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Managing local dependencies in asymptotic theory for maxima of stationary random fields

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Abstract
In the paper we solve the limit problem for partial maxima of m-dependent stationary random fields and we extend the obtained solution to fields satisfying some local mixing conditions. New methods for describing the limitting distribution of maxima are proposed. A notion of a phantom distribution function for a random field is investigated. As an application, several original formulas for calculation of the extremal index are provided. Moving maxima and moving averages as well as Gaussian fields satisfying the Berman condition are considered.

Keywords Stationary random fields · Extremes · m-dependence · Extremal index · Phantom distribution function · Moving averages · Moving maxima · Berman's condition

AMS 2000 Subject Classifications 60G70 · 60G60 · 60F05

1 Introduction

Leadbetter and Rootzén (1998) showed that the class of limit distributions for suitably centered and normalized partial maxima of stationary weakly dependent *random fields* coincides with the corresponding class for i.i.d. *sequences*. This observation, however, does not mean that also the normalizing and centering constants are the same as in the i.i.d case, for they may be heavily dependent on the local structure of the random field under consideration.

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For stationary weakly dependent *sequences* this phenomenon has been well understood for long time and can be easily demonstrated within the simplest non-i.i.d. class of m-dependent stationary random variables (Newell 1964). So it is rather surprising that there seems to be no existing limit theory for m-dependent stationary random *fields*. To the authors' knowledge the only attempts in this direction have been made by Turkman (2006) and Ferreira and Pereira (2008), but their results do not yield any concise method of calculating the limits and the extremal index for m-dependent random fields. In particular, the formula proposed in the latter paper does not work for the simple 1-dependent random field given in Example 5.5 below.

In Section 2 we study the impact of local dependencies on asymptotics of maxima by means of the Bonferroni-like inequality due to Jakubowski and Rosiński (1999). Theorem 2.1 obtained this way completely explains the m-dependent case in a way analogous (but not identical) to Newell's (1964) result for sequences.

The developed machinery proves to be equally effective in a substantially larger class of stationary random fields. In Proposition 2.3 we provide a condition similar to Condition $D^{(m+1)}(v_n)$ of Chernick et al. (1991) guaranteeing that the asymptotics of partial maxima of a random field can be determined on the base of tail properties of joint distribution of a *fixed finite dimension* (like in the case of m-dependent random fields). Section 3.2 serves appropriate non-m-dependent examples.

Another form of local dependency is exhibited by moving averages or moving maxima, for which detailed calculations can be found in Basrak and Tafro (2014). The asymptotics of maxima or the extremal index of such random fields involve infinitely many parameters, that cannot, in general, be deduced from any fixed finite dimensional distribution. Nevertheless the maxima of the original random field can be approximated by maxima of suitable m-dependent random fields and the limit parameters for the original random field can be obtained as limits of parameters of the approximating sequence.

The above general scheme is very close in spirit to so called L^p-m-approximability, a notion formally introduced in Hörmann and Kokoszka (2010), but possessing both long history and important applications. Of course, the idea of L^p-m-approximability, originally created for sums, could not be directly adopted to the needs of the limit theory for maxima of random fields. Therefore we first propose the proper notion of *max-m-approximability* in Definition 2.4, and then, in Theorem 2.5, we show that our abstract framework works. We also show that both moving averages and moving maxima fit with our formalism (see Section 3.1).

In Section 4 we define and discuss the notion of a phantom distribution function for random fields. Phantom distribution functions for sequences, closely related to extremal indices, were introduced by O'Brien (1987) as follows. For $\{X_n : n \in \mathbb{Z}\}$, a stationary sequence with partial maxima $M_n := \max\{X_k : 1 \le k \le n\}$, we say that any distribution function G satisfying

$$\sup_{x\in\mathbb{R}} \left| P(M_n \le x) - G(x)^n \right| \to 0, \quad \text{as} \quad n \to \infty, \tag{1}$$

is a *phantom distribution function*. In the special case, when condition (1) holds with $G(x) = P(X_0 \le x)^\theta$ for some $\theta \in (0, 1]$, we call θ the *extremal index*, following Leadbetter (1983). It is worthy to note that sometimes a phantom distribution function exists while the extremal index $\theta \in (0, 1]$ does not (see, e.g., Doukhan et al. 2015 and references therein). We refer to Doukhan et al. (2015) for the recent results on existence of phantom distribution functions.

Sections 5 and 6 are devoted to extremal indices of random fields. Basing on the results presented in Sections 2 and 4, we provide formulas for the extremal index for some classes of fields. Then we apply these formulas to calculate the extremal index for moving maxima and moving averages. It is also shown that the extremal index of a centered stationary Gaussian field satisfying the Berman condition equals 1.

In the paper, we consider a d-dimensional stationary random field $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ and its partial maxima of the form

$$M_{\mathbf{n}} := \max\{X_{\mathbf{k}} : \mathbf{1} \leq \mathbf{k} \leq \mathbf{n} \text{ coordinatewise}\}.$$

An element $\mathbf{n} \in \mathbb{Z}^d$ is often denoted by $(n_1, n_2, \ldots, n_d)$; we write $n^* := n_1 n_2 \cdots n_d$ and $\|\mathbf{n}\|$ is the sup norm; we put $\mathbf{0} := (0, 0, \ldots, 0)$, $\mathbf{1} := (1, 1, \ldots, 1)$, $\boldsymbol{\infty} := (\infty, \infty, \ldots, \infty)$.

2 Maxima of random fields satisfying some local conditions

In the forthcoming section we investigate the asymptotic behaviour of maxima under different local mixing conditions. First we present a limit theorem for m-dependent case. Then we extend the obtained result to some other classes of random fields. The notion of max-m-approximability is introduced.

2.1 Asymptotics for maxima of m-dependent fields and some generalization

We recall that $\{X_{\mathbf{n}}\}$ is *m-dependent* for some $m \in \mathbb{N}$, if families $\{X_{\mathbf{n}} : \mathbf{n} \in A\}$ and $\{X_{\mathbf{k}} : \mathbf{k} \in B\}$ are independent for every pair of finite sets $A, B \subset \mathbb{Z}^d$ satisfying

$$\min_{\mathbf{n} \in A,\, \mathbf{k} \in B} \|\mathbf{n} - \mathbf{k}\| > m,$$

where $\|\mathbf{n} - \mathbf{k}\| = \max\{|n_1 - k_1|, |n_2 - k_2|, \ldots, |n_d - k_d|\}$.

In the following, we will assume that $\{X_{\mathbf{n}}\}$ is m-dependent and the condition

$$\limsup_{n \to \infty} n^d P(X_{\mathbf{0}} > v_n) < \infty \tag{2}$$

holds for some sequence $\{v_n\} \subset \mathbb{R}$. It is not difficult to show that then for $\mathbf{N}(n) \to \boldsymbol{\infty}$ satisfying $N^*(n) := N_1(n) N_2(n) \cdots N_d(n) = O(n^d)$ we have

$$P\left(M_{\mathbf{N}(n)} \leq v_n\right) = P\left(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} \leq v_n\right)^{k_n^d} + o(1)$$

for every sequence $\{k_n\} \subset \mathbb{N}$ tending to infinity such that $k_n = o(N_l(n))$ for each l, with $\lfloor \cdot \rfloor$ the floor function. Combining the above equality with the classical fact (see, e.g., O'Brien 1987):

$$(a_n)^n - \exp(-n(1 - a_n)) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for} \quad a_n \in [0, 1],$$

we obtain that

$$P\left(M_{\mathbf{N}(n)} \leq v_n\right) = \exp\left(-k_n^d P\left(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} > v_n\right)\right) + o(1). \tag{3}$$

Theorem 2.1 *Assume that the stationary field* $\{X_{\mathbf{n}}\}$ *is* m*-dependent and* (2) *holds with some* $\{v_n\} \subset \mathbb{R}$.

(a) *If* $\mathbf{N}(n) \to \boldsymbol{\infty}$ *is such that* $N^*(n) = O(n^d)$*, then*

$$P\left(M_{\mathbf{N}(n)} \leq v_n\right) = \exp\left(-N^*(n) \cdot D(m, v_n)\right) + o(1) \tag{4}$$

with $D(m, v)$ *defined as*

$$D(m, v) := \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^d} (-1)^{\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_d} P\left(M_{(m+1,\, m+1, \ldots,\, m+1) - \boldsymbol{\varepsilon}} > v\right). \tag{5}$$

(b) *If the condition*

$$\limsup_{n \to \infty} P(M_{(n,n,\ldots,n)} \leq v_n) < 1 \tag{6}$$

is satisfied, then the asymptotic behaviour (4) *holds for every* $\mathbf{N}(n) \to \boldsymbol{\infty}$.

Proof To prove (a), let us consider $\mathbf{N}(n) \to \boldsymbol{\infty}$ and $T > 0$ such that $N^*(n) \leq T n^d$. Then (3) holds, provided that $k_n \to \infty$ slowly. Applying the Bonferroni-type inequality given by Theorem 2.1 of Jakubowski and Rosiński (1999), we can approximate the exponent in (3) as follows

$$\begin{aligned}
&\left| k_n^d P\left(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} > v_n\right) - N^*(n) D(m, v_n)\right| \\
&\leq a(m) \cdot k_n \sum_{l=1}^{d} \frac{N^*(n)}{N_l(n)} P(X_{\mathbf{0}} > v_n) \cdot (1 + o(1)) \\
&\quad + b(m) \cdot k_n^d \sum_{\substack{\mathbf{i}, \mathbf{j} \in \prod_{l=1}^{d} \{1, 2, \ldots, \lfloor N_l(n)/k_n \rfloor\}, \\ \|\mathbf{i} - \mathbf{j}\| > m}} P(X_{\mathbf{i}} > v_n, X_{\mathbf{j}} > v_n) + o(1) =: R(n)
\end{aligned} \tag{7}$$

with $a(m) := 2^{d+1} m((m+1)^d - 1)$ and $b(m) := 2^{-1} + 2^{d-1}(2m+1)^d$. Using m-dependence and other assumptions of the theorem, we conclude that the right-hand side denoted by $R(n)$ satisfies

$$\begin{aligned}
R(n) &\leq a(m) \cdot T n^d P(X_{\mathbf{0}} > v_n) \sum_{l=1}^{d} \frac{k_n}{N_l(n)} \cdot (1 + o(1)) \\
&\quad + b(m) \cdot k_n^d \left(\frac{T n^d}{k_n^d}\right)^2 P(X_{\mathbf{0}} > v_n)^2 + o(1) = o(1),
\end{aligned}$$

whenever $k_n \to \infty$ slowly. Combining the above approximation with equation (3), we complete the proof of part (a) of the theorem.

