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ORIGINAL RESEARCH

Critical Math Kinds: A Framework for the Philosophy of Alternative Mathematics

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Abstract
Mathematics, even more than the other sciences, is often presented as essentially unique, as if it could not be any other way. And yet, prima facie alternative mathematics are all over the place, from non-Western mathematics to mathematics based on nonclassical logics. Taking inspiration from Robin Dembroff's analysis of critical gender kinds, and from Andrew Aberdein and Stephen Read's analysis of alternative logics, in this paper I will introduce a practice-centered framework for the study of alternative mathematics based on the notion of critical math kind. After sketching a model of mainstream mathematics, I will provide examples of how deviation along several distinct dimensions can occur, and how deviations can vary in their gravity. I will then discuss how the framework can be used to think of questions concerning the alternativeness status and philosophical implications of alleged alternative mathematics, and help us in identifying alternatives that suit our purposes.

1 Introduction

Mathematics appears to be a very special field. It is the deductive science par excellence; it seems to provide necessary and universal truths; and the degree to which there is a consensus on methodology and standards appears to be simply unparalleled among other disciplines.

And yet, prima facie substantial disagreement does exist, not only across history and cultures, but also within the contemporary scene. We have "nonstandard" mathematics challenging received conceptions of basic mathematical concepts, e.g. non-well-founded set theory, nonstandard analysis, Petr Vopěnka's alternative set

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theory, and non-Cantorian counting.[1] We have "nonclassical" mathematics adopting different underlying logics, e.g. intuitionistic and constructive mathematics, quantum mathematics, relevant arithmetic, and inconsistent mathematics.[2] We have experimental mathematics, suggesting a world where deduction might be left behind.[3] More conjecturally, Van Bendegem (2005) has developed several sketches of ways in which mathematical practice could function fundamentally differently, e.g. vague mathematics, random mathematics, and non-compact mathematics.

Of course, it is controversial whether any of this constitutes a *real* alternative. There are two general strategies to counter any alleged alternative: denying that it is alternative (*reappropriation*), or denying that it is mathematics (*exclusion*). As Bloor (1991) puts it: "One of the reasons why there appears to be no alternative to our mathematics is because we routinely disallow it. We push the possibility aside, rendering it invisible or defining it as error or as nonmathematics" (p.180). Exclusion is the fate of much that was once considered to be part of mathematics, e.g. numerology. Reappropriation is even more popular: the very idea of mathematics as cumulative is built on it, since the general conception of mathematics has hardly stayed the same across the centuries, and definitions change all the time. Reappropriation can work in two ways: by arguing that the alleged alternative is just *more* standard mathematics, or by arguing that it is just *worse* standard mathematics. The second route plays a lot into how mainstream mathematics deals with non-Western mathematical practices: tangled with the social, yet translatable as primitive applications of the Western perspective.[4] These are two faces of the same medal: if mathematical progress is seen as a linear necessary cumulative process, any alleged alternative will be seen as either less than current mathematics or a natural extension of current mathematics.

The alternativeness question is important due to the appearance of alternative mathematics, in one form or the other, in many ongoing discussions. For example, Bloor (1991) and Ernest (1998) both take (different kinds of) alternative mathematics as evidence of social constructivism in mathematics. Building on this, Burton (1995) suggests alternative mathematics is one of the reasons why a feminist epistemology of mathematics is needed, while (Ohara, 2006) explores its value from a pedagogical perspective. Van Bendegem (2016) interrogates the conditions under which alternative mathematics could serve as evidence against inevitabilism. Several versions of logical pluralism, while usually not using the expression explicitly, take some sort of alternative mathematics to count as evidence, e.g. Shapiro (2014); Kouri Kissel (2018), and Caret (2021); conversely, Williamson (2018) takes the non-existence (in some other sense) of alternative mathematics to count as evidence *against* logical pluralism.

As this brief sketch suggests, there isn't much agreement in the literature on what the notion of alternativeness can or cannot do; in fact, there is barely any theoretical discussion of it at all. My goal in this paper is to start filling this gap by proposing

[1] See respectively Aczel (1988); Robinson (2016); Vopěnka (1991), and (Mancosu 2016, ch.3).

[2] See respectively Bridges and Richman (1987); Dunn (1980); Meyer (2021), and Weber (2021).

[3] See e.g. Horgan (1993) and Zeilberger (1993).

[4] The field of ethnomathematics, on which see e.g. Ascher (2017) and Selin (2001), tries to avoid this by using Western mathematics as a mere "model" of outside practices, with all the intended limitations. There can still be problems, e.g. the imposition of a mathematical reading on practices that the practitioners themselves would not treat as such.

a general framework. The central idea is to understand alternativeness as a cluster of deviations from a standard model, where each deviation can have a different "criticality degree". Ideally, this will allow us to systematically compare alleged alternatives, clarify debates on genuine alternativeness, clarify philosophical debates in which alternatives play a role, and individuate alternatives suited for specific purposes.

The paper is organized as follows. Section 2 sets the stage by defining math kinds in terms of a basic notion of mathematical practice. Section 3 sketches a standard model of dominant mathematical practices to serve as a background against which to define alternativeness. Section 4 introduces the central notion of critical math kind, and provides some examples of deviation along different dimensions. Section 5 introduces the notion of critical level. Section 6 showcases some of the ways in which this framework can be applied: diagnosing non-criticality, highlighting particular phenomena related to exclusion and reappropriation, and individuating the kind of alternatives required to achieve certain goals. Section 7 wraps things up.

2 Math Kinds

By a *math kind* I mean a collection of mathematical practices sharing some common properties.[5] For example, intuitionistic mathematics is a math kind: a collection of mathematical practices relying on intuitionistic logic. I understand practice in a very broad sense here, including not only the outcome of the practice (e.g. the theorems and proofs accepted by the community), but also the standards regulating their genesis and acceptance, the goals of the practice, and the way results are interpreted. I borrow from Ferreirós (2015) the crucial distinction between frameworks and the agents which interpret them, together with the idea that we should not be too rigid in delimiting what a framework can be. This is partly to account for the fact that practices constantly evolve in non-revolutionary ways, and we want to preserve practice identity through such changes: it would be unwieldy and unenlightening to, say, talk about a different practice every time a new theorem is proved or a definition is changed.

