ORIGINAL RESEARCH

Models as Fundamental Entities in Set Theory: A Naturalistic and Practice-based Approach

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Abstract

This article addresses the question of fundamental entities in set theory. It takes up J. Hamkins' claim that models of set theory are such fundamental entities and investigates it using the methodology of P. Maddy's naturalism, Second Philosophy. In accordance with this methodology, I investigate the historical case study of the use of models in the introduction of forcing, compare this case to contemporary practice and give a systematic account of how set-theoretic practice can be said to introduce models as new entities. In conclusion, I argue for a view that takes both sets and models to be fundamental entities in set theory.

1 Introduction

The debate between a multiverse and universe conception of set theory has garnered a lot of attention in recent years.[1] This debate has its origins in the first independence results (Gödel, 1940; Cohen, 1963a) and in the categoricity results for second-order set theory (Zermelo, 1930). These results show not only that important mathematical sentences can be undecidable in a chosen axiomatization of mathematics[2] but they also raise the fundamental question whether such sentences have determinate truth-values. The universe/multiverse debate provides a way to capture a fundamental disagreement on how to answer this question: very roughly, the *universist view* (universism, for short) claims that there is one intended model of set theory, the universe *V*, that decides the truth value of such statements. The *multiversist view* (multiversism, for short) denies this by claiming that we have to accept a multiplicity of models of set theory on equal footing.

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[1] See for example Woodin (2011), Hamkins (2012), Koellner (2013), Maddy (2017), Steel (2014).

[2] The standard example is the Continuum Hypothesis (CH) and its independence from the axiom system *ZFC*.

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One of the most frequently discussed multiverse views is the one of Hamkins (2012), who advocates a quite radical position. One of his central claims is that the development of set theory in the last decades has led to models becoming the fundamental objects of set theory. This stands in contrast to the classical account of set theory as being solely about sets. Hamkins sees this classical account, which focuses on set-related practices from a universe viewpoint, as not taking into consideration the modern model-practices of set theorists (Hamkins, 2012, p. 418). The classical account can counter this by subsuming the model-related practices under the set-related ones, pointing out that, as models are themselves sets, they are already included in the study of sets.[3]

In the last years this particular debate about fundamental entities seems to have stalled, branching out instead to other aspects of the universe/multiverse discussion. In this article I want to take up the question of fundamental entities in set theory yet again, using a methodological and practice-based approach.[4] Under this approach, methodology provides the basis for discussions about ontology; by analysing the former in detail, I hope to likewise provide the basis for further research in the latter, thus also delivering new input for the universe/multiverse debate.

In the article, I investigate Hamkins' claim from the point of view of Second Philosophy. This is a naturalistic program developed by Penelope Maddy that targets mathematics in general and set theory in particular.[5] Both Hamkins and Maddy base their investigations on the underlying practice of set theory, but while Hamkins mostly reflects on it as a practicing set theorist, Maddy develops a general framework for investigating practice and drawing philosophical conclusions from methodologically informative parts of it.

Interestingly, despite the fact that both of them focus on practice, Maddy and Hamkins come to very different foundational conclusions: Hamkins endorses a radical multiverse, whereas Maddy argues for a universe view (Maddy, 2017). At the same time, both positions have been criticized with regards to their arguments from practice: Ternullo (2019) and Rittberg (2016) argue that Maddy neglects certain parts of set-theoretic practice in her study of set-theoretic methodology, in particular domains that are connected to multiverse-flavoured mathematics. And while Hamkins focuses on these kinds of model-related practices in set theory, a detailed methodological argument of why these practices are so fundamental as to be considered on a par with the set-related practices is missing.[6]

With regards to these points, the present article has several aims: In investigating set-theoretic practice with the methods of Second Philosophy, I will strengthen Hamkins' observations and provide a complete argument for considering models

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[3] This in turn raises the issue of how to deal with proper class-sized models; I discuss this problem in Sect. 3.1.

[4] This is the approach of Second philosophy that is discussed in detail in Sects. 2.1 and 2.2.

[5] See for example (Maddy, 2007) and (Maddy, 2011).

[6] Hamkins (2012) indicates how such an argument could be made by many and varied references to recent results and practices, e.g. the importance of independence results and research programs such as set-theoretic geology. This provides a strong motivation for his multiverse view; however, it gives no clear argument on why the practices justify such a view.

as fundamental entities. To do this, I investigate the model-building practice in set theory from a historical and systematic point of view. I will break down this practice to one of its underlying methods, namely, the method of introducing models as entities; and use the account provided by Second Philosophy to argue for the claim that this method is of justificatory value for the philosophy of set theory. At the same time, Maddy (2011)'s argument for sets as fundamental entities still holds. Therefore the argument provided here will give support to a position that recognizes both sets *and* models as fundamental entities in modern set theory.

Furthermore, I also address the criticisms that both accounts face in the wake of the way they handle practice by supplying a more detailed argument for (a version of) Hamkins' conclusions regarding a change in set-theoretic practice; and by broadening Maddy's considerations to parts of the practice that seem to give rise to a multiverse view. Finally, this conclusion will show that both Hamkins' and Maddy's accounts of practice can interact in a fruitful manner when focusing on methodology, despite their main proponents' disagreement on foundational issues. While a detailed discussion on how the methodological results of this paper inform the foundational and ontological debate has to be left to future research, I will point out some possible implications for Maddy's and Hamkins' positions at the end of the article.

The structure of the paper is as follows: I will start by introducing the question of models as fundamental entities and presenting the main methodology of my analysis in Sect. 2: First, I outline the approach from Second Philosophy needed to carry out my investigation (Sect. 2.1); and, second, I discuss in detail what the method of introducing something as a new entity means for Second Philosophy (Sect. 2.2). Then, I discuss the concrete example from practice I will study, namely the use of models in set theory, in Sect. 3. I give an overview of the model-related practices in set theory (Sect. 3.1) and analyse it in depth in the historical setting of the introduction of forcing (Sect. 3.2); compare this to its contemporary setting (Sect. 3.3); and, finally, argue from the perspective of Second Philosophy that we can accept the method of introducing models as fundamental entities as part of set-theoretic methodology (Sect. 3.4). Sect. 4 provides an outlook on how these results from methodology may affect foundational questions in the universe/multiverse debate and addresses some open questions for further research.

2 Investigating Models: How and Why?

Hamkins' claim about models as fundamental entities has been voiced most explicitly in Hamkins (2012):

> A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible, often to exhibit precise features or to have specific relationships with other models. [...] As a result, the fundamental objects of study in set theory have become the models of set theory [...]. While group theorists study groups, ring theorists study rings

and topologists study topological spaces, set theorists study the models of set theory. (Hamkins, 2012, p. 417)

Hamkins bases his conclusions on his observations of set-theoretic practice: He refers to the development of set theory in the last decades to make the point that working with models has become one of the central practices of set theory. Hamkins also provides further evidence for his claim by pointing out the massive increase of set-theoretic work done not only *within* the models of set theory, but also *about* the models themselves, i. e. how they behave and relate to one another. Hamkins (2012) gives examples for such a way of using models in recent research programs, such as the modal logic of forcing (e.g. Hamkins & Löwe, 2005) and set-theoretic geology (e.g. Reitz, 2007).

However, Hamkins does not provide a full argument from set-theoretic practice.[7] Instead, he observes from practice that models are fundamental entities and takes this as evidence for his actual main claim, namely that each model that is set-theoretic enough "exists independently in [a] Platonic sense" (Hamkins, 2012, p. 417). Consequently, the main discussions in the universe/multiverse debate mention the question of fundamental entities only as an aside and focus instead on other parts of Hamkins' argument. Koellner (2013) for example only mentions the question of fundamental entities in passing, in the broader context of the discussion about set theory as a formal or abstract theory. Overall, Koellner's article is devoted to raising problems for the broad multiverse view in general and Hamkins' arguments for it in particular, discussing issues such as possible multiversist background theories, the ontology of different accounts of forcing and the (in)determinateness of truth values of independent sentences.[8] All of these are major points of discussion in the universe/multiverse debate.

