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ORIGINAL RESEARCH

Axioms and Postulates as Speech Acts

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Abstract
We analyze axioms and postulates as speech acts. After a brief historical appraisal of the concept of axiom in Euclid, Frege, and Hilbert, we evaluate contemporary axiomatics from a linguistic perspective. Our reading is inspired by Hilbert and is meant to account for the assertive, directive, and declarative components of modern axiomatics. We will do this by describing the constitutive and regulative roles that axioms possess with respect to the linguistic practice of mathematics.

1 Introduction

Mathematics strives for clarity. Its rigor consists in making explicit each single step of an argument. In doing so mathematics is therefore rigidly structured in premises and conclusions, hypotheses and theses. But to avoid infinite regress a mathematical argument needs a starting point, able to offer the ultimate bedrock of its rigor. This is normally presented in the form of an *axiom* or a *postulate*.

At the center stage of mathematical practice, the premise-argument structure presented in mathematical texts plays an important justificatory role, as mathematicians are often required to provide reasons for their claims. From this perspective, axioms and postulates play an important role since they provide a starting point for a mathematical argument. Like the *incipit* of any discourse, they can set the stage for the rest of the discussion, serving the argumentative goals of the speaker.

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There is much more to the mathematical practice than the argumentation provided by mathematical texts and much more to the premise-conclusion structure than the axiomatic method.[1] Nonetheless, the use of axioms and postulates features a key aspect of mathematical practice since antiquity and is still representative of how mathematical texts are structured today. Therefore, our goal is to analyze their peculiar function from a linguistic perspective. Building on the preliminary work of Ruffino (2021), our starting point considers mathematical arguments as a complex and well-structured set of speech acts in which axioms and postulates are key elements.

The traditional understanding of axioms as self-evident propositions and of postulates as auxiliary hypotheses has received severe criticisms in a recent debate in the philosophy of mathematics.[2] Moreover, a careful historical analysis of the notion of axiom shows a plethora of different uses.[3] But then, how can we attempt to describe such a peculiar and multifaceted character of axioms and postulates, by analyzing their linguistic roles?

Luckily we do not have to start from scratch, since the philosophy of language has already developed sharp tools for such analysis. We can indeed apply Speech Act Theory to the study of axioms and postulates in order to understand what we do with words when we axiomatize and postulate.

Speech Act Theory is roughly the study of the illocutionary acts and the *illocutionary forces* they express. The main goal of Speech Act Theory is to unveil such a variety of linguistic acts and to describe the structure underlying each speech act. Here, we follow Searle's analysis as exposed in Searle (1969, 1979), and (jointly with Vanderveken) in Searle and Vanderveken (1985). From these classical texts, we will extract a taxonomy of five different speech act classes: *assertives, directives, declaratives, commissives* and *expressives*. As we shall see, for the analysis of axioms and postulates the first three classes suffice.

The paper is structured as follows. We start in Sect. 1 considering the notion of axiom in Euclid, Frege, and Hilbert. We show that one can find substantial differences between these authors simply by considering the different illocutionary forces in play. Moreover, we will argue that this analysis suggests not only a shift from a more procedural to a more ontological conception of mathematics, but also a progressive identification of three different notions: postulates, axioms, and definitions. Then, in sect.2 we propose our own reading of contemporary axiomatics. We argue that in laying down axioms and postulates it is possible to find both assertive and declarative illocutionary forces (together with some directive elements). This will not only show a hybrid type of speech act, but will also explain the current linguistic use in accordance with a Hilbertian picture of the axiomatic method.

[1] See Lakatos, (1976) for a famous attack on the formalist and deductivist argumentation method. For a more recent survey on the informal logic behind the dialectics of mathematical proofs, and the different dialogue roles it may take, see Aberdein, (2006).

[2] See, for example, Maddy (1988, p. 481).

[3] A summary of the different roles played by axioms in history can be found in Schlimm (2013).

2 Axioms and Postulates in Euclid, Frege, and Hilbert

The historical examples discussed in this section are not meant to provide a thorough and complete story of the use of these notions; they only serve as paradigmatic examples of three different conceptions linked to three different linguistic practices.

2.1 Euclid

The birth of the axiomatic method goes back to ancient Greece and to Euclid's *Elements*.[4] The basic principles that Euclid laid down at the foundations of geometry were divided into three groups called, respectively, *Definitions*, *Postulates*, and *Common Notions*.

Common notions correspond to axioms in the Aristotelian sense. As a matter of fact, Aristotle not only represents the cultural background that shaped the *Elements*, but also his use of the terms 'axiom' and 'postulate' seems to be derived from the mathematical use.[5] Common notions and postulates seem to differ not only with respect to their scope of application - while common notions were the most basic premises for any form of argument, the postulates were specific to a domain of knowledge[6] - but also with respect to the way these notions were applied. Indeed, postulates had a constructive component that common notions lacked.

The latter difference clearly bears on the centrality that constructions have in the geometry of Euclid's *Elements*. Indeed, Euclid's postulates are usually interpreted as instructions to be followed for the performance of a geometrical construction, which in turn grounds the truth of a geometric proposition. This reading is supported by the linguistic form of postulates, which are written in the infinitive form.[7]

Let it have been postulated *to draw* a straight-line from any point to any point.

On the other hand, common notions are written in the general neutral.

Things equal to the same thing *are* also equal to one another.

Euclid's understanding of these two notions would dominate geometrical practice until the XIX century, when a new conception of axioms will take over, determining a more abstract (and realist) conception of mathematics. For that reason, we will take a huge step forward to late XIX century Germany and consider two events of fundamental importance for the history of axiomatics.

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[4] We will refer to Euclids's Elements as translated by Richard Fitzpatrick in Fitzpatrick (2008).

[5] For a detailed discussion on this point see (Einarson, 1936).

[6] This explains why Euclid's common notions can *also* be applied to geometry, while any talk about points, lines, or planes, which is the content of the postulates, can *only* be applied to geometry.

