p^{(2)}(g_1, g_2) + \kappa(F_1) \kappa(F_2) [\exp(-\|g_1 \vee g_2\|_\alpha^\alpha) - p^{(2)}(g_1, g_2)]. \end{aligned} \quad (4.12)$$

where

$$p^{(2)}(g_1, g_2) := \int_{\{g_1(s) \wedge g_2(s) > 0\}} \left(\frac{\exp(-(g_1(s) \vee g_2(s))^\alpha \int_E (\frac{g_1(u)}{g_1(s)} \vee \frac{g_2(u)}{g_2(s)})^\alpha m(du))}{\int_E (\frac{g_1(u)}{g_1(s)} \vee \frac{g_2(u)}{g_2(s)})^\alpha m(du)} \right) m(ds). \quad (4.13)$$

Note that $p^{(2)}(g_1, g_2)$ is well-defined, since we demand that $\|g_1\|_\alpha, \|g_2\|_\alpha > 0$ throughout.

Proof First assume that g_1 and g_2 are simple functions. Find a common partition and write

$$g_1(s) = \sum_{j=1}^k \alpha_j \mathbb{1}_{A_j}(s), \quad g_2(s) = \sum_{j=1}^k \beta_j \mathbb{1}_{A_j}(s) \quad (4.14)$$

for some $k \in \mathbb{N}$, where $\alpha_j, \beta_j \geq 0$ (for $j = 1, \dots, k$) and $A_1, \dots, A_k \in \mathcal{E}_0$ are disjoint. In this case, recall that

$$I(g_1) = \bigvee_{j=1}^k \alpha_j M(A_j), \quad I(g_2) = \bigvee_{j=1}^k \beta_j M(A_j).$$

Hence, without loss of generality, we can assume that $m(A_j) > 0$ for $j = 1, \dots, k$. It follows that there are no ties and that $\Theta_j := M(A_j)/f(M(A_j))$ is well-defined a.s. Moreover, since we assume that $\|g_1\|_\alpha, \|g_2\|_\alpha > 0$, we have that $\bigvee_{j=1}^k \alpha_j > 0$ as well as $\bigvee_{j=1}^k \beta_j > 0$. If we define the sets

$$\begin{aligned} C_{i,j} : &= \{ \bigvee_{v \neq i} \alpha_v f(M(A_v)) \leq \alpha_i f(M(A_i)) \leq 1, \Theta_i \in F_1, \\ &\quad \bigvee_{v \neq j} \beta_v f(M(A_v)) \leq \beta_j f(M(A_j)) \leq 1, \Theta_j \in F_2 \}, \end{aligned}$$

we see that the relation

$$\mathbb{P}(I(g_1) \in D(1, F_1), I(g_2) \in D(1, F_2)) = \sum_{1 \leq i, j \leq k} \mathbb{P}(C_{i,j})$$

holds true. Fix $i \in \{1, \dots, n\}$. If $\alpha_i \wedge \beta_i = 0$ we obtain that $\mathbb{P}(C_{i,i}) = 0$. Else, recall Definition 2.1 to compute by independence that

$$\begin{aligned} \mathbb{P}(C_{i,i}) &= \mathbb{P} \left(\bigvee_{v \neq i} \left(\frac{\alpha_v}{\alpha_i} \vee \frac{\beta_v}{\beta_i} \right) f(M(A_v)) \leq f(M(A_i)) \leq (\alpha_i \vee \beta_i)^{-1}, \Theta_i \in F_1 \cap F_2 \right) \\ &= \kappa(F_1 \cap F_2) \mathbb{P} \left(\bigvee_{v \neq i} \left(\frac{\alpha_v}{\alpha_i} \vee \frac{\beta_v}{\beta_i} \right) f(M(A_v)) \leq f(M(A_i)) \leq (\alpha_i \vee \beta_i)^{-1} \right). \end{aligned}$$

In addition, using similar ideas as in the proof of Lemma 3.3, it can be verified that the last mentioned probability (i.e. without the term $\kappa(F_1 \cap F_2)$) equals

$$\begin{aligned} & \left(1 - \frac{\sum_{v \neq i} \left(\frac{\alpha_v}{\alpha_i} \vee \frac{\beta_v}{\beta_i} \right)^\alpha m(A_v)}{m(A_i) + \sum_{v \neq i} \left(\frac{\alpha_v}{\alpha_i} \vee \frac{\beta_v}{\beta_i} \right)^\alpha m(A_v)} \right) e^{-(m(A_i) + \sum_{v \neq i} \left(\frac{\alpha_v}{\alpha_i} \vee \frac{\beta_v}{\beta_i} \right)^\alpha m(A_v))(\alpha_i \vee \beta_i)^\alpha} \\ &= m(A_i) \frac{e^{-(\alpha_i \vee \beta_i)^\alpha \sum_{j=1}^k \left(\frac{\alpha_j}{\alpha_i} \vee \frac{\beta_j}{\beta_i} \right)^\alpha m(A_j)}}{\sum_{j=1}^k \left(\frac{\alpha_j}{\alpha_i} \vee \frac{\beta_j}{\beta_i} \right)^\alpha m(A_j)}. \end{aligned}$$

A combination of our previous findings yields that

$$\begin{aligned} \sum_{i=1}^k \mathbb{P}(C_{i,i}) &= \kappa(F_1 \cap F_2) \sum_{i=1}^k \mathbb{P} \left(\bigvee_{v \neq i} \alpha_v f(M(A_v)) \leq \alpha_i f(M(A_i)) \leq 1, \right. \\ &\quad \left. \bigvee_{v \neq i} \beta_v f(M(A_v)) \leq \beta_i f(M(A_i)) \leq 1 \right) \\ &= \kappa(F_1 \cap F_2) \sum_{i=1}^k m(A_i) \left(\frac{e^{-(\alpha_i \vee \beta_i)^\alpha \sum_{j=1}^k \left(\frac{\alpha_j}{\alpha_i} \vee \frac{\beta_j}{\beta_i} \right)^\alpha m(A_j)}}{\sum_{j=1}^k \left(\frac{\alpha_j}{\alpha_i} \vee \frac{\beta_j}{\beta_i} \right)^\alpha m(A_j)} \right) \mathbb{1}_{\alpha_i \wedge \beta_i > 0} \\ &= \kappa(F_1 \cap F_2) p^{(2)}(g_1, g_2) \end{aligned}$$

holds true due to (4.14). In a similar way we obtain that

$$\begin{aligned} \sum_{i \neq j} \mathbb{P}(C_{i,j}) &= \kappa(F_1) \kappa(F_2) \sum_{i=1}^k \mathbb{P}(\bigvee_{v \neq i} \alpha_v f(M(A_v)) \leq \alpha_i f(M(A_i)) \leq 1, \\ &\quad \bigvee_{v \neq j} \beta_v f(M(A_v)) \leq \beta_j f(M(A_j)) \leq 1). \end{aligned}$$

Actually, since

$$\sum_{1 \leq i, j \leq k} \mathbb{P} \left(\bigvee_{v \neq i} \alpha_v f(M(A_v)) \leq \alpha_i f(M(A_i)) \leq 1, \bigvee_{v \neq j} \beta_v f(M(A_v)) \leq \beta_j f(M(A_j)) \leq 1 \right)$$

equals $\exp(-\|g_1 \vee g_2\|_\alpha^\alpha)$ by Lemma 4.7, this gives (4.12) for simple functions g_1, g_2 . In the general case consider sequences $(g_{1,n})$ and $(g_{2,n})$ of simple functions such that $g_{i,n} \uparrow g_i$ for $i = 1, 2$ and as $n \rightarrow \infty$. Using Theorem 3.10, it follows that $(I(g_{1,n}), I(g_{2,n})) \rightarrow (I(g_1), I(g_2))$ in probability, particularly in distribution. Now fix κ -continuity sets $F_1, F_2 \in \mathcal{B}(S)$, that is $\kappa(\partial F_i) = 0$, where ∂F_i denotes the boundary of F_i (with respect to the relative topology on S) for $i = 1, 2$. This is always possible, since we can choose $F_1 = F_2 = S$ for example. Recall that the radial part of $I(g_i)$ is $f(I(g_i))$, having a continuous Fréchet distribution. Therefore it is easy to verify that $D(1, F_i)$ is a continuity set with respect to the distribution of $I(g_i)$ in this case, respectively. Moreover, for $A, B \subset \mathbb{R}^d$, it is generally known that $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$, where $\overline{A}$ denotes the closure of A . We derive the following computation:

$$\mathbb{P}((I(g_1), I(g_2)) \in \partial(D(1, F_1) \times D(1, F_2))) \leq \mathbb{P}(I(g_1) \in \partial D(1, F_1)) + \mathbb{P}(I(g_2) \in \partial D(1, F_2)),$$

which shows that $D(1, F_1) \times D(1, F_2)$ is a continuity set with respect to $\mathcal{L}((I(g_1), I(g_2)))$. Using the Portmanteau-theorem (cf. Theorem 11.1.1 in Dudley (2002)) together with the first part of this proof, we obtain that

$$\begin{aligned} & \mathbb{P}(I(g_1) \in D(1, F_1), I(g_2) \in D(1, F_2)) \\ &= \lim_{n \rightarrow \infty} \left(\kappa(F_1 \cap F_2) p^{(2)}(g_{1,n}, g_{2,n}) + \kappa(F_1) \kappa(F_2) [\exp(-\|g_{1,n} \vee g_{2,n}\|_\alpha^\alpha) - p^{(2)}(g_{1,n}, g_{2,n})] \right). \end{aligned}$$