In the present article, however, I want to come back to the claim that models are fundamental entities in set-theoretic practice. I want to provide a full argument of this claim by using the framework of Maddy's Second Philosophy. This naturalistic account is particularly suited for the purposes of this article as it places a major emphasis on the analysis of mathematical practice and provides a detailed and tested tool for such an analysis, in particular with respect to set theory. For instance, Maddy (2011, Ch.II.2) analyses the introduction of sets as fundamental entities and I will

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[7] He does discuss in detail two *analogies* between his observations about set-theoretic practice and the practices of algebra (especially group theory) and, more historically, of non-Euclidean geometries. These analogies—and the related question whether set theory might have become an algebraic or non-algebraic field of research in mathematics—have been discussed more broadly since the development of forcing in the 1960s. Because I aim to give an argument from set-theoretic practice itself, I will not go into detail about these discussions. For early discussions of analogies between the development of set theory and algebra, see the contributions of Mostowski (1967) and Kalmar (1967). The article (Körner, 1967) in the same volume is also of interest for the question of models as fundamental entities. The discussions about algebraic/non-algebraic (or abstract/concrete) field of mathematics is a broad one and set theory is only one of the fields discussed there. Shapiro (1997) introduced a first distinction, that he changed in Shapiro (2005), Hellman (2003) provides a different distinction on the basis of characteristics of axiomatic systems.

[8] I would like to thank one of the anonymous reviewers for pressing me to clarify the context and main aim of Koellner's criticism of Hamkins.

use this as a template for my analysis of models. In the following, I will therefore concentrate on a naturalistic investigation of set-theoretic practice; the foundational and ontological questions of the universe/multiverse debate will therefore not be discussed in detail. Yet I hope that the analysis of methodology can eventually be used to inform these broader debates.

To set-up my investigation into practice and methodology, I will first give an account of the naturalistic framework of Second Philosophy (Sect. 2.1) and then apply it to the question of models considered here (Sect. 2.2).

2.1 Set-Theoretic Practice and Second Philosophy

Penelope Maddy developed her account of Second Philosophy as a form of naturalism in the philosophy of mathematics, with a special focus on the philosophy of set theory.[9] Here the Second Philosopher is interested in two main questions: *What* is the right methodology for a certain area of pure mathematics; and *why* are these the proper methods, leading us to traditional philosophical questions in ontology, etc.[10]

To answer the first question about methodology, Maddy bases her investigation on the practice of the discipline which she studies via means-ends relations: We isolate a practice of a certain area of mathematics and investigate whether it is an "effective means to particular desirable ends" (Maddy, 1997, p. 194). Here one identifies a certain method in the practice of (some area of) mathematics and then examines if there are goals in the practice for which this method serves as an effective means. Considering the method and goal as a means-ends relation, one can finally judge if it is "rational" (in Maddy's words) to accept the method to be part of the discipline's methodology. Second Philosophy therefore 'selects' parts of the practice that can be argued to be part of its methodology.

Maddy also gives examples of which parts of the practice will not be part of methodology, such as "heuristics". One such example is Dedekind's famous remark about natural numbers as "free creations of the human mind". Maddy posits that the Second Philosopher should treat such remarks "as colorful asides or heuristic aides, but not as part of the evidential structure of the subject" (Maddy, 2011, pp. 52–53). Heuristics are still an important part of doing mathematics, e.g. by providing intuition for reasoning, but—according to Maddy—they have no justificatory value for broader philosophical questions.[11]

After getting clear about methodology, the Second Philosopher aims to answer "the second group of questions—about the nature of set-theoretic activity, about

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[9] As a rough timeline, Maddy's Naturalism was already presented in Maddy (1997); a broader philosophical investigation led to the formulation of Second Philosophy in Maddy (2007), which was in turn utilized to study set theory in Maddy (2011). As some parts of the framework changed between these presentations, I will focus on the newest presentation of Second Philosophy given for set theory in Maddy (2011).

[10] See in particular the two parts of Maddy (2011) as outlined in Maddy (2011, p. 41).

[11] See for example the discussion in Maddy (2019, pp. 75–76).

its subject matter and the reliability of its methods—in a way that [...] respects the actual methods of set theory" (Maddy, 2011, p. 59). For Maddy this leads to a rejection of what she calls "Robust Realism", as this position does not arise in mathematical reasoning itself (Maddy, 2011, p. 59). She then discusses a realist alternative, dubbed "Thin Realism", which is "a version of realism that genuinely accounts for the nature of set-theoretic language and practice, that respects the actual structure of set-theoretic justifications" (Maddy, 2011, p. 61); and a non-realist position called "Arealism". As an Arealist, one focuses on the effectiveness and fruitfulness of set theory. As Maddy writes, an arealist Second Philosopher "would see no reason to think that sets exist or that set-theoretic claims are true [...], but she does regard set theory, and pure mathematics with it, as a spectacularly successful enterprise, unlike any other." (Maddy, 2011, p. 89).[12] One of the most interesting outcomes of this discussion is that a Second Philosopher endorsing Thin Realism and another endorsing Arealism do agree on set theoretic practice and methodology (see Maddy, 2011, Ch.IV.4 and Ch.IV.5). From this Maddy concludes that Second Philosophy really supports what she calls a "post-metaphysical" form of objectivity (Maddy, 2011, p. 116). So while the study of methodology is necessary for the investigation of further philosophical questions, metaphysical question do not have to be necessarily addressed. Instead, Maddy (2011, p. 116) proposes "a form of objectivity in mathematics that doesn't depend on the existence of mathematical objects or the truth of mathematical statements, or even on the non-existence of mathematical objects or the rejection of mathematical claims." But no matter how and if one approaches these second type of questions, the study of the first type is essential to the Second Philosopher's endeavour.[13]

The investigation I aim at in this article is therefore firmly set in the first group of questions about the methodology of set theory. I want to show that introducing models as fundamental entities is part of set-theoretic methodology and not just a heuristic aid. Being about methodology, this claim then becomes eligible to inform further philosophical questions, namely the questions in the second group.[14] Showing that something is part of set-theoretic methodology is therefore of major significance for studying questions of e.g. truth and ontology of set theory. But before we can investigate what the introduction of models as fundamental entities might mean or imply for questions of this second type, we have to show that they are indeed relevant for these kinds of questions in the first place—and that means showing that they are part of set-theoretic methodology. In this article I therefore focus on questions of methodology; I will give an outlook on possible foundational implications of my methodological investigation in Sect. 4.

Let us now look more closely at the method of means-ends relations as the deciding factor about which parts of practice count as methodology. Candidates for means comprise a variety of things that mathematicians use to aim at and eventually reach goals in their mathematical work. These can be their mathematical

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[12] Note that Arealism is not the same as standard Nominalism, see Maddy (2011, pp. 96–98).

[13] I want to thank one of the anonymous reviewers for suggesting to clarify this point.

[14] If, that is, one wants to tackle all of these questions. As we have seen in the last paragraph, Maddy does not consider the metaphysical issues to be of importance to the Second Philosophers project.

bread-and-butter, such as 'using certain kinds of reasoning (proof types, but also analogies etc.)', 'using certain entities (e.g. in definitions)' or larger-scale, such as 're-structuring parts of the subject area'. Maddy (2011, Ch.II.2) gives several case studies for means such as 'introducing new entities' and 'adding new axioms'.

Goals that serve as ends are typically ordinary mathematical goals such as 'proving a certain theorem', 'developing a new technique' or 'give an appropriate definition'; but they can also be foundational goals such as 'providing a court of final appeal for claims of existence and proof in classical mathematics' (Maddy, 2007, p. 354).[15] To give an example, the goal of "unifying diverse mathematical phenomena into one coherent theory" has foundational relevance for the introduction of a *group* as a new entity in modern algebra (Maddy, 2007, pp. 352–353). For set theory, the pursuit of the foundational goal of a unified basis can be found in Dedekind's work on the construction of the real numbers as well as later work by Cantor, Zermelo and others.

It is important to note that Maddy, working in the Second Philosophy framework, does not investigate such means as part of the philosophical debate about e.g. foundations, but as part of the mathematical practice in set theory. It is therefore crucial that also the foundational goals can be traced back to set-theoretic practice. Second Philosophy does not consider philosophical goals as proper foundational goals unless they were made mathematically tangible by the historical actors.[16]

Let me exemplify this procedure by presenting one of Maddy's case studies that lies very close to the heart of what I aim at in this article: Maddy (2011, II.2.(i)) discusses the method of 'introducing sets as new entities', examining the method in the context of one of the first times it arose in the practice, namely when Cantor first uses sets as distinctive entities in his mathematical work on derived point sets (Cantor, 1872). In this context Cantor's goal is to prove a general theorem on representing functions by trigonometric series. As a goal in the methodological perspective, Maddy positively evaluates the resulting means-ends relation: "a new entity—a set—has been introduced as an effective means toward an explicit and concrete mathematical goal: extending our understanding of trigonometric representations" (Maddy, 2011, p. 42).