[7] The last two postulates are not, which suggests a later addition, as argued in Einarson (1936).

2.2 Gottlob Frege

The year 1879 is commonly set as a turning point in the history of logic. It is in this year that Frege published the first edition of the *Begriffsschrift* (Frege, 1967). Even by today's standards, Frege's conceptual-notation is considered *sui generis*. But for what concerns us here, Frege was the first to offer an explicit linguistic analysis of axioms. His axioms, which he called judgements of pure thought, and later basic laws, are not mere sentences but *assertions*.

In Begriffsschrift's notation, axioms are prefixed with the special sign "$\vdash$" in which the vertical line on the left was called the 'judgement-stroke'. While in modern notation, an axiom is considered a complex propositional sentence – a syntactic entity – instead, for Frege, it is an example of an utterance: an *assertion* of some self-evident content.[8] Thus, "$\vdash$" is, in all respects and with a modern terminology, an illocutionary force indicating device for an *assertive speech act.*

Although Tarski's semantical conception of truth is now standard, back in the pre-formal days of logic, the concept of truth was treated indirectly through judgements. As Frege realized, a sentence needs to be conceptually separated from its truth. Thus, Frege proposed to separate the assertion from the assertible content. This means, in modern terminology, to separate the illocutionary force of an assertion from the asserted propositional content. To formally indicate the assertoric force, Frege used the judgement-stroke.

Frege's axioms can be compared to Euclid's common notions. They are both written in the present tense (as Frege's assertions read that way) and both have general content. Naturally, Frege's universal conception of logic pushed him to read even Euclid's postulates as assertions[9]; contrary to their directive form. Take, for example, Euclid's first postulate. Surely, a straight-line can be drawn from two given points, but this is not a question of our ability in geometrical constructions. Indeed, in Frege's eyes, between two given points *there exists* a line.[10]

Frege's point can be seen as a prime example of the objectification of mathematics undergone in the XIX century, as postulates were naturally translated into existential statements about geometrical objects. Postulates serve only auxiliary purposes, as they "[...] merely bring to attention, apprehend, what is already there. What is essential to a proof is only that there be such a thing." Frege (2013, I, §66). So, under Frege's reading, postulates are expressed in the infinitive form only for matters of heuristics. But, of course, logic deals with justification and not heuristics.

[8] Frege's view on axioms can be found in Frege and Posthumou (1979, p.205).

[9] On Frege's universal conception of logic, see Van Heijenoort, (1967) and Goldfarb, (2010).

[10] See Frege and Posthumou(1979, p.207).

A proof can rely on the construction of some auxiliary line, or point, or number, but its truth ultimately relies on the existence of such a line, point, or number. To construct is just a way of grasping their existence. Ultimately, postulates are, still, assertions in disguise: “In proofs, Euclid’s postulates thus have the force of axioms that *assert*[11] that there are certain lines, certain points.” Frege (2013, I,§66). This existential shift explains why, and around when, our current practice starts treating axioms and postulates as synonymous, and this change is easily grasped as a shift in the illocutionary point.

To sum up, two things should be highlighted about Frege’s analysis. The first one is linguistic, as axioms are seen intrinsically as assertive utterances, making Frege the first to realize the importance of the illocutionary force of axioms and postulates.[12] The second one is logical, and it justifies why Frege saw axioms in this way: his universal conception of logic. But only twenty years later, this conception will clash with the most important revolution in the axiomatic conception since Euclid: the birth of the modern axiomatic method.

2.3 David Hilbert

The year 1899 marks an important dividing line in the history of mathematics. Many ideas that were roaming Europe in the second half of the XIX century found a clear and powerful expression in the innovative presentation of geometry that Hilbert gave in the *Grundlagen der Geometrie*. And from the perspective of how axiomatics developed since then, Hilbert’s influence far surpasses Frege’s.

For what concerns axioms, if Frege made a first step in identifying axioms and postulates, in Hilbert’s *Grundlagen*, the three basic notions that we find in Euclid (i.e., definitions, postulates, and common notions) are further unified into one: axioms. Hilbert’s axioms for Geometry are divided into five groups: connection, order, parallels, congruence, and continuity. These axioms do not represent self-evident truths, but “Each of these groups expresses certain related facts basic to our intuition.” Hilbert (1971, p. 3) For example, the first axiom of the first group reads as follows.

> **I, 1** For every two points A, B there exists a line a that contains each of the points A, B.

Although Hilbert is often considered the champion of formalism, there is, at least in the explanation of the origin of the axioms of geometry, a reference to our geometric intuition. However, the origin is not to be confused with the justification. Hilbert’s axioms are not self-evident facts of our geometric intuition, but only formal principles that *express* fundamental facts of our intuition. In this sense, axioms

[11] The emphasis is ours.

[12] Although his analysis also produced a reduction of postulates to axioms.

only represent, without necessarily being identified with, facts of geometrical intuition. This separation between formal representation and concrete subject matter is, for Hilbert, the force and the innovative character of his axiomatic method. Formal principles do not come with an intended interpretation, even if they might originate from it.

The indicative tense of Hilbert's axioms assimilates them to Euclid's common notions. However, as is the case with Frege and contrary to Euclid's usage, the verb "to be" is charged with existential meanings and often used in expressions like "there exists" or "there is".[13] Therefore the question of existence of mathematical objects becomes essential to understand Hilbert's axiomatics. Indeed, Hilbert's work is clearly immersed in a set-theoretic context with a clear realist flavour.[14]

But despite this realist flavour, Hilbert's axioms are not simply assertions but also definitions. In correspondence with Frege, he maintains that the axioms of the *Grundlagen der Geometrie* define the objects of geometry. However, as Frege rightly points out in his response, Hilbert's axioms determine the properties of geometrical objects and not the geometrical objects themselves.[15] Following Frege's correct assessment, the axioms of the *Grundlagen* specify the conditions for a group of objects to be geometrical. In other words, they define what it means to be an object of geometry,[16]. But still, Hilbert's idea that consistency implies existence reinforced the existential character of the axioms.