Now, let us assume that (6) holds. Observe that, since (a) is true, it is sufficient to show that (4) is satisfied for every $\mathbf{N}(n) \to \boldsymbol{\infty}$ such that $N^*(n) = T_n n^d$ with some

$T_n \to \infty$, to establish part (b) of the theorem. We will prove, using (a), that both the left and the right sides of (4) tend to zero in this case. Indeed, for the right-hand side we have

$$\exp\left(-N^*(n)\cdot D(m,v_n)\right) = \exp\left(-T_n\cdot n^d\cdot D(m,v_n)\right)$$
$$= \left(P\left(M_{(n,n,\ldots,n)}\le v_n\right)+o(1)\right)^{T_n} = o(1).$$

To show that the left-hand side also tends to zero, let us consider $\boldsymbol{\psi}(n)\to\boldsymbol{\infty}$ from Lemma 7.1 and note that

$$\limsup_{n\to\infty} P\left(M_{\mathbf{N}(n)}\le v_n\right) \le \limsup_{n\to\infty} P\left(M_{(\psi_1(n),\psi_2(n),\ldots,\psi_{d-1}(n),\ \lfloor T\cdot\psi_d(n)\rfloor)}\le v_n\right)$$
$$= \limsup_{n\to\infty}\exp\left(-T\cdot n^d\cdot D(m,v_n)\right)$$
$$= \limsup_{n\to\infty} P\left(M_{(n,n,\ldots,n)}\le v_n\right)^T$$

for arbitrary $T>0$, where we applied part (a) of the theorem in the last two relations. Hence we get $P\left(M_{\mathbf{N}(n)}\le v_n\right)=o(1)$ and the proof is complete. □

Remark 2.2 As an anonymous reviewer noticed, one can find in Theorem 2.1 the shape of standard compound Poisson approximation theorems (see, e.g., Arratia et al. 1989, Barbour and Chryssaphinou 2001). According to this approach, the random variable

$$\Lambda_n := \sum_{\mathbf{1}\le\mathbf{k}\le\mathbf{N}(n)}\ \sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d}(-1)^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_d}\,\mathbb{1}_{\{M_{\mathbf{k},\mathbf{k}+(m,m,\ldots,m)-\boldsymbol{\varepsilon}}>v_n\}},$$

with $M_{\mathbf{k},\mathbf{n}} := \max\{X_{\mathbf{j}} : \mathbf{k}\le\mathbf{j}\le\mathbf{n}\}$ and $\lambda_n := E\Lambda_n = N^*(n)\cdot D(m,v_n)$, approximates the number of clusters of exceedances over v_n in $\{\mathbf{j} : \mathbf{1}\le\mathbf{j}\le\mathbf{N}(n)\}$.

In fact, one can assume some weaker than m-dependence conditions on $\{X_{\mathbf{n}}\}$ and, by fully analogous arguments to those from the proof of Theorem 2.1, calculate limits for maxima by knowledge of $(m+1)^d$ terms.

Proposition 2.3 *Assume the stationary field $\{X_{\mathbf{n}}\}$ fulfills (2) with $\{v_n\}\subset\mathbb{R}$. Suppose that*

(i) *condition (3) and*
(ii) *the local mixing condition:*

$$k_n^d \sum_{\substack{\mathbf{i},\mathbf{j}\in\prod_{l=1}^d\{1,2,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{i}-\mathbf{j}\|>m}} P\left(X_{\mathbf{i}}>v_n, X_{\mathbf{j}}>v_n\right)\to 0, \quad as \quad n\to\infty, \tag{8}$$

are both satisfied for all $\mathbf{N}(n)\to\boldsymbol{\infty}$ such that $N^*(n)=O(n^d)$, with some $k_n\to\infty$ such that $k_n=o(N_l(n))$ for each $l\in\{1,2,\ldots,d\}$. Then statements (a) and (b) from Theorem 2.1 hold.

Proof We mime the proof of Theorem 2.1. To approximate the exponent in the right-hand side of equality (3) (guaranteed by (i)) we apply Theorem 2.1 of Jakubowski and Rosiński (1999). It is crucial to observe that the right-hand side of (7) tends to zero due to local condition (ii). □

Note that assumption (8) is in the same spirit as Condition $D^{(m+1)}(v_n)$ proposed by Chernick et al. (1991). Examples of non-m-dependent fields fulfilling (8) are given in Section 3.2.

2.2 Asymptotics of maxima of max-*m*-approximable fields

Definition 2.4 Suppose that the stationary field $\{X_{\mathbf{n}}\}$ satisfies condition (2) with some $\{v_n\} \subset \mathbb{R}$. We call $\{X_{\mathbf{n}}\}$ *max-m-approximable* if it admits the representation

$$X_{\mathbf{n}} := f(\{Z_{\mathbf{n}+\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}), \quad \text{for} \quad \mathbf{n} \in \mathbb{Z}^d, \tag{9}$$

with $\{Z_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$ a family of i.i.d. random variables and $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ a measurable function, and, moreover, for $\delta(m, v)$ defined as

$$\delta(m, v) := P\left(f(\{Z_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}) \le v,\ f(\{Z_{\mathbf{j}} : \|\mathbf{j}\| \le m\} \cup \{Z'_{\mathbf{j}} : \|\mathbf{j}\| > m\}) > v\right),$$

with $\{Z'_{\mathbf{j}}\}$ an independent copy of $\{Z_{\mathbf{j}}\}$, the following property

$$\limsup_{m\to\infty} \limsup_{n\to\infty} n^d \delta(m, v_n) = 0 \tag{10}$$

holds.

Investigating max-m-approximable fields, we will often deal with the case when there exists a sequence $r_n \to \infty$ such that $a(m_n, n) \to 0$, as $n \to \infty$, holds for every $m_n \to \infty$ satisfying $m_n \le r_n$, for some $a(m, n) \in \mathbb{R}$. In this case we will say that $a(m_n, n) \to 0$ for all $m_n \to \infty$ *sufficiently slowly*.

From Lemma 7.2 we know that assumption (10) ensures that $n^d \delta(m_n, v_n) \to 0$, as $n \to \infty$, for $m_n \to \infty$ sufficiently slowly.

In fact, partial maxima $M_{\mathbf{n}}$ of max-m-approximable $\{X_{\mathbf{n}}\}$ may be approximated by maxima $M_{\mathbf{n}}^{[m]}$ of the field $\{X_{\mathbf{n}}^{[m]}\}$ defined for $m \in \mathbb{N}$ as follows

$$X_{\mathbf{n}}^{[m]} := f\left(\{Z_{\mathbf{n}+\mathbf{j}} : \|\mathbf{j}\| \le m\} \cup \{Z^{(\mathbf{n})}_{\mathbf{n}+\mathbf{j}} : \|\mathbf{j}\| > m\}\right), \quad \text{for} \quad \mathbf{n} \in \mathbb{Z}^d,$$

where $\{Z^{(\mathbf{n})}_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}$, for $\mathbf{n} \in \mathbb{Z}^d$, are independent, also of $\{Z_{\mathbf{j}}\}$, copies of $\{Z_{\mathbf{j}}\}$. The field $\{X_{\mathbf{n}}^{[m]}\}$ is $(2m)$-dependent and stationary, moreover, has the same marginal distribution as $\{X_{\mathbf{n}}\}$. Observe that applying the elementary inequality

$$\left| P\left(\bigcap_{\mathbf{1}\le\mathbf{n}\le\mathbf{N}(n)} A_{\mathbf{n}}\right) - P\left(\bigcap_{\mathbf{1}\le\mathbf{n}\le\mathbf{N}(n)} B_{\mathbf{n}}\right)\right| \le \sum_{\mathbf{1}\le\mathbf{n}\le\mathbf{N}(n)} P(A_{\mathbf{n}} \triangle B_{\mathbf{n}}),$$

with $\triangle$ the symmetric difference, for $A_{\mathbf{n}} := \{X_{\mathbf{n}} \le v_n\}$ and $B_{\mathbf{n}} := \{X_{\mathbf{n}}^{[m]} \le v_n\}$, then using stationarity of the fields $\{X_{\mathbf{n}}\}$ and $\{X_{\mathbf{n}}^{[m]}\}$ and the fact that random vectors $(X_{\mathbf{0}}, X_{\mathbf{0}}^{[m]})$ and $(X_{\mathbf{0}}^{[m]}, X_{\mathbf{0}})$ are equal in distribution, we obtain that

$$\begin{aligned}
&\left|P(M_{\mathbf{N}(n)} \le v_n) - P\left(M_{\mathbf{N}(n)}^{[m]} \le v_n\right)\right| \\
&\le N^*(n)P\left(X_{\mathbf{0}} \le v_n, X_{\mathbf{0}}^{[m]} > v_n\right) + N^*(n)P\left(X_{\mathbf{0}} > v_n, X_{\mathbf{0}}^{[m]} \le v_n\right) \qquad (11)\\
&= 2N^*(n)P\left(X_{\mathbf{0}} \le v_n, X_{\mathbf{0}}^{[m]} > v_n\right)\\
&= 2N^*(n)P\left(f(\{Z_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}) \le v_n, f(\{Z_{\mathbf{j}} : \|\mathbf{j}\| \le m\} \cup \{Z'_{\mathbf{j}} : \|\mathbf{j}\| > m\}) > v_n\right)\\
&= 2N^*(n)\delta(m, v_n).
\end{aligned}$$

Note that for $N^*(n) = O(n^d)$ and $m_n \to \infty$ sufficiently slowly, under assumption (10), the right-hand side of (11) tends to zero.