More importantly, being too strict with one's definition of practice is particularly dangerous in the context of analysing or conceiving mathematical practices that may be very different from what we are used to. For example, the notion of practice in Van Bendegem and Van Kerkhove (2004) - while certainly adequate for many particular studies - takes for granted the existence of a notion of proof. But the very idea behind the random mathematics of Van Bendegem (2016) is to do without proof! In other words, there is a risk that too rigid a notion of practice may hide some possibilities in what can constitute a mathematical practice. Of course, this is to some degree inevitable when attempting to develop a general framework. In this paper, rather than make my assumptions at the level of practices, I will forefront them in the choice of standard model, i.e. the choice of properties taken to characterize dominant practices for the purpose of determining what counts as alternative.

[5] I am not infusing the word "kind" with any metaphysical baggage here: for my purposes, a math kind is just a kind of mathematics.

In this paper I will only be concerned with *salient* math kinds, meaning that I will only consider properties which seem to have an implication on how mathematics is actually practiced. For example, I do not see much point in looking at "realist maths": even if we could somehow divide the space of practices this way based on practitioners' metaphysical views, it is not clear that such views have any systematic effect on how mathematicians do their job. Intuitively, it seems possible for a realist and an anti-realist to carry out exactly the same mathematics in exactly the same way.

While math kinds showing up in the actual world will of course take precedence, it will be interesting to look at possible math kinds as well: this will be particularly relevant for the problem of identifying alternatives suited to a particular goal. I will not attempt a general classification of math kinds here; I am skeptical of both the feasibility and the usefulness of such an endeavour, without fixing a standard model first as reference. To this I now turn.

3 Building a Standard Model

Before we can study critical math kinds, we need to provide an appropriate standard model of dominant mathematics. A *standard model* is a salient covering of the space of mainstream mathematical practices. What counts as mainstream mathematical practices is of course dependent on both the historical time and the society of reference. In this paper I will understand mainstream mathematics as mainstream Western mathematics in the present time. Critical math kinds will then be construed as deviations from the chosen model.

So, what is mainstream mathematics all about? What characterizes the standard picture of mathematics we are exposed to in schools, not to mention in mathematics and philosophy departments? One preliminary difficulty in answering such questions is that the standard picture is not necessarily consistent across all these different contexts. For example, in education (and in publications) the process of discovering theorems and forming concepts is often completely hidden. Consider also the role of formal logic: while some mathematics departments may fail to even have logic courses - let alone mandatory ones - mainstream analytic philosophy of mathematics has long taken logic to be central to the field. Furthermore, it is of course the case that mathematicians are not a hivemind, and diverging views of the practice - not to mention diverging practices - exist even within the mainstream.

Still, we have to start from somewhere. In this paper I am less concerned with identifying the perfect standard model than I am in showcasing a general strategy for studying alternativeness. So I take my following specification to be very much incomplete and open to revision. It is not necessary, for the framework to be useful, that all philosophers agree on the choice of model; on the contrary, by asking that everyone be explicit about their chosen model we can make it easier to diagnose disagreements about alternativeness.

As a preliminary modeling choice, I am going to focus on features of *pure* mathematics; and in order to incorporate in the model the widespread "no real alternatives" stance, I will assume there is *one* salient math kind covering the whole (mainstream) field.[6] Call this MATH. Of course MATH will have all sorts of salient subkinds - particular branches of mathematics being the obvious example - but they need not concern us for the time being. Now the question is: what are the core features of MATH?

Mathematics is often characterized, among sciences, by its core mode of reasoning: deductive, rather than inductive. The discovery of results may proceed otherwise, of course; but it is generally accepted that in order to *justify* the obtained results, in order to be able to call them theorems, a deduction from previously accepted results must be exhibited.[7] Furthermore, it is generally assumed that every theorem must in principle admit a fully formal proof. While the last few decades of work on computer-assisted proofs have made much progress in exhibiting this, for the vast majority of mathematical practices the expectation simply manifests into a certain standard of (informal) proof writing that makes the existence of a formal proof plausible.[8]

Next, we have what is maybe the most common observation in philosophical environments: standard mathematics goes with classical logic. This is why standard mathematics is often referred to as *classical mathematics*; it is mathematics based on classical logic, the study of structures with an underlying classical logic, and so on. For our purposes, the crucial point is just that definitions and theorems use classical connectives, e.g. Boolean negation and the classical material conditional, and that proofs use typically classical inference rules, e.g. reductio and proof by cases.[9] The claim that classical logic is the *only* logic that would fit mainstream informal mathematics is stronger, and need not be incorporated in the model.

The next point concerns the role of set theory, which is generally accepted as *the* foundation of mathematics. This means that "All standard mathematical objects may be viewed as sets, and all classical mathematical theorems can be proved from [the axioms of set theory] using the usual logical rules of proof" (Gowers et al., 2008, Sect IV.22). Again, this is not something that needs to be spelled out every time. The way the foundational status of set theory works in practice is rather this: given a piece of mathematics, it is always allowed (albeit not always necessary) to ask for more precise definitions and principles up until the set-theoretic level, at which point it is okay (for mathematicians) to stop.[10]

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6 The fact that it makes sense to focus on pure mathematics is not a given, and will in fact be incorporated in the model.

7 To be more precise: enough evidence must be exhibited that there exists such a deduction.

8 "The ideal is to write in as friendly and approachable a way as possible, while making sure that the reader [...] can see easily how what one writes could be made more formal if it became important to do so." (Gowers et al., 2008, Sect I.2). To be sure, formal results are sometimes given informal glosses which reach further than the formal would allow: this is what happens e.g. in the context of axiom choice in set theory. However, in practice only the formal results are built on in subsequent research, any statements which are arrived at informally serve at best as heuristics or as new socially acceptable assumptions. It's also worth noting that the (stronger) claim that informal proofs have corresponding formal proofs is controversial, and the epistemic value - if any - of the formalizability assumption is unclear: see e.g. Tanswell (2015) and De Toffoli and Giardino (2015).