In effect, axioms are not only meant to describe an independent reality, but they play an active role in the construction of a new theory for geometry. Indeed, either axioms impose a logical structure on a subjacent independent reality, or they bring about the very reality they are meant to capture, thanks to a consistency proof. Therefore for Hilbert, axioms are not simply assertions whose truth can be assessed by a comparison with some pre-existent reality, neither are they directions for the construction of mathematical objects, but, similarly to definitions, they play an active role in structuring our image of mathematics.[17]

[13] The most notable exception is the axiom of completeness, which is of a meta-theoretical character and which does not appear in the first edition of the *Grundlagen der Geometrie*.

[14] For example, in Hilbert (1971, p. 3), Hilbert invites the reader to consider a class of objects that the axioms of the *Grundlagen der Geometrie* will structure into the objects of a geometrical theory, suggesting that the objects of geometry come before the characteristic marks given by the axioms.

[15] See letter from Frege to Hilbert, dated January 6th, 1900, in Frege (1980, p. 39).

[16] Hilbert's axioms can, in fact, be considered implicit definitions of the primitive terms of the theory being axiomatized. But, following (Giovannini & Schiemer, 2019) it is more precise to label them as structural definitions since there is no uniqueness condition on the interpretation of the terms. Although the axioms are holistically specifying a class of models for the given theory, no particular model is fixed, and thus, the meanings are left undetermined. This condition is crucial in analyzing the linguistic role that axioms play in Hilbert's and contemporary axiomatics, as will be shown later.

[17] The creative part of Hilbert's axiomatics very much resembles Dedekind's definitions, in Dedekind (2008). Dedekind's view on axioms is much closer to Frege's than it is to Hilbert's, which explains why he opted for an axiomless presentation in the *Was sind und was sollen die Zahlen?* as he saw numbers as "free creations" of the human mind. This reliance on the creative powers of definitions has its roots both in Christian Wolff's deductive methodology and Kant's principle of the spontaneity of the understanding, as argued in Ferreirós and Lassalle-Casanave (2022). Moreover, we can apply this heritage to Hilbert, given the influence of Dedekind on his work.

2.4 The Frege-Hilbert Controversy

Hilbert's axiomatic innovation was very successful, although it faced the fierce opposition from Frege. For what concerns us here, the dispute is interesting in one important respect: the controversy can be viewed as a clash of two opposite views on the illocutionary nature of axioms.

As we saw, Hilbert's axioms did not come with a unique interpretation, although their arrangement in five groups served the purpose of representing our geometric intuitions; and Hilbert's (relative) consistency and independence proofs showed how useful this perspective is, in practice. But the strategy of proving the consistency of the axioms, by interpreting geometry in the real field, or the construction of models for different *ad hoc* geometries, could not be reconciled with a perspective like Frege's, according to which "Axioms do not contradict one another, since they are true; this does not stand in need of proof" Frege (1984, p.275). Hilbert famously responded that he believed in the exact opposite: "if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist."[18]

Hilbert's model-theoretic thinking is utterly alien to Frege's universal conception of logic. And it makes perfect sense for Hilbert to say that a set of axioms *define* a geometry. As a matter of fact, definitions are not considered anymore to capture some essential aspects of a given field of knowledge, as Frege thought they should. On the contrary, for Hilbert, definitions are blueprints that are used to determine concepts, whose content may vary according to the field of application of the definitions. The definition of a 'point' in Hilbert's *Grundlagen* is general and ambiguous enough to allow different references within different geometries. What Frege saw as a limit, Hilbert saw as a fruitful possibility.

This sets the background for Frege's criticism that interests us here: Hilbert's axioms have mixed (confused, for Frege) illocutionary points. Frege's view on definitions roughly takes them as mere stipulations, like when a word, or a sign, receives a meaning by an act of baptism.[19] From the perspective of speech act theory, definitions are declarative speech acts, written using the declarative mood. Since assertives and declaratives are different types of illocutionary acts, and since axioms are inherently assertives, axioms and definitions are, for Frege, distinguished on the ground of their illocutionary force. As he claimed, "To definitions that *stipulate* something, I opposed principles and theorems that *assert*[20] something" Frege (1984, p.294). Whereas for Frege they are not supposed to stipulate anything, this is exactly what Hilbert's axioms do. In Frege's perspective, Hilbert's axioms are: "[...] saddled with something that is the function of definitions." Frege (1984, p.275).

The differences between Frege and Hilbert therefore can be precisely cast in terms of their different views on the function of language in axiomatic theories. A definition, in Frege's mind, is just a linguistic convention that only serves the goal

[18] Letter from Hilbert to Frege, dated December 29th, 1899; in Frege (1980), p.39.

[19] See Frege (1967, p.26) and Frege (2013, p.27).

[20] The emphasis is ours.

of naming a concept. One can only find out the success of one's system of axioms and definitions in proving theorems. But the background for the theory is given: the unique realm of geometry, which is the only place to evaluate the success of the choice of an axiomatic system.[21] In Frege's mind, axioms and definitions serve entirely different purposes within this unique realm.

Hilbert's view is completely different. Since a given formal theory has no unique interpretation, what is defined as 'point', 'line' and 'space' can vary according to the models of the theory. Therefore, one can see that, ultimately, axioms *do* define something.

We take it that Frege was correct (and to the point) in saying that Hilbert's axioms are also definitions. But this was far from being Hilbert's Achilles' heel. In fact, as is well known, a model-theoretic perspective dominated subsequent investigations both in mathematics and logic. If we are to assess the present-day axiomatics from the perspective of speech act theory, therefore, we must listen to Frege's precise diagnosis and recognize the mixed role that axioms had, for Hilbert, and still have, for us.