Recall that $g_{i,n} \uparrow g_i$. On the one hand, we obtain that $\exp(-\|g_{1,n} \vee g_{2,n}\|_\alpha^\alpha) \rightarrow \exp(-\|g_1 \vee g_2\|_\alpha^\alpha)$ by the monotone convergence theorem. On the other hand, this shows that $\{g_{1,n}(s) \wedge g_{2,n}(s) > 0\} \uparrow \{g_1(s) \wedge g_2(s) > 0\}$. Hence, for fixed $s \in E$ fulfilling $g_1(s) \wedge g_2(s) > 0$, we can assume without loss of generality that $g_{1,1}(s) \wedge g_{2,1}(s) > 0$ and obtain that

$$\forall u \in E \quad \forall n \in \mathbb{N} : \quad \left(\frac{g_{1,n}(u)}{g_{1,n}(s)} \vee \frac{g_{2,n}(u)}{g_{2,n}(s)} \right)^\alpha \leq (g_{1,1} \wedge g_{2,2}(s))^{-\alpha} (g_1(u) \vee g_2(u))^\alpha.$$

In a similar way it follows for every $s \in S$ and $n \in \mathbb{N}$ that

$$\begin{aligned} & \left(\frac{\exp(-(g_{1,n}(s) \vee g_{2,n}(s))^\alpha \int_E (\frac{g_{1,n}(u)}{g_{1,n}(s)} \vee \frac{g_{2,n}(u)}{g_{2,n}(s)})^\alpha m(du))}{\int_E (\frac{g_{1,n}(u)}{g_{1,n}(s)} \vee \frac{g_{2,n}(u)}{g_{2,n}(s)})^\alpha m(du)} \right) \mathbb{1}_{\{g_{1,n}(s) \wedge g_{2,n}(s) > 0\}} \\ & \leq \left(\int_E (\frac{g_{1,n}(u)}{g_{1,n}(s)} \vee \frac{g_{2,n}(u)}{g_{2,n}(s)})^\alpha m(du) \right)^{-1} \mathbb{1}_{\{g_{1,n}(s) \wedge g_{2,n}(s) > 0\}} \leq \|g_{1,1} \vee g_{2,1}\|_\alpha^{-\alpha} (g_1(s) \vee g_2(s))^\alpha \end{aligned}$$

holds true. Then, using the dominated convergence theorem two times, we derive that $p^{(2)}(g_{1,n}, g_{2,n}) \rightarrow p^{(2)}(g_1, g_2)$. Actually, this gives (4.12), whenever F_1, F_2 are κ -continuity sets. Finally, fix an arbitrary κ -continuity set F_2 and define

$$\mu(F_1) := \mathbb{P}(I(g_1) \in D(1, F_1), I(g_2) \in D(1, F_2)), \quad F_1 \in \mathcal{B}(S)$$

as well as

$$\tilde{\mu}(F_1) := \kappa(F_1 \cap F_2) p^{(2)}(g_1, g_2) + \kappa(F_1) \kappa(F_2) [\exp(-\|g_1 \vee g_2\|_\alpha^\alpha) - p^{(2)}(g_1, g_2)], \quad F_1 \in \mathcal{B}(S).$$

Obviously, μ and $\tilde{\mu}$ are finite measures on $\mathcal{B}(S)$, respectively. Moreover, we have just seen, that they coincide on $\{F_1 \in \mathcal{B}(S) : \kappa(\partial F_1) = 0\}$. Using Proposition 8.2.8 in Bogachev (2007) it is easy to verify that this remains true for all $F_1 \in \mathcal{B}(S)$. By swapping the roles of F_1 and F_2 , this finishes the proof. $\square$

Remark 4.9 (a) The proof of Theorem 4.8 revealed something more. To be specific, the event $\{f(I(g_1)) \leq 1, f(I(g_2)) \leq 1\}$ has probability $\exp(-\|g_1 \vee g_2\|_\alpha^\alpha)$ and can be written as $A_1 \cup A_2$, where A_1 and A_2 are disjoint sets such that $\mathbb{P}(A_1) = p^{(2)}(g_1, g_2)$. Moreover, on A_1 , the angular parts of $I(g_1)$ and $I(g_2)$ (that is $I(g_1)/f(I(g_1))$ and $I(g_2)/f(I(g_2))$) are the same. Conversely, on A_2 , they are independent.

(b) A generalization of Theorem 4.8 for the case $l \geq 3$ remains an open problem.

In spite of Remark 4.9 (b) and as a counterpart of Lemma 4.7, it is possible to characterize the angular part of $(I(g_1), \dots, I(g_l))$ by means of a recursion formula. However, this formula and its notation are still very involved. Hence, we state the following example instead. Its proof uses ideas of the foregoing one and is therefore left to the reader.

Corollary 4.10 *In the case $l = 3$ the following relation holds true for all $F_1, F_2, F_3 \in \mathcal{B}(S)$:*

$$\begin{aligned} \mathbb{P}(I(g_j)/f(I(g_j))) &\in F_j : j = 1, 2, 3) \\ &= \lim_{r \rightarrow \infty} \mathbb{P}(I(g_j/r) \in D(1, F_j) : j = 1, 2, 3) \\ &= \kappa(F_1 \cap F_2 \cap F_3) q^{(3)}(g_1, g_2, g_3) \\ &\quad + \kappa(F_1) \kappa(F_2 \cap F_3) [q^{(2)}(g_2, g_3) - q^{(3)}(g_1, g_2, g_3)] \\ &\quad + \kappa(F_2) \kappa(F_1 \cap F_3) [q^{(2)}(g_1, g_3) - q^{(3)}(g_1, g_2, g_3)] \\ &\quad + \kappa(F_3) \kappa(F_1 \cap F_2) [q^{(2)}(g_1, g_2) - q^{(3)}(g_1, g_2, g_3)] \\ &\quad + \kappa(F_1) \kappa(F_2) \kappa(F_3) [1 + 2q^{(3)}(g_1, g_2, g_3) - q^{(2)}(g_2, g_3) - q^{(2)}(g_1, g_3) - q^{(2)}(g_1, g_2)], \end{aligned}$$

where

$$q^{(2)}(g_1, g_2) := \int_{\{g_1(s) \wedge g_2(s) > 0\}} \left(\frac{\exp(-\int_E (\frac{g_1(u)}{g_1(s)} \vee \frac{g_2(u)}{g_2(s)})^\alpha m(du))}{\int_E (\frac{g_1(u)}{g_1(s)} \vee \frac{g_2(u)}{g_2(s)})^\alpha m(du)} \right) m(ds)$$

and where

$$q^{(3)}(g_1, g_2, g_3) := \int_{\{\bigwedge_{j=1}^3 g_j(s) > 0\}} \left(\frac{\exp(-\int_E (\bigvee_{j=1}^3 \frac{g_j(u)}{g_j(s)})^\alpha m(du))}{\int_E (\bigvee_{j=1}^3 \frac{g_j(u)}{g_j(s)})^\alpha m(du)} \right) m(ds).$$

5 Implicit max-stable processes: Examples and outlook

Generally, it is known that stochastic integrals are often used for the representation of stochastic processes. In doing so the properties of the considered integral somehow determine the range of possible representations (see Section 1). Hence, in view of Theorem 4.2, it should not surprise that we introduce the following notion, which is due to Definition 3.0.1 in Goldbach (2016). Let the loss function f be as before.

Definition 5.1 Assume that $T \neq \emptyset$ is an index set. Then an $\mathbb{R}^d$ -valued stochastic process $\mathbb{X} = \{X(t) : t \in T\}$ is called *f-implicit max-stable* if, for all $l \in \mathbb{N}$ and $a_1, \dots, a_l \geq 0$ as well as $t_1, \dots, t_l \in T$, the random vector

$$\xi := \bigvee_{j=1}^l a_j X(t_j)$$

is *f-implicit max-stable* (in the sense of (1.4)).