9 See any standard textbook introducing mathematical proofs, e.g. Roberts (2009) and Solow (2013).

10 While the language of categories is also very widespread, category theory is not very popular qua foundation in the sense just described: pick almost any textbook introducing categories, and you'll find them to be defined set-theoretically. This is reflected in the claim by Maddy (2019b) that category theory (and, for that matter, homotopy type theory) is serving a *different* kind of foundational goal. This is not to say that category theory *couldn't* replace set theory on its turf; I will come back to this in the next section.

Next, we have the world-independence of mathematics. By this I mean the acceptance of a distinction between pure and applied mathematics such that the former can in principle develop *independently* of the latter. This is not to say that applications could never be found, or that they could not inspire new research avenues; rather, the point is that there are many areas of mathematics where applicability is simply not a concern in everyday research, insofar as the standards by which new mathematics is assessed are internal to pure mathematics itself.[11] Granted, there is a sense in which applicability is always at the forefront, via the requirement that new mathematics be consistent with old applied mathematics; but this requirement alone is hardly sufficient to drive research, insofar as wildly different theories can extend a common core.

Objectivity is another property commonly ascribed to mathematics: the truth of a theorem and the correctness of a proof are not taken to depend on any particular mathematician. The interaction between world-independence and objectivity is often taken to be one of the central issues in the philosophy of mathematics: how can mathematics depend neither on the individual nor on the material world? Answers range from the social to the transcendental, but they need not concern us here. What matters is that objectivity shows up in practice through the demand that the (proper) product of the individual mathematician be assessable by others, with disagreement being reducible to human error.[12]

Finally, there is the relationship between mathematics and ethics: the development of mathematics should not involve any ethical considerations. By this I mean that the choice of logics, axioms, foundations, proof standards, etc. can depend on a variety of theoretical virtues, from simplicity to fruitfulness; yet ethics should have nothing to do with it. Now, this should not be taken to mean that mathematics is completely insulated from ethical concerns. Mathematics is a social practice: as such, it involves an ethical rule of conduct, and is just as susceptible to power dynamics as any other practice. This can have a significant influence on the production and distribution of mathematical knowledge.[13] However, at least officially, this is all treated as interference: while social factors may influence the direction mathematics goes sometimes, *in principle* they should not, and it is not standard practice to explicitly argue for a particular development on such grounds.[14]

11 This was not always the case: for the story of how we got here, see e.g. Hacking (2015) and (Maddy, 2011, ch.1). Note that traditional Platonist views of mathematics involving the one true mathematical universe do not quite contradict the self-sufficiency of pure mathematics, insofar as the Platonic world is understood to be accessible only through the standards of pure mathematics itself.

12 Even in cases of seemingly deep disagreement, like the one described by Aberdein (2023), one finds the parties unwilling to accept that it may be a subjective matter. Of course this is compatible with *some* parts of the mathematics-making process being purely subjective.

13 See e.g. Hunsicker and Rittberg (2022) and Rittberg et al. (2020).

14 Wagner (2023) argues that, as a matter of fact, choices of frameworks and standards can have - and have had - a significant societal impact. So it would be a mistake to say that mathematics is ethically neutral. But I think it is still fair to say that it is mainstream mathematical practice to behave and argue as if it were: ethical arguments are very much a rarity in mathematics lectures or publications. At best, ethics can enter the picture when it comes to the choice of pursuing certain applications: it is good to look into certain potential applications because of what they could do for humankind, while on the other hand there have been some prominent cases of mathematicians campaigning against military sponsorship for their research (see (Bell, 2021, Afterword)). Yet the acceptance of the mathematics underlying these applications is not in question.

To recap, we can identify seven dimensions along which the dominant math kind MATH can be specified:

1. *justification*: mathematical results must be justified deductively
2. *form*: mathematical results can in principle be given a formal proof
3. *logic*: the underlying logic of mathematics is classical
4. *grounding*: set-theoretic language and ontology is treated as a safe bedrock
5. *worldliness*: there is a distinction between pure mathematics and applied mathematics, and the former has its own internal world-independent standards
6. *objectivity*: mathematical correctness should be in principle agreeable upon by everyone
7. *ethicality*: ethical considerations should not play any role in the development of mathematics

It is worth noting that not every coordinate has the same cultural relevance; in fact, the reader may worry that some of these features *just are* part of the definition of mathematics, at which point one might simply want to say that deviations would not be mathematics. I will discuss this in Section 6.

4 Critical Math Kinds

I can now introduce the core notion of the framework: a math kind is a *critical math kind* relative to a given society if and only if its practitioners collectively destabilize one or more core elements of the dominant mathematics ideology in that society.[15]

A few comments are in order. First, destabilization cannot occur merely in one's mind: if there is a community of mathematicians believing that there exist only finitely many objects, but this does not influence their work *at all*, then that does not constitute a critical math kind. Second, the focus on *collective* destabilization is there to allow for critical kinds in which it is specifically the interaction between practitioners that leads to destabilization; it also lets us set aside concerns about the ability of a single individual to destabilize much of anything. Third, relativization to society is important because it constrains the choice of standard model representing the dominant ideology. MATH would have looked very different at other points in history: the connection between formal logic and mathematics is quite recent, and there was nothing resembling set theory before the 19th century; the pure and applied distinction (in its current form, at least) is also relatively recent.[16]

[15] This definition is borrowed from (Mangraviti, 2023b, Sect 4.6), which more narrowly deploys it to distinguish effectiveness with respect to a particular liberatory purpose. Compare the definition of critical gender kinds from (Dembroff, 2019, p.12): "For a given kind X, X is a critical gender kind relative to a given society iff X's members collectively destabilize one or more core elements of the dominant gender ideology in that society". I will set aside the analogy between critical math kinds and critical gender kinds here, but the reader is welcome to entertain themself by carrying it in their heart, as I certainly have while writing this.

[16] This suggests that contemporary mathematics had to be at some point critical with respect to older mathematics: I will come back to this in Section 6.