3 Contemporary Axiomatics and Speech Acts

If we can learn a lesson from this very brief historical overview, it is that axioms can play different linguistic roles in the construction of our mathematical theories. We find directive speech acts as a set of instructions for the construction of mathematical entities, assertive speech acts meant to describe an independent reality, or declarative ones aimed to create a new layer of mathematical reality. We believe that this linguistic point of view not only provides a taxonomy of the different speech acts we find in mathematical language, but that it also provides a vantage perspective for evaluating and clarifying different positions in the philosophy of mathematics. Our next task, then, is to put to good use this connection between linguistic and theoretical ideas and to shed new light on contemporary axiomatics.

The analysis we propose might be interpreted in two ways: as a linguistic analysis of mathematical practice or as a reconstruction of Hilbert's axiomatic perspective in linguistic terms. Although both interpretations are, in our view, partially correct, neither constitutes a basic claim of our proposal. There is no doubt that Hilbert's view is essential to our contemporary practice, but it is also true that the former cannot be identified with the latter. If our proposal is considered historically inaccurate or divergent from practice, still, we believe it is of theoretical value as an example of a practice-oriented proposal inspired by a Hilbertian perspective.

In order to evaluate contemporary axiomatics there are two important aspects to be considered; both of which have clear and direct roots in Hilbert's thought. First of all, axioms are syntactic items deprived of an intended interpretation and, second, only within a model do axioms acquire meaning. Based on these two simple

[21] The axioms of geometry, thus, are truths from "a source which might be called spatial intuition," Frege writes in a Letter to Hilbert, December 27th, 1899. Frege (1980, p.37).

observations, we will develop a proposal that is Hilbertian in spirit and Tarskian in truth.

Given these premises, what can the illocutionary point of an axiom be? As our reading of the Frege-Hilbert controversy made clear, the crux of the dispute was the illocutionary role of axioms. Frege's realist position viewed axioms as pure and self-evident assertions about a well determined reality, whereas Hilbert's model-theoretic view saw them as definitions.[22]

In our proposal, we will explore the possibility that axioms have a double illocutionary role: partially declarative, and partially assertive. Moreover, we will also find some directive components in their declarative dimension. Axioms seem to define the context in which they are successively interpreted as asserting features of that same context. But how to disentangle this double role and, thus, how to determine the illocutionary point of contemporary axioms?

Our strategy is to take modern axioms' illocutionary point to vary with the scale we use to evaluate them. Individually, axioms are assertives, but jointly they are declaratives. In other terms, a set of axioms has as a whole a definitional role, while individual axioms, on the other hand, allow assertions about what has been defined (or its possible interpretations). Both aspects must be detailed.

3.1 Axioms' Definability

We are now in the position of addressing the main issue connected with the definitional character of axioms. Do they really bring something into existence? We believe that the answer to this question rests on a modality of being more than on existence itself. Concretely, the definitional dimension of axioms amounts to a new *way* to look at mathematical reality.

A few important clarifications are needed. The first one consists in acknowledging the explicit theoretical separation between language and reality that underlies this proposal. This is an important aspect of speech act theory that finds here a natural exemplification. Indeed, it is only by conceptually keeping separate language and reality that it is possible to account for an active component of language. A second important aspect to be clarified is the status of the alleged reality that counterbalances this separation. On this point, it is also important to be explicit that no claim is here advanced on the ultimate constituents of mathematical reality. The present account is *neutral* on the ontological status of mathematical reality. Mathematics might be a made-up fiction or the study of an independent reality of eternal objects, but still there would be a creative component of language in the construction of our mathematical theories. To negate this would consist in misunderstanding an essential component of language. As a matter of fact, this is already the case when considering natural language and its role in shaping our (social) reality. If language is able to create a new layer of reality over and above the one we concretely experience in our everyday life, why should this not be the case when dealing with a reality

[22] There have been many views on how to differentiate both trends. See, for example, Coffa (1991, Ch. 7), Detlefsen (2014), Shapiro (2005), Shapiro (2009) and Potter (2004).

whose independent existence is even thinner (often even up for discussion)? If the thin character of mathematical reality is of any advantage with respect to the thick reality of physical bodies, this will surely be the case in assessing the creative component of language.

A possible objection would consist in pointing out the obvious differences between natural and formal languages. This objection, however, is based on a misleading idealized picture of formal language. To explain why this is the case, notice, first of all, that formal languages (as those defined by logicians) are not those used in practice in journal papers. The notion of deduction, which is normally formalized as a sequence of statements that starts with axioms and advances with inference rules, is a highly idealized one, clearly not representing the language that is actually used in the everyday work of mathematicians.[23] The presence of illocutionary force indicator devices and different kinds of linguistic acts in mathematics talks for itself. The recognition of a strong and deep connection between natural and formal languages represents a basic presupposition of the present approach that is motivated by its fruitfulness in assessing discussions like the one between Frege and Hilbert and in proposing a more complete picture of mathematics in continuation with any other human activity based on language.

It is now time to explain in which sense the definitional aspect of an axiomatic system provides a new way to look at mathematical reality. By this, we mean that axioms introduce a conceptual scaffolding that allows one to carve up mathematical reality according to the concepts involved. This idea is not peculiar to mathematics but is a fundamental aspect of a linguistic conceptualization of reality. Consider, for example (of Fregean inspiration), a deck of cards on a table. According to the use of the concept of deck or of that of card, we would be inclined to say that there are either one or fifty two objects on the table. And of course the same reading applies to the table and its atoms. The same happens when considering the relationship between language and mathematical reality (disregarding the ultimate nature of the latter). As in the case of natural language, the conceptual framework that we use to represent a collection of facts allows us to see reality in new ways by building into one's perspective the same concepts used to express it. In the case of axioms, the declarative dimension of axioms allows us to impose a new conceptual apparatus on mathematics and, therefore, axioms provide new ways to look at a collection of (previously given) mathematical facts. For example, it took the genius of Galois to see the connection between finding the roots of a polynomial and the group of their permutations. In this case, the study of the solutions of polynomial equations provided a collection of mathematical facts that were later conceptualized in terms of group theory. Consequently, from a purely axiomatic perspective, the axioms of group theory can be seen as the definition of a group structure that is not only realized by, but also imposed on, the collection of permutations of the roots of a polynomial.