Of course, we will try to benefit from the theory that we developed within the foregoing sections by defining each random vector $X(t)$ as an appropriate *f-implicit extremal integral*. Note that our examples will mostly deal with the cases $T \subset \mathbb{R}$ and $T \subset \mathbb{R}^n$ (for some $n \geq 1$), leading to $\mathbb{R}^d$ -valued stochastic processes and random fields, respectively. Often the choice of T also affects the choice of the underlying space $(E, \mathcal{E}, m)$ and vice versa.

The following observation can be easily concluded from Theorem 4.2 and the fact that f -implicit α -Fréchet distributions are always f -implicit max-stable ones (see Section 1 again). The details are left to the reader.

Proposition 5.2 *Fix $T \neq \emptyset$ and let M be an f -implicit α -Fréchet sup-measure as before. Moreover, consider a family $(g_t)_{t \in T} \subset L_+^\alpha(m)$ of functions and define $X(t) := I(g_t)$ for every $t \in T$. Then the resulting process $\mathbb{X} = \{X(t) : t \in T\}$ is f -implicit max-stable. More precisely, for $a_1, \dots, a_l \geq 0$ and $t_1, \dots, t_l \in T$ as above, we a.s. have that*

$$\bigvee_{j=1}^l a_j X(t_j) = \int_E^{\vee_f} \left(\bigvee_{j=1}^l a_j g_{t_j}(s) \right) dM(s) \sim \Phi_{\alpha, \kappa}^f \left(\left\| \bigvee_{j=1}^l a_j g_{t_j} \right\|_\alpha \right).$$

We want to proceed with a list of examples that are inspired by Proposition 5.2 and that illustrate how to *play* with the different parameters. But first recall that two $\mathbb{R}^d$ -valued processes $\mathbb{X} = \{X(t) : t \in T\}$ and $\mathbb{Y} = \{Y(t) : t \in T\}$ have the same *finite-dimensional distributions* if we have that

$$\forall l \in \mathbb{N} \, \forall t_1, \dots, t_l \in T : \quad (X(t_1), \dots, X(t_l)) \stackrel{d}{=} (Y(t_1), \dots, Y(t_l)). \quad (5.1)$$

However, since we determined (4.11) only for $l \leq 2$, it is complicated to consider all finite-dimensional distributions of the occurring processes in the sequel. Instead we will mostly use the following notation, which is slightly weaker.

Definition 5.3 As before let $\mathbb{X} = \{X(t) : t \in T\}$ and $\mathbb{Y} = \{Y(t) : t \in T\}$ be two stochastic processes with common index set $T \neq \emptyset$. Then we write $\mathbb{X} \stackrel{2-fdd}{=} \mathbb{Y}$ if (5.1) is accordingly fulfilled for $l \leq 2$.

The remark given below suggests a further possibility to understand the probabilistic structure of f -implicit max-stable random fields.

Remark 5.4 Assume that the process $\mathbb{X} = \{X(t) : t \in T\}$ has the form $X(t) = I(g_t)$ with $0 < \|g_t\|_\alpha < \infty$ for every $t \in T$. Then it can also be useful to consider $Z(t) := f(X(t))$ as well as $\Theta(t) := X(t)/Z(t)$. This leads to the corresponding *radial process* $\{Z(t) : t \in T\}$ and *angular process* $\{\Theta(t) : t \in T\}$, respectively. Note that both have been studied in Lemma 4.7 and Corollary 4.10.

Let us start with the announced collection of examples and observe that the first one has already been addressed in (1.6). Throughout note that, if not specified or stated otherwise, all measures and parameters are as general as before. Conversely, we use a consistent notation, although the described objects can be different within each occurrence. For instance, the set E in part (a) of the following example is different from that in part (b).

Example 5.5 (a) Let $m(ds) = ds$ be the 1-dimensional Lebesgue measure on $(E, \mathcal{E}) = ([0, \infty), \mathcal{B}([0, \infty)))$ together with $\mathbb{X} = \{X(t) : t \geq 0\}$, defined by

$X(t) := M([0, t]) = I(\mathbb{1}_{[0, t]})$. Then $\mathbb{X}$ is an f -implicit max-stable process that, in view of Theorem 4.3, is also stochastically continuous. Moreover, using Theorem 4.8, it is easy to verify (compare the proof of Example 5.6 below) that the following relation holds true:

$$\forall c > 0 : \quad \{X(ct) : t \geq 0\} \stackrel{2-fdd}{=} \{c^{1/\alpha} X(t) : t \geq 0\}.$$

- (b) Consider a further σ -finite measure space $(E', \mathcal{E}', m')$ and let M be the f -implicit sup-measure on $(E, \mathcal{E}) := (\mathbb{R}^n \times E', \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{E}')$ with control measure $m(dx, ds) = dx m'(ds)$ (in the sense of Proposition 2.3). Here, dx denotes the n -dimensional Lebesgue measure. Fix some $g \in L_+^\alpha(m)$. Then, similarly as in Stoev and Taqqu (2005), we can define a random field $\mathbb{X} = \{X(t) : t \in \mathbb{R}^n\}$ by means of

$$X(t) := \int_{\mathbb{R}^n \times E'}^{\vee f} g(t - x, s) M(dx, ds). \quad (5.2)$$

In this case we say that $\mathbb{X}$ has an *f -implicit mixed moving-maxima integral representation*. In the special case, where m' is a probability measure and where $g(\cdot, s) = g(\cdot)$ does not depend on s , we simply say that $\mathbb{X}$ has a *f -implicit moving-maxima representation*. Note that, besides Proposition 5.2, one benefit of this model is that we obtain a stationarity property. More precisely, by a change of variables one can verify for every $h \in \mathbb{R}^n$ that

$$\{X(t + h) : t \in \mathbb{R}^n\} \stackrel{2-fdd}{=} \{X(t) : t \in \mathbb{R}^n\} \quad (5.3)$$

holds true (again see the proof of Example 5.6 below).

Recall that Stoev and Taqqu (2005) already presented interesting parallels between classical max-stable processes on the one hand and α -stable processes ($0 < \alpha \leq 2$) on the other hand. In the sequel we want to pursue this idea and therefore need some preparation: We will consider an $(n \times n)$ -matrix A with real-valued entries and such that the real parts of the eigenvalues of A are strictly positive. Denote the trace of A by $\text{tr}(A)$. Moreover, recall the well-known matrix exponentials,

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!} \quad \text{and} \quad c^A := \exp(A \log c),$$

and that the inverse of c^A is given by c^{-A} for every $c > 0$. Then a continuous function $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ is called *A -homogeneous* if we have that

$$\forall c > 0 \forall x \in \mathbb{R}^n \setminus \{0\} : \quad \phi(c^A x) = c \phi(x), \quad (5.4)$$

which obviously generalizes the concept of 1-homogenous functions. Moreover, using a further parameter $\beta > 0$, the continuity of such a function ϕ is sometimes specified by a term called *(β, A) -admissibility*. For details and satisfying examples of such functions we refer the reader to Biermé et al. (2007).

Example 5.6 Let M be an f -implicit α -Fréchet sup-measure M on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), ds)$. Moreover, for some $\beta > 0$, let $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ be an A -homogeneous and

(β, A) -admissible function such that $\phi(x) = 0 \Leftrightarrow x = 0$. Then, for $0 < H < \beta$, define the functions

$$g_t(s) := |\phi(t - s)^{H - \frac{\text{tr}(A)}{\alpha}} - \phi(-s)^{H - \frac{\text{tr}(A)}{\alpha}}|, \quad s \in \mathbb{R}^n.$$

It follows that $\mathbb{X} = \{X(t) : t \in \mathbb{R}^n\}$, defined by $X(t) := I(g_t)$, is an f -implicit max-stable random field, which is stochastically continuous and which fulfills (5.3) accordingly. Furthermore, we have the following scaling property:

$$\forall c > 0 : \quad \{X(c^A t) : t \in \mathbb{R}^n\} \stackrel{2\text{-}f\text{d}d}{=} \{c^H X(t) : t \in \mathbb{R}^n\}. \quad (5.5)$$

Remark 5.7 Note that $\mathbb{X}$ does not exactly fit into the pattern of (5.2). However, Biermé et al. (2007) still speak of *moving-average representations* within their framework, which tempts us to do the same within our setting. Also note that, in view of (5.5), H is often referred to as the so-called *Hurst-index*. In order to avoid misunderstandings, let us emphasize again that H is a real number, while A is a matrix. Finally, in case $n = 1$, we can always use $A = 1$ together with $\phi(\cdot) = |\cdot|$ and choose $0 < H < 1$.