In this example (as well as in the others given in Maddy, 2011), historical case studies constitute the backbone of Second Philosophy's investigation into methods. Such case studies provide the Second Philosopher with the method and goals to formulate the relevant means-ends relation and serve as evidence for the acceptance or rejection of the method under consideration. It is an essential part of Maddy's philosophy that the Second Philosopher justifies her acceptance of means *extrinsically*, i.e. connected to the usefulness, fruitfulness and mathematical relevance of the method.[17] If this can be shown, the means can be accepted as a method and we

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[15] In this context, Maddy (2017) elaborates on foundational goals, proposing a list that a good foundation should adhere to.

[16] See also the difference between methodology and heuristics discussed above.

[17] This directive is embedded in a fundamental distinction between intrinsic vs. extrinsic justification, see for example Maddy (2011, CH.V.4). For further discussion see Carolin and Kant (forthcoming) and in particular Maddy's answer in the same volume.

are confronted with an instance of a practice that can now be considered to be an integral part of the methodology of the relevant mathematical field.

2.2 Introducing Models as Fundamental Entities

Each time a definition of the type "Call x the mathematical object that has the following properties ..." is given, we have prima facie evidence that a new entity was introduced and this is often done to serve a specific mathematical purpose, thus supplying the Second Philosopher with ends for a means-ends relation that she can then investigate. This kind of definition is, then, a *marker* for the introduction of a (new) entity. However, in each mathematical subdiscipline, there are some such definitions that seem to be more fundamental to the practice of the discipline than others.

Maddy discusses one such a fundamental occurrence, namely the case of sets as new entities in the work of Cantor (Maddy, 2011, Ch.II.2.(i)) and Dedekind (Maddy, 2011, Ch.II.2.(ii)). Here Maddy employs several criteria that show that sets were indeed introduced as new entities, building on Cantor's and Dedekind's original work as well as drawing from historical studies, in particular (Ferreirós, 2007) and (Avigad, 2006). In the following, I discuss these criteria and collect them into two groups for the introduction of new entities. In Sect. 3.4 I will then check them against the case of models in set-theoretic practice.

The key phrase used in the case study of sets is that the new entities are treated as entities *in their own right*.[18] Maddy takes the following as criteria for such a treatment: they are susceptible to general mathematical operations (Maddy, 2011, pp. 42, 43); one is encouraged to speak about 'arbitrary' such entities;[19] and the way in which they are used is independent of the way in which they are represented (see in particular the quotes of (Avigad, 2006) in Maddy (2011, p. 43); also Maddy (2011, p. 45)). For example, Cantor's treatment of point sets as he operates on them by taking the derived set and generally employing the "language of sets" (Ferreirós (2007) as cited in Maddy, 2011, p. 42) satisfies these criteria. Another example is Dedekind's work that defines sets of numbers independent from the previously used construction via algorithms (Maddy, 2011, p. 43).

In her discussion of Dedekind's work, Maddy (2011, pp. 44–45) also points at broader issues that accompany the introduction of new entities: Such an introduction can bring about new styles of reasoning; attempt new types of foundations; provide new kinds of explanations; and work towards a superior presentation of previously known facts or theories, where superiority can take on different forms such

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[18] See Maddy (2011, pp. 42, 43).

[19] Here the word 'arbitrary' is not meant in a technical sense, such as it is used in theories of arbitrary objects like (Horsten, 2019). Such an interpretation is quite interesting especially when certain types of set-theoretic models, the so-called Boolean valued models are investigated as arbitrary objects (see Horsten, 2020). However, this meaning is arguably not the one intended by Maddy (2011), nor does it fit the Second Philosopher's methodology, as it is tied to a specific metaphysical interpretation of models of set theory. Therefore, there word 'arbitrary' is used here to simply mean that the entities are not tied to a specific criterion or constraint.

as improved generality, unification, rigor or simplicity.[20] These broader issues can be the explicit target of the mathematical work under consideration, such as in the case of Dedekind, who was aiming to give a "perfectly rigorous foundation for the principles of infinitesimal analysis" (Dedekind, as cited in Maddy (2011, p. 44)), but they can also be part of later implications of the mathematical work, as in the case of Cantor (Maddy, 2011, CH.II.2.(i)).

It important to note that these broader criteria do not point exclusively to foundational questions. Providing a coherent foundational theory certainly is a major goal, yet other questions also play a decisive role in the case of the introduction of fundamental entities: Finding new styles of reasoning or explanations in the practice provides evidence for new entities and the broader implications they can have; they also highlight how the means under consideration are functionally entrenched in the practice apart from giving rise to foundational theories. In the case studies of Maddy (2011) these criteria were already satisfied before Cantor (and later Zermelo, Fraenkel and others) provided a foundational theory of set theory. If these criteria are also fulfilled for the case of models, it supports the case of their introduction as new, fundamental entities.

For clarity's sake, I will subdivide these criteria into two groups: The first group, concerned with questions of representation of and operation on entities, consists of methodological explications. They focus on the changes with regards to the entities themselves and I will call them *specific criteria*. The second group consists of consequences for the whole area of mathematics they appear in; and/or more widespread phenomena that go hand-in-hand with the introduction of these new entities. These focus on questions of reasoning, explanation, rigor and foundations and I will call them *broad criteria*.

As it is apparent from Maddy (2011, Ch.II.2)'s discussion, the case of introducing sets as new entities satisfies all of these criteria. To this I would like to add an aspect that is not explicitly mentioned in Maddy's work but, I think, is implicitly included: It seems reasonable to say that episodic uses of some means, as surfaced in the historical case studies, cannot be enough to concede them status as general methods of the discipline under consideration. The means under consideration should also be shown to have circulated and spread through sufficiently large parts of the community as to make them relevant to general practice.[21]

To test this general relevance for methodology, I propose to consider contemporary practice in the discipline to see if the method analysed in the case study remains to be important. The reason why Maddy does not explicitly consider current research in the study of sets might be that the relevance and importance of sets for set-theoretic practice today is so obvious it does not need mentioning. This reason also suggests itself when comparing the case of sets to the case study of

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[20] This last point collects together several remarks of Maddy (2011, Ch.II.2) on rigour, simplicity and generality and Maddy (2007, pp. 352–353)'s discussion of the introduction of groups in algebra.

[21] I read this off Maddy's comments on the relevance and force the means have in the practice of the discipline (see for example Maddy, 2011, p. 53) and similar consideration connected to a consensus in the community concerning its methods that Maddy discusses in Maddy (1997, p. 194).

determinacy (also given by Maddy (2011, Ch.ii.2.(iv))): Although there is a strong line of argument for the methodological and foundational relevance of projective determinacy, it is not as obvious and beyond dispute as the case of sets. Maddy therefore presents not only one case study of the use of projective determinacy, but gives a more extensive discussion following the historical development of (projective) determinacy until very recent work (e.g. Woodin, 2001). I therefore think that especially in controversial cases, such as the case of models as fundamental entities, contemporary practice should be considered carefully and I believe this to be in the spirit of Second Philosophy.

Taking this framework of specific and broader criteria for granted, it is natural to ask whether we cannot make the same case for other entities in the practice, such as functions, ordinals, etc.[22] If we want to make such a case, we have to provide detailed case studies for each of these entities along the same lines as we do here for models. Conducting such case studies is outside the purview of this article, but let me briefly give some *prima facie* considerations as to why we can expect these case studies to fail for the respective cases of functions and ordinals:[23] First, in the case of functions it might not be possible to find the appropriate means. Functions were present in set-theoretic practice from the beginning and have a rigorous definition as a certain kind of set. Differently from models, however, their treatment in set-theoretic practice has not changed since the beginning of set theory. There are simply no common, general practices in which means like "introducing functions as new, fundamental entities in set-theoretic methodolgy" could be investigated; their treatment and use in set theory has remained quite stable and unchanging. In particular, there is no comparable discussion[24] to what we have seen in the case of models at the beginning of Sect. 2, at least not in set-theoretic practice.[25]

Second, let us turn to the case of ordinals: Here, we can find the means relevant to build a case study, namely the introduction of ordinals by Cantor (1883). But problems present themselves when checking the specific and broad criteria. In particular, I submit that ordinals aren't used independently of the way in which they are represented; instead, they are tied to their representation as specific kinds of sets. Also, I don't know of attempts to provide foundations for set theory with ordinals

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[22] I would like to thank both anonymous reviewers for insisting on discussing this question in more detail. I think it is an important point overall and I would be interested to see if the methodology of Second Philosophy as applied here can justify other cases of fundamental entities (although I don't presently see an obvious candidate).