Theorem 2.5 *Assume that the stationary field* $\{X_{\mathbf{n}}\}$ *fulfills* (2) *and* (10) *with some sequence* $\{v_n\} \subset \mathbb{R}$.

(a) *If* $\mathbf{N}(n) \to \boldsymbol{\infty}$ *satisfies* $N^*(n) = O(n^d)$, *then*

$$P\left(M_{\mathbf{N}(n)} \le v_n\right) = \exp\left(-N^*(n) \cdot \tilde{D}(2m_n, v_n)\right) + o(1) \qquad (12)$$

for every sequence $\{m_n\} \subset \mathbb{N}$ *tending to infinity sufficiently slowly, with* $\tilde{D}(2m, v)$ *given by*

$$\tilde{D}(2m, v) := \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^d} (-1)^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_d} P\left(M_{(2m+1,2m+1,\ldots,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v\right). \qquad (13)$$

(b) *If the condition*

$$\limsup_{n\to\infty} P(M_{(n,n,\ldots,n)} \le v_n) < 1$$

holds, then (12) *is true for all* $\mathbf{N}(n) \to \boldsymbol{\infty}$.

Proof To prove (a), let us consider $\mathbf{N}(n) \to \boldsymbol{\infty}$ such that $N^*(n) \le Tn^d$ for some $T > 0$. Then, by (11), we know that

$$\left|P(M_{\mathbf{N}(n)} \le v_n) - P\left(M_{\mathbf{N}(n)}^{[m]} \le v_n\right)\right| \le 2Tn^d\delta(m, v_n).$$

Since $X_{\mathbf{0}} \stackrel{d}{=} X_{\mathbf{0}}^{[m]}$,

$$\limsup_{n\to\infty} n^d P\left(X_{\mathbf{0}}^{[m]} > v_n\right) = \limsup_{n\to\infty} n^d P\left(X_{\mathbf{0}} > v_n\right) < \infty$$

holds. Applying Theorem 2.1 for the field $\{X_{\mathbf{n}}^{[m]}\}$, we obtain

$$P\left(M_{\mathbf{N}(n)}^{[m]} \le v_n\right) - \exp\left(-N^*(n) \cdot \tilde{D}(2m, v_n)\right) = o(1).$$

Combining the above results with (10) gives

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{n\to\infty}\left|P\left(M_{\mathbf{N}(n)}\le v_n\right)-\exp\left(-N^*(n)\cdot\tilde{D}(2m,v_n)\right)\right|\\
&\le \limsup_{m\to\infty}\limsup_{n\to\infty}\left|P\left(M_{\mathbf{N}(n)}\le v_n\right)-P\left(M^{[m]}_{\mathbf{N}(n)}\le v_n\right)\right|\\
&\qquad+\ \limsup_{m\to\infty}\limsup_{n\to\infty}\left|P\left(M^{[m]}_{\mathbf{N}(n)}\le v_n\right)-\exp\left(-N^*(n)\cdot\tilde{D}(2m,v_n)\right)\right|\\
&\le \limsup_{m\to\infty}\limsup_{n\to\infty} 2Tn^d\delta(m,v_n)+0\\
&= 0.
\end{aligned}$$

By Lemma 7.2, it follows that

$$\lim_{n\to\infty}\left|P\left(M_{\mathbf{N}(n)}\le v_n\right)-\exp\left(-N^*(n)\cdot\tilde{D}(2m_n,v_n)\right)\right|=0$$

for every $m_n\to\infty$ sufficiently slowly. This completes the proof of (a). By analogous arguments to those from the proof of Theorem 2.1(b), one can show that also part (b) holds. □

Corollary 2.6 *If* (2) *and* (10) *are satisfied and, moreover,*

$$\lim_{n\to\infty}P\left(M^{[m]}_{(n,n,\ldots,n)}\le v_n\right)=\gamma_m \quad \textit{and} \quad \lim_{m\to\infty}\gamma_m=\gamma$$

for some $\gamma,\gamma_1,\gamma_2,\ldots\in(0,1)$, *then*

$$\lim_{n\to\infty}P\left(M_{(n,n,\ldots,n)}\le v_n\right)=\gamma.$$

Corollary 2.7 *If* (2) *and* (10) *hold and, moreover, D and $\tilde{D}$ given by* (5) *and* (13), *respectively, satisfy*

$$\limsup_{m\to\infty}\limsup_{n\to\infty} N^*(n)\cdot|\tilde{D}(2m,v_n)-D(2m,v_n)|=0, \tag{14}$$

then, for every sequence $\{m_n\}\subset\mathbb{N}$ *tending to infinity sufficiently slowly,*

$$P\left(M_{\mathbf{N}(n)}\le v_n\right)=\exp\left(-N^*(n)\cdot D(2m_n,v_n)\right)+o(1).$$

Property (14) *is guaranteed by, e.g., the strengthened condition* (10)*:*

$$\limsup_{m\to\infty}\limsup_{n\to\infty} m^dn^d\delta(m,v_n)=0.$$

In Section 3.1 we show that both moving maxima and moving averages belong to the class of max-m-approximable fields.

3 Examples: random fields fulfilling local conditions

3.1 Moving maxima and moving averages as max-m-approximable fields

We will show that random fields of moving maxima and moving averages, built on fields of i.i.d. regular variables, are max-m-approximable.

In the following, $\{Z_{\mathbf{n}}\}$ is an array of i.i.d. random variables satisfying for some index $\alpha > 0$ and slowly varying function L

$$P(|Z_{\mathbf{0}}| > x) = x^{-\alpha} L(x) \tag{15}$$

and the tail balance condition

$$\frac{P(Z_{\mathbf{0}} > x)}{P(|Z_{\mathbf{0}}| > x)} = p \quad \text{as} \quad x \to \infty, \quad \text{for some} \quad p \in [0, 1], \tag{16}$$

is assumed. We define $a_n := \inf\{y > 0 : P(|Z_{\mathbf{0}}| > y) \le n^{-d}\}$ and $v_n := a_n v$ with some fixed $v > 0$. Then we have

$$n^d P(|Z_{\mathbf{0}}| > v_n) \to v^{-\alpha} \quad \text{as} \quad n \to \infty.$$

For definitions of moving maxima and moving averages, we will need some weights $c_{\mathbf{j}} \in \mathbb{R}$, $\mathbf{j} \in \mathbb{Z}^d$, satisfying appropriate conditions. The trivial case $c_{\mathbf{j}} = 0$ for all $\mathbf{j} \in \mathbb{Z}^d$ is excluded.

3.1.1 Moving maxima

Let $\{X_{\mathbf{n}}\}$ be defined by (9) with f given by $f(\{z_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}) := \max_{\mathbf{j} \in \mathbb{Z}^d} c_{\mathbf{j}} z_{\mathbf{j}}$, i.e.

$$X_{\mathbf{n}} = \max_{\mathbf{j} \in \mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{n}+\mathbf{j}},$$

where

$$\sum_{\mathbf{j} \in \mathbb{Z}^d} |c_{\mathbf{j}}|^{\beta} < \infty \quad \text{for some} \quad 0 < \beta < \alpha.$$

Then $\{X_{\mathbf{n}}\}$ is well defined and

$$\lim_{x \to \infty} \frac{P(\max_{\mathbf{j} \in \mathbb{Z}^d} |c_{\mathbf{j}} Z_{\mathbf{j}}| > x)}{P(|Z_{\mathbf{0}}| > x)} = \lim_{x \to \infty} \frac{1 - \prod_{\mathbf{j} \in \mathbb{Z}^d} P(|c_{\mathbf{j}} Z_{\mathbf{j}}| \le x)}{P(|Z_{\mathbf{0}}| > x)} = \sum_{\mathbf{j} \in \mathbb{Z}^d} |c_{\mathbf{j}}|^{\alpha} < \infty,$$

due to Lemma 2.2 from Cline (1983). Moreover, as a corollary, we obtain that the convergence

$$\lim_{x \to \infty} \frac{P(X_{\mathbf{0}} > x)}{P(|Z_{\mathbf{0}}| > x)} = \lim_{x \to \infty} \frac{P\left(\max_{\mathbf{j} \in \mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{j}} > x\right)}{P(|Z_{\mathbf{0}}| > x)} = p \sum_{c_{\mathbf{j}} > 0} c_{\mathbf{j}}^{\alpha} + q \sum_{c_{\mathbf{j}} < 0} |c_{\mathbf{j}}|^{\alpha} \tag{17}$$

with $q := 1 - p$ holds.

The field $\{X_{\mathbf{n}}\}$, called the *moving maximum field*, is max-m-approximable. Indeed, applying the above properties, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} n^d \delta(m, v_n) &\le \limsup_{n\to\infty} n^d P\left(\max_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n\right)\\ &\le \limsup_{n\to\infty} n^d P\left(\max_{\|\mathbf{j}\|>m} |c_{\mathbf{j}} Z_{\mathbf{j}}| > v_n\right)\\ &= \limsup_{n\to\infty} n^d P(|Z_{\mathbf{0}}| > v_n) \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^\alpha\\ &= v^{-\alpha} \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^\alpha.\end{aligned}$$

Since the right-hand side tends to zero as $m \to \infty$, condition (10) holds.