A classification of critical math kinds is a classification of "deviations from the norm". For a start, we can distinguish critical kinds by the coordinates of the standard model they appear to undermine.[17] To exemplify, I will now go through some possible deviations from MATH. Let us start from the easiest case: the *logic* coordinate. There are many contemporary examples of mathematics built on a different logic: some notable examples are intuitionistic and constructive mathematics (built on intuitionistic logic), quantum mathematics (built on quantum logic), relevant arithmetic (built on relevant logic), and inconsistent mathematics (built on paraconsistent logics). In fact, there are so many examples of this deviation that some would use it as a counterpoint to the idea that the standard logic coordinate is enforced at all. This is, I think, a bit naive. A quick look at mathematics curricula around the world will make it clear that classical mathematics is the *only* one taught in the vast majority of departments - and I have never heard of *any* mathematics department where nonclassical mathematics gets more than a couple courses of attention. It doesn't take shooting intuitionists behind the barn to establish a hegemony. I am not saying that this is an unfair or unjustified state of affairs; I am merely pointing out that it is the *current* state of affairs. The very idea of alternative logics remains a niche thing that most students never even get to hear about.

The *grounding* coordinate is maybe the most obvious candidate for future toppling. Category theory and homotopy type theory have both been proposed as alternative languages to encompass all of mathematics. Talk of sets may well end up being someday replaced at the primitive level by talk of categories and types, and plenty of work has already been dedicated to fleshing out the pros and cons of such a switch. This might go together with a logic change; for example, some category theorists have argued that topoi - a certain kind of categories - may replace the usual notion of set-theoretic universe, and topoi have an internal intuitionistic logic. Conversely, a change of logic need not affect the idea that the language of mathematics should be set-theoretic: for example, much of the literature on inconsistent mathematics focuses on the possibility of grounding mathematics in a naive set theory.

The *worldliness* coordinate has been questioned in some recent literature, although the criticality of this view is often severely underplayed by its own proposers. For example, Weber (2021) works under the assumption that mathematics depends on metaphysics for its validity, and that it should be reformed in case of misfit with our metaphysical intuitions. This is putting a weight on metaphysics - on how the world is *really* like - that is nowhere to be found in contemporary mathematical practice. It is simply *not a worry*, within the mathematics department, whether the current conception of the continuum is metaphysically adequate or not; at best, its fruitfulness can be taken as evidence it is, but no arguments to the effect that it isn't would ever

[17] Another way to be critical is to *add* coordinates along new dimensions. This can be reduced to the first way by incorporating into the standard model the new dimension in question, with the relevant coordinate being set to indifference. For example, I could have skipped including ethicality into the model, and modelled ethical deviations as arguing for the inclusion of such a dimension; my choice was for ease of exposition, although I reckon there might be advantages to emphasizing the non-salience of certain dimensions.

be taken to undermine classical analysis.[18] A more empirical example of this kind of deviation comes from constructive mathematics. According to Bishop (1975), classical mathematics is in a “crisis” due to its extensive reliance on an implicit assumption of determinacy - or “limited omniscience” as he calls it - which is unjustified insofar as it limits its applicability; the solution is a mathematics where the assumption is avoided as much as possible, and its use clearly marked. Again, the world-independence of mathematics is rejected: the needs of applied mathematics are taken to constrain, rather than merely inspire, the discipline.

Let’s move on to wilder horizons. Is there any mathematics challenging the *justification* coordinate? Experimental mathematics may seem to be going in that direction. Zeilberger (1993) enthusiastically suggests we may be entering a time when theorems no longer need to be deductively established in order to be treated as such, either because they are gained from physical experiments or because their likelihood has been computed to be sufficiently high. However, while there is much work on this, up to now it certainly hasn’t affected the standard notion of theorem: deductive reasoning remains the standard, and everything else remains a conjecture, even within the field of experimental mathematics itself. As Epstein and Levy (1995) puts it: “the objective [...] is to play a role in the discovery of formal proofs, not to displace them” (p.671). A more explicit example of deviating from this coordinate is the *random mathematics* discussed in Van Bendegem (2016). In this thought experiment, mathematics functions without a notion of proof: theorems are not proven, they are (defeasibly) computed based on empirical information. At least for arithmetic, it can be shown that this process would, given infinite time, converge to first-order Peano Arithmetic.[19]

Is there any mathematics deviating along the *objectivity* dimension? The relevant difference in practice would be for some mathematics to be recognized as such and accepted while still denying that it should be correct (or in principle assessable) to everyone. While there are ongoing controversies concerning the proofs of certain theorems, they are hardly treated as a subjective matter: *both* sides paint the other as failing to understand the objective proof / problems thereof. While I do not know of any concrete proposals, it is worth noting that a deviation from objectivity in this sense was identified by Bloor (1991) already as a possibility for alternativeness: “it could be that lack of consensus was precisely the respect in which the alternative was different to ours. For us agreement is of the essence of mathematics. An alternative might be one in which dispute was endemic” (p.108).

I am similarly at a loss in thinking of current counterexamples to the *form* coordinate: I am not aware of any results that the community accepts *as theorems* while also believing no formal proof could in principle exist.[20] The seeming lack of alternatives

[18] Fletcher (2017) levels a similar accusation at Maudlin (2014). One referee wondered whether the universe/multiverse debate in set theory might be a mainstream exception, since it is sometimes framed as a metaphysical debate. While I cannot elaborate here, I am unconvinced this is a serious deviation, because both parties argue that no results would be lost either way: it is part and parcel of defences of the true universe that it would be possible to emulate whatever is done in a multiverse framework. In fact, one could argue there is no particular pressure to choose: “since these intuitive pictures, universist and multiversist, are playing a merely heuristic role, there’s no reason at all not to exploit them both” (Maddy, 2019a, p.76).

[19] Although showing this would, presumably, not be part of the envisioned practice.