[23] These conjectured failings can be taken to be exemplary for problems that arise with other entities as well.

[24] We investigate the basis of this discussion further in Sect. 3.1.

[25] There were attempts at using functions as fundamental entities, such as (von Neumann, 1925). However despite being of theoretical significance, they never fundamentally influenced set-theoretic practice: usual renderings of the Gödel-Bernays-von Neumann axiomatization and variants thereof use sets as fundamental entities. Of course, one could still make the case that functions should play a bigger role in foundations because they are so fundamental for mathematics in general. From such a point of view, other foundations such as category theory may seem to be more appropriate than set theory. But this lies outside the scope of our investigation of set-theoretic practice.

as fundamental objects on par with sets (or perhaps instead of sets).[26] I hope that it becomes clear from these examples that the methodology of Second Philosophy for introducing new entities is not a dispensation to view all kinds of entities from the practice as fundamental. Even very useful concepts such as functions or ordinals in set theory can still not be regarded as fundamental entities according to the methodology presented here.

Taking this and last section's discussion together, we can now see more clearly what it means to argue for models as fundamental entities: First, we have to show that the corresponding method of introducing models as new entities is an effective means to some mathematical or foundational end by checking the specific and broad criteria given above. Furthermore, we have to confirm the stability of the entities in question, the lasting effect of its introduction in contemporary practice. Only if we succeed on both accounts, we have shown that this method is part of set-theoretic methodology. This closes the gap between what we have observed in set-theoretic practice and what is in the purview of methodological consideration (and therefore is not only part of heuristics).

In the next section I will therefore first investigate the historical case study of the introduction of forcing (Sect. 3.2), compare the findings of the case study against contemporary practice (Sect. 3.3) and then show that the criteria given in this section are fulfilled for the case of models (Sect. 3.4).

3 The Case Study of Models and Forcing

To conduct our investigation into the model-building practice in accordance with the approach described in Sect. 2.1, in this section I will investigate the means of introducing models of set theory as a new type of fundamental entity in set-theoretic practice. I will analyse if this means is effective in achieving a mathematical end to be found in the practice, namely, the goal of *proving various independence and consistency results*.[27]

I conduct this investigation in four steps: First, I provide an outline of how and why the practice with models changed in the last decades and how this is connected to the forcing technique (Sect. 3.1). Then I give an detailed study of in the historical development of forcing by Paul Cohen in Sect. 3.2 as forcing is one of the most productive model-building tools set theorists ever developed, leading to far-reaching

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[26] Note that this discussion is targeted at set-theoretic practice. There are practices outside of set theory for which ordinals might be considered as fundamental entities, for example the way in which they are used in non-standard analysis (I want to thank one of the anonymous reviewers for pointing this out to me). This is another example where we can see that the notion of "fundamental entity of a practice" really is dependent on the practice under consideration. As we have seen in the previous Footnote about functions, considering these different practices within mathematics might then be used to provide arguments for or against different proposals for a foundation of mathematics.

[27] Note, that it is not obvious that this is the case, as one might argue that a syntactical formulation of forcing is enough to do this. I will address this point in the following sections.

changes within the discipline.[28] In a third step, I will give a systematic account of how models are used in forcing, incorporating the relevant contemporary practice (Sect. 3.3). To conclude, I will give a measure of its fundamentality in methodology by discussing the criteria developed in Sect. 2.2 and show that the use of models satisfies them (Sect. 3.4).

3.1 The Rise of Models as Fundamental Entities

The construction of different set-theoretic models originally developed when set theorists began to investigate statements likely to be independent. To prove whether a mathematical statement A is independent from some axiomatization (usually ZF or ZFC), one has to show that the statement as well as its negation are consistent with the chosen axiomatization. This is usually done by building a model for each scenario, i.e. a model for $ZFC + A$ and another for $ZFC + \neg A$.

Up to the late 1950s, model-building was carried out mostly at a *local* level, meaning that specific models were built for specific purposes, yet the methods by which they were built was tied to very specific outcomes. Examples are the construction of the constructible universe, Gödel's L (1940); von Neumann's model of well-founded sets (1925); and models with urelements (Zermelo, 1930; Mirimanoff, 1917; Fraenkel, 1922). During this time, important partial independence results were achieved, in particular showing that the Axiom of Choice and the Continuum Hypothesis are consistent with ZF through the construction of Gödel's model L.[29] However, most of these model-building methods were so specialized as to give rise only to specific consistency results and could not be generalized to tackle different mathematical statements or provide full independence results.

As an example, consider the construction method for Gödel's model L: Gödel's construction is based on the idea that instead of building a model that consists of all the arbitrary sets that are usually considered, we only allow sets that are definable via formulas. The model L is then built up by transfinite induction: We start with the empty set and then add step-by-step the sets that are definable over the model already built.[30] Gödel proved that L is a model of ZF plus an additional axiom, the Axiom of Constructibility. This axiom states that every set is constructible, i.e. $V = L$. L is an important model in set theory, leading to much mathematical research (see for example Jensen, 1972; Devlin, 2017). However, the construction method itself is in a certain sense limited when it comes to model building, because it does not give rise to a more general procedure allowing to construct other models of set theory. If we build up a model via definable sets, we simply get L (or with a similar construction we can produce $L[A]$ for some parameter A). We can use this way of building a model to show that CH is consistent with ZF (and indeed CH follows

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[28] This does not imply that forcing is the only instance of such a use of models. The case of inner model theory could be investigated as well.

[29] "Partial" in this context means that only one direction could be shown, for example a model had been produced for ZFC+A but not for ZFC+$\neg A$.

[30] For a detailed definition see for example Jech (2003, 175 ff.).

from the axiom $V = L$), but we cannot vary the construction in a way that shows the same for not-CH. Therefore, with regards to CH, the method of building a model via definable sets can only show us one part of the independence of CH.[31]

This state of affairs changed drastically in the 1960s, when more *general* model-building techniques were developed, most important among them Paul Cohen's forcing technique (Cohen, 1963a, 1964).[32] A major strength of forcing is its wide applicability, which is due to the general setup of the technique: with forcing, one is able to start from some model of (a fragment of) $ZF(C)$,[33] the so-called ground model, and produce a different model, called the extension, that satisfies additional statements (such as not-CH). Forcing takes care of all the problematic issues involved in such a construction by making sure that the extension will again be a model of $ZF(C)$ and that truth in it can be "seen" up to a certain point in the ground model[34]. Forcing allows a lot of variations regarding the kind of additional statements which hold in the extension, making this a matter of precise mathematical choices in the ground model. One major possibility of variance lies in the choice of the partial order considered in the ground model: For example, in Cohen forcing it consists of the finite partial functions from ω to 2 ordered by reverse inclusion. The new model produced by this forcing then contains a partial function from ω to 2 that was not an element of the ground model and therefore gives rise to a subset of the natural numbers (or a real) that does not belong to the ground model. In this way we can add "new" reals to ground models, which can lead to a failure of *CH*. However we can choose the partial order quite differently: In Sacks forcing its elements are certain perfect trees, where the extension model produced is minimal in the sense that no smaller extension of the ground model can be produced. In a yet different form of forcing, called the Lévy collapse, the elements can be finite sequences of ordinals less than some cardinal κ, resulting in a model where κ is collapsed to ω; and there are many other forms of forcing.

In this way we can use forcing to both produce models that satisfy not-CH as well as models that satisfy CH. Indeed, with forcing and other techniques like inner models, a wide range of statements were shown to be independent. Of note is also that independence results come up not only in the area of set theory, but in other areas of mathematics as well, such as algebra (e.g. Whitehead's Problem in group theory, which was shown to be undecidable by Shelah (1974)), or analysis (e.g. Kaplansky's

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[31] I want to emphasize that despite this "restrictiveness" of the construction method of L, the model L itself is used broadly in set theoretic research. For instance, it also contributes to model-centered research in set theory as can for example be seen in work about the covering lemma, which was initiated by Devlin and Jensen (1975) (I want to thank one of the anonymous reviewers for pointing this out to me). See Mitchell (2010b) for a general account.