3.1.2 Moving averages

Consider $\{X_{\mathbf{n}}\}$ defined by (9) with f given by $f(\{z_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^d\}) := \sum_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}} z_{\mathbf{j}}$, i.e.

$$X_{\mathbf{n}} = \sum_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{j}},$$

where

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} |c_{\mathbf{j}}|^\beta < \infty \quad \text{for some} \quad 0 < \beta < \min\{\alpha, 1\}.$$

Then, due to Theorems 2.1 and 2.3 from Cline (1983), the field $\{X_{\mathbf{n}}\}$ is well defined and we have

$$\lim_{x\to\infty} \frac{P(|X_{\mathbf{0}}| > x)}{P(|Z_{\mathbf{0}}| > x)} = \lim_{x\to\infty} \frac{P(|\sum_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{j}}| > x)}{P(|Z_{\mathbf{0}}| > x)} = \sum_{\mathbf{j}\in\mathbb{Z}^d} |c_{\mathbf{j}}|^\alpha < \infty. \tag{18}$$

Moreover, by Lemma A3.26 from Embrechts et al. (2003),

$$\lim_{x\to\infty} \frac{P(X_{\mathbf{0}} > x)}{P(|Z_{\mathbf{0}}| > x)} = \lim_{x\to\infty} \frac{P(\sum_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{j}} > x)}{P(|Z_{\mathbf{0}}| > x)} = p \sum_{c_{\mathbf{j}}>0} c_{\mathbf{j}}^\alpha + q \sum_{c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^\alpha. \tag{19}$$

It will be shown that $\{X_{\mathbf{n}}\}$, called the *moving average field*, is max-m-approximable. First, we note that

$$\begin{aligned} n^d \delta(m, v_n) &\le n^d P\left(\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n, \sum_{\mathbf{j}\in\mathbb{Z}^d} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n\right)\\ &\quad + n^d P\left(\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z'_{\mathbf{j}} > v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \qquad (20)\\ &=: R_1(m, v_n) + R_2(m, v_n), \end{aligned}$$

with $\{Z'_{\mathbf{n}}\}$ an independent copy of $\{Z_{\mathbf{n}}\}$. For $m \in \mathbb{N}$, let us choose $t_m > 0$ so that

$$(t_m)^{2\alpha} = \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^{\alpha}.$$

For the first summand in (20), we have

$$\begin{aligned}
R_1(m, v_n) &= n^d P\left(\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \\
&\le n^d P\left(\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} > (1+t_m)v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \\
&\quad + n^d P\left((1+t_m)v_n \ge \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \\
&\le n^d P\left(\sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}} < -t_m v_n\right) + n^d P\left((1+t_m)v_n \ge \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n\right) \\
&\le n^d P\left(\left|\sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z_{\mathbf{j}}\right| > t_m v_n\right) + n^d P\left((1+t_m)v_n \ge \left|\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right| > v_n\right).
\end{aligned}$$

Then, by property (18) and the choice of v_n, we conclude that for all $m \in \mathbb{N}$

$$\begin{aligned}
R_1(m, v_n) &\le n^d P(|Z_{\mathbf{0}}| > t_m v_n) \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^{\alpha} \\
&\quad + n^d (P(|Z_{\mathbf{0}}| > v_n) - P(|Z_{\mathbf{0}}| > (1+t_m)v_n)) \sum_{\|\mathbf{j}\|\le m} |c_{\mathbf{j}}|^{\alpha} + o(1) \\
&= v^{-\alpha} t_m^{-\alpha} \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^{\alpha} + v^{-\alpha}(1 - (1+t_m)^{-\alpha}) \sum_{\|\mathbf{j}\|\le m} |c_{\mathbf{j}}|^{\alpha} + o(1),
\end{aligned}$$

as $n \to \infty$. Using similar arguments for the second summand in (20), we get

$$\begin{aligned}
R_2(m, v_n) &= n^d P\left(\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} \le (1-t_m)v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z'_{\mathbf{j}} > v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \\
&\quad + n^d P\left((1-t_m)v_n < \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n, \sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z'_{\mathbf{j}} > v_n - \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right) \\
&\le n^d P\left(\sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z'_{\mathbf{j}} > t_m v_n\right) + n^d P\left((1-t_m)v_n < \sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}} \le v_n\right) \\
&\le n^d P\left(\left|\sum_{\|\mathbf{j}\|>m} c_{\mathbf{j}} Z'_{\mathbf{j}}\right| > t_m v_n\right) + n^d P\left((1-t_m)v_n < \left|\sum_{\|\mathbf{j}\|\le m} c_{\mathbf{j}} Z_{\mathbf{j}}\right| \le v_n\right).
\end{aligned}$$

Applying (18) and the definition of v_n again, we obtain that

$$\begin{aligned}R_2(m, v_n) &\le n^d P(|Z_{\mathbf{0}}| > t_m v_n) \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^\alpha \\ &\quad + n^d (P(|Z_{\mathbf{0}}| > (1-t_m)v_n) - P(|Z_{\mathbf{0}}| > v_n)) \sum_{\|\mathbf{j}\|\le m} |c_{\mathbf{j}}|^\alpha + o(1) \\ &= v^{-\alpha} t_m^{-\alpha} \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^\alpha + v^{-\alpha}((1-t_m)^{-\alpha} - 1) \sum_{\|\mathbf{j}\|\le m} |c_{\mathbf{j}}|^\alpha + o(1),\end{aligned}$$

as $n \to \infty$. Summarizing,

$$\begin{aligned}\limsup_{n\to\infty} n^d \delta(m, v_n) &\le \limsup_{n\to\infty} (R_1(m, v_n) + R_2(m, v_n)) \\ &\le 2v^{-\alpha} t_m^{-\alpha} \sum_{\|\mathbf{j}\|>m} |c_{\mathbf{j}}|^\alpha \\ &\quad + v^{-\alpha} \left((1-t_m)^{-\alpha} - (1+t_m)^{-\alpha}\right) \sum_{\mathbf{j}\in\mathbb{Z}^d} |c_{\mathbf{j}}|^\alpha.\end{aligned}$$

To complete the proof, let us observe that the choice of t_m entails that the right-hand side of the above inequality tends to zero, as $m \to \infty$, which implies (10).

3.2 Weakly dependent Gaussian field and moving maxima built on it

In this section, we give three examples of random fields which are not m-dependent for any $m \in \mathbb{N}$, but satisfy the local condition (8) with some $m \in \mathbb{N}$.

3.2.1 Weakly dependent Gaussian field

Let $\{W_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ be a centered stationary Gaussian random field with correlation $r(\mathbf{n}) := Cov(W_{\mathbf{0}}, W_{\mathbf{n}})$ satisfying the multidimensional Berman condition

$$r(\mathbf{n}) \cdot \log \|\mathbf{n}\| \to 0 \quad \text{as} \quad \|\mathbf{n}\| \to \infty. \tag{21}$$

We denote $\Phi(x) := P(W_{\mathbf{0}} \le x)$ and $\bar{\Phi}(x) := 1 - \Phi(x)$. The following adaptation of Lemma 4.3.2 in Leadbetter et al. (1983) will play an important role.

Lemma 3.1 *Let $r(\mathbf{n})$ defined above satisfy (21) and let a sequence $\{u_n : n \in \mathbb{N}\} \subset \mathbb{R}$ be such that $\{n^d \bar{\Phi}(u_n) : n \in \mathbb{N}\}$ is bounded. Then*

$$n^d \sum_{\mathbf{j}\in\prod_{l=1}^d \{-N_l(n),\dots,N_l(n)\}\setminus\{\mathbf{0}\}} |r_{\max}(\mathbf{j})| \exp\left(-\frac{u_n^2}{1+|r_{\max}(\mathbf{j})|}\right) \to 0, \quad \text{as} \quad n \to \infty,$$

for every $\mathbf{N}(n) = O(n^d)$, with $r_{\max}(\mathbf{j}) := \max\{r(\mathbf{n}) : \|\mathbf{n}\| \ge \|\mathbf{j}\|\}$.

Let $\{v_n\}$ be chosen so that $\limsup_{n\to\infty} n^d P(W_{\mathbf{0}} > v_n) < \infty$. Put $X_{\mathbf{n}} := W_{\mathbf{n}}$. Then (3) holds for every $\mathbf{N}(n) \to \boldsymbol{\infty}$ with some $k_n \to \infty$. Moreover, condition (8) is satisfied with $m = 0$. We omit the proof of these facts since they can be easily deduced from considerations (involving Lemma 3.1) presented in Section 3.2.3.

It follows that the extremal index of the weakly dependent Gaussian field $\{W_{\mathbf{n}}\}$ exists and equals 1; for details see Section 6.3.

3.2.2 Moving maxima built on a weakly dependent Gaussian field

Consider $\{W_{\mathbf{n}}\}$ the Gaussian field from Section 3.2.1. We define $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ as

$$X_{\mathbf{n}} := \max\{c_{\mathbf{j}} W_{\mathbf{n}+\mathbf{j}} : \mathbf{0} \le \mathbf{j} \le (m, m, \dots, m)\},$$

for some $c_{\mathbf{j}} \in \{-1, 0, 1\}$ and $m \in \mathbb{N}$, and assume that $c_{\max} := \max\{|c_{\mathbf{j}}|\} = 1$.