Proof of Example 5.6 Under the given assumptions the proof of Theorem 3.1 in Biermé et al. (2007) shows that $g_t \in L_+^\alpha(ds)$ for every $\alpha > 0$ and $t \in \mathbb{R}^n$. Hence, $\mathbb{X}$ is well-defined. The recently quoted proof also shows how Theorem 4.3 can be used to verify that $\mathbb{X}$ is stochastically continuous. Therefore we only prove (5.5) in the sequel. For this purpose fix $t_1, t_2 \in \mathbb{R}^n$. Also fix $s \in \mathbb{R}^n$ such that $g_{c^A t_1}(s) \wedge g_{c^A t_2}(s) > 0$ holds true. Then we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\bigvee_{j=1}^2 \frac{g_{c^A t_j}(u)}{g_{c^A t_j}(s)} \right)^\alpha du &= \int_{\mathbb{R}^n} \left(\bigvee_{j=1}^2 \frac{c^{H - \frac{\text{tr}(A)}{\alpha}} |\phi(t_j - c^{-A} u)^{H - \frac{\text{tr}(A)}{\alpha}} - \phi(-c^{-A} u)^{H - \frac{\text{tr}(A)}{\alpha}}|}{c^{H - \frac{\text{tr}(A)}{\alpha}} |\phi(t_j - c^{-A} s)^{H - \frac{\text{tr}(A)}{\alpha}} - \phi(-c^{-A} s)^{H - \frac{\text{tr}(A)}{\alpha}}|} \right)^\alpha du \\ &= c^{\text{tr}(A)} \int_{\mathbb{R}^n} \left(\bigvee_{j=1}^2 \frac{|\phi(t_j - u)^{H - \frac{\text{tr}(A)}{\alpha}} - \phi(-u)^{H - \frac{\text{tr}(A)}{\alpha}}|}{|\phi(t_j - c^{-A} s)^{H - \frac{\text{tr}(A)}{\alpha}} - \phi(-c^{-A} s)^{H - \frac{\text{tr}(A)}{\alpha}}|} \right)^\alpha du \end{aligned}$$

by A -homogeneity of ϕ and after a change of variables. If we recall (4.13), it similarly follows that $p^{(2)}(g_{c^A t_1}, g_{c^A t_2})$ equals

$$\begin{aligned} &\int_{\{\bigwedge_{j=1}^2 g_{t_j}(c^{-A} s) > 0\}} \left(\frac{\exp(-(\bigvee_{j=1}^2 c^{H - \frac{\text{tr}(A)}{\alpha}} g_{t_j}(c^{-A} s))^\alpha c^{\text{tr}(A)} \int_E \left(\bigvee_{j=1}^2 \frac{g_{t_j}(u)}{g_{t_j}(c^{-A} s)} \right)^\alpha du)}{c^{\text{tr}(A)} \int_E \left(\bigvee_{j=1}^2 \frac{g_{t_j}(u)}{g_{t_j}(c^{-A} s)} \right)^\alpha du} \right) ds \\ &= \int_{\{\bigwedge_{j=1}^2 g_{t_j}(s) > 0\}} \left(\frac{\exp(-(\bigvee_{j=1}^2 c^H g_{t_j}(s))^\alpha \int_E \left(\bigvee_{j=1}^2 \frac{g_{t_j}(u)}{g_{t_j}(s)} \right)^\alpha du)}{\int_E \left(\bigvee_{j=1}^2 \frac{g_{t_j}(u)}{g_{t_j}(s)} \right)^\alpha du} \right) ds. \end{aligned}$$

Hence, for $r_1, r_2 > 0$ arbitrary, we obtain that

$$p^{(2)}(r_1^{-1} g_{c^A t_1}, r_2^{-1} g_{c^A t_2}) = p^{(2)}((c^{-H} r_1)^{-1} g_{t_1}, (c^{-H} r_2)^{-1} g_{t_2}). \quad (5.6)$$

Obviously, using similar arguments as before, it is easy to verify that

$$\exp(-\|r_1^{-1} g_{c^A t_1} \vee r_2^{-1} g_{c^A t_2}\|_\alpha) = \exp(-\|(c^{-H} r_1)^{-1} g_{t_1} \vee (c^{-H} r_2)^{-1} g_{t_2}\|_\alpha) \quad (5.7)$$

holds true. Finally, fix $F_1, F_2 \in \mathcal{B}(S)$ and observe that

$$\begin{aligned} & \mathbb{P}(c^H I(g_{t_1}) \in D(r_1, F_1), c^H I(g_{t_2}) \in D(r_2, F_2)) \\ &= \mathbb{P}(I(g_{t_1}) \in D(c^{-H} r_1, F_1), I(g_{t_2}) \in D(c^{-H} r_2, F_2)). \end{aligned} \quad (5.8)$$

Combine (5.6)-(5.8) to verify the accuracy of (5.5) due to Theorem 4.8. $\square$

We turn over to the last example, which states something about component-wise powers of f -implicit max-stable processes.

Example 5.8 Fix $\alpha > 0$ as well as $T \neq \emptyset$ and let $(E, \mathcal{E}, m)$ be a σ -finite measure space.

- (a) Fix $1 \leq q < \infty$ as well as $0 < p \leq q$ and define $f(\cdot) := \|\cdot\|_q$ being the q -norm on $\mathbb{R}^d$. As usual consider κ to be an arbitrary probability measure on $S := \{f = 1\}$. Also let $f'(\cdot) := \|\cdot\|_{q/p}$ and

$$h_p(x) := (\text{sgn}(x_1)|x_1|^p, \dots, \text{sgn}(x_d)|x_d|^p), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

where $\text{sgn}(x_j) := \mathbb{1}_{[0, \infty)}(x_j) - \mathbb{1}_{(-\infty, 0)}(x_j)$. Then we denote the push-forward measure of κ with respect to h_p by κ' . It is essential to note that $\|h_p(x)\|_{q/p} = \|x\|_q^p$ holds true for every $x \in \mathbb{R}^d$. Because on the one hand this implies that κ' is supported on $S' := \{f' = 1\}$. Therefore let M and M' be the f -implicit Fréchet sup-measures from Proposition 2.3 such that

$$M(A) \sim \Phi_{\alpha, \kappa}^f(m(A)^{1/\alpha}) \quad \text{and} \quad M'(A) \sim \Phi_{\alpha/p, \kappa'}^{f'}(m(A)^{p/\alpha}) = \Phi_{\alpha', \kappa'}^{f'}(m(A)^{1/\alpha'}),$$

where $\alpha' := \alpha/p$. On the other hand, if we define the sets

$$D'(r, F) := \{x \in \mathbb{R}^d \setminus \{0\} : f'(x) \leq r, x/f'(x) \in F\}, \quad r > 0, F \in \mathcal{B}(S'),$$

we observe for every $x \in \mathbb{R}^d \setminus \{0\}$ that

$$h_p(x) \in D'(r, F) \Leftrightarrow x \in D(r^{1/p}, h_p^{-1}(F)), \quad (5.9)$$

where $h_p^{-1}(\cdot)$ denotes the pre-image of h_p . Finally, this suggests to define

$$X(t) = \int_E^{\vee f} g_t(s) dM(s) \quad \text{and} \quad Y(t) = \int_E^{\vee f'} g_t(s)^p dM'(s)$$

for every $t \in T$, where $(g_t)_{t \in T} \subset L_+^\alpha(m)$ are arbitrary kernel functions. Then, using (5.9) and Theorem 4.8, it is easy to verify that

$$\{h_p(X(t)) : t \in T\} \stackrel{2\text{-}fdd}{=} \{Y(t) : t \in T\}$$

holds true. The details are left to the reader.

- (b) Observe that, for any $0 < p < \infty$, part (a) remains accordingly true in case $q = \infty$ with $f(\cdot) = f'(\cdot) = \|\cdot\|_\infty$ being the maximum norm on $\mathbb{R}^d$. In particular, we obtain for every $x \in \mathbb{R}^d$ that $\|h_p(x)\|_\infty = \|x\|_\infty^p$ holds true, which implies that $S = S' = \{x : \|x\|_\infty = 1\}$. So in this case a change of the underlying loss function is no longer needed. Moreover, the present example emphasizes that we are actually dealing with a multivariate extension of Proposition 2.9 in

Stoev and Taququ (2005). In this context, also recall the setting in Example 4.6 for which $h_p(X(t))$ is nothing else than $X(t)^p$.

We want to finish the main content of this paper with some kind of outlook. That is we complement the list of open problems, which we already initiated by part (b) of Remark 4.9 before.

Remark 5.9 • According to Li and Xiao (2011) for instance, it could be nice to consider kernel functions g taking values in the space of non-negative $(d \times d)$ -matrices. Then, on the one hand, we will probably have to consider more general loss functions f (like the A -homogeneous functions from (5.4)). On the other hand, properties like (5.5) could even remain valid for suitable matrix-valued scaling in *space*.