[32] Another example for such more general techniques are Ultraproduct constructions and Scott's Ultrafilter method (see for example Scott, 1961; Kunen, 1970, Silver, 1971). This was groundbreaking work for the area of inner models theory. A very comprehensive introduction and advanced results can be found in Volume III of (Foreman and Kanamori, 2010), in particular (Mitchell, 2010a) and (Steel, 2010) may be of interest.

[33] There are some meta-mathematical problems one has to deal with here, however they can be overcome; see also Sect. 3.3.

[34] For general presentations of forcing see (Kunen, 1980) or (Jech, 2003).

conjecture, see (Solovay, 1976)[35], and one can even find applications of forcing in mathematical physics (see e.g. Benioff, 1976).

Nowadays independence results are one of the core areas in set theory; furthermore, model-building techniques like forcing turned out to be useful for much more than merely proving independence results. Forcing is used in very different areas of set theory as a tool for theorem-proving (see for example Todorčecić and Farah (1995) and Miller (1995)) and even gives rise to classes of axioms, the so-called forcing axioms.[36] Because of these wide range of applications, building a variety of set-theoretic models has become an integral part of regular set-theoretic practice.

This development of set-theoretic practice is the basis for Hamkins' conclusions about the change of fundamental entities of set theory that we encountered in the introductory quote of Sect. 2: He rejects the view that the fundamental entities of set theory are the sets of one specific universe V of set theory. Instead he perceives this development to be evidence for the claim that there is no unique concept of set associated with such a universe, but a multiplicity of them—and the way in which set-theorists study these different conceptions is by studying the models that give rise to these conceptions of set (Hamkins, 2012, p. 417). Models are therefore fundamental for the study of set theory as they provide the specific contexts that one needs to study sets. In turn, one studies models through the set theory that arises in them.[37] The consideration of the model-related practices in set theory therefore seem to lead Hamkins to a position where both sets and models have the status of fundamental entities.

This stands in contrast to the classical view of fundamental entities in set theory, which considers them to be the sets. A very clear expression of this view is given by Koellner (2013, pp. 12–13):[38]

> To [the claim that models have become the fundamental objects of set theory] one is bound to protest that the fundamental objects are still what they always were, namely, sets. Some of those sets happen to be models of certain theories. At certain times set theorists focus on sets that are models [...]. At other times set theorists focus on sets in general [...]. But in all cases they are studying sets.

Under this view, models are just some other specific kind of set that is investigated in set theory. However, this does not cover all appearances of models in set theoretic practice, as reference to proper class-sized models frequently occur. This can

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[35] Woodin and Dales (1987) dedicates a whole book to showing that there are interesting independence results in analysis.

[36] One of the earliest is Martin's Axiom, see (Martin and Solovay, 1970).

[37] Hamkins expressed this position in private correspondence with the author (e-mail communication, September 13th, 2021).

[38] As mentioned at the beginning of Sect. 2, this is not the major thrust of Koellner's overall argument. Nonetheless, I will discuss it here as representative for the most common argument the classical view provides against models as fundamental entities. Of course, even if we manage to reject this particular argument, we have not provided a counterargument to Koellner's general criticism that is directed against Hamkins' ontological position. See also Sect. 4.

be taken as evidence that considering sets as the only fundamental entities might not be sufficient for a faithful representation of set-theoretic practice.[39] Note that this observation does not necessarily provide us with an argument for considering *models* as fundamental entities. Instead, we could argue that class-sized models provide a reason to view *classes* as fundamental entities in addition to sets. This can then be made explicit by choosing a class theory as foundation; examples of this are the von Neumann-Bernays-Gödel axiomatization (*NBG*), or perhaps Kelley-Morse class theory (*KM*) if we want to include non-definable classes as well. From this point of view, one could argue that models simply are not only sets but also classes in the same way in which Koellner argues for models as sets, therefore making the inclusion of models as fundamental entities redundant.

The reason why I do not take this line of argument is that in my view both the models-as-sets and the models-as-sets-or-classes argument do not really address the question of fundamental entities of the practice. While both arguments provide a way to address the *occurrence* of models in the practice, they do not deal with the way in which the models are *used* in, and the significance models have for, set-theoretic practice. To incorporate these latter viewpoints, criteria for what fundamental entities in the practice are should include not only foundational, but also epistemological and methodological aspects, like the issue whether the entities are flexible and general tools in the practice; whether they provide explanations; and whether they improve generality or rigor. This accords with the criteria for the introduction of new entities that I collected together from Maddy (2011)'s account of the introduction of sets in Sect. 2.2.

To clarify my point: Assume that all models considered in the practice actually are set-sized. Even in this case I think that an argument can be made for models as fundamental entities, because the fact that models are sets is only one (namely the foundational) aspect of the question of fundamental entities. But this aspect alone does not do justice to the way models are used in the practice, as it represents only one aspect of what fundamental entities are. So the existence of class-sized models in set-theoretic practice certainly serves as a motivation for looking into the question of models as fundamental entities and perhaps also as evidence that models are indeed special, but it does not answer the question by itself.

Let us briefly consider the possible objection that there are areas of set theory where set-practices are much more prevalent than model-practices and that it therefore is not true that models are fundamental entities for these areas. In response to this, let's note that for models to be fundamental entities does not imply that every part of set theory has to study models directly. If, for example, parts of set theory take place in a certain model, e.g. a standard model of *ZFC* and we study the behaviour of sets in this model, without considering any other model (such as forcing extensions, inner models, etc.), this is still compatible with the view of

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[39] Interestingly, the same evidence exists for the case of functions and ordinals, as class-versions of both are used in set-theoretic practice. However, the misgivings I presented in Sect. 2.2 that speak against regarding functions and ordinals as fundamental entities in set-theoretic practice still hold. This might provide additional motivation for conducting a more thorough investigation for these cases.

models as fundamental entities. This is because also in this area the study of the sets proceeds under the assumption that they are the sets in a certain model, meaning that considering models is part of the fundamental set-up for the practice of studying sets. Therefore, the focus on set-related practices in some areas of set theory does not refute the importance of models as fundamental entities. In the same way the focus on model-related practices in some areas of set theory does not refute the importance of sets as fundamental entities.

We have now seen an overview of how the practice of models developed in set theory.[40] In particular, we saw that the introduction of large-scale model building techniques massively increased the use of models in the practice. In the following, I want to study the influence of these techniques in more detail, as they seem to have contributed in an essential manner to the use of models as fundamental entities. I will therefore present the case study of the introduction of forcing by Paul Cohen in the next section. This also serves as the historical case study the Second-Philosophical investigation can be based on.

3.2 Cohen's Use of Models in the Introduction of Forcing

At the beginning of the 1960s, Paul Cohen was working on proving the independence of AC from *ZF* and CH from *ZF*(*C*), two questions that had been open since the axiomatization of set theory. Being aware of the results by Gödel regarding the consistency of AC with *ZF* and CH with *ZF*(*C*), he took on the difficult task of showing the consistency of their respective negations.[41] Cohen's first concrete result towards developing forcing was a rather essential fact about minimal models of *ZF*, which showed that it is impossible to construct an inner model that will satisfy *ZF* + not-*AC* and *ZFC* + not-*GCH* (Cohen, 1963b).[42] To overcome the problem of the existence of suitable models, Cohen assumed the additional axiom *SM* that states that there is a standard model of *ZF*. In his earliest presentation of forcing, he only mentions this fact in passing (Cohen, 1963a, pp. 1143–1144) and refers to an alternative way in which forcing can be set up that does not rely on *SM* (Cohen, 1964, p.

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[40] A related issue is how model-based practices developed in other mathematical practices. The obvious candidate here is model theory that indeed has models as its primary fundamental entities. The development of model theory as a separate discipline began in the 1950s with work by Tarski and Robinson and it always had strong connections within set theoretic research (take for example the Löwenheim-Skolem theorems). A detailed analysis of the interconnections of these practices would be too extensive for this article. However as a starting point let me mention Baldwin (2018)'s "Model theory and the philosophy of mathematical practice" that might be of particular importance here. Baldwin not only gives detailed insight into the development of model theory, he also argues for a paradigmatic shift in model theory in the 1960s, around the same time I'm arguing that a shift happened in the treatment of models in set-theoretic practice. He also shows more generally what importance model considerations have for general mathematics. So it seems that making a connection to model theory can strengthen my arguments further.