The idea that we would like to convey is that the process of axiomatization (at least in this Hilbertian spirit that we are evoking) not only consists in fixing the

[23] For a thorough analysis of the mixture of formal and informal language used in mathematical texts, see Ganesalingam, (2013).

logical structure of a set of given mathematical facts, but it is also constitutive of a new way to look at the same facts. The new ways of looking at mathematics, provided by the definitional component of axioms, suggest an active component of the axiomatizer, realized by linguistic means. The suggestion that language has a constitutive role in the construction of our mathematical theories is an important insight that is based, ultimately, on the acceptance of a methodological similarity between natural and formal languages. Once again, this connection suggests importing linguistic tools for the clarification of the functioning of mathematical language. In order to provide such an analysis we can recall the distinction between *constitutive* and *regulative* rules.[24]

3.1.1 The Constitutive and the Regulative Roles of Axioms

The recognition of a creative dimension for language is a traditional philosophical theme that finds in Searle's work (Searle, 1969, 1995) its modern and standard formulation, in terms of the distinction between constitutive and regulative rules.

In order to exemplify this distinction, we can follow Searle's paradigmatic examples of the rules of chess and the rules of etiquette. While the former are constitutive of the game of chess, in the sense that it is only by following the correct rules that we can be said to be playing chess, the latter sanction a pre-existing set of good behaviours that only subsequently are normed by explicit rules.

Searle's distinction attracted many criticisms (Ransdell, 1971; Warnock, 1971; Giddens, 1984; Ruben, 1997; Hindriks, 2009). One in particular is relevant to our discussion and it consists in arguing that it is far from obvious how to trace a precise dividing line to separate constitutive from regulative rules. On the one hand, it has been suggested that any constitutive rule regulates the behaviour of those who follow it, and thus that all rules are regulative (Warnock, 1971; Giddens, 1984). On the other hand, it has been argued that regulative rules are also able to set up a web of norms which, alone, are constitutive of a new layer of reality (Hindriks, 2009). The main content of this discussion, thus, is whether Searle's distinction is only a linguistic one or, instead, if it is able to capture some genuinely independent aspects of language. For what concerns us here, it is not essential to take a stand on this debate, since we are only interested in axioms and their definitional component. But if we can infer anything from this specific case, this is a partial confirmation of the criticism directed to Searle's distinction, since we can find in axioms both constitutive and regulative aspects.

In a nutshell, we can present our view on the definitional character of axioms as follows. While a set of axioms *constitute*, as a whole, a new way to look at a collection of mathematical facts, each axiom individually *regulates* how mathematical

[24] There is here an important connection with the Wittgensteinian notion of "seeing as". We will not discuss the connection between Wittgenstein's and our position, since this would lead us to exegetic matters that are beyond the scope of the present work. However, we acknowledge an already existing connection, in the literature, between this Wittgensteinian theme and mathematical creativity in Beaney and Clark (2017)

objects should be, in order to be rightful elements of this new perspective expressed by the axioms. Before presenting the specific linguistic acts connected with these constitutive and regulative roles of axioms, a few important clarifications are needed. First of all, the reference to mathematical objects in the regulative component of axioms might seem to contradict the neutral perspective of our linguistic analysis, with respect to ontology. At a closer look, however, we notice that this reference to objects is a by-product of the language at use: that of first-order logic. In other terms, it is a consequence of the basic feature of first-order language that axioms are meant to express facts about mathematical objects. Nonetheless, we take this aspect of logic to be constitutive of our mathematical practice, but not informative about mathematical ontology. A second important point to be clarified concerns the novelty of the new perspective offered by axioms. Since we are putting forward a neutral perspective on mathematical ontology, what axioms constitute are new roles or statuses that the (possible) referents of our theories play in our linguistic practice.

Let us exemplify this talk of roles or statuses with a concrete case. Consider the collection of all rational numbers with the sum operation $(\mathbb{Q}, +)$. It is a basic algebraic fact that $(\mathbb{Q}, +)$ has a group structure, but it is one thing to list the basic mathematical facts according to which the sum of a rational q with 0 gives the same number ($q + 0 = q$), but another to say that 0 is the neutral element of the group $(\mathbb{Q}, +)$. The former is a collection of mathematical facts that pre-dated the definition of a group and the axioms of group theory, while the latter is a direct product of the group language, which constitutes a new role for the number 0. A third important point of clarification, directly connected to the second, is that, although we talk about roles or status, we do not consider any social dimension of the linguistic constructions performed by axioms. As a matter of fact, when a mathematical definition is felicitous and well accomplished, it does not need a collective agreement for its validation. Any mathematician has at her disposal the constitutive tools of definitions in isolation from the rest of the mathematical community. Notice that we are not claiming that mathematics does not have a social dimension and does not produce institutions, as any other human linguistic activity. We are only claiming that at the level of definitions, and thus at the level of a definitional account of axioms, this social component is absent, since a mathematician is free in the constitution of a new linguistic item.[25] Fourth, the neutral perspective towards ontology that motivates this account does not only apply to the existence of mathematical objects, but it also extends to the existence of the linguistic constructions that originate from mathematical definitions in general, and in particular from axioms. There is a sense in which the products of our linguistic practices participate in a (very light) notion of existence (Thomasson, 2015) and our scientific practice is full of defined terms, whose weak form of existence can therefore be justified internally

[25] There are important differences between this view and other accounts which consider the social dimension of mathematics relevant. The more developed one is that of Cole (2013). The most notable and important difference with Cole's account is that, on the one hand, we do not identify linguistic constructions with mathematical objects or structures (and therefore we are not bound to claim that linguistic constructions are necessary; and indeed are in our view partially contingent), while on the other hand, contrary to Cole's view, we do not consider here the social dimension of linguistic constructions.

from the perspective of our linguistic practices. Lewis even argued that the possibility to define away a scientific term "serves the cause of scientific realism" (Lewis, 1970). Although a similar argument can be applied to the implicit definitions conveyed by axiomatic presentations, we refrain from making this further step. As a matter of fact, our analysis is linguistic and our central claim is that axiom systems can be seen as constitutive definitions of new perspectives on mathematics. Thus, we leave open for now the question of whether the products of our linguistic declarations exist and in which sense.