- Suppose that $\mathbb{X} = \{X(t) : t \in T\}$ is an f -implicit max-stable process in the sense of Definition 5.1. Then, following Stoev and Taququ (2005) (there only for $m(ds) = ds$ on $(E, \mathcal{E}) = ([0, 1], \mathcal{B}([0, 1]))$), the question arises under which (minimal) conditions there exist $(g_t)_{t \in T} \subset L_+^\alpha(m)$ such that $\mathbb{X}$ and $\mathbb{Y} = \{Y(t) : t \in T\}$, defined by $Y(t) := I(g_t)$, have the same finite-dimensional distributions. And, in this case, how can we choose these kernel functions g_t as well as the sup-measure M ?
- Actually, Stoev and Taququ (2005) managed to solve the last-mentioned question by using an alternative approach, which is somehow based on *Poisson random measures*. Let us make a hint how this could be adopted to our setting: For convenience, assume that the control measure m is a probability measure on $(E, \mathcal{B}(E))$. Then we consider an i.i.d. sequence (U_j) of uniformly distributed random variables on $(0, 1)$. Let (Θ_j, T_j) be a further i.i.d. sequence of $(S \times E)$ -valued random vectors being independent from (U_j) and having common distribution

$$\nu(d\theta, ds) := \kappa(d\theta)m(ds). \quad (5.10)$$

Fix $g \in L_+^\alpha(m)$ and note that, exactly as in the proof of Proposition 3.1 in Stoev and Taququ (2005), we obtain that

$$\mathbb{P}(f(U_1^{-1/\alpha} \Theta_1 g(T_1)) > r) = \mathbb{P}(U_1^{-1/\alpha} g(T_1) > r) \sim r^{-\alpha} \mathbb{E}(g(T_1)^\alpha)$$

as $r \rightarrow \infty$, where $\mathbb{E}(g(T_1)^\alpha) = \|g\|_\alpha^\alpha$. A standard calculation yields that

$$\forall x > 0 : \quad n \mathbb{P}(f(U_1^{-1/\alpha} \Theta_1 g(T_1)) > x) \xrightarrow{(n \rightarrow \infty)} \|g\|_\alpha^\alpha x^{-\alpha}$$

holds true. At the same time, (5.10) shows that Θ_1 and $g(T_1)$ are independent, too. Hence, we can combine Proposition 3.10, Theorem 3.14 and Proposition 3.19 in Scheffler and Stoev (2017) to conclude the following convergence in distribution:

$$Y_n(g) := n^{-1/\alpha} \bigvee_{j=1}^n U_j^{-1/\alpha} \Theta_j g(T_j) \Rightarrow Y(g) \quad (n \rightarrow \infty), \quad (5.11)$$

where $Y(g) \sim \Phi_{\alpha, \kappa}^f(\|g\|_\alpha)$. Note that $Y_n(g)$ and its behavior for $n \rightarrow \infty$ are closely related to a Poisson random measure with *intensity measure*

$\mu(dr, d\theta, ds) := r^{-\alpha-1} dr \nu(d\theta, ds)$. At this point, we refer the reader to Resnick (2013) and Stoev and Taqqu (2005) for details. Anyway, since

$$M(A) \stackrel{d}{=} Y(\mathbb{1}_A) \quad \text{as well as} \quad I(g) \stackrel{d}{=} Y(g),$$

one can speculate that there exist Poisson based alternatives to define the sup-measure M as well as the corresponding implicit integral. However, in both cases and besides the problem stated in Remark 4.9 (b), there will be technical questions again, which we already encountered in Section 3 before.

- It could be tempting to allow the angular part κ of $M(A)$ (within Proposition 2.3) to depend on A . More precisely, provided that the above-mentioned Poisson approach is successful, one could modify the measure ν in (5.10) such that $\nu(d\theta, ds) = \pi_s(d\theta)m(ds)$, where $(\pi_s)_{s \in E}$ is a suitable family of probability measures on $S = \{f = 1\}$.

Acknowledgments The author would like to emphasize that this paper is inspired by the fundamental results in Goldbach (2016), which my former colleague Johannes Goldbach developed during his PhD time under the supervision of Hans-Peter Scheffler. Moreover, particular thanks are due to Marco Oesting for many fruitful discussions that were particularly helpful in the context of Lemma 3.3. Finally, the author would like to thank two anonymous referees for their very detailed suggestions, which helped to improve the paper. For instance, their remarks stimulated the examination of (4.11) as well as parts of Section 5.

Funding Open Access funding provided by Projekt DEAL.

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Appendix: Some proofs and auxiliary results

Proof of Proposition 3.6 Obviously, we can always assume that $\emptyset$ is not an element of the occurring partitions. This allows to define $\beta_j^{(n)} := \min\{h_{2,n}(s) : s \in A_j^{(n)}\}$ and in view of $g_n \leq h_{2,n}$ we obtain that $\alpha_j^{(n)} \leq \beta_j^{(n)}$ for every $n \in \mathbb{N}$ and $1 \leq j \leq k_n$. If we let $h_n \sim (A_j^{(n)}, \beta_j^{(n)})_{j=1, \dots, k_n}$ together with

$$Y_n := I(h_n) = \bigvee_{j=1}^{k_n} \beta_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad Y_n^* := \bigvee_{j=1}^{k_n} \beta_j^{(n)} M(A_j^{(n)}),$$

it follows for every $n \in \mathbb{N}$ that $f(X_n^*) \leq f(Y_n^*)$ a.s. Also note that the sequence (h_n) is increasing, since the same holds true for $(h_{2,n})$ by assumption. In particular, we deduce that $f(Y_n^*)$ is increasing due to part (b) of Remark 3.5. Let

$Y^* := \sup_{n \in \mathbb{N}} f(Y_n^*)$ and verify that $h_{1,n} \leq g_n \leq h_n \leq h_{2,n}$. Then Proposition 3.2.4 (together with (1.3.2)) in Goldbach (2016) states that

$$\forall n \in \mathbb{N}: \quad f(I(h_{1,n})) \leq f(X_n) \leq f(Y_n) \leq f(I(h_{2,n})) \quad \text{a.s.} \quad (5.12)$$

However, Remark 2.5 and Proposition 2.7 in Stoev and Taqqu (2005) imply that the increasing sequences $f(I(h_{1,n}))$ and $f(I(h_{2,n}))$ have the same limit a.s., say Y . It follows that $f(Y_n) \rightarrow Y$ a.s. Also note that $Y \geq Y^*$ and that $0 < Y < \infty$ a.s., provided that $\|g\|_\alpha > 0$ (otherwise we conclude that $g_n = 0$ m -a.e. and (3.5) is true anyway).

The next step is to prove that $Y - Y^* > 0$ holds true a.s. Conversely, assume that there exists a set $B \in \mathcal{A}$ with $p := \mathbb{P}(B) > 0$ and $Y(\omega)/Y^*(\omega) = 1$ for every $\omega \in B$. Then we obtain that $(f(Y_n)/f(Y_n^*) - 1)\mathbb{1}_B \rightarrow 0$ a.s. (and particularly in probability). Hence, for $\gamma, \gamma' > 0$ arbitrary, it follows that

$$\mathbb{P}((f(Y_n)/f(Y_n^*) - 1)\mathbb{1}_B \leq \gamma) \geq 1 - \gamma' \quad \text{for almost all } n,$$

which also implies that $\mathbb{P}(f(Y_n) \leq (1 + \gamma)f(Y_n^*)) \geq p - \gamma'$ for those n . Observe that this gives a contradiction to Lemma 3.3, when choosing $0 < \gamma' < p + (1 + \gamma)^{-\alpha} - 1$, which is always possible as long as we have that $p = 1$ or $0 < \gamma < (1 - p)^{-1/\alpha} - 1$, respectively.