Let $\{v_n\}$ satisfy $\limsup_{n\to\infty} n^d P(W_{\mathbf{0}} > v_n) < \infty$. Then (3) holds for each $\mathbf{N}(n) \to \boldsymbol{\infty}$ with some $k_n \to \infty$ and (8) is true with m. The proof of this fact follows from the results of Section 3.2.3.

3.2.3 Moving maxima built on a transformed weakly dependent Gaussian field

Let $\{W_{\mathbf{n}}\}$ be the Gaussian field from Section 3.2.1. For fixed $\alpha > 0$ and for $h : \mathbb{R} \to \mathbb{R}$ an increasing, odd, bijective function given by

$$h(x) := \begin{cases} (2\bar{\Phi}(x))^{-1/\alpha} - 1 & \text{, if } x \ge 0; \\ 1 - (2\Phi(x))^{-1/\alpha} & \text{, if } x < 0, \end{cases}$$

we define $\{Z_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ by $Z_{\mathbf{n}} := h(W_{\mathbf{n}})$. Observe that then we have

$$P(Z_{\mathbf{0}} > x) = P(Z_{\mathbf{0}} < -x) = \frac{(1+x)^{-\alpha}}{2}, \quad \text{for any} \quad x > 0,$$

and hence conditions (15) and (16) are fulfilled.

We define $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ as $X_{\mathbf{n}} := \max\{c_{\mathbf{j}} Z_{\mathbf{n}+\mathbf{j}} : \mathbf{0} \le \mathbf{j} \le (m, m, \dots, m)\}$, for some $c_{\mathbf{j}} \in \mathbb{R}$, and assume that $c_{\max} := \max\{|c_{\mathbf{j}}|\} > 0$.

Let a sequence $\{v_n\} \subset \mathbb{R}$ be chosen so that $\limsup_{n\to\infty} n^d P(Z_{\mathbf{0}} > v_n) < \infty$ holds. Then the condition $\limsup_{n\to\infty} n^d \bar{\Phi}(u_n^{(c)}) < \infty$ is also satisfied for $u_n^{(c)} := h^{-1}(v_n/c)$, for every $c > 0$.

From the results for moving maxima recalled in Section 3.1.1 we know that if $\{Z_{\mathbf{n}}\}$ were independent then (2) would be true. To prove property (2) in the considered weakly dependent Gaussian setting, one can apply The Normal Comparison Lemma (see, e.g., Theorem 4.2.1 and its Corollary 4.2.4 in Leadbetter et al. 1983).

The field $\{X_{\mathbf{n}}\}$ defined above fulfills conditions (3) and (8) with m, for each $\mathbf{N}(n) \to \boldsymbol{\infty}$ with some $k_n \to \infty$. Both of them may be successfully verified by use of Lemma 3.1 (compare with the proof of Lemma 4.4.1 in Leadbetter et al. 1983).

Here we shall show (8). In order to do this, let us consider arbitrary $\mathbf{N}(n) \to \boldsymbol{\infty}$ such that $N^*(n) = O(n^d)$ and $k_n \to \infty$. Observe that for large n we have

$$
\begin{aligned}
k_n^d & \sum_{\substack{\mathbf{i},\mathbf{j}\in\prod_{l=1}^d\{1,2,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{i}-\mathbf{j}\|>m}} P(X_\mathbf{i} > v_n, X_\mathbf{j} > v_n) \\
\le\ & C_1 n^d \sum_{\substack{\mathbf{j}\in\prod_{l=1}^d\{-\lfloor N_l(n)/k_n\rfloor,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{j}\|>m}} P(X_\mathbf{j} > v_n, X_\mathbf{0} > v_n) \\
\le\ & C_1 n^d (1+m)^{2d} \sum_{\substack{\mathbf{j}\in\prod_{l=1}^d\{-\lfloor N_l(n)/k_n\rfloor,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{j}\|>m}} \sup_{\substack{\mathbf{i}\in\mathbb{Z}^d,\\ \|\mathbf{i}\|\ge\|\mathbf{j}\|-m}} P\left(|Z_\mathbf{i}| > \frac{v_n}{c_{\max}}, |Z_\mathbf{0}| > \frac{v_n}{c_{\max}}\right) \\
\le\ & C_2 n^d \sum_{\substack{\mathbf{j}\in\prod_{l=1}^d\{-\lfloor N_l(n)/k_n\rfloor,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{j}\|>m}} \sup_{\substack{\mathbf{i}\in\mathbb{Z}^d,\\ \|\mathbf{i}\|\ge\|\mathbf{j}\|-m}} P\left(|W_\mathbf{i}| > u_n^{(c_{\max})}, |W_\mathbf{0}| > u_n^{(c_{\max})}\right) \\
\le\ & C_3 n^d \frac{n^d}{k_n^d} \bar{\Phi}\left(u_n^{(c_{\max})}\right)^2 \\
& + C_4 n^d \sum_{\substack{\mathbf{j}\in\prod_{l=1}^d\{-\lfloor N_l(n)/k_n\rfloor,\ldots,\lfloor N_l(n)/k_n\rfloor\},\\ \|\mathbf{j}\|>m}} \sup_{\substack{\mathbf{i}\in\mathbb{Z}^d,\\ \|\mathbf{i}\|\ge\|\mathbf{j}\|-m}} |r(\mathbf{i})| \exp\left(-\frac{\left(u_n^{(c_{\max})}\right)^2}{1+|r(\mathbf{i})|}\right) \\
\le\ & \frac{C_5}{k_n^d} + C_6 n^d \sum_{\mathbf{j}\in\prod_{l=1}^d\{-N_l(n),\ldots,N_l(n)\}\setminus\{\mathbf{0}\}} |r_{\max}(\mathbf{j})| \exp\left(-\frac{\left(u_n^{(c_{\max})}\right)^2}{1+|r_{\max}(\mathbf{j})|}\right),
\end{aligned}
$$

for some constants $C_1, C_2, C_3, C_4, C_5, C_6 > 0$, where the first three relations follow from the definition of $\{X_\mathbf{n}\}$, the fourth relation is a consequence of Corollary 4.2.4 in Leadbetter et al. (1983) and in the last one we use the fact that $n^d\bar{\Phi}(u_n^{(c_{\max})})$ is bounded. Finally, since Lemma 3.1 implies that the right-hand side tends to zero as $n \to \infty$, condition (8) holds.

4 Phantom distribution function

In this section a notion of a phantom distribution function is introduced and some consequences of the results from Section 2 are concluded.

Definition 4.1 We call any distribution function G a *phantom distribution function* for $\{X_\mathbf{n}\}$, whenever

$$
\sup_{x\in\mathbb{R}} \left| P\left(M_\mathbf{n} \le x\right) - G(x)^{n_1 n_2 \cdots n_d} \right| \to 0 \quad \text{as} \quad \mathbf{n} \to \boldsymbol{\infty} \quad \text{coordinatewise.} \tag{22}
$$

We will assume that the field $\{X_{\mathbf{n}}\}$ satisfies the following condition

$$P\left(M_{\boldsymbol{\psi}(n)} \leq v_n\right) \to \gamma \quad \text{as} \quad n \to \infty, \quad \text{for some} \quad \gamma \in (0, 1), \tag{23}$$

with $\{v_n\} \subset \mathbb{R}$ a *nondecreasing* sequence, where

$$\boldsymbol{\psi} = \{\boldsymbol{\psi}(n)\} = \{(\psi_1(n), \psi_2(n), \ldots, \psi_d(n))\} \subset \mathbb{N}^d$$

is a fixed sequence such that

$$\boldsymbol{\psi}(n) \to \boldsymbol{\infty} \text{ and } \psi^*(n) := \psi_1(n)\psi_2(n)\cdots\psi_d(n) \sim n^d. \tag{24}$$

Remark 4.2 If $\{X_{\mathbf{n}}\}$ is m-dependent or max-m-approximable and satisfies (23), then condition (2) holds.

Proof Observe that for m-dependent $\{X_{\mathbf{n}}\}$ we have

$$P\left(M_{\boldsymbol{\psi}(n)} \leq v_n\right) \leq P(X_{\mathbf{0}} \leq v_n)^{\prod_{i=1}^d \lfloor \psi_i(n)/(m+1) \rfloor} = \exp\left(-n^d P(X_{\mathbf{0}} > v_n) \frac{1+o(1)}{(m+1)^d}\right)$$

and the remark easily follows. If $\{X_{\mathbf{n}}\}$ is max-m-approximable, then (23) combined with (11) entails that maxima of the $(2m)$-dependent field $\{X_{\mathbf{n}}^{[m]}\}$ satisfy

$$0 < \liminf_{n\to\infty} P\left(M_{\boldsymbol{\psi}(n)}^{[m]} \leq v_n\right) \leq \limsup_{n\to\infty} P\left(M_{\boldsymbol{\psi}(n)}^{[m]} \leq v_n\right) < 1,$$

for every large m. By already used arguments, keeping in mind that $X_{\mathbf{0}}$ and $X_{\mathbf{0}}^{[m]}$ are equal in distribution, we obtain

$$\limsup_{n\to\infty} n^d P(X_{\mathbf{0}} > v_n) = \limsup_{n\to\infty} n^d P\left(X_{\mathbf{0}}^{[m]} > v_n\right) < \infty$$

and thus (2) is satisfied. □

Both (23) and (24) provide the following construction of a candidate for a phantom distribution function for $\{X_{\mathbf{n}}\}$:

$$G(x) := \begin{cases} 0 & \text{for} \quad x < v_1; \\ \gamma^{1/n^d} & \text{for} \quad x \in [v_n, v_{n+1}); \\ 1 & \text{for} \quad x \geq v_\infty := \sup\{v_n : n \in \mathbb{N}\}. \end{cases} \tag{25}$$

If $d = 1$, the above formula reduces to the recipe for the phantom distribution function given in Theorem 1.3 of Jakubowski (1991) (see also O'Brien 1987 and Jakubowski 1993). Observe that G defined by (25) is *regular* in the sense of O'Brien (1974), i.e.

$$G(G_*-) = 1 \quad \text{and} \quad \lim_{x\to G_*-} \frac{1-G(x-)}{1-G(x)} = 1,$$

with G_* the right end of G. One should also notice that G is strictly tail-equivalent to its *continuous* modification $\tilde{G}$ that can be defined similarly as in Theorem 2 by Doukhan et al. (2015).