[20] Which is not to say mathematicians never *believe* mathematical statements without proof. Most believe set theory to be consistent, for example. But such beliefs are treated very differently from theorems.

may well be why Azzouni (2007) takes this - or, to be more precise, "mechanical recognizability" - to be the very essence of contemporary mathematics. That being said, it is not hard to at least *imagine* criticalities along this dimension: for example, while independence results from set theory show there can be no formal proof of the continuum hypothesis or its negation, one could conceive of a mathematical community taking existing informal arguments towards their acceptance - based on e.g. intuitiveness or fruitfulness - to constitute genuine mathematical proofs.[21]

Finally, challenges to the supposed neutrality of mathematics have led to some proposals to shift the *ethicality* coordinate. For example, Chiodo and Müller (2023) provides various pointers to all practising mathematicians wishing to "avoid or minimise the harm that might come from their own direct mathematical work" (p.3). Another example: Mangraviti (2023a) discusses the incorporation into mathematics of a moral duty to avoid the naturalization of its underlying logic, which in practice can be cashed out as a duty to inconsistentize existing mathematics and to explore the conceptual possibilities revealed by said inconsistentizations. Communities dedicated to inconsistentization, or taking Chiodo and Müller's pointers seriously, would generate an ethically critical math kind.

5 Critical Levels

Now, which parts of the standard model are affected is not the only characteristic feature of a critical kind: there is also the matter of *how* they are affected. We can classify the criticality of a kind X along a certain dimension as follows:

- *inert*: X can be fully absorbed by the standard model with no change;
- *conservative*: X requires extending the standard model so that some previously excluded practices are allowed;
- *progressive*: X requires modifying (and possibly extending) the standard model in ways that change how some standard practices fit;
- *radical*: X requires modifying the standard model so that some standard practices no longer fit.[22]

[21] A notorious example of such an argument is the one by Freiling (1986) against the continuum hypothesis, based on a (seemingly non-formalizable) probabilistic intuition. The intuition can be argued against by relying on formal conceptions of the continuum: if the two clash, the formal wins. Going back in time, one could similarly imagine, say, the Banach-Tarski paradox being used as a justification to reject the axiom of choice.

[22] This is inspired by the distinction by Aberdein and Read (2009) between (respectively) reactionary right, centre right, centre left, and radical left attitudes towards recapture in logic (p.629), although they apply this more narrowly to the relationship between two logical theories. One can also see some overlap with their classification of revolutions in logic (pp.618-619), roughly as follows: their null and radical revolutions correspond to my inert and radical criticalities respectively; their glorious revolutions are always progressive; while their paraglorious revolutions may be progressive or conservative, depending on whether the key components that are preserved change in their significance or not. However, while I cannot go into detail here, this overlap is somewhat superficial: my use of a standard model does not quite match Aberdein & Read's "key components", and it might not always be appropriate to think of alternatives as revolutions. Thanks to a referee for pushing me on this.

Conservative kinds can be divided into *positivist* kinds, which are accommodated into the standard model by broadening the existing kinds, and *liberal* kinds, which are accommodated by adding new kinds. Note that a radical kind may well be a reactionary kind, taking us back to an earlier understanding of mathematics. Every critical math kind is associated with a vector of critical levels, one per dimension. I call this vector the *critical profile* of a kind.

As an example, let us see how these levels can manifest in deviations from the logic coordinate.[23] One curious example of inert deviation is the quantum mathematics of Dunn (1980): even if we replace classical logic by quantum logic, adding basic mathematical axioms (either the Peano axioms or the ZFC axioms) will suffice to *derive* classical mathematics anyway. So insofar as no axiom change is advocated for, mathematical practice remains the same.

A conservative liberal deviation may be instantiated by claiming that mathematics can study different structures, and different structures can have different logics. This is the picture envisioned by the logical pluralism of Shapiro (2014): classical practices remain utterly untouched - classical structures remain perfectly valid - but the practice at large is expanded by the added possibility of looking at other structures.[24] Meanwhile, a conservative positivist deviation may suggest working with weaker logics than classical logic, but in such a way that all classical mathematics is recovered as a special case. This is what happens in much paraconsistent and inconsistent mathematics: for example, Carnielli and Coniglio (2013) propose a paraconsistent extension of ZFC which allows for inconsistent sets.

A progressive deviation can occur if the new logics allow us to derive classical theorems, but only under some particular assumptions undermining their central status. Something like this can be found in Bishop (1975) and Bauer (2017), where constructive mathematics is presented as a *refinement* of classical mathematics. While classical theorems are, strictly speaking, false, they are still helpful approximations, so there is no real need to reject them, although the impact on classical practices of this attitude is still quite major - classical mathematicians are asked to not assume their theorems to be true just in virtue of having proven them. Meanwhile, under a radical deviation, the new logics may not allow us to derive some classical theorems at all, with the consequence that we should reject them. This is what old-school intuitionistic mathematics claims, but also more recently the dialetheic mathematics of Weber (2021).

6 Applying the Framework

So, here is my proposed strategy for thinking about alternative mathematics:

1. select an appropriate standard model
2. identify the critical profile of the alternative under scrutiny

[23] The assignment of critical levels to existing alternatives is to some degree an interpretive matter. While I think my readings that follow are quite plausible, they might be disagreed with. In that case, I think the framework is helpful in identifying where interpretive disagreement is occurring.

[24] A similar shift is also suggested by Priest (2013): all kinds of mathematical practices are in principle legitimate, and they may characterize objects to be realized in worlds whose underlying logic is nonclassical.

3. analyse the relationship between the critical profile and exclusion/reappropriation

The strategy may be applied not only to the study of alternative mathematics "in the wild", but also to the creation and assessment of alternative mathematics for specific purposes. The strategy seems in principle adequate to study alternatives not only in the here and now, but also in historical times when the standard model was quite different. It might also be interesting to try out imaginary standard models to see how the idea of alternativeness shifts, or to analyse alternativeness at the level of small social groups rather than entire societies.

The most obvious application of this framework is, of course, comparing different alternatives. Fun as it would be, I am not going to engage in an attempt to fit all alternatives showing up in the literature into my scheme. Rather, in the following I want to showcase several other ways in which the framework can help making some progress in our understanding of alternativeness, and contribute to debates in the literature.

6.1 Non-Criticality

A choice of standard model corresponds to a choice of what counts as critical and what does not. In particular, we would like the model to not recognize as critical proposed alternatives that the community does not seem to consider critical.