Fix $\varepsilon > 0$. By what we have just seen there exist some $0 < \delta' < 1$ and a set $A_1 \in \mathcal{A}$ with $\mathbb{P}(A_1) \geq 1 - \varepsilon/2$, fulfilling the relation $Y \geq (1 + \delta')Y^*$ on A_1 . In a similar way and using that $f(I(h_{1,n})) \uparrow Y$ a.s. (see above), we obtain some $N \in \mathbb{N}$ and a further set $A_2 \in \mathcal{A}$ with $\mathbb{P}(A_2) \geq 1 - \varepsilon/2$ and such that $f(I(h_{1,n}))(\omega)/Y(\omega) \geq 1 - \delta'/2$ holds true for every $\omega \in A_2$ and $n \geq N$. Let $A := A_1 \cap A_2$ and observe that $\mathbb{P}(A) \geq 1 - \varepsilon$. Finally, recall (5.12) and that $f(X_n^*) \leq f(Y_n^*) \leq Y^*$. Then, for $n \geq N$, the following computation is valid on A , where we can assume that $f(X_n^*) > 0$ (else (3.5) is true anyway again):

$$\frac{f(X_n)}{f(X_n^*)} \geq \frac{f(I(h_{1,n}))}{Y^*} = \frac{Y}{Y^*} \cdot \frac{f(I(h_{1,n}))}{Y} \geq (1 + \delta')(1 - \delta'/2).$$

Letting $\delta := (1 + \delta')(1 - \delta'/2) - 1 > 0$ for instance, this gives the assertion. $\square$

Lemma 6.1 Let $h : E \rightarrow \mathbb{R}_+$ be measurable and assume that (h_n) is a sequence of simple functions with $h_n \leq h$ and such that h_n converges to h uniformly on E . Then there exist further sequences $(h_{1,n}), (h_{2,n})$ of simple functions with $h_{1,n} \leq h_n \leq h_{2,n}$ and such that $h_{i,n} \uparrow h$ for $i = 1, 2$ as $n \rightarrow \infty$.

Proof By assumption we can find a strictly increasing sequence of naturals $(N_l)_l$ such that, for any $n \geq N_l$ and $s \in E$, we have $h(s) - h_n(s) \leq 1/l$. In case $n < N_1$ let $h_{1,n} := 0$. Else we define

$$h_{1,n} := \max\{0, \max\{h_1(s), \dots, h_n(s)\} - 1/l\}, \quad \text{if } N_l \leq n < N_{l+1}.$$

Now it is easy to verify that this gives a sequence $(h_{1,n})$ of simple functions as desired. Conversely, we can simply choose $h_{2,n} := \max\{h_1, \dots, h_n\}$ for every $n \in \mathbb{N}$. $\square$

Proof of Proposition 3.8 Since the measure m is σ -finite, we can use Egorov's theorem (see Chapter VI, Exercise 3.1 in Elstrodt (2006)) to obtain increasing sets $E_1, E_2, \dots \in \mathcal{E}$ with $m(E \setminus \bigcup_{l=1}^{\infty} E_l) = 0$ and such that, for any $l \in \mathbb{N}$, the convergence $g_n \mathbb{1}_{E_l} \rightarrow g \mathbb{1}_{E_l}$ holds uniformly as $n \rightarrow \infty$. Using the σ -finiteness of m again and by a little abuse of notation, we can even assume that $(E_l) \subset \mathcal{E}_0$. Moreover, note that the proof of the present statement is obvious in case $\|g\|_{\alpha} = 0$. Hence, without loss of generality, we can even assume that $\|g \mathbb{1}_{E_l}\|_{\alpha} > 0$ for every $l \in \mathbb{N}$.

Fix $l \in \mathbb{N}$ and consider the sequence $(g_n \mathbb{1}_{E_l})_n$, where $g_n \mathbb{1}_{E_l}$ is still simple. Denote by $\mathcal{P}'_n$ a partition of $g_n \mathbb{1}_{E_l}$ for every $n \in \mathbb{N}$ and define $\mathcal{P}_1 = \mathcal{P}'_1$. Then, using the construction from Remark 3.5 (a), we obtain a common partition for $g_1 \mathbb{1}_{E_l}$ and $g_2 \mathbb{1}_{E_l}$, denoted by $\mathcal{P}_2$ and which, in addition, fulfills $\mathcal{P}_1, \mathcal{P}'_2 \leq \mathcal{P}_2$. Based on $\mathcal{P}_2$ and $\mathcal{P}'_3$, we do the same to obtain $\mathcal{P}_3$. Inductively, this gives a sequence $(\mathcal{P}_n)$ of partitions such that, on the one hand, we have $\mathcal{P}_{n-1}, \mathcal{P}'_n \leq \mathcal{P}_n$. On the other hand, $g_{n-1} \mathbb{1}_{E_l}$ and $g_n \mathbb{1}_{E_l}$ can be both represented by using the common partition $\mathcal{P}_n$ for every $n \geq 2$. In particular, if we assume that $\mathcal{P}_n$ consists of $A_1^{(n)}, \dots, A_{k_n}^{(n)} \in \mathcal{E}_0 \setminus \{\emptyset\}$ (which is always possible, see the proof of Proposition 3.6 above), there exist $\alpha_1^{(n)}, \dots, \alpha_{k_n}^{(n)} \geq 0$ such that we have $g_n \mathbb{1}_{E_l} \sim (A_j^{(n)}, \alpha_j^{(n)})_{j=1, \dots, k_n}$, i.e.

$$g_n \mathbb{1}_{E_l} = \sum_{j=1}^{k_n} \alpha_j^{(n)} \mathbb{1}_{A_j^{(n)}}, \quad n \in \mathbb{N}. \quad (5.13)$$

At the same time, whenever $m > n$, the previous construction also allows us to find suitable coefficients $\beta_{m,1}^{(n)}, \dots, \beta_{m,k_m}^{(n)} \geq 0$, which only depend on $\alpha_1^{(n)}, \dots, \alpha_{k_n}^{(n)}$ and which fulfill

$$g_n \mathbb{1}_{E_l} = \sum_{j=1}^{k_m} \beta_{m,j}^{(n)} \mathbb{1}_{A_j^{(m)}}. \quad (5.14)$$

Based on (5.13), we define

$$X_n := I(g_n \mathbb{1}_{E_l}) = \bigvee_{j=1}^{k_n} \alpha_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad X_n^* := \bigvee_{j=1}^{k_n} \alpha_j^{(n)*} M(A_j^{(n)})$$

for every $n \in \mathbb{N}$. In view of Lemma 1.1 (with $h := g \mathbb{1}_{E_l}$) we can apply Proposition 3.6 (with $g_n \mathbb{1}_{E_l}$ instead of g_n) to X_n and X_n^* in this case. Hence, for fixed $\varepsilon > 0$, there exist a set $A_0 \in \mathcal{A}$ with $\mathbb{P}(A_0) \geq 1 - \varepsilon/3$ as well as some $\delta > 0$ and $N_0 \in \mathbb{N}$ fulfilling

$$f(X_n)(\omega) \geq (1 + \delta)f(X_n^*)(\omega) \quad \text{for every } n \geq N_0 \text{ and } \omega \in A_0. \quad (5.15)$$

The fundamental idea is to use Lemma 3.7 now. However, its assumptions are not fulfilled yet. As a way out, recall the proof of Proposition 3.6 and that, in a very similar way, $f(X_n)$ converges to a random variable Y a.s. In addition, Proposition 2.7 in Stoev and Taqqu (2005) states that $Y \sim \Phi_{\alpha}(\|g \mathbb{1}_{E_l}\|_{\alpha})$. Moreover, since $\|g \mathbb{1}_{E_l}\|_{\alpha} > 0$, we have that $Y > 0$ a.s. If we combine both results, there exist a set $A_1 \in \mathcal{A}$ with $\mathbb{P}(A_1) \geq 1 - \varepsilon/3$ as well as some $\tau > 0$ and $N_1 \in \mathbb{N}$ such that

$$f(X_n)(\omega) \geq \tau \quad \text{for every } n \geq N_1 \text{ and } \omega \in A_1.$$

Let $j_0 = j_0(n, \omega)$ be the (random) index fulfilling $X_n(\omega) = \alpha_{j_0}^{(n)} M(A_{j_0}^{(n)})(\omega)$. Here, without loss of generality, we can assume that $A_j^{(n)} \subset E_l$ for every $n \in \mathbb{N}$ and $1 \leq j \leq k_n$. Using Proposition 3.2.4 in Goldbach (2016) again, this implies for those j and n that

$$f(M(A_j^{(n)})) = f(I(\mathbb{1}_{A_j^{(n)}})) \leq f(I(\mathbb{1}_{E_l})) = f(M(E_l)) \quad \text{a.s.}, \quad (5.16)$$

where $f(M(E_l)) \sim \Phi_\alpha(m(E_l)^{1/\alpha})$ due to (2.5). Note that $m(E_l) < \infty$, since $E_l \in \mathcal{E}_0$. Hence, there finally exist a set $A_2 \in \mathcal{A}$ with $\mathbb{P}(A_2) \geq 1 - \varepsilon/3$ and some $K > 0$ such that $f(M(E_l))(\omega) \leq K$ for every $\omega \in A_2$. Let $A := A_0 \cap A_1 \cap A_2$ and observe that $\mathbb{P}(A) \geq 1 - \varepsilon$. Moreover, for any $\omega \in A$ and $n \geq N := \max\{N_0, N_1\}$, we obtain that $\alpha_{j_0}^{(n)} \geq \tau/K$. Let

$$J_n = \{j \in \{1, \dots, k_n\} : \alpha_j^{(n)} \geq \tau/2K\} \quad \text{and} \quad \tilde{g}_n \sim (A_j^{(n)}, \alpha_j^{(n)})_{j \in J_n}.$$