Using the results from Section 2, we establish the following theorem on phantom distribution functions.

Theorem 4.3 *If* $\{X_{\mathbf{n}}\}$ *satisfies* (23) *and*

(i) *fulfills the assumptions of Proposition 2.3, in particular if it is* m*-dependent,*

or

(ii) *is max-*m*-approximable,*

then G *given by formula* (25) *is a phantom distribution function for* $\{X_{\mathbf{n}}\}$.

Proof Let us assume (23) and (i). Observe that it is sufficient to show that

$$P\left(M_{\mathbf{N}(n)} \leq x_n\right) - G(x_n)^{N^*(n)} \to 0, \quad \text{as} \quad n \to \infty, \tag{26}$$

holds, whenever $\mathbf{N}(n) \to \boldsymbol{\infty}$ and $\{x_n\} \subset \mathbb{R}$ is an arbitrary nondecreasing sequence. Note that both in the case $\lim_{n\to\infty} x_n < v_\infty$ and when $\lim_{n\to\infty} x_n > v_\infty$ convergence (26) is obvious.

Let us consider $\mathbf{N}(n) \to \boldsymbol{\infty}$ and nondecreasing $x_n \to v_\infty$. Let $k(x_n) \in \mathbb{N}$ be chosen so that $v_{k(x_n)} \leq x_n < v_{k(x_n)+1}$. Then $k(x_n) \to \infty$ and we have

$$P\left(M_{\mathbf{N}(n)} \leq v_{k(x_n)}\right) \leq P\left(M_{\mathbf{N}(n)} \leq x_n\right) \leq P\left(M_{\mathbf{N}(n)} \leq v_{k(x_n)+1}\right).$$

By Proposition 2.3 we obtain

$$\begin{aligned} P\left(M_{\mathbf{N}(n)} \leq v_{k(x_n)}\right) &= \exp\left(-N^*(n) \cdot D\left(m, v_{k(x_n)}\right)\right) + o(1) \\ &= \left(\exp\left(-k(x_n)^d \cdot D\left(m, v_{k(x_n)}\right)\right)\right)^{N^*(n)/k(x_n)^d} + o(1) \\ &= \gamma^{\,N^*(n)/k(x_n)^d} + o(1). \end{aligned}$$

Similarly, one can show that

$$P\left(M_{\mathbf{N}(n)} \leq v_{k(x_n)+1}\right) = \gamma^{\,N^*(n)/(k(x_n)+1)^d} + o(1).$$

Since $\gamma^{\,N^*(n)/(k(x_n)+1)^d} - \gamma^{\,N^*(n)/k(x_n)^d} = o(1)$, we conclude that

$$P\left(M_{\mathbf{N}(n)} \leq x_n\right) = \gamma^{\,N^*(n)/k(x_n)^d} + o(1) = G(x_n)^{N^*(n)} + o(1)$$

and thus (26) holds.

By similar arguments to those above, applying Theorem 2.5, one can prove the theorem under assumption (ii). □

Remark 4.4 One could also investigate, instead of (22), the convergence of maxima *along a fixed sequence* $\mathbf{N}(n) \to \boldsymbol{\infty}$ (compare with Theorem 4.1 in Leadbetter and Rootzén 1998) and consider directional phantom distribution functions. We do not know if there exists a stationary field with a directional phantom distribution function which is not its phantom distribution function in the sense of Definition 4.1.

5 Extremal index

In this section we combine results obtained in Sections 2 and 4 to establish new formulas for calculation of the extremal index for random fields.

Definition 5.1 We say that $\theta \in (0, 1]$ is the *extremal index* for $\{X_{\mathbf{n}}\}$, if the function G given by $G(x) := P(X_{\mathbf{0}} \leq x)^{\theta}$, $x \in \mathbb{R}$, is a phantom distribution function for $\{X_{\mathbf{n}}\}$.

We note here that some definition of the extremal index for random fields was also proposed by Choi (2002).

Applying some results established in previous sections, we immediately obtain the following theorem for a class including m-dependent fields.

Theorem 5.2 *Suppose that the stationary field $\{X_{\mathbf{n}}\}$ satisfies the assumptions of Proposition 2.3 and $n^d P(X_{\mathbf{0}} > v_n) \to \tau$ holds with some $\tau > 0$. Let $\theta \in (0, 1]$. Then the following three conditions are equivalent.*

(a) *The number θ is the extremal index for $\{X_{\mathbf{n}}\}$.*
(b) *Relation (23) holds with $\gamma := \exp(-\theta\tau)$ and some $\boldsymbol{\psi}$ satisfying (24).*
(c) *The statement*

$$\frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d} (-1)^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_d} P\left(M_{(m+1,m+1,\ldots,m+1)-\boldsymbol{\varepsilon}} > v_n\right)}{P(X_{\mathbf{1}} > v_n)} \to \theta, \quad \text{as} \quad n \to \infty,$$

is true.

Proof Let $\{X_{\mathbf{n}}\}$ satisfy the assumptions of the theorem. Observe that condition (a) implies

$$\begin{aligned} P(M_{\boldsymbol{\psi}(n)} \leq v_n) &= P(X_{\mathbf{0}} \leq v_n)^{\theta\psi^*(n)} + o(1) \\ &= \exp(-\theta n^d P(X_{\mathbf{0}} > v_n)) + o(1) = \exp(-\theta\tau) + o(1) \end{aligned}$$

for every $\boldsymbol{\psi}$ satisfying (24) and thus (b) follows. If (b) holds, then from Theorem 4.3 we get that G given by (25) is a phantom distribution function for $\{X_{\mathbf{n}}\}$. Applying the theorem again we obtain that $\hat{G}(x) := G(x)^{1/\theta}$ is a phantom distribution function for the field $\{\hat{X}_{\mathbf{n}}\}$ of i.i.d. random variables with $\hat{X}_{\mathbf{0}} \overset{d}{=} X_{\mathbf{0}}$. Then we have

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left| P(M_{\mathbf{n}} \leq x) - P(X_{\mathbf{0}} \leq x)^{\theta n_1 n_2 \cdots n_d} \right| \\ &\leq \sup_{x\in\mathbb{R}} \left| P(M_{\mathbf{n}} \leq x) - G(x)^{n_1 n_2 \cdots n_d} \right| + \sup_{x\in\mathbb{R}} \left| (\hat{G}(x)^{n_1 n_2 \cdots n_d})^{\theta} - (P(X_{\mathbf{0}} \leq x)^{n_1 n_2 \cdots n_d})^{\theta} \right| \\ &= o(1). \end{aligned}$$

Hence $P(X_{\mathbf{0}} \leq x)^{\theta}$, $x \in \mathbb{R}$, is a phantom distribution for $\{X_{\mathbf{n}}\}$ and condition (a) holds. The equivalence of (b) and (c) is due to Proposition 2.3. □

Similarly, we can prove the following theorem for max-m-approximable fields.

Theorem 5.3 *Let the stationary field $\{X_{\mathbf{n}}\}$ satisfy $n^d P(X_{\mathbf{0}} > v_n) \to \tau$ with some $\tau > 0$ and be max-m-approximable. Then conditions (a), (b) from Theorem 5.2 and (c') given below are equivalent.*

(c') *We have, as* $n \to \infty$,

$$\frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^d}(-1)^{\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_d} P\left(M^{[m_n]}_{(2m_n+1,2m_n+1,\ldots,2m_n+1)-\boldsymbol{\varepsilon}} > v_n\right)}{P(X_{\mathbf{1}} > v_n)} \to \theta,$$

for every $\{m_n\} \subset \mathbb{N}$ *tending to infinity sufficiently slowly.*

Proof The proof is fully analogous to the argumentation from Theorem 5.2. Here, one shall use Theorem 2.5 instead of Proposition 2.3. □

Remark 5.4 For $d = 1$ Theorem 5.2 gives the formula:

$$\theta = \lim_{n\to\infty} \frac{P(M_{m+1} > v_n) - P(M_m > v_n)}{P(X_1 > v_n)},$$

after simple transformations

$$\theta = \lim_{n\to\infty} \frac{P(M_{m+1} > v_n) - P(M_{2,m+1} > v_n)}{P(X_1 > v_n)} = \lim_{n\to\infty} P(M_{2,m+1} \le v_n \mid X_1 > v_n),$$

with $M_{2,m+1} := \max\{X_k : 2 \le k \le m+1\}$. This is the well known method of calculating the extremal index θ for sequences satisfiying some local mixing conditions, including m-dependent sequences, with the knowledge of the joint distribution of $(m+1)$ consecutive terms (see, e.g., Chernick et al. 1991).

Theorems 5.2 and 5.3 provide formulas for calculation of the extremal index. We present a simple motivating example below. Further examples with the new tools succesfully applied can be found in Section 6.