Maybe the clearest example of this is non-well-founded set theory. Classical set theory always includes one form or the other of the Foundation axiom: there are no non-well-founded sets, i.e. sets that involve membership cycles. In particular, no set belongs to itself. In non-well-founded set theory, as presented by Aczel (1988), there is a so-called Anti-Foundation axiom stating that every consistent non-well-founded set exists. We seem to have a clear contradiction. Yet noone really talks of non-well-founded set theory as an alternative mathematics. Why is that?

First of all, we can see that non-well-founded set theory contradicts none of the coordinates of MATH. It is a formal axiomatic system based on classical logic. It is more general than ZFC set theory, so no rejections are implied; one is free to single out the well-founded sets within a non-well-founded universe. Now, depending on whether one understands the grounding coordinate as fixing the dominant conception of set or not, one could understand this as a conservative positivist shift in grounding, insofar as we are working with an *expanded* conception of set compared to the usual one. However, I think this is unnecessary for the following reason: in order to take non-well-founded set theory seriously as a mathematical theory, there is *no need at all* to understand its axioms as contradicting those of ZFC. Aczel (1988, ch.3) shows that non-well-founded sets can be understood as equivalence classes of classical graphs. Non-well-founded set theory can be modelled within ZFC; but if so, then there is no need for it to replace ZFC as a bedrock. As Jon Barwise puts it in his foreword to Aczel's book, this is but a "*linguistic* obstacle [...] arising out of the dominant conception of set" (p.xii, emphasis mine). Non-well-founded sets lose any criticality they could have had by simply not being treated as sets in the usual sense. Since thinking of them as "real" sets is not at all necessary for the success of the theory, no criticality is instantiated.

The case of nonstandard analysis is similar. In classical analysis there is no such thing as infinitesimals, i.e. positive numbers smaller than any positive real number. Nonstandard analysis extends the standard continuum by adding infinitesimals (and infinite quantities). This creates a new way of doing analysis, but one that is perfectly compatible with the old one: we are simply producing different proofs by working with a different structure, which ZFC is perfectly able to comprehend and in fact compare to the standard continuum. As Robinson (2016) puts it: "the non-standard methods that have been proposed to date are conservative relative to the commonly accepted principles of mathematics [...] This signifies that a non-standard proof can always be replaced by a standard one, even though the latter may be more complicated and less intuitive. Thus, the present writer holds to the view that the application of non-standard analysis to a particular mathematical discipline is a matter of choice and that it is natural for the actual decision of an individual to depend on his early training" (p.xv).

Based on these two relatively mainstream examples, it is interesting to look at the case of a practice that *is* often sold as critical, but by the same lights can be deduced to not be. Mangraviti (2023a) suggests that a practice belongs to inconsistent mathematics if "it accepts inconsistent theorems, is about inconsistent concepts, or allows proofs to detour through inconsistency" (p.280). Now, consistency was not mentioned explicitly in the standard model I picked. It does implicitly feature in the logic coordinate, insofar as classical logic does not countenance contradictions due to the Explosion law: inconsistency leads to triviality. But whether this actually forbids inconsistency in any of the three senses above depends on many things, because - much like non-well-founded sets and infinitesimals - inconsistency can be represented consistently.

For example, it is coherent with the standard model to maintain that mathematics can study inconsistent concepts. This need not undermine *any* of the coordinates, which I think accurately tracks how this kind of inconsistent mathematics need not be very critical. Consider for example the study of "inconsistent" models. These are classical structures modelling - in the sense of model theory - inconsistent theories: some sentences will be understood as both true and false in the model.[25] But "true in the model" and "false in the model" are classically defined mathematical concepts, so strictly speaking a mathematician is not doing anything unusual by studying such structures. To see this as a study of inconsistent concepts amounts to little more than a realist attitude about said models. Now, this is not necessarily to say that from a philosophical perspective this view is not radical; rather, the point is that a radical shift in the philosophy of mathematics need not generate a corresponding radical shift - or any shift, really - in the practice of mathematics.

6.2 Exclusion Revisited

In the introduction I talked about two strategies used to dismiss alternatives: exclusion, and reappropriation. Now, to be non-critical makes exclusion impossible on pain of rejecting some existing features of standard mathematics. On the other hand, it makes

[25] See e.g. Mortensen (1995) and Paris and Sirokofskich (2008).

reappropriation absolutely trivial, as the "alternative" already fits the standard model after all. But what about critical math kinds?

Exclusion is the natural reaction when a practice deviates from the model along a dimension that is taken to be essential to the very definition of mathematics. Historically speaking, it seems clear enough that at least some dimensions do not have this power. For example, mathematics managed to survive without a set-theoretic language for a long time. There might be disagreement on whether grounding mathematics in a purely category-theoretic language would be *better*, but I doubt anyone would say it would make it not mathematics.

An example of deviation that sparks debates on exclusion concerns the centrality of deduction. Now, as already mentioned, noone would ever claim that mathematical results can only be discovered deductively. However, the idea that physical experiments or statistical computations may *suffice* to justify mathematical results makes folks a lot more nervous. If we started doing that, one might say, then we are just no longer doing mathematics.

One thing to note is that if we have no other name for it and it looks like mathematics, it is somewhat vacuous to insist it is not mathematics. This is not to say there can be no demarcation between mathematics and other disciplines; but it just begs the question to demarcate mathematics so as to exclude anything beyond standard mathematics.[26] Considering alternatives seriously may well lead to the creation of new *distinct* disciplines, but it all has to start somewhere.

Now, some readers may be worried about the analogy with pseudoscience, or indeed "alternative science". It might be thought that exclusion is a safety measure: we should not call this mathematics, lest people start thinking it is as legitimate as standard mathematics. Sympathetic as one may be to this worry, this cannot be an a priori judgement based on the mere existence of deviation from the standard model, on pains of locking mathematics away from any sort of structural criticism. In other words, the *legitimacy* of a critical math kind should, for the sake of discovery, be a matter of its legitimacy qua practice rather than its legitimacy qua mathematics. This doesn't mean relying entirely on the practice's internal standards, but simply to be open to various avenues of justification, while keeping track of the fact that the adequacy of a justification will partly depend on what the practitioners are aiming for.[27] And besides: worst case scenario, we can call it bullshit mathematics.