[41] See Moore (1987) and Kanamori (2008) for a detailed description of Cohen's career until this point and his endeavour of proving the consistency of AC and CH.

[42] The earlier proof of the same fact by Shepherdson (1953) seems not to have been well-known at the time, cf. (Moore, 1987, p. 155).

109). These different set-ups of forcing produce the same mathematical outcome, meaning that the desired consistency results can be achieved in both ways. However they differ on their meta-mathematical assumptions with the main difference that Cohen added the axiom *SM* to the usual axiomatization of *ZFC*. The alternative syntactical approach outlined in (Cohen, 1964) does not require this additional assumption. In his later, much more comprehensive presentation of forcing and the independence phenomenon, Cohen (1966) emphasises that he regards the approach by assuming *SM* as the more natural one. He would go on to use this as the main set-up when presenting forcing in 1966 and also include an informal argument relying on the Löwenheim-Skolem theorems for why *SM* should be assumed (Cohen, 1966, pp. 78–79). He includes outlines for alternative approaches, but only after his presentation of forcing and all the results are concluded (see Cohen, 1966, pp. 147–148).

So, in Cohen's published presentations of forcing, he chose an approach that uses models in a central way, namely, assuming their existence in an additional axiom, although he was aware that alternative approaches to forcing existed. This preference might be connected to a revealing change in Cohen's original approach in the development of forcing. Cohen first thought about forcing in a syntactic way: "The question I faced was this: How to perform any kind of induction on the length of a proof. [...] It was at this point that I realized the connection with the models, specifically standard models. Instead of thinking about proofs, I would think about the formulas that defined sets [...]" (Cohen, 2002, p. 1089). Interestingly, Cohen describes the work with models as the more "mathematical" of the two approaches: "[...] the feeling of elation was that I had eliminated many wrong possibilities by totally deserting the proof-theoretic approach. I was back in mathematics, not in philosophy" (Cohen, 2002, p. 1089).

In his last publication shortly before his death, (Cohen, 2011, p. 436) reinforces the importance of models for his development of forcing, positioning himself in the tradition of Skolem as well as Gödel. Here he calls his assumption of *SM* "most natural" and claims that "its truth would intuitively be accepted by everyone". Such statements may seem surprising when compared to Cohen's strong leaning towards formalism.[43] A similar observation has been made by Giaquinto (1983, p. 130), who detects "an inner tension, if not a self-contradiction, in Cohen's philosophical view".[44] So, what can we take away from this? Second Philosophy actually allows us quite clearly to tell which parts of Cohen's work and thoughts count in an analysis of means-ends relations. Similar to how (Maddy, 2011, Ch.ii.2.(i)) relies upon everything anchored in Dedekind's mathematical work, including his foundational pursuits, but rejects his philosophical thinking as heuristics (Maddy, 2011, pp. 52–53), we can take under consideration everything that can be found in Cohen's mathematical work as well as comments by Cohen on it. For example, it is significant that the approach via models was ultimately mathematically successful and led to the development of forcing, whereas Cohen's initial syntactical approach was not. It is

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[43] I would like to thank one of the anonymous referees for pointing this out.

[44] A close analysis of Cohen's philosophical views and his practice in the development of forcing is given in (Wagner, forthcoming).

further a fact of mathematical practice that forcing was developed within the model-approach and not the alternative syntactical approach.[45] In contrast, wider questions asking whether models are just formal tools, really existing or similar, count as heuristic aids.[46]

To further judge the importance of models for forcing, let us look at historical accounts. Here we find that they support Cohen's own recollections: In his historical reconstruction of the discovery of forcing, Moore (1987, p. 156) remarks on the fact that Cohen's turn towards models enabled him to recognize the key ideas of forcing, while Cohen was not able to do this under his syntactic approach. Similarly, the mathematician and historian Akihiro Kanamori regards Cohen's turn towards a semantic approach as an important step towards forcing: "[The] view of forcing as a way of actually extending models with conditions held the reservoir of mathematical sense and the promise of new discovery" (Kanamori, 2008, p. 370).

He ties this in with a larger development towards model-theoretic methods in the research in logic done at UC Berkeley starting in the 1950s (cf. Kanamori, 2008, p. 356). In a way similar to Cohen's, Kanamori also perceives this as part of a development of set theory away from philosophical considerations and towards a mathematical approach: "With Cohen there was an infusion of mathematical thinking and of method and a proliferation of models, much as in other modern, sophisticated fields of mathematics" (Kanamori, 2008, pp. 370–371).

So in both historical accounts, the use of models in Cohen's mathematical work on forcing is judged to be relevant and fruitful for the mathematical outcome. This is exactly the kind of evidence the Second Philosopher is searching for when evaluating means-ends relations: The success and mathematical effectiveness of the model-related approach of forcing is the deciding factor, which is made even more significant when considering that approaches that don't rely on models were not effective in practice. Also, Cohen's work with models is not only relevant to the specific area of forcing, it is also judged to have been formative to set theory in general, contributing to the full mathematization of set-theoretic practice. All in all, we can conclude that in the case of forcing the means of introducing models as entities into set theory was an effective means towards reaching the goal of proving consistency and independence results.

3.3 Forcing Approaches and Contemporary Practice

Let us now turn to the investigation of contemporary practice. After all, we not only want to show the effectiveness of the means of introducing models as entities for

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[45] This does not make the syntactic approach less mathematical. Quite the contrary, one could argue that not relying on the assumption of an additional axiom could be preferable. But this is simply not what happened in set-theoretic practice, and therefore cannot be the evidential basis for Second Philosophy. In the next section we discuss what approaches to forcing are used in contemporary practice and we will see that the syntactical approach has still not gained importance as of yet.

[46] At least at this juncture of the Second Philosopher's investigation; as pointed out in the last section, after getting clear about methodology, we can tackle questions of ontology. But for the methodological investigation we aim at here, this does not come into play.

the historical development, but we also want to show that models have retained this relevance until this day.

As we saw in the last section, Cohen was successfully able to develop forcing within an approach based on models. In particular, he assumed the additional axiom *SM*, claiming that a standard model of *ZFC* exists. When analysing contemporary forcing practice, we do not find such an assumption any more.[47] Neither are syntactical approaches, such as the one comprised in (Cohen, 1964), in use today.[48] Instead, the forcing practice quite quickly focused on the use of two approaches, the *countable transitive model* approach (CTM) and the *Boolean valued models* approach (BVM).[49]

The CTM approach is close to Cohen's original work.[50] However, instead of adding *SM*, one considers countable transitive models that satisfy some finite parts of *ZFC* to circumvent the problem that we are not able to show that a full model of *ZFC* exists in *ZFC*.[51] In practice this restriction to fragments of *ZF(C)* is not problematic; one simply assumes that the fragment is sufficiently large to conduct all of the necessary mathematics in it. This leads to a practice where set theorists operate with these models as if they were models of full *ZF(C)*, even if they are not when carefully spelling out the set-up. It is essential to note that the CTM approach gives rise to the same mathematical results as the other approaches, in particular all of the independence and consistency results can be obtained as well. So it is a fully satisfactory set-up for forcing is used extensively in modern practice.

Like Cohen's approach, the CTM approach relies in essence on the use of models. It is mathematically highly fruitful, being used constantly and successfully in set theory. Still, it has been criticised from a philosophical point of view as unsatisfactory. For example Hamkins (2012, p. 421) remarks, that it "provides an understanding of forcing over only some models of set theory, whereas other accounts of forcing allow one to make sense of forcing over any model of set theory". Just like in the case of Cohen's view, such criticisms are not relevant for the Second Philosophers investigations into methodology. What counts is that the CTM provides a mathematically satisfactory account for the practice of forcing, that it is extensively used as such and therefore produces mathematical results. Furthermore,

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[47] Cohen has been criticised for the assumption of *SM* from very early on. For example, Moschovakis "wrote to Cohen, asking why Cohen had made such a 'ridiculous assumption' ", i.e. that of the existence of a standard model. (Moore, 1987, p. 159).

