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CRITICAL DISCUSSION

Luck and Proportions of Infinite Sets

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Hill (2020, §1)[1] argues that the proportional view of luck[2] is "nonsensical" because "it does not make mathematical sense to say" that "there are *more* nearby worlds, a *wide enough set* of nearby worlds, or a *higher proportion* of nearby worlds" where a proposition is true than where it is false, because there are infinitely many such worlds. (More precisely: both sets of worlds are infinite and of the same cardinality. That is, the two sets can be put in one-to-one correspondence, and therefore are of the same size, for a certain meaning of "same size".) In this note, I point out a mathematically well-defined way of non-trivially comparing the sizes of uncountable sets of equal cardinality.[3]

Here's a counterexample. The set of real numbers between 0 and 1 and the set of real numbers between 0 and 2 are of the same cardinality. Therefore, there is a mathematically well-defined sense in which they are the same size. Nevertheless, there is also a mathematically well-defined sense in which the former is half the size of the latter. If there were not, analytic geometry (geometry using the real numbers for coordinates) would be in trouble.

How could this be? Since the two sets are of the same cardinality, we could break the line from 0 to 1 down to its constituent parts and move them around to constitute the line from 0 to 2. This is the sense in which the two sets have the same size. As Tao (2011, §1.1) puts it, this is the *problem of measure*; measure theory is the branch of mathematics responsible for saving analytic geometry from Cantor.

In brief, a *measure* on a set is a function from subsets to nonnegative real numbers; the values of this function are the sizes of the subsets. The size of a disjoint

[1] Hill cites Carter and Peterson (2017, p. 2177 n9), who attribute the following claim to a reviewer: "The problem is that the set of nearby possible worlds is an infinite set, meaning that the term 'most' has no mathematically well-defined meaning." Carter and Peterson reply to the reviewer by appeal to (arithmetic) *density*, although they seem to accept the reviewer's claim.

[2] See Pritchard (2004, 2005), Levy (2011), Coffman (2015). Note that my aim in this note is to provide an interpretation of these philosophers' claims, not to defend a modification of their views.

[3] See also Mancosu (2009), more controversially, on doing the same thing for countable infinite sets.

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union of countably many sets is the sum of the sizes of each set. More formally: Let X be a set (e.g., of possible worlds) and let Σ be a σ-algebra on X.[4] Then a *measure* (on X) is a function $\mu : \Sigma \to [0, +\infty]$ such that (i) $\mu(\emptyset) = 0$, (ii) μ is countably additive.[5] If μ is a measure, then we call the triple (X, Σ, μ) a *measure space*.

Nor is measure theory an arcane area of mathematics unlikely to be of use to philosophers: it is foundational for the mathematical theory of probability. A *probability space* is a special case of a measure space, one where the measure of the universe is 1. More formally: if, in addition to (i) and (ii) above, (iii) $\mu(X) = 1$ then μ is a *probability measure* and (X, Σ, μ) is a *probability space*.

The connection between measure spaces and probability spaces gives us another way to see that Hill's argument must be mistaken. To see why, let's work through the example Hill gives.[6] We have two independent variables: the outcome of a lottery draw and the position of Smith's car in his driveway. The former has finitely many possible values[7] (say, between 1 and n); the latter could reasonably be modelled with an interval on the real number line—say, $[0, 1]$.[8] Then we are interested in comparing the sizes (measures) of two sets: the set of worlds where Smith's lottery ticket wins and the set of worlds where it loses. Both sets are uncountably large, since both include all possible positions of Smith's car in his driveway—but we want to determine their measures, not their cardinalities.

Suppose for *reductio* that it does not make mathematical sense to say the set W of worlds where Smith's ticket wins is smaller than the set L of worlds where Smith's ticket loses. Then it must not be possible to define a measure μ such that $\mu(W)$ is much smaller than $\mu(L)$. Therefore, since a probability measure is a measure, it must not make mathematical sense to say that the probability of Smith's ticket winning is much lower than the probability of its losing. But, clearly, it is much more probable that Smith's ticket loses than that it wins.

So we must be able to make mathematical sense of W being much smaller than L; let's spell it out in detail. Our universe of possible worlds can be represented as the product of two sets: $X = \{1, \dots, n\} \times [0, 1]$. That is, we can represent each possible world $w \in X$ as an ordered pair (x, y), where x represents the lottery outcome and y represents the car's position. Suppose Smith holds ticket number 1. Then the two subsets of this universe we are interested in are $W = \{1\} \times [0, 1]$ and $L = \{2, \dots, n\} \times [0, 1]$.

[4] A σ-algebra on X is a set of subsets of X which (i) includes X, (ii) is closed under complementation, and (iii) is closed under countable unions.

[5] μ is countably additive iff the measure of the union of countably many disjoint sets $s_n \in \Sigma$ is the sum of the measures of each s_n.

[6] Note that in the following, I will appeal to both probabilities and modal claims, but this is not because I mean to defend a hybrid probabili-modal account of luck. Rather, I mean to show that *if* one thinks these probability claims make sense, *then* one must not also think the modal claims about proportions are nonsensical.

[7] But there are infinitely many possible worlds where the variable takes each of these finitely many values.

[8] I assume for simplicity that Smith's driveway is linear. Nothing important changes if Smith's driveway has multiple dimensions.

There are standard measures for each sort of space: the *counting measure* μ_c on a finite set and the *Lebesgue measure* μ_L on the real numbers. For our purposes, it suffices to note that (a) $\mu_c(S) = |S|$, i.e., the counting measure of a set is the number of elements it contains, and (b) $\mu_L([a,b]) = b - a$, i.e., the Lebesgue measure of an interval is the length of the interval. A standard way to define a measure on a product set like our universe is by using the *product measure* μ_Π, according to which $\mu_\Pi(A \times B) = \mu_c(A) \cdot \mu_L(B)$. That is, our product measure says that multiplying the counting measure of $A \subseteq \{1, \ldots, n\}$ and the Lebesgue measure of $B \subseteq [0,1]$ gives the measure of the product set $A \times B$.

In our case, then, $\mu_\Pi(W) = 1 \cdot 1 = 1$ and $\mu_\Pi(L) = (n-1) \cdot 1 = n - 1$. To turn this into a probability measure, we normalize by setting $\mu_N(X) = 1$; thus, we set $\mu_N(S) = \mu_\Pi(S)/\mu_\Pi(X) = \mu_\Pi(S)/n$ for all $S \in \Sigma$. Then we get the probabilities we'd want: the probability Smith wins is $1/n$ and the probability he loses is $\frac{n-1}{n}$. Thus, it does make mathematical sense to say that there are many more worlds where Smith loses than where Smith wins.[9]

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