Example 5.5 Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$ be given as $X_{\mathbf{n}} := \max\{Z_{(n_1+1,n_2)}, Z_{(n_1,n_2+1)}\}$, where $\{Z_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$ is a field of i.i.d. random variables and $n^2 P(Z_{\mathbf{0}} > v_n) \to \tau/2$, as $n \to \infty$, holds with some $\{v_n\} \subset \mathbb{R}$ and $\tau > 0$. Then $\{X_{\mathbf{n}}\}$ is 1-dependent and

$$n^2 P(X_{\mathbf{0}} > v_n) \to \tau,$$

$$P\left(M_{(n,n)} \le v_n\right) = P(Z_{\mathbf{0}} \le v_n)^{n^2} + o(1) = e^{-\tau/2} + o(1).$$

From the equivalence of (a) and (b) in Theorem 5.2 we obtain that $\theta = 1/2$ is the extremal index for $\{X_{\mathbf{n}}\}$. Moreover, the calculation of θ based on the formula from part (c) of the theorem looks as follows:

$$\theta = \lim_{n\to\infty} \frac{7P(Z_{\mathbf{0}} > v_n) - 4P(Z_{\mathbf{0}} > v_n) - 4P(Z_{\mathbf{0}} > v_n) + 2P(Z_{\mathbf{0}} > v_n)}{2P(Z_{\mathbf{0}} > v_n)} = \frac{1}{2}.$$

On the contrary, the method proposed by Ferreira and Pereira (2008) gives $\theta = 1$.

6 Examples: calculation of extremal indices

6.1 Extremal index for moving maxima

Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$ be the moving maximum field from Section 3.1.1. Let $\{v_n\} \subset \mathbb{R}$ be the sequence defined therein for some $v > 0$. Then, as we already know, $\{X_{\mathbf{n}}\}$ is max-m-approximable and

$$n^2 P(X_{\mathbf{0}} > v_n) \to \tau, \quad \text{as} \quad n \to \infty,$$

for

$$\tau := \left(p \sum_{\mathbf{j}\in\mathbb{Z}^d,\, c_{\mathbf{j}}>0} c_{\mathbf{j}}^{\alpha} + q \sum_{\mathbf{j}\in\mathbb{Z}^d,\, c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^{\alpha} \right) v^{-\alpha}. \tag{27}$$

Our goal is to apply Theorem 5.3 to calculate the extremal index for $\{X_{\mathbf{n}}\}$.

Let $m \in \mathbb{N}$ be arbitrary. Then maxima of the field $\{X_{\mathbf{n}}^{[m]}\}$, associated with $\{X_{\mathbf{n}}\}$, satisfy for every $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \{\mathbf{0}, (0,1), (1,0), \mathbf{1}\}$ the following condition

$$\begin{aligned} &P\left(M^{[m]}_{(2m+1,2m+1)-\boldsymbol{\varepsilon}} > v_n\right)\\ &\sim \sum_{I\in\mathcal{I}_{\boldsymbol{\varepsilon}}} P\left(\max_{\mathbf{i}\in I}(c_{\mathbf{i}}Z_{\mathbf{0}}) > v_n\right) + (2m+1-\varepsilon_1)(2m+1-\varepsilon_2)P\left(\max_{\|\mathbf{i}\|>m}(c_{\mathbf{i}}Z_{\mathbf{i}}) > v_n\right), \end{aligned}$$

where $\mathcal{I}_{\boldsymbol{\varepsilon}} := \left\{(\mathbf{j} + I_{\boldsymbol{\varepsilon}}) \cap I_{\mathbf{0}} : \mathbf{j} \in \mathbb{Z}^2\right\}$, $I_{\boldsymbol{\varepsilon}} := \{\mathbf{j} \in \mathbb{Z}^2 : (-m,-m) \le \mathbf{j} \le (m,m) - \boldsymbol{\varepsilon}\}$. Since we have

$$(\mathcal{I}_{\mathbf{0}} \backslash \mathcal{I}_{(0,1)}) \backslash (\mathcal{I}_{(1,0)} \backslash \mathcal{I}_{\mathbf{1}}) = \{I_{\mathbf{0}}\}$$

and, moreover,

$$\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2} (-1)^{\varepsilon_1+\varepsilon_2}(2m+1-\varepsilon_1)(2m+1-\varepsilon_2) = 1$$

holds, we conclude that

$$\begin{aligned} &\frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2} P\left(M^{[m]}_{(2m+1,2m+1)-\boldsymbol{\varepsilon}} > v_n\right)}{P(X_{\mathbf{1}} > v_n)}\\ &= \frac{P\left(\max_{\mathbf{i}\in I_{\mathbf{0}}}(c_{\mathbf{i}}Z_{\mathbf{0}}) > v_n\right) + P\left(\max_{\|\mathbf{i}\|>m} c_{\mathbf{i}}Z_{\mathbf{i}} > v_n\right)}{P(X_{\mathbf{1}} > v_n)} + o(1). \end{aligned}$$

Applying the results recalled in Section 3.1.1, we obtain

$$\begin{aligned} \frac{P\left(\max_{\mathbf{i}\in I_{\mathbf{0}}}(c_{\mathbf{i}}Z_{\mathbf{0}}) > v_n\right)}{P(X_{\mathbf{1}} > v_n)} &= \frac{P\left((\max_{\mathbf{i}\in I_{\mathbf{0}}} c_{\mathbf{i}}^{+})Z_{\mathbf{0}} > v_n\right) + P\left((\max_{\mathbf{i}\in I_{\mathbf{0}}} c_{\mathbf{i}}^{-})Z_{\mathbf{0}} < -v_n\right)}{P(X_{\mathbf{1}} > v_n)}\\ &= \frac{p \cdot (\max_{\|\mathbf{i}\|\le m} c_{\mathbf{i}}^{+})^{\alpha} + q \cdot (\max_{\|\mathbf{i}\|\le m} c_{\mathbf{i}}^{-})^{\alpha}}{p \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}>0} c_{\mathbf{j}}^{\alpha} + q \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^{\alpha}} + o(1), \end{aligned}$$

with $c_{\mathbf{i}}^+ := \max\{c_{\mathbf{i}}, 0\}$ and $c_{\mathbf{i}}^- := \max\{-c_{\mathbf{i}}, 0\}$, and

$$\frac{P\left(\max_{\|\mathbf{i}\|>m} c_{\mathbf{i}} Z_{\mathbf{i}} > v_n\right)}{P\left(X_{\mathbf{1}} > v_n\right)} \leq \frac{P\left(\max_{\|\mathbf{i}\|>m} |c_{\mathbf{i}} Z_{\mathbf{i}}| > v_n\right)}{P\left(X_{\mathbf{1}} > v_n\right)} = \frac{\sum_{\|\mathbf{i}\|>m} |c_{\mathbf{i}}|^\alpha}{p \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}>0} c_{\mathbf{j}}^\alpha + q \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^\alpha} + o(1).$$

All the above approximations are for an arbitrary $m \in \mathbb{N}$ and for $n \to \infty$.

Observe that if $m \to \infty$, then we get

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2} (-1)^{\varepsilon_1+\varepsilon_2} P\left(M^{[m]}_{(2m+1,2m+1)-\boldsymbol{\varepsilon}} > v_n\right)}{P\left(X^{[m]}_{\mathbf{1}} > v_n\right)} = \theta$$

with

$$\theta := \frac{p(c^+)^\alpha + q(c^-)^\alpha}{p \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}\geq 0} c_{\mathbf{j}}^\alpha + q \cdot \sum_{\mathbf{j}\in\mathbb{Z}^d, c_{\mathbf{j}}<0} |c_{\mathbf{j}}|^\alpha} \in (0, 1], \tag{28}$$

where $c^+ := \max_{\mathbf{i}\in\mathbb{Z}^2} c_{\mathbf{i}}^+$ and $c^- := \max_{\mathbf{i}\in\mathbb{Z}^2} c_{\mathbf{i}}^-$. By Lemma 7.2, for every sequence $\{m_n\} \subset \mathbb{N}$ tending to infinity sufficiently slowly the convergence from Theorem 5.3(c') holds. We conclude that the number θ is the extremal index for $\{X_{\mathbf{n}}\}$.

6.2 Extremal index for moving averages

Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$ be the moving average field from Section 3.1.2, built on $\{Z_{\mathbf{n}}\}$, with partial maxima denoted by $M(X)_{\mathbf{n}}$. Let $\{v_n\} \subset \mathbb{R}$ be the sequence defined therein for some $v > 0$. Then $\{X_{\mathbf{n}}\}$ is max-m-approximable and

$$n^2 P(X_{\mathbf{0}} > v_n) \to \tau, \quad \text{as} \quad n \to \infty,$$

with τ given by (27). Below we combine Theorem 5.3 and the result from Section 6.1 to calculate the extremal index for $\{X_{\mathbf{n}}\}$. In the following, $\{Y_{\mathbf{n}}\}$ denotes the moving maximum field built on $\{Z_{\mathbf{n}}\}$ and $M(Y)_{\mathbf{n}}$ are its partial maxima.