6.3 Reappropriation Revisited

Reappropriation is a more diplomatic tactic than exclusion, insofar as it avoids outright rejection of alternatives. Nowadays it is a widespread and complex phenomenon, and the framework can do much to help us understand it. First of all, I think the focus on

[26] In this sense I am following Bloor (1991), who is skeptical of exclusion charges on the grounds that they prevent discussion from happening by fiat. Of course there is no social component to mathematics if we exclude everything social from mathematics by definition!

[27] There is of course the matter of when we should bring a practice up for consideration as a critical math kind. But I doubt there can (or should) be any deeper answer than "when it looks enough like mathematics, and not enough like anything else".

practice helps a lot in dealing with unwanted trivial reappropriations. Insofar as math kinds are made out of practices, it doesn't really matter if the output of one practice can be conceived as the possible output of another, as long as the two communities are practising mathematics differently. So, for example, formal translations should not count as direct evidence against the existence of a critical kind unless they come with an explanation of why they undermine the specific criticality in question.

The difficulty in reappropriating a critical kind is roughly proportional to its criticality. Non-critical and inert critical kinds are of course easy reappropriations. Conservative criticality is also quite straightforward: by design, acknowledging its existence does not require any kind of shift on the part of standard practitioners. Practitioners of this kind of criticality will just keep working at the very edge of the field, their work not being quite part of the standard model but also inoffensive enough that exclusion is unnecessary; and maybe one day their work will be such that it will be encompassed in the standard model. This is an uncontroversial form of historical progress in mathematics, quite independent of any belief in the occurrence of revolutions.

Now, as a matter of fact, it is not the case that *every* critical kind is conservative or inert. Many critical mathematicians have been quite explicit in their progressive/radical aims. Despite this, there is a tendency in the history of mathematics to turn any attempted progressive or radical shift into a conservative one. Azzouni (2007) puts it quite well: "Brouwer wasn't interested in developing more mathematics, nor were (and are) the other kinds of constructivists that followed; he wanted to change the practice, including his earlier practice. But he only succeeded in developing more mathematics [...] This is common. Fads in mathematics often arise because someone (or a group, e.g. Bourbaki) thinks that some approach can become *the* tradition of mathematics - the result, invariably, is just more (additional) mathematics." (p.8). This is directly connected to the idea that there can be no revolutions in mathematics: every attempt at a revolution is just going to create more mathematics. I am going to call this tendency *conservative reappropriation*.

There are two kinds of conservative reappropriation, based on the two types of conservative kinds. Positivist reappropriation maintains the idea of mathematical knowledge as safe and cumulative: this is what happens when a critical kind is subsumed into the standard model by expanding what some coordinates allow for. This is so common that some take it to be a necessary condition for a successful critical kind: for example, much of the literature on paraconsistent and naive set theory - which is usually critical along the logic dimension - takes for granted the requirement of *classical recapture*, i.e. the idea that classical mathematics should be recoverable as a hopefully straightforward subcase.[28]

Meanwhile, liberal reappropriation defuses any threat alternatives could pose to the mainstream by simply accepting them as separate. Following Azzouni again, the 20th century proliferation of alternative logics led to "mathematical liberalism: the side-by-side noncompetitive existence of (logically incompatible) mathematical systems" (p.22). While this may sound like a nice compromise for nonclassical mathematicians, it is worth emphasizing how much of a *distortion* it is to throw every proposed

[28] See e.g. Carnielli and Coniglio (2013), (Istre 2017, chs.5-6), and Priest (2017).

logic change in the liberalism bin. This is noticeable not only in the move from a progressive/radical shift to a conservative one on the logical side, but also in the complete erasure of *other* parallel shifts that were supposed to accompany the logical one. For example, as already mentioned, Bishop (1975) deviates progressively along both the logic and worldliness dimensions: the assumptions that let us recapture classical mathematics within intuitionistic logic are the same that limit its applicability, so classical practices are only acceptable qua idealizations or approximations, and should be treated as such. Meanwhile, Weber (2021) deviates radically from the logic and worldliness coordinates: mathematics should be guided and constrained by metaphysical intuition, which demands a paraconsistent logic and sometimes prevents the recapture of classical theorems, meaning that many of them should be rejected. To see these proposals as a mere change of logical systems, or even worse as an exploration of logical systems "over there", independently of what classical mathematicians do, is simply to misinterpret them.

Even setting aside what critical mathematicians actually say, Azzouni's mathematical liberalism is also quite reductive when it comes to accounting for the *practice* of nonclassical mathematicians. If I abstract from concrete procedures and think in terms of following rules to derive consequences from axioms, then sure, constructive and classical mathematics function more or less the same; but the abstraction is too removed from reality. I think this is also evidenced by just how difficult it can be for classical mathematicians to "get" the constructive mindset; it feels like learning an entire new way to do mathematics.[29] For the same reason, I think Van Bendegem (2005) is a bit too quick in dismissing the alternativeness of intuitionistic and inconsistent mathematics on the grounds that they "share too many properties" (p.351) with standard practice; it would be more accurate to say that they are *presented* as sharing too many properties through liberal reappropriation.

Now, the reader may nevertheless join Azzouni in his apparent enthusiasm here. Isn't it cool that, through conservative reappropriation, mathematics can incorporate all kinds of novelty without losing its cumulative flavor? Sure, some mathematicians are not being taken very seriously, but they can still do the work they want, over there in the corner; and if the corner grows big enough, well, they may be welcome into the standard model. It is truly the land of mathematical opportunity![30] What this optimistic outlook misses is the fact that conservative reappropriation is, at its core, a way to *avoid confrontation*. Of course this may be more or less appropriate depending on the specifics; but insofar as constructivists, inconsistent mathematicians, and whoever else are trying to *criticize* to some extent current mathematical practice, conservative reappropriation shoves that criticism in the corner under a pretense of tolerance. If constructivists are right that there is a problem with classical mathematics, the problem is not addressed by granting that it's an acceptable feature of mathematics that

[29] The fact that the converse isn't the case is simply a consequence of the standard model being what it is. If constructive mathematicians want a career, they must become proficient in classical mathematics first.