We can now present a more precise analysis of axiom systems and axioms through the lens of speech act theory.

3.1.2 Axioms as Declaratives

Let $\mathcal{S}$ be a set of axioms. We will characterize the linguistic act of laying down $\mathcal{S}$ using Searle's taxonomy and notation. The illocutionary point of $\mathcal{S}$ is declarative and its mode of achievement is $\mathcal{S}$ itself. If we take $\mathfrak{M}^{\mathcal{S}}$ to be the propositional content expressed by $\mathcal{S}$, then the utterance of $\mathcal{S}$ has the following speech act structure:

$$D \updownarrow \varnothing \; (\mathfrak{M}^{\mathcal{S}})$$

Where D is the illocutionary point[26] of a declaration, $\updownarrow$ is the direction of fit of a declaration,[27] $\varnothing$ is the null sincerity-condition,[28] and $\mathfrak{M}^{\mathcal{S}}$ expresses the content of the declaration. In other words, by uttering $\mathcal{S}$, one is defining the way in which a collection of objects that satisfy $\mathcal{S}$ should be.

The double direction of fit implies that each successful and non-defective definition manages to perform some change in the world. But notice that by uttering or writing down $\mathcal{S}$ one is not bringing any domain of objects into existence, but only defining a way in which a collection of entities should be. In Hilbert's approach, consistent systems imply the existence of a domain of objects satisfying the axioms. On the contrary, here, the existence of a domain of objects satisfying the conditions imposed by $\mathcal{S}$ is *not* part of the definitional role of $\mathcal{S}$. Exactly as the creation of a chair in a department does not create a new person to fill it; and indeed such a chair is perfectly defined before anybody can fill it.

Given our neutral stance towards existence in mathematics, this definitional account of axioms takes into account that axioms can have no realization. Although a set of axioms can be shown to be inconsistent, still as far as its definitional aspect is concerned, a set of axioms represents a genuine definition. As a matter of fact, this separation between definition and realization is what allows us a meaningful use of reasoning by contradiction; thus keeping apart the success of a definition from the

[26] The point of an utterance is the intent of the speaker with the speech act.

[27] The direction of fit is the relation between the propositional content and the utterance. The direction is word-to-world ($\downarrow$) whenever the speaker tries to describe how the world is (assertions), and it is world-to-word ($\uparrow$) whenever the speaker tries to alter the world to match his words (directives). A double direction of fit ($\updownarrow$), as the above, is the attempt to alter the world by representing it as being so altered.

[28] The sincerity condition of a speech act is the psychological state expressed by the speaker.

truth of the propositional content that it expresses. For example, it is exactly the possibility to define Russell's set that permits the statement of Russell's paradox.

3.1.3 Axioms as Directives

As a whole, a set of axioms $\mathcal{S}$ is constitutive of a new perspective on mathematics. The regulative role played by a single axiom follows accordingly.[29] But there are some caveats. In chess, the set of rules constitutes the game, as it defines what certain events must obtain in order to be in the presence of a move of chess. From this constitutive role, one derives a regulative one. For example, the bishop may move to any square along a diagonal on which it stands. This rule does not only constitute what a bishop is, but it gives players permission to move it diagonally if the relevant conditions are met. In speech act terms, we are dealing with permissives: a specific kind of directive speech acts that Searle and Vanderveken call denegations of prohibitions (Searle & Vanderveken, 1985).

We need to introduce some details in order to understand what a denegation is. Searle and Vanderveken (1985, p.4), a denegation is the negation of an illocutionary force, not to be confused with the negation of a propositional content. Consider the following cases:

(1) "I promise not to P"
(2) "I do not promise to P"

While in (1) the speaker commits himself to a negative propositional content (not P), in (2) the speaker is denying a commitment, *viz.* denying an illocutionary force, not a propositional content. For that reason, a denegation is the negation of a speech act. Thus, while (1) is a commissive speech act with negative propositional content, (2) is the denegation of a commissive speech act. Although they are also speech acts, a denegation of a speech act A is not performed automatically simply because A is not performed, as Searle and Vanderveken put it Searle and Vanderveken (1985, p.77). In order for a denegation to happen, the speaker must do something: to make explicit the denial. This distinction mirrors the difference between the denial of an act, and the truth-functional denial of a propositional content. In order to distinguish the two cases, a denegation will be expressed with the symbol $\neg$, while the truth-functional negation (of a propositional content) with the symbol $\sim$.

With this distinction in mind, let us now see how to define permissives from directives. First, "Do P" is a simple directive. Using the truth-functional negation, we obtain a prohibition, "Do $\sim P$", or more simply $!\sim P$.[30] And with the denegation, we obtain a permission $\neg!\sim P$, that is, the denegation of the prohibition of P.[31] Following this notation we can now see how the present case is similar to that of chess.

[29] Again, we do not delve into the dispute over which kind of rule is foundational with respect to the other.

[30] The exclamation sign ! marks the illocutionary point of a directive.