[48] To be more precise, fully syntactical approaches are not used in the practice but sometimes syntatical set-ups are used to prove theorems about forcing when developing it in a textbook. One example for this is Kunen (1980), who introduces the syntactical notion of the ⊢$_*$-relation, but then latter drops it in favour of working with models. He mentions an alternative fully syntactical approach in his appendix (Kunen, 1980, pp. 234–235) that is similar to what Cohen outlines in Cohen (1964, p. 109) and Cohen (1966, pp. 147–148).

[49] The historical development of both approaches is quite interesting and involves many of the defining figures of set theory in the second half of the 21st century, such as Cohen, Scott, Shoenfield, Solovay and others. See (Moore, 1987) for a detailed historical account.

[50] For an introduction to CTM, see Kunen (2011).

[51] This impossibility is due to Gödel's incompleteness results, see for example discussion in Kunen (1980).

to make our case it is important to highlight that the use of models is essential for CTM and therefore also for its mathematical practice. With the above quote Hamkins makes a judgement about what is the "right" type of model to consider for his foundational and ontological purposes, but as we have seen in Sect. 2.1, this does not figure into the methodological investigation of the Second Philosopher.

Let us turn to the second approach used in contemporary set theory: The Boolean valued models (BVM) approach was developed shortly after Cohen's introduction of forcing by Dana Scott and Robert Solovay.[52] In this approach, truth values from a complete Boolean algebra $\mathbb{B}$ are assigned to formulas, therefore producing a Boolean-valued model $V^{\mathbb{B}}$ of *ZFC*.[53] Here we do not have to restrict ourselves to countable transitive models; we can see this as forcing over general models of *ZFC*, with the only caveat that the model produced will not be two-valued. However, as for the CTM approach, this does not influence the mathematical results provided by forcing via BVM; we still arrive at the same independence and consistency results as with the other approaches. So we can see that the use of models is essential for the BVM account as it relies on the *ZFC* models $V^{\mathbb{B}}$.

Analogously to our conclusions for the CTM approach, the BVM approach can be seen to be mathematically highly successful and fruitful. It produces the desired consistency and independence results and has the added advantage that forcing can be expressed in algebraic terms, therefore connecting it more closely to other areas of mathematics. Its philosophical significance can either be praised[54] or criticised.[55] Yet these criticisms are aimed at their ability to solve metaphysical questions, such as the discussions if and how these models can be said to be "universes" of set theory. These discussions do not diminish the relevance of these model-based approaches to set-theoretic practice and their usefulness in getting mathematical results.

Let's take stock: In the last section I have shown that the use of models was essential to the development of forcing. In this section I have investigated the contemporary uses of forcing and shown that both of the ways of doing forcing still employ models in this manner. In contrast, approaches to forcing not based on models never had the same importance for the practice, nor were they particularly fruitful.[56]

I have also shown that it does not matter mathematically whether the models employed in the set-up of forcing are countable transitive ones, standard models or Boolean-valued ones. In each of these cases we arrive at the same mathematical results: the approaches are mathematically equivalent. The forcing practice itself is not even restricted to only use models of *ZF* or *ZFC*; sometimes models of weaker

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[52] For an introduction to BVM, see Jech (2003).

[53] This problem of many truth values can be addressed by using an ultrafilter over $\mathbb{B}$ to form the new model. Recently, this has been used to develop the so-called "naturalist account of forcing" (Hamkins, 2012, p. 423); however, the approach used in the practice is most often the "simple" BVM approach.

[54] As done in (Hamkins, 2012), especially when developed further towards the Naturalist approach.

[55] As done in Koellner (2010, p. 217) for not producing two-valued models.

[56] For the sake of completeness, let me point out that there are variants of the approaches discussed in this article as well as some other approaches, that are theoretically possible but not really used in the practice. An overview can be found in Kunen (1980, pp. 232–235).

theories such as *ZF⁻* are used and sometimes models of *ZFC* plus additional axioms such as large cardinal axioms are used—indeed, a broad variety of models is used in the set-theoretic practice of forcing.

The conclusion about the variety of models in the practice can further be strengthened by the observations made in Sect. 3.1: Forcing itself is not only used for independence results but for various other tasks as well, therefore being fruitful for set theory over and above independence results. And to add yet another aspect, forcing is not the only model-building technique used in set theory: there are models in use that are not produced by forcing. Last, there are important areas in set theory that study models themselves, such as inner model theory. This leads us to the conclusion that, while our main case study is forcing, not only the specific models built with forcing are essential to the practice, but models in general. From all of this we can judge the means of introducing models of set theory as new entities in the practice as a effective means towards a wide range of set-theoretic goals over decades of mathematical research.

3.4 Models as Fundamental Entities

In the last sections we have collected evidence from historical and contemporary practice about the use of models in set theory. We will now refer to this evidence when we check if the criteria for fundamental entities given in Sect. 2.2 are supported.

Let us first look at the specific criteria: we have to show that models are treated as entities *in their own right*. One marker for this is that models should be susceptible to general mathematical operations. In the case of sets this was ascertained in Cantor's practice of taking the derived set over point sets. In the case of models a similar change can be detected: With forcing one studies how models can be extended, what operation techniques can be applied to models (for example different kinds of forcing such as set- or class-forcing) and how models behave under these operations. Moreover, research is now being conducted into the interrelations between models, such as in the modal logic of forcing, set-theoretic geology[57] or work on generic absoluteness (see e.g. Bagaria, 2006). Models are now operated on directly, therefore becoming the focus of attention, instead of just serving as vehicles for studying sets contained in them.

Turning to the marker of *arbitrary entities* and of *independent representations*, we have seen in Sect. 3.2 how building of models changed from using specific constructions to applying general and flexible model-building techniques such as forcing: Whereas models like *L* are very specifically tied to their construction, forcing models can vary greatly despite being built with the same technique. This is because forcing models are tied to very few specific constraints. From their construction we know that they retain the base axioms of the ground model (usually some fragment of *ZFC*) and that they contain the generic. Other properties of the

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[57] See the Appendix to (Hamkins, 2012).

model however are not determined by the general method of forcing[58] but vary with the choices made in the ground model about the partial order under consideration. In this sense models are arbitrary as they are not tied to specific constraints and largely independent from the way in which they are built by forcing.

By satisfying the specific criteria of the case of introducing something as a new entity, we can conclude that models changed in a methodologically salient manner. This leaves us with the broad criteria, which show themselves in larger scale changes in foundations and reasoning.

First, let us consider if the use of models is tied to particularly fruitful new styles of reasoning. In Sect. 3.1 we saw how model-related reasoning led to the development of forcing when Cohen switched to a semantic approach in proving the independence of CH and AC, as the use of models was essential to Cohen's mathematical reasoning. We also observed that this model-based reasoning is still prevalent in contemporary approaches to forcing. This also shows itself in general set-theoretic practice: "Let M be a model of set theory..." is a standard sentence for setting up definitions, theorems or proofs and is deeply ingrained in set-theoretic reasoning. Moreover, some parts of set-theoretic reasoning would simply not be possible without the consideration of models. This holds in particular for the study of the properties of sets under different axiomatizations or the behaviour of independent sentences. This shows that the reasoning with models is not only fruitful and relevant, it is essential for set-theoretic practice.

Next, we see from our investigation that models figure prominently into explanations and understanding of mathematical facts: For example, the way in which a number of large cardinals can be defined via model embeddings explains the connection and degrees of consistency strength between them in a unifying manner.[59] Also, forcing explains why large cardinal axioms cannot solve the Continuum Hypothesis (because we can change the value of the continuum via small forcings) and it provides a deep understanding of the independence phenomenon,[60] for example by tracing the behaviour of independent statements over forcing models (Hamkins and Löwe, 2005). From Cohen's own and Moore's account about the change in Cohen's understanding once he switched to a semantic approach, one can also infer that the success of the semantic approach in forcing is at least partly related to its greater explanatory value than that of the syntactic approach. In turn, this offers an explanation for why the model-related approaches to forcing were so successfully used whereas the syntactic approaches were not.

Lastly, the marker for a superior presentation in terms of improved generality, unification, rigor or simplicity obtains: Forcing provides a well-defined methodological framework for studying independence. Before forcing, it was

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[58] Although forcing provides us with the means to investigate what holds in the models via the forcing relation, it does not itself pre-determine what holds in them.

[59] Unification is the basis for explanations in the account of Kitcher (1989); in this case I only assume that unification is one indicator (but maybe not the only one) for explanatory value.

[60] As in the case of unification, one can take understanding to be the sole basis for explanation (see for example Inglis & Mejía-Ramos, 2019), but here I take understanding to be one indicator amongst others for explanatory value.

difficult to make independence approachable in general; each case of attempted consistency and independence proofs had to be handled individually and models had to be built "by hand" to fit the cases under consideration. In contrast, forcing provides a general and unified way of dealing with the phenomenon of independence. It also adds rigour because all of the mathematical and meta-mathematical requirements are laid out in a general and rigorous set-up. Interestingly, despite forcing being a quite difficult technique to master, it nonetheless adds simplicity to the question of independent statements: Once the set-up of forcing is done and proved, its complete meta-mathematical background as well as the main theorems about axiomatization and definability can be as a black box. The set theorist who wants to use forcing only has to focus on choosing a partial order appropriate to her task without having to take care of actually building up a model by hand.

We have now seen that the use of models and the use of techniques based on models such as forcing have wide-spread consequences for set theory in questions of reasoning, explanations, rigour and generality. That leaves the last of the broad criteria, namely the attempts at new kinds of foundations. Such attempts have indeed been made. One example is (Gitman and Hamkins, 2010) where possible axioms for multiversism are discussed. Note that, though closely related to (Hamkins, 2012), the discussion in (Gitman and Hamkins, 2010) does not require metaphysical assumptions, such as the posit that all models are "real" universes of set theory. A more widely discussed non-metaphysical take at foundations for sets and models is provided by Steel (2014). He develops a theory that has sets and models as fundamental entities and then discusses several issues connected to foundations. Recently, Maddy and Meadows (2020) have revisited and expanded on this approach, emphasizing the compatibility of Steel's program with the aims of Second Philosophy (see Maddy & Meadows, 2020, p. 38). These theories can certainly count as attempts at foundations with sets and models as fundamental entities and these attempts occur in the practice of set theory. This is what counts for a Second-Philosophical investigation into methodology, even if it is open issue if these attempts will succeed and how they are assessed in the universe/multiverse debate.[61] So while the question remains largely open as how to deal with two such types of entities on a foundational level, on the level of practice we observe that attempts at foundations including models have been made and continue to be investigated. They have not yet succeeded, but they also have not yet been declared a failure.

Let me emphasize again, that it does not matter to the investigation we are carrying out into set-theoretic methodology what advantages or drawbacks such foundations might have for metaphysical theories such as universism or (Hamkins') multiversism. For instance, many of the arguments provided in (Maddy, 2017) against multiversism are not aimed against foundations including models, but the "high metaphysics" (Maddy, 2017, pp. 312) she detects in multiverse positions, such as Hamkins.

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[61] For example, (Maddy and Meadows, 2020) claims that Steel's multiverse in the end comes down to a variant of universism, not multiversism.

My takeaway from the current state of play is that models are being regarded as fundamental enough to initiate a discussion about new foundations, even if this discussion has not yet come to a conclusion.

Taking stock, we have seen that the case of models can indeed be shown to fulfill the criteria for the introduction of fundamental entities. The specific criteria are satisfied in full and the same holds for most of the broad criteria. With regard to foundations, the case is not resolved; however, clear evidence is available that models will play a part in further foundational discussions. Using the methodology of Second Philosophy we have therefore shown that the means of introducing models as new fundamental entities should be considered to be part of set-theoretic methodology.

4 Conclusion and Outlook

The main result of the present article is that there is a detailed and methodologically salient argument from set-theoretic practice for the existence of two fundamental entities in set theory: sets and models. This analysis shows that the classical position discussed in Sect. 3.1 stating that fundamental entities in set theory are only sets has to be revised to include models as well.

This opens up many more questions: How do these two entities interact? How does the fundamental status of models impact the current discussions about foundations and metaphysics of set theory? In particular, what does it mean for the positions of Hamkins and Maddy on which we based our investigation? I will have to leave many of these and related questions open in this article. Let me, however, sketch the significance of my investigation for Maddy's and Hamkins' projects and provide some outlook on possible further work in this area.

The present investigation shows that Hamkins' claim about the fundamental status of models for set-theoretic practice can be justified from a naturalistic, Second Philosophy point of view. Considering models as fundamental entities is part of set-theoretic methodology. This provides an argument for Hamkins' claim that has been missing until now. However, it does not, by itself, justify all the ontological conclusions Hamkins draws from it. Instead, when following Second Philosophy, a Robust Realism of the pluralistic kind cannot be supported. What could be supported, however, are multiverse views that are along the lines of Arealism or perhaps Thin Realism.[62]

Regarding Second Philosophy, we have seen that the naturalist position can agree with multiversist positions on the level of practice and methodology. This means that the practice-related parts of multiversism cannot be disregarded as heuristics, but have to be factored in when making conclusions based on methodology. By including multiversist-flavored set theory into the investiagtion of Second Philosophy, we also address the criticisms by (Ternullo, 2019) and (Rittberg, 2016) pointed out in Sect. 1. However, Second Philosophy can still disagree significantly with multiverse positions in questions of ontology. For example, even if the Maddian

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[62] Maddy (2019, Footnote 6) hints at such a possibility for an arealist multiverse.

naturalist, who argues for a universist position (Maddy, 2017), accepts models as fundamental entities in set-theoretic methodology, she does not have to change her view to multiversism: models can be seen as fundamental entities under a universist picture if they are considered as models inside a larger model *V*. However, the analysis given in this paper may make it harder to argue for such a universist position, as it might weaken standard arguments in its favour.

Let us consider two such arguments, namely the argument from foundational goals and the argument from simplicity. In (Maddy, 2017), Maddy formulates foundational goals for set theory and uses them to test possible set-theoretic foundations, namely universism, multiversism and category theory. She argues that the consideration of foundational goals confirms universism as the most convincing foundation for set theory. Here, I conjecture that accepting the method of models as fundamental entities has the potential to change the foundational goals by adding a little more multiversist flavour. Interestingly, Maddy (2017) provides an example on how the acceptance of multiversist practice can change a foundational goal, namely her **Final Court of Appeal**. This goal "provides decisive answers to questions of ontology and proof: if you want to know whether or not a so-and-so exists, see whether one can be found in *V*" (Maddy, 2017, p. 296). But later on she notes that this may seem too restrictive: "Why does 'final court' insist that we restrict ourselves to exactly what happens in *V*?" (Maddy, 2017, p. 297) If one asks a mathematical question where the answer is dependent on the models under consideration, then the answer should reflect this richness and not only adhere to what holds in (or is independent from) *ZFC* alone. Maddy therefore changes the **Final Court of Appeal** criterion to the new goal of **Generous Arena**. Later she points out that, with this revised goal, "a strong prima facie case against a multiverse of any kind" (Maddy, 2017, p. 310) is not available anymore. I hypothesize that the method of accepting models as fundamental entities could change the foundational goals even further, leading to a possible re-evaluation of which set-theoretic foundations best fit the resulting goals.

A more speculative conjecture is that the result about models presented here might influence the debate about the simplicity of universe or multiverse foundations: A common argument in the multiverse/universe debate is that the default position should be universism, because there already is a good foundational theory in universism and multiversism seems to require more assumptions (for example about the existence of various models). Maddy phrases this in the following manner:

> [Given] that we could work with inner models and forcing extensions from within the simple confines of *V*, as described by our best universe theory, what mathematical motivation is there to move to a more complex multiverse theory? (Maddy, 2017, p. 316)

But is *V* really 'simple' and the multiverse theory 'more complex'? From a metaphysical point of view, this is certainly correct. Positing the existence of a not well-delineated number of models is more complex than simply stating that one universe exists. However, from a methodological point of view, the use of models of set theory as fundamental entities seems to suggest otherwise: Here multiversism offers a very natural interpretation for these models, namely as entities in their own right. Universism, instead, offers an interpretation for the models that requires an

additional argumentative step by interpreting them as parts of an overall *V*. The universist would, for example, have to argue why *V* as a model of set theory itself has a different *methodological* status than the other models.

I think it is of interest for further work to explore in detail how the naturalist argument given here about the fundamental status of models in set-theoretic methodology can have wider implications for the core areas of the universe/multiverse debate.

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