[30] In fact, some might suggest that this sort of pluralism is *already* a part of the standard model. I am wary of framing things like this, however, because it overlooks the de facto dominance of certain coordinates. Still, my points in this paper remain largely unchanged if the standard model is so modified. The thing about conservative criticalities is precisely that very little changes whether they are included in the standard model (i.e. in the mainstream) or not.

everyone is free to study their own structures no matter the underlying logic. Progressive and radical criticalities cannot be ignored by the community at large; conservative ones can.[31] None of this is to say that every criticism of standard mathematics is automatically valid and deserves uptake, of course; the point is simply that conservative reappropriation is not, in general, a way to take that criticism seriously, so there are reasons to be wary of its pervasiveness.

6.4 Conceiving Alternatives

One last application of the framework I want to highlight is the individuation of particular alternatives. Given a certain philosophical question, what kind of alternatives would serve as evidence for one answer or the other? Another version: given a certain problem with standard mathematics, what kind of alternatives would help us addressing it? In order to showcase how the framework can help in answering such questions, I will focus on one particular example, namely the search for a feminist mathematics, i.e. a mathematics that would answer to criticisms of mainstream mathematics from a feminist standpoint.

Let us go through this step-by-step. First question: which features of the standard model are problematic from a feminist perspective? Maybe the most obvious target is the ethicality coordinate: insofar as choices concerning the development of mathematics have social consequences, the mainstream attitude of not paying attention to said consequences goes against the feminist aim of accounting for and possibly correcting them.[32] Meanwhile, a notorious argument by Plumwood (1993) concludes that classical logic is problematic due to its naturalization of the logical structure of dualisms like man/woman, human/nature, etc.; Mangraviti (2023a) extends this line of argument to classical mathematics, from which it follows that a feminist mathematics needs to at least be critical along the logic dimension.

Second question: what critical level is required to properly address the issue? In unpublished work, Thomas Ferguson and Jitka Kadlečíková follow Plumwood in arguing for a rejection of classical logic, and draw the conclusion that some classical methods and theorems should no longer be accepted. This is a radical shift of the logic coordinate. In contrast, Mangraviti (2023b) argues in favor of a proliferation of logics as tools for problem-solving with no one logic sitting above the others, *and* a shift in the ethical attitude of mathematicians such that logical experimentation and

[31] Appeals to unfettered freedom are also a bit disingenuous here: it makes a difference where the funding goes, and it won't go to the corners so easily. This creates a situation where critical mathematics, which of course starts underdeveloped compared to mainstream mathematics, has fewer chances to develop than mainstream mathematics, thus reinforcing its underdeveloped status. Could the situation not change once the alternative is recognized to be sufficiently useful? Maybe in some cases; but it is hardly a *given* that such a recognition would ever occur, because the alternative may never receive enough support to prove itself as promising as the mainstream. More importantly, in general there is no assumption that an alternative has the potential to be *more* useful than mainstream practices, nor that it is competing with the mainstream on grounds the mainstream accepts as legitimate. See Soler (2015) for discussion of this issue around alternative physical theories. Thanks to a referee for pushing me on this point.

[32] See e.g. Shulman (1996).

destabilization of existing results need not require justification on epistemic grounds.[33] This is a progressive - not radical - shift along both the logic and ethicality dimensions: nothing has to be rejected, but all of standard mathematics is nevertheless affected.

Third question: is there any known alternative with this profile? Plumwood argues that relevant logics are some of the best alternatives for the job of fixing the problem with classical logic. Ferguson and Kadlečíková take this to its logical conclusion, and point at Bob Meyer's relevant arithmetic $\mathbf{R}^\sharp$ as an example of feminist arithmetic in its infancy.[34] Meanwhile, Mangraviti (2023a) argues towards a new alternative altogether, but at the same time suggests that the existing frameworks produced by inconsistent mathematics suffice as an ideal groundwork: the required criticality lies entirely in the way they are used, i.e. in the agents. This is not to say that new frameworks would not arise; the point is just that these new frameworks would not necessarily be different in kind from the ones already existing.

There is a fourth question, which does not serve the purpose of individuation but rather that of assessment: how does the required critical profile fare w.r.t. reappropriation and exclusion? By experience, we know that radical proposals like $\mathbf{R}^\sharp$ are likely to suffer liberal reappropriation.[35] And we can see here how in this case liberal reappropriation would completely defeat the purpose: the alternative was argued to be a desirable feminist alternative only insofar as it constituted a radical shift - if classical logic is allowed to preserve its default role, it's all for nothing. The proposal in Mangraviti (2023b) is slightly less in danger: progressive shifts are naturally going to encounter less pushback than radical ones. But once again, the danger of conservative reappropriation is one to be taken seriously, as it risks undermining the very justification for pursuing the alternative. We see here how the framework is not only useful in pinpointing areas of disagreement or comparing proposed alternatives: it can also drive research by highlighting questions that need answering.

7 Conclusion

There is much to be gained from a systematic study of alternative mathematics. By looking at alternatives as graded deviations from a multi-dimensional standard model, we can make sense of the myriad of alternatives in the literature, analyse patterns of exclusion and reappropriation, clarify debates that involve alternatives, and individuate alternatives that could serve as solutions to existing problems. There is an entire field - a *philosophy of alternative mathematics* - waiting to be explored. I hope this paper provided some helpful tools and insights to begin the journey.

33 See Sections 2.8 and 6.4 in particular.

34 Actually, Plumwood argues in favor of *weak* relevant logics, rather than the strong relevant logic underlying $\mathbf{R}^\sharp$. Ferguson (2023) addresses just this issue.

35 $\mathbf{R}^\sharp$ was not originally meant to be that radical at all: the hope was that it would turn out to be an extension of Peano Arithmetic, and serve simply as a different formalization of the same informal practices. Recapture of Peano Arithmetic was shown to fail by Friedman and Meyer (1992), and the fairly basic counterexample is easily seen as a product of informal practices.

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