[31] Once again, the truth-functional negation is defined over propositional contents, while the denegation is defined over illocutionary forces. With the former, we can make the transition from a directive (Do P) to a prohibition (Do not P). With the latter, we can make the transition from a prohibition (Do not P) to

The regulative role played by the axioms in $\mathcal{S}$ is expressed in terms of permissions of assertability of each axiom, *modulo* $\mathcal{S}$ as a context of assertion. The idea is fairly simple: the definitional and constitutive role played by the axioms collectively sets forward what it means for a domain of objects to satisfy $\mathcal{S}$, and, at the same time, lay down permissions for each axiom to be asserted individually in such context. Now, because of the internal structure of a speech act, these assertability conditions are expressed as a propositional content of a permissive speech act. We do so by taking the possible future assertion of A, *qua* member of $\mathcal{S}$, as the content of the act. This possible future assertion is not, by itself, an assertion. It just expresses the general conditions for such assertions to be performed successfully and non-defectively by an agent.[32]

When specifying general conditions for directive speech acts, Searle and Vanderveken take the propositional content as representing the state of affairs that the speaker wants the hearer to perform. By uttering "Leave the room", the speaker wants the hearer to act in such a way that the resulting state of affairs includes the hearer as being out of the room. The propositional content P in this case must express that an action is expected to happen in the future, but at the same time, it should provide reasons to the hearer for doing so. Similarly, the regulative role played by an axiom system expresses a possible future course of actions (i.e. the assertion of an axiom A *qua* member of $\mathcal{S}$), together with reasons for an agent to assert it (i.e. that whenever we are in the position to give an interpretation to $\mathcal{S}$, then, A expresses a true[33] fact about that interpretation).

Offering reasons for an assertion is a common event in the context of a speech act, even when an assertion is not an expected action. If a speaker utters "I need water!", she is primarily giving reasons for the hearer to bring her water, but she is also giving enough reasons for asserting that "The speaker is thirsty". Something similar happens with respect to the regulative role of axioms. The permission for an

Footnote 31 (continued)
a permission, in denying the illocutionary force of the directive. In this case, the speaker makes it clear that if a prohibition for P is not being performed, then the hearer has the permission to make P the case.

[32] Here we are considering the conditions of assertability in general, since, as we will show next, the specific conditions of an axiom, *qua* an assertion, are dependent on the specific interpretations under consideration for the assertion (i.e. a model in the model-theoretic sense). Since the concrete act of assertion, then, depends on the possibility to provide an interpretation for an axiomatic system, these conditions are concretely realized only when the assertion is successful and non-defective. In case of a contradictory set of axioms, however, the permission that derives from the declarational role of axioms would only grant assertability conditions with respect to a false statement: i.e. a statement that when interpreted in a (any) model, would not be valid.

[33] The truth regulated by this definitional dimension of an axiom is a trivial one, since it depends directly on its status of definition. This is what is normally taken to be the (analytical) truth by convention expressed by implicit definition. However, this notion of trivial correspondence generated by the double direction of fit of a definition, together with the truth that emerges from this correspondence, should not be confused with an autonomous and non-conventional notion of truth that acts in the assertive dimension of an axiom and that is based on the (formal) notion of satisfaction within a model. A similar point, although not in a linguistic context, was made in Ben-Menahem (2006), in the attempt to defend a notion of conventionalism that does not accept a notion of truth by convention.

assertion means both to open the possibility for a given future course of action, the assertion proper, and to provide reasons for doing it truthfully.

In order to express this permission, we take a few notational liberties. Following Searle and Vanderveken, we will take δP to denote a future action which brings about P.[34] But since δ leaves the action unspecified, and since we are speaking about future assertions, we will denote by $\delta_{\vdash} P$ the future assertion of a content P.[35] With this in mind, and for a given axiom $A^{\mathcal{S}}$, the regulative role is expressed generally by the following structure:

$$\neg ! \uparrow W \sim (\delta_{\vdash} A^{\mathcal{S}})$$

Breaking down the expression, $\neg$ is the denegation of the illocutionary point, ! is the illocutionary point of a directive speech act, while $\uparrow$ is the direction of fit from world to words and W is the sincerity condition of desire.[36] And, as we explained, the content being denied by the truth-functional negation is the future assertion of $A^{\mathcal{S}}$, that is of A *qua* member of $\mathcal{S}$.

This explains the regulative aspect that axioms may have according to the declarative effects the axiomatic system puts forward. But allowing an assertion to be made in the future is a different act from the actual uttering of the assertion. For this reason we now turn to analyze the assertability conditions of the axioms.

3.2 Axioms' Assertability

We are now in the position to account for the assertive component of axioms. The idea is simple: axioms assert a content only when an interpretation is provided. Consequently, an uninterpreted axiom is not an assertion, but, following the previous discussion, only a permission of assertability, when considered as part of an axiomatic system. Now, since an axiom can be seen as an assertion only when an interpretation is provided, we can restrict our attention to axioms belonging to consistent

[34] The notation follows from the distinction between actions and reasons in Searle and Vanderveken (1985, pp.34-5). In Searle and Vanderveken's reading, a proposition P expresses a course of action if $P = \delta utQ$, where u is a hearer, t is a time, Q is the propositional content that u is supposed to bring about. Truth conditions are given in terms of possible worlds, meaning that δutQ is true in a world if u is successful in making Q true at t in that world. In this sense, a future course of action is any such action in which $t > t'$, for t' being the time of the utterance. On the other hand, a proposition P will express a reason for a given action if $P = \rho utQ$, which is true in a given world if u has theoretical or practical reasons to take Q as true at t in that world. Here, we'll take a simplified notation, dropping u and t. Another reason for simplifying this notation is that we do not commit to a notion of time within mathematical practice. Indeed, we leave the notion of possibility that arises from the regulative role of axioms undefined.

[35] Notice the important distinction between a future assertion $\delta_{\vdash} P$, which has no illocutionary force, from the actual assertion $\vdash P$, which has one. This is so because a future assertion is not properly an action: it is a proposition that roughly states that some given action will or will not be performed in the future.