Then it is clear that $X_n = I(g_n)$ and $I(\tilde{g}_n)$ coincide for every $n \geq N$ on A . In addition, if we introduce

$$Y_n := I(\tilde{g}_n) = \bigvee_{j \in J_n} \alpha_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad Y_n^* := \bigvee_{j \in J_n}^* \alpha_j^{(n)} M(A_j^{(n)}), \quad (5.17)$$

relation (5.15) can be preserved. More precisely, for every $n \geq N$ and $\omega \in A$, we have that

$$f(Y_n)(\omega) = f(X_n)(\omega) \geq (1 + \delta)f(X_n^*)(\omega) \geq (1 + \delta)f(Y_n^*)(\omega). \quad (5.18)$$

On the other hand, since the convergence $g_n \mathbb{1}_{E_l} \rightarrow g \mathbb{1}_{E_l}$ holds uniformly, we can choose some $N' \geq N$ such that, for every $m, n \geq N'$ and $s \in E$, the estimation

$$\|g_m \mathbb{1}_{E_l} - g_n \mathbb{1}_{E_l}\|_\infty := \sup_{s \in E} |g_m \mathbb{1}_{E_l}(s) - g_n \mathbb{1}_{E_l}(s)| < \min \left\{ \frac{\tau}{4K}, \frac{\varepsilon}{CK}, \frac{\tau}{4K} (\sqrt{1 + \delta} - 1) \right\} \quad (5.19)$$

is valid with C being defined as in Lemma 3.7. Moreover, we claim that

$$\forall m, n \geq N' : \quad \mathbb{P}(\|I(g_m \mathbb{1}_{E_l}) - I(g_n \mathbb{1}_{E_l})\| \geq \varepsilon) \leq \varepsilon \quad (5.20)$$

holds true. Recall that $\varepsilon > 0$ was arbitrary. Hence, it is well-known that (5.20) would imply that the sequence $(I(g_n \mathbb{1}_{E_l}))_n$ is Cauchy with respect to convergence in probability (see Corollary 6.15 in Klenke (2013) for instance) and would therefore complete the proof. In order to prove (5.20), let us assume that $m > n \geq N'$ are fixed naturals. Since $\mathbb{P}(A) \geq 1 - \varepsilon$, it suffices to show for every $\omega \in A$ that $\|I(g_m \mathbb{1}_{E_l})(\omega) - I(g_n \mathbb{1}_{E_l})(\omega)\| < \varepsilon$. For this purpose, we additionally fix $\omega \in A$ and recall that $X_m(\omega) = I(g_m \mathbb{1}_{E_l})(\omega)$ according to (5.17). At the same time, we can also use another representation for $g_n \mathbb{1}_{E_l}$, which is given by (5.14). More precisely, we have that $g_n \mathbb{1}_{E_l} \sim (A_j^{(m)}, \beta_{m,j}^{(n)})_{j=1, \dots, k_m}$, where $\beta_{m,1}^{(n)}, \dots, \beta_{m,k_m}^{(n)} \geq 0$ are appropriate coefficients. Recall that $\emptyset \notin \mathcal{P}_m$. Hence, by definition of J_m and in view of (5.19), we obtain for every $j \in \{1, \dots, k_m\} \setminus J_m$ the estimation

$$\beta_{m,j}^{(n)} \leq |\alpha_j^{(m)} - \beta_{m,j}^{(n)}| + \alpha_j^{(m)} < \|g_m \mathbb{1}_{E_l} - g_n \mathbb{1}_{E_l}\|_\infty + \frac{\tau}{2K} < \frac{3\tau}{4K} < \frac{\tau}{K}. \quad (5.21)$$

Similarly to (5.17), the previous observation suggests to consider the truncation

$$Z_n := Z_n^{(m)} := \bigvee_{j \in J_m} \beta_{m,j}^{(n)} M(A_j^{(m)})$$

and to conclude that $I(g_n \mathbb{1}_{E_l})(\omega) = Z_n(\omega)$. At this point, we neglect the fact that $I(g_n \mathbb{1}_{E_l})$ could vary on a $\mathbb{P}$ -null set by using the representation from (5.14) now. Anyway, let us summarize that the equality

$$\|I(g_m \mathbb{1}_{E_l})(\omega) - I(g_n \mathbb{1}_{E_l})(\omega)\| = \left\| \bigvee_{j \in J_m} \alpha_j^{(m)} M(A_j^{(m)})(\omega) - \bigvee_{j \in J_m} \beta_{m,j}^{(n)} M(A_j^{(m)})(\omega) \right\| \quad (5.22)$$

holds true. Then, a similar calculation as performed in (5.21), using (5.19) and the reverse triangle equality, ensures that $\beta_{m,j}^{(n)}, \alpha_j^{(m)} \geq \frac{\tau}{4K}$ for every $j \in J_m$. Hence, if we let

$$\gamma := \frac{\tau}{4K} \quad \text{as well as} \quad \rho := \|g_m \mathbb{1}_{E_l} - g_n \mathbb{1}_{E_l}\|_\infty$$

and recall (5.16), we can use Lemma 3.7 together with (5.18) and (5.19) again to conclude that (5.22) is smaller than ε . As justified before already, this gives the assertion. $\square$

Proof of Lemma 3.9 Consider $n, l \in \mathbb{N}$. Since $g_n = g_n \mathbb{1}_{E_l} \vee g_n \mathbb{1}_{E_l^c}$, Proposition 3.2.4 in Goldbach (2016) reveals that the random vectors $I(g_n \mathbb{1}_{E_l})$ and $I(g_n \mathbb{1}_{E_l^c})$ are independent and that

$$I(g_n) = I(g_n \mathbb{1}_{E_l}) \vee_f I(g_n \mathbb{1}_{E_l^c}) = I(g_n \mathbb{1}_{E_l^c}) \vee_f I(g_n \mathbb{1}_{E_l}) \quad \text{a.s.} \quad (5.23)$$

Recalling (1.3), we see that $I(g_n) \neq I(g_n \mathbb{1}_{E_l})$ is equivalent to $f(I(g_n \mathbb{1}_{E_l})) < f(I(g_n \mathbb{1}_{E_l^c}))$ in this case and that (3.6) would follow if we can prove that

$$\mathbb{P}(f(I(g_n \mathbb{1}_{E_l})) < f(I(g_n \mathbb{1}_{E_l^c}))) \rightarrow 0 \quad (\text{as } n, l \rightarrow \infty). \quad (5.24)$$

Note that $f(I(g_n \mathbb{1}_{E_l})) \sim \Phi_\alpha(\|g_n \mathbb{1}_{E_l}\|_\alpha)$ and $f(I(g_n \mathbb{1}_{E_l^c})) \sim \Phi_\alpha(\|g_n \mathbb{1}_{E_l^c}\|_\alpha)$, respectively. On the one hand, this shows that we can assume that $\|g_n \mathbb{1}_{E_l^c}\|_\alpha > 0$ (which particularly implies that $\|g\|_\alpha > 0$). On the other hand, a similar computation as performed in (3.1) yields

$$\mathbb{P}(f(I(g_n \mathbb{1}_{E_k})) < f(I(g_n \mathbb{1}_{E_k^c}))) = \frac{\|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha}{\|g_n \mathbb{1}_{E_l}\|_\alpha^\alpha + \|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha} = \left(1 + \frac{\|g_n \mathbb{1}_{E_l}\|_\alpha^\alpha}{\|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha}\right)^{-1}.$$

Hence, instead of (5.24) it suffices to show that

$$1 + \frac{\|g_n \mathbb{1}_{E_l}\|_\alpha^\alpha}{\|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha} = \frac{\int_E g_n^\alpha dm}{\int_{E_l^c} g_n^\alpha dm} = \frac{\|g_n\|_\alpha^\alpha}{\|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha} \rightarrow \infty \quad (\text{as } n, l \rightarrow \infty). \quad (5.25)$$

For this purpose, observe that we have $\|g_n\|_\alpha^\alpha \rightarrow \|g\|_\alpha^\alpha > 0$ (as $n \rightarrow \infty$) by the dominated convergence theorem. Conversely, we obtain (for every $n \in \mathbb{N}$) that

$\|g_n \mathbb{1}_{E_l^c}\|_\alpha^\alpha \leq \|g \mathbb{1}_{E_l^c}\|_\alpha^\alpha \rightarrow 0$ (as $l \rightarrow \infty$), since $E_l^c \downarrow$ with $m(\cap_{l=1}^\infty E_l^c) = 0$ and since $g \in L_+^\alpha(m)$. This implies (5.25). $\square$