For $n \in \mathbb{N}$, we define $A_n := \{X_{\mathbf{0}} > v_n\} = \{\sum_{\mathbf{j}\in\mathbb{Z}^2} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n\}$ and $B_n := \{Y_{\mathbf{0}} > v_n\} = \{\max_{\mathbf{j}\in\mathbb{Z}^2} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n\}$. Then $n^2 P(A_n) = n^2 P(B_n) + o(1)$, as $\to \infty$, by properties (17) and (19) and the definition of $\{v_n\}$. We will show that also

$$n^2 P(A_n \triangle B_n) \to 0, \quad \text{as} \quad n \to \infty, \tag{29}$$

is true. To do this, for arbitrary $\varepsilon > 0$ we choose $t > 0$ so that $P(\sum_{\mathbf{i}\in\mathbb{Z}^2} |c_{\mathbf{i}} Z_{\mathbf{i}}| > t) < \varepsilon$ and put $B_{n,t} := \{\max_{\mathbf{j}\in\mathbb{Z}^2} c_{\mathbf{j}} Z_{\mathbf{j}} > v_n + t\}$. Then, by the long-tail property guaranteed by regularity (see, e.g., Embrechts et al. 2003, p. 50), $n^2 P(B_n) = n^2 P(B_{n,t}) + o(1)$ holds as $n \to \infty$. Since $B_{n,t} \subset B_n$, it follows that $n^2 P(B_n \triangle B_{n,t}) \to 0$. Furthermore,

$$n^2 P(B_{n,t} \backslash A_n) \leq n^2 \sum_{\mathbf{j}\in\mathbb{Z}^2} P\left(c_{\mathbf{j}} Z_{\mathbf{j}} > v_n + t, \sum_{\mathbf{i}\in\mathbb{Z}^2, \mathbf{i}\neq\mathbf{j}} c_{\mathbf{i}} Z_{\mathbf{i}} < -t\right)$$

$$\leq n^2 \sum_{\mathbf{j}\in\mathbb{Z}^2} P(c_{\mathbf{j}} Z_{\mathbf{j}} > v_n + t) \cdot P\left(\sum_{\mathbf{i}\in\mathbb{Z}^2} |c_{\mathbf{i}} Z_{\mathbf{i}}| > t\right) \leq \sum_{\mathbf{j}\in\mathbb{Z}^2} |c_{\mathbf{j}}|^\alpha \cdot v^{-\alpha} \cdot (1 + o(1)) \cdot \varepsilon.$$

Since ε is arbitrary and $n^2P(A_n) = n^2P(B_{n,t}) + o(1)$, we get $n^2P(A_n \triangle B_{n,t}) \to 0$. We conclude that (29) is satisfied.

Now, observe that condition (29) implies that for every $m \in \mathbb{N}$

$$\lim_{n\to\infty} n^2 P\left(\left\{X_{\mathbf{k}}^{[m]} > v_n\right\} \triangle \left\{Y_{\mathbf{k}}^{[m]} > v_n\right\}\right) = 0,$$

provided that $X_{\mathbf{k}}^{[m]}$ and $Y_{\mathbf{k}}^{[m]}$ are constructed with the same $\{Z_{\mathbf{n}}^{(\mathbf{k})}\}$. Moreover, for every $\boldsymbol{\varepsilon} \in \{\mathbf{0}, (0,1), (1,0), \mathbf{1}\}$ we have

$$\limsup_{n\to\infty} n^2 P\left(\left\{M(X)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right\} \triangle \left\{M(Y)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right\}\right)$$
$$\leq \limsup_{n\to\infty} n^2(2m+1)^2 P\left(\left\{X_{\mathbf{0}}^{[m]} > v_n\right\} \triangle \left\{Y_{\mathbf{0}}^{[m]} > v_n\right\}\right) = 0,$$

in particular, as $n \to \infty$,

$$n^2 P\left(M(X)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right) = n^2 P\left(M(Y)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right) + o(1).$$

Applying the above fact and the result for moving maxima from Section 6.1, we get

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2} P\left(M(X)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right)}{P\left(X_{\mathbf{1}} > v_n\right)}$$
$$= \lim_{m\to\infty}\lim_{n\to\infty} \frac{\sum_{\boldsymbol{\varepsilon}\in\{0,1\}^2}(-1)^{\varepsilon_1+\varepsilon_2} P\left(M(Y)_{(2m+1,2m+1)-\boldsymbol{\varepsilon}}^{[m]} > v_n\right)}{P\left(X_{\mathbf{1}} > v_n\right)} = \theta$$

with θ given by (28). Since for all sequences $\{m_n\} \subset \mathbb{N}$ tending to infinity sufficiently slowly the convergence from Theorem 5.3(c') holds (see Lemma 7.2), the number θ is the extremal index for $\{X_{\mathbf{n}}\}$.

6.3 Extremal index for a Gaussian field satisfying the Berman condition

Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ be a centered stationary Gaussian random field with correlation $r(\mathbf{n})$ satisfying the Berman condition (21). Then the extremal index of $\{X_{\mathbf{n}}\}$ equals $\theta = 1$.

To show it, let us consider $\{v_n\}$ satisfying $\limsup_{n\to\infty} n^d P(X_{\mathbf{0}} > v_n) \to \tau$ with some $\tau > 0$. From Section 3.2.1 we know that assumptions (3) and (8) with $m = 0$ are fulfilled. Applying Theorem 5.2(c) we get that $\theta = 1$ is the extremal index for $\{X_{\mathbf{n}}\}$.

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Appendix

Two self-serving lemmas applied in the proof of Theorem 2.1 and in Section 2.2 are given below.

Lemma 7.1 *Let* $\mathbf{N}(n) \to \infty$ *be such that* $N^*(n)/n^d \to \infty$. *Then there exists a sequence* $\{\boldsymbol{\psi}(n)\} \subset \mathbb{N}^d$ *satisfying* $\boldsymbol{\psi}(n) \to \infty$, $\psi_1(n)\psi_2(n)\cdots\psi_d(n) \sim n^d$, $\psi_i(n) \le N_i(n)$ *for* $i \in \{1, 2, \ldots, d-1\}$ *and* $\psi_d(n)/N_d(n) \to 0$.

Proof Let the sequences $\{t_i(n)\} \subset \mathbb{R}$, for $i \in \{1, 2, \ldots, d-1\}$, be chosen so that

$$t_1(n) \to \infty, \quad t_1(n) = o(n^d), \quad t_1(n) = o(N_2(n)N_3(n)\cdots N_d(n)),$$

and

$$t_i(n) \to \infty, \quad t_i(n) = o\left(t_{i-1}(n)\right), \quad t_i(n) = o(N_{i+1}(n)N_{i+2}(n)\cdots N_d(n))$$

for $2 \le i \le d-1$. Let us consider $a_i(n) \in \mathbb{R}$ defined as follows

$$a_i(n) := \begin{cases} \min\left\{N_1(n), \frac{n^d}{t_1(n)}\right\} & \text{if } i = 1; \\ \min\left\{N_i(n), \frac{n^d}{a_1(n)a_2(n)\cdots a_{i-1}(n)t_i(n)}\right\} & \text{if } i \in \{2, 3, \ldots, d-1\}; \\ \frac{n^d}{a_1(n)a_2(n)\cdots a_{d-1}(n)} & \text{if } i = d. \end{cases}$$

Then, we easily get that $a_i(n) \to \infty$ and $a_i(n) \le N_i(n)$ for $i \in \{1, 2, \ldots, d-1\}$. We will show that also $a_d(n) \to \infty$ and $a_d(n)/N_d(n) \to 0$. Indeed, by the definition of $a_{d-1}(n), a_{d-2}(n), \ldots$, we have

$$\begin{aligned} a_d(n) &= \frac{n^d}{a_1(n)\cdots a_{d-1}(n)} \\ &= \max\left\{\frac{n^d}{a_1(n)\cdots a_{d-2}(n)N_{d-1}(n)},\ t_{d-1}(n)\right\} \\ &= \max\left\{\frac{n^d}{a_1(n)\cdots a_{d-3}(n)N_{d-2}(n)N_{d-1}(n)},\ \frac{t_{d-2}(n)}{N_{d-1}(n)},\ t_{d-1}(n)\right\} \\ &= \ldots = \max\left\{\frac{n^d}{N_1(n)\cdots N_{d-1}(n)},\ \max_{2\le i\le d-1}\frac{t_{i-1}(n)}{N_i(n)\cdots N_{d-1}(n)},\ t_{d-1}(n)\right\}. \end{aligned}$$

Since $t_{d-1}(n) \to \infty$ and $a_d(n) \ge t_{d-1}(n)$, the condition $a_d(n) \to \infty$ holds. Moreover, applying $n^d = o(N^*(n))$ and $t_{i-1}(n) = o(N_i(n)N_{i+1}(n)\cdots N_d(n))$, we conclude that $a_d(n)/N_d(n) \to 0$. To complete, we shall define $\psi_i(n) := \lfloor a_i(n)\rfloor \in \mathbb{N}$. □

Lemma 7.2 *For* $a(m,n) \ge 0$, $b(m,n) \in \mathbb{R}$ *(*$m \in \mathbb{N}_+$, $n \in \mathbb{N}$*)*, $b \in \mathbb{R}$,

(i) *if* $\limsup_{m\to\infty} \limsup_{n\to\infty} a(m,n) = 0$ *holds, then* $\lim_{n\to\infty} a(m_n, n) = 0$*;*
(ii) *if* $\lim_{m\to\infty} \lim_{n\to\infty} b(m,n) = b$ *holds, then* $\lim_{n\to\infty} b(m_n, n) = b$*,*

for some sequence $r_n \to \infty$ *and all* $m_n \to \infty$ *satisfying* $m_n \le r_n$.

Proof To prove (i), observe that for every m and $a(m) := \limsup_{n\to\infty} a(m,n)$, there exists $N_m \in \mathbb{N}$ such that $a(m,n) - a(m) \leqslant 1/m$ for all $n \ge N_m$. Let us choose N_m for each m so that the sequence $\{N_m\}$ is increasing. Define $r_n := \sum_{k=1}^{\infty} k \cdot \mathbb{1}_{[N_k, N_{k+1})}(n)$. Then $r_n \to \infty$ and for every $m_n \to \infty$ satisfying $m_n \leq r_n$ we obtain

$$a(m_n, n) - a(m_n) \leq \frac{1}{m_n},$$

since $n \geq N_{r_n} \geq N_{m_n}$ holds, and consequently

$$0 \leq a(m_n, n) = (a(m_n, n) - a(m_n)) + a(m_n) \leq \frac{1}{m_n} + o(1) = o(1).$$

The proof of part (ii) is fully analogous. □

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