[36] These permissions are consequences of the system $\mathcal{S}$. It may sound odd to say that $\mathcal{S}$ is expressing a desire as sincerity condition. One possible solution would be to maintain that such condition is carried over from the declarational act that poses $\mathcal{S}$ in the first place, in the same way that a written law may carry the sincerity condition of its promulgator as a directive.

axiomatic systems, which, by the completeness theorem have a model. Once a model is fixed, we can then evaluate the assertive component of an axiom. In order to exemplify these ideas, let us consider the following example.[37]

Given a countable set of first-order variables x, y, z, ... and an unspecified relation R, we take a miniature system $\mathcal{S}$ comprised of just the following axioms:

A1: $\forall x(xRx)$
A2: $\forall x\forall y\forall z((xRy \wedge xRz) \rightarrow yRz)$

From both, it follows as theorems:

T1: $\forall x\forall y(xRy \rightarrow yRx)$
T2: $\forall x\forall y\forall z((xRy \wedge yRz) \rightarrow xRz)$

What are the axioms in $\mathcal{S}$ asserting? At first sight, this cannot be answered without offering a domain for interpreting each variable and the relation R. For instance, we can take $\mathcal{S}$ as a fragment of geometry, by taking x, y, z... to denote line segments in a domain $\mathcal{G}$, and R to be the relation of congruence $\cong$. In this case, $A1$ and $A2$ are asserting, respectively, that every line segment is congruent to itself, and that two segments congruent to the same segment are congruent to each other. Likewise, theorems $T1$ and $T2$ show that the congruence relation is also symmetric and transitive. Thus, we say that $\langle \mathcal{G}, \cong \rangle$ is a model for $\mathcal{S}$. But $\mathcal{S}$ may have different readings. Let x, y, z, ... be integers in $\mathbb{Z}$, and R the equivalence relation $\equiv$ holding between two numbers x, y just when their difference $x - y$ is also an integer. In this case, $A1$, $A2$, $T1$ and $T2$ are true assertions about integers, and $\langle \mathbb{Z}, \equiv \rangle$ is a model for $\mathcal{S}$.

Since we have at least two alternatives in interpreting $\mathcal{S}$, what are we actually asserting when $A1$ or $A2$ are uttered? Truths about line segments or truths about integers? Given that axioms are assertions, *modulo* the interpretations of the theory, what are their propositional contents? Call this the problem of determining the scope of assertability, or the *assertability problem* for short. Our solution consists in relativizing the assertive component of an axiom to a given interpretation. In this way we are faithful to a model-theoretic approach to axiomatics which, by investigating formal systems from an abstract perspective, allows one to gain knowledge about all its possible interpretations.[38] In this way the formal study of an axiomatic system is clearly separated from its use in expressing a specific mathematical content. As a matter of fact, $T1$ and $T2$ can be proved regardless of which interpretation we choose. Once proved, they become truths about any model of $A1$ and $A2$; specifically about line segments and integers. Similarly $A1$ and $A2$ can be seen both as assertions about $\langle \mathcal{G}, \cong \rangle$ and about $\langle \mathbb{Z}, \equiv \rangle$.

[37] Taken and adapted from Tarski (1994).

[38] It is exactly this movement from the abstract to the "concrete" that provoked Frege's criticism of Hilbert's axiomatics: "If a general proposition contain a contradiction, then every particular proposition included under it will do likewise. Therefore from the consistency of the latter we can infer that of the general one, but not vice versa." Frege (1980, p. 19).

Now, if we choose an interpretation of a system of axioms $\mathcal{S}$ in terms of a model M, we can evaluate the speech act structure of an axiom A of $\mathcal{S}$ along the following lines.

$$\vdash\downarrow_M B(A^M)$$

Here, $\vdash$ marks the illocutionary point of an assertion and $\downarrow_M$ is the direction of an assertion about the model M. Given the possibility of multiple interpretations of a system of axioms, we have to specify which is the (semantic) reality described by our language. Moreover, the content A^M is given by the standard notion of interpretation and satisfaction of a sentence in a model.

Coming back to the example that opened this section, if we choose the model $\langle\mathcal{G},\cong\rangle$ as the world to which our language refers, then uttering $A1$ as an assertive speech act would have the following structure.

$$\vdash\downarrow_{\langle\mathcal{G},\cong\rangle} B(\forall x(xRx)^{\langle\mathcal{G},\cong\rangle})$$

Therefore, in concrete cases, the assertive component of axioms coincides with their model-theoretic interpretation, which is a natural realization of the idea that axioms (and theorems) say something about their interpretation.

4 Conclusion

We hope we have convinced the reader that an illocutionary perspective can help not only to clarify the roles of axioms and postulates in the history of mathematics, but also in expressing different philosophical views about mathematics. Surely, axioms and postulates have been generally considered as starting points for mathematical reasoning. Yet, mathematicians intended to do different things as these notions evolved: from postulates as directions for geometrical constructions, to axioms as assertions about a previously given domain of mathematical objects, to finally the modern idea according to which axioms are both assertions and definitions.

This double reading of modern axioms is precisely what makes Hilbert's model-theoretic perspective so peculiar and innovative. If history shows a progressive identification of directive, assertive, and declarative components, the complex notion of axioms that we inherited from Hilbert still manifests an interesting mixture of these three components. The possibility that axioms may convey different assertions in the context of different interpretations is what drives axiomatic investigations. Its richness lies in this inherent multifaceted aspect. On the other hand, the possibility to use axioms for the definitions of new fruitful perspectives on (a possibly independent) mathematical reality is a fundamental component of the creative dimension of mathematical language, the recognition of which serves to bring mathematical knowledge closer to its human dimension. Speech Act Theory serves exactly this purpose: to offer a concrete description of the many things we can do with language in mathematics and to draw a more complete picture of our mathematical practices.

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