Proof of Lemma 4.1 Recall (1.3) and the beginning of the proof of Lemma 3.9 above. Then, letting

$$A := \{I(g_1) \neq I(g_2) \text{ and } f(I(g_1)) = f(I(g_2))\}, \quad (5.26)$$

we have to show that $\mathbb{P}(A) = 0$. Since $g_1 \leq g_2$, there exist sequences $(g_{1,n})$ and $(g_{2,n})$ of simple functions such that $g_{1,n} \leq g_{2,n}$ and $g_{i,n} \uparrow g_i$ for $i = 1, 2$ as $n \rightarrow \infty$. Moreover, Remark 3.5 allows us to find a common sequence of partitions (each not containing $\emptyset$, see above) for $g_{1,n}$ and $g_{2,n}$, which, in addition, is consistent. More precisely, let us assume that

$$g_{1,n} \sim (A_j^{(n)}, \alpha_j^{(n)})_{j=1, \dots, k_n} \quad \text{and} \quad g_{2,n} \sim (A_j^{(n)}, \beta_j^{(n)})_{j=1, \dots, k_n},$$

respectively. In view of $g_{1,n} \leq g_{2,n}$ and $A_j^{(n)} \neq \emptyset$, we necessarily have that $\alpha_j^{(n)} \leq \beta_j^{(n)}$ for every $n \in \mathbb{N}$ and $1 \leq j \leq k_n$. Let

$$X_n := I(g_{1,n}) = \bigvee_{j=1}^{k_n} \alpha_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad Y_n := I(g_{2,n}) = \bigvee_{j=1}^{k_n} \beta_j^{(n)} M(A_j^{(n)})$$

together with

$$X_n^* := \bigvee_{j=1}^{k_n^*} \alpha_j^{(n)} M(A_j^{(n)}) \quad \text{and} \quad Y_n^* := \bigvee_{j=1}^{k_n^*} \beta_j^{(n)} M(A_j^{(n)}).$$

Anyway, Theorem 3.10 states that $I(g_{i,n})$ converges to $I(g_i)$ in probability and therefore, by passing to a suitable subsequence, a.s. Without loss of generality, we omit the consideration of this subsequence in the sequel and therefore obtain a set $B \in \mathcal{A}$ such that $\mathbb{P}(B) = 1$ and

$$\forall \omega \in B \quad \forall i = 1, 2 : \quad I(g_{i,n})(\omega) \rightarrow I(g_i)(\omega) \quad (n \rightarrow \infty). \quad (5.27)$$

Now, if we assume that $\mathbb{P}(A) =: p > 0$, we can apply Proposition 3.6 to (Y_n) , providing a set $C \in \mathcal{A}$ with $\mathbb{P}(C) \geq 1 - p/2$ as well as some $\delta > 0$ and $N \in \mathbb{N}$ fulfilling

$$f(Y_n)(\omega) \geq (1 + \delta)f(Y_n^*)(\omega) \quad \text{for every } n \geq N \text{ and } \omega \in C. \quad (5.28)$$

Note that, for certain (random) indices $j_1 = j_1(n, \omega)$ and $j_2 = j_2(n, \omega)$, we can always write

$$X_n(\omega) = \alpha_{j_1}^{(n)} M(A_{j_1}^{(n)}) \quad \text{and} \quad Y_n(\omega) = \beta_{j_2}^{(n)} M(A_{j_2}^{(n)}).$$

Moreover, observe that $\mathbb{P}(A \cap B \cap C) > 0$. Then, for fixed $\omega \in A \cap B \cap C$, we have to distinguish two cases. In the first case the indices j_1 and j_2 differ. Then, using (5.28) and $\alpha_j^{(n)} \leq \beta_j^{(n)}$, we obtain for every $n \geq N$ that

$$f(X_n)(\omega) \leq f(Y_n^*)(\omega) \leq (1 + \delta)^{-1} f(Y_n)(\omega). \quad (5.29)$$

However, by definition of the set A and by using the continuity of f together with (5.27), we verify that $f(X_n)(\omega)/f(Y_n)(\omega) \rightarrow 1$. This means that (5.29) can only

happen for finitely many n . Else, the second case occurs, where $j_1 = j_2$. By the homogeneity of f this yields

$$f(X_n(\omega) - Y_n(\omega)) = f(X_n(\omega)) - f(Y_n(\omega)) \quad \text{for almost all } n.$$

Using similar arguments as before, it follows that $f(X_n(\omega) - Y_n(\omega)) \rightarrow 0$. However, in view of Lemma 3.1.14 in Goldbach (2016), this implies that $(X_n(\omega) - Y_n(\omega)) \rightarrow 0$. Remembering that

$$I(g_1)(\omega) - I(g_2)(\omega) = \lim_{n \rightarrow \infty} (X_n(\omega) - Y_n(\omega)) \quad (n \rightarrow \infty),$$

we finally obtain that $I(g_1)(\omega) = I(g_1)(\omega)$, which is a contradiction to the claim $\omega \in A$. $\square$

Proof of Lemma 4.4 Let (g'_n) be a sequence of simple functions fulfilling $g'_n \uparrow g$, which particularly means that $I(g) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g'_n)$. As in the proof of Theorem 3.10, define a new sequence $(h_v)_{v \in \mathbb{N}}$ that alternates between (g_n) and (g'_n) . Again it follows that $(I(h_v))$ converges in probability. Actually, this gives the assertion, since all subsequences yield the same limit. More precisely,

$$\mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g_n) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(h_{2n-1}) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(h_{2n}) = \mathbb{P}\text{-}\lim_{n \rightarrow \infty} I(g'_n) = I(g).$$

$\square$

Proof of Lemma 4.5 Using homogeneity and the f -implicit monotonicity from Theorem 4.2, we first obtain that $f(I(h_{2,n})) \leq \gamma_n f(M(A))$ a.s. (recall (4.1)), which shows that $f(I(h_{2,n})) \rightarrow 0$ a.s. In view of Lemma 3.1.14 in Goldbach (2016) this also implies that $I(h_{2,n}) \rightarrow 0$ a.s. Moreover, since $I(h_{1,n} \vee h_{2,n}) = I(h_{1,n}) \vee_f I(h_{2,n})$ a.s. due to (4.2), we merely need that the $\vee_f$ -operation provides continuity a.s. For this purpose, recall the proof of the f -implicit max-linearity above or use Lemma 1.1.9 in Goldbach (2016), respectively. $\square$

References

- Biermé, H., Meerschaert, M.M., Scheffler, H.-P.: Operator scaling stable random fields. *Stoch. Process. Appl.* **117**(3), 312–332 (2007)
- Billingsley, P.: Probability and measure. Wiley, New York (2008)
- Bogachev, V.I.: Measure theory, vol. 2. Springer Science & Business Media (2007)
- de Fondeville, R., Davison, A.C.: High-dimensional peaks-over-threshold inference. *Biometrika* **105**(3), 575–592 (2018)
- Dombry, C., Ribatet, M.: Functional regular variations, pareto processes and peaks over threshold. *Stat. Interface* **8**(1), 9–17 (2015)
- Dudley, R.M.: Real analysis and probability, vol. 74. Cambridge University Press (2002)
- Elstrodt, J.: Maß- und Integrationstheorie. Springer, Berlin (2006)
- Goldbach, J.: A new approach to multivariate extreme value theory: f -implicit max-infinitely divisible distributions and f -implicit max-stable processes. PhD thesis, University of Siegen (2016)
- Klenke, A.: Probability theory: a comprehensive course. Springer Science & Business Media (2013)
- Kremer, D., Scheffler, H.-P.: Multivariate stochastic integrals with respect to independently scattered random measures on δ -rings. *Publ. Math. Debr.* **95**(1-2), 39–66 (2019)

- Li, Y., Xiao, Y.: Multivariate operator-self-similar random fields. *Stoch. Process. Appl.* **121**(6), 1178–1200 (2011)
- Rajput, B.S., Rosinski, J.: Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields* **82**(3), 451–487 (1989)
- Resnick, S.I.: Extreme values, regular variation and point processes. Springer, Berlin (2013)
- Samoradnitsky, G., Taqqu, M.S.: Stable non-Gaussian random processes: stochastic models with infinite variance. CRC press (1994)
- Scheffler, H.-P., Stoev, S.: Implicit extremes and implicit max-stable laws. *Extremes* **20**(2), 265–299 (2017)
- Stoev, S.A., Taqqu, M.S.: Extremal stochastic integrals: a parallel between max-stable processes and α -stable processes. *Extremes* **8**(4), 237–266 (2005)

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