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ORIGINAL RESEARCH

Paraconsistent Belief Revision: An Algebraic Investigation

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Abstract
This paper offers a logico-algebraic investigation of AGM belief revision based on the logic of paradox (LP). First, we define a concrete belief revision operator for LP, proving that it satisfies a generalised version of the traditional AGM postulates. Moreover, we investigate to what extent the Levi and Harper identities, in their classical formulation, can be applied to a paraconsistent account of revision. We show that a generalised Levi-type identity still yields paraconsistent-based revisions that are fully compatible with the AGM postulates. The main outcome is that, once the classical AGM framework is lifted up to an appropriate level of generality, it still appears as a regulative ideal for treating of paraconsistent-based epistemic operators.

Keywords AGM belief revision · Algebraic logic · Paraconsistent logic · Epistemic operators · Kleene lattices · Levi identity · Harper identity

1 Introduction

Belief revision theory thrives during the 80s and 90s, reflecting the growing concern in the same years of database updates in computer science and artificial intelligence. Among all the belief revision theories, AGM theory is widely recognised as a milestone. AGM belief revision theory was initialised by Alchourrón and Makinson (1982) and Alchourrón et al. (1985) and developed by Gärdenfors (1992). Briefly, the AGM belief revision theory primarily answers how to get

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rid of the inconsistency and minimise the information loss to accommodate new information, sometimes contradictory information to an agent's own beliefs.

To achieve this, a formal epistemology of belief revision theory is required. Basically, such a theory should possess the following essential components: a classification of the epistemic attitudes; a formal representation of epistemic states; an account of the epistemic inputs and a classification of epistemic changes; and, finally, a general criterion of rationality. Such a criterion should be a principle of information economy, which requires that an agent could accommodate new information and concurrently minimise the loss of the original beliefs. This criterion is induced because data are valuable, and it is better to preserve them as much as possible.

Following the above requirements, the main framework of AGM theory includes an agent's belief state, which is closed under a consequence operation Cn. Since AGM theory adopts a classical propositional logic, the consequence operation is $\vdash_{\mathsf{CL}}$ in this regard.

In this paper, we develop a logico-algebraic analysis of an AGM-based paraconsistent belief operator. We deeply rely on the framework introduced in Fazio and Baldi (2021), where it is shown that the machinery of (abstract) algebraic logic can be fruitfully applied to study the semantic aspects of a general contraction operator. The fundamental technical tool is the theory of generalised matrices, which allows for a language-independent formulation on the AGM contraction postulates, therefore generalizing the usual axiomatisation which strongly relies on the classical vocabulary. In such a way, a contraction can be performed not only over theories belonging to the syntactical level, but also over the intended semantics of a logic. The main advantage of this perspective is that it provides a unique framework for studying the properties of contraction in non-classical logics. The necessary notions of this perspective are recalled in Sects. 2 and 3. However, in the AGM framework, contraction is just one side of the coin, the other comprising the fundamental epistemic action, i.e., the revision operator.

The debate on AGM belief revision and paraconsistency originates in Priest (2001), where several issues concerning the AGM treatment of inconsistencies within an epistemic process are addressed. A different investigation on the topic is contained in Bueno and da Costa (1998). There, the authors provide a detailed construction, based on the hierarchy of the so-called C_n paraconsistent logics, where the AGM framework is capable of dealing with specific inconsistent epistemic states, without leading to triviality. A further proposal, closely connected with the previous one, consists in Mares (2002). Here, the key idea is to replace the AGM constraint of consistency as well as non-triviality with the requirement of coherency between what an agent accepts and what she does reject. In this case, the investigation is based on the relevant logic R and it relies on a modification of the primitive notion of belief set. More recently, the authors of Testa et al. (2017) tackle the problem of a paraconsistent AGM account of revision at a greater level of generality, showing how to develop a meaningful notion of belief change for a family of super-classical paraconsistent logics, as well as for logics of formal inconsistency. Each of the above mentioned proposals is forced to embrace at least one of the following limitations:

(i) rejecting many of the AGM postulates;
(ii) rejecting the primitive AGM notion of belief set;
(iii) dealing with logics that have classical logic as fragment.

Here, we show that, once the AGM framework is appropriately generalised by using the techniques of Fazio and Baldi (2021), it allows for a satisfactory treatment of a paraconsistent revision operator without relying on (i)-(iii). As proved in Sect. 4, it is possible to formulate a concrete paraconsistent belief revision operator for the Logic of Paradox (LP) that satisfies all the traditional AGM postulates (Definition 31 and Theorem 33). Even if we do not claim that the mentioned operator is ideal regarding every possible situation, its treatment reveals that there is nothing intrinsic to the AGM framework preventing us to model contradictory, and non-trivial epistemic situations. The explicit motivation for this is highlighted by the construction of Lemma 27. Similarly, we further extend the investigation to other two milestones of the AGM tradition: namely the Levi and the Harper identity (Sect. 5). We determine sufficient and necessary conditions to recapture the classical interdefinability between contraction and revision *modulo* these identities (Theorems 45, 48). As for the AGM potulates for a revision operator, it becomes clear that an appropriate reformulation of the Levi identity can fully represent revisions (for our operator) in terms of contractions and expansions (Proposition 40, (i)). This is not true, in general, for the Harper identity (Proposition 40, (ii)). There is a radical difference between these two identities. While Levi's identity encodes a logic-independent process to dene revisions, Harper's identity is intrinsically connected to precise properties of classical logic. This is because in a Boolean algebra, the identity $x \wedge \neg x \approx 0$ holds. The overall outcome is that, once we replace the classical interpretation of *negation* with the concept of *trivialiser* (Definition 17), a considerable portion of the AGM framework perfectly fits with a paraconsistent account of revision.

The emerging picture suggests that it is possible to deviate from the route specified by the above mentioned proposals (Priest, 2001; Mares, 2002; Bueno & da Costa, 1998), where, for different reasons, the standard AGM account appears unsatisfactory when it comes to model paraconsistency.

2 Preliminaries

Let **Fm** be the absolutely free algebra (the term algebra) of a fixed type built up over a countably infinite set *Var* of variables.

A *consequence relation* on *Fm* is a relation $\vdash \subseteq \mathscr{P}(Fm) \times Fm$ s.t. for all $\Gamma, \Delta \in \mathscr{P}(Fm)$ and $\varphi \in Fm$,

R. If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$
M. If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$
C. If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$.

A *logic* is a consequence relation $\vdash\, \subseteq \mathscr{P}(Fm) \times Fm$, which is *substitution-invariant* in the sense that for every substitution (i.e. endomorphism) $\sigma : \mathbf{Fm} \to \mathbf{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma\Gamma \vdash \sigma\varphi.$$

A *generalised matrix* (*g-matrix* for short), is a pair $\langle \mathbf{A}, \mathscr{C} \rangle$, where $\mathbf{A}$ is an algebra, and $\mathscr{C} \subseteq \mathscr{P}(A)$ is a closure system on A (see Font, 2016, Definition 1.24). Given a g-matrix $\langle \mathbf{A}, \mathscr{C} \rangle$, we denote by $\mathrm{C}_g^{\mathbf{A}^{\mathscr{C}}}(\cdot)$ the closure operator (Font, 2016, Definition 1.19) on A associated with $\mathscr{C}$ (cf. Font, 2016, Theorem 1.26). We often omit the superscript $\mathscr{C}$ when the context is clear. A g-matrix $\mathcal{M} = \langle \mathbf{A}, \mathscr{C} \rangle$ induces a logic $\vdash_{\mathcal{M}}$ as follows:

$$\Gamma \vdash_{\mathcal{M}} \varphi \iff \text{for every homomorphism } h : \mathbf{Fm} \to \mathbf{A}, h(\varphi) \in \mathrm{C}_g^{\mathbf{A}}(h[\Gamma]).$$

A class of g-matrices M induces a logic $\vdash_{\mathsf{M}}$ upon setting:

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff (\Gamma, \varphi) \in \bigcap_{\mathcal{M} \in \mathsf{M}} \vdash_{\mathcal{M}} .$$

A g-matrix $\mathcal{M}$ is a model of $\vdash$ when $\vdash\, \leq\, \vdash_{\mathcal{M}}$ i.e. when $\vdash_{\mathcal{M}}$ is an extension of $\vdash$ (see Font, 2016, p. 27; see also Burris and Sankappanavar 1981). A logic $\vdash$ is complete with respect to a class of g-matrices M when $\vdash\, =\, \vdash_{\mathsf{M}}$. Given a logic $\vdash$ and an algebra $\mathbf{A}$ of the same type, $\mathcal{F}i_{\vdash}^{\mathbf{A}}$ denotes the closure system on $\mathbf{A}$ whose members are the $\vdash$-filters over $\mathbf{A}$ (see Font, 2016, Def. 2.18). We will denote by $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(\cdot)$ the closure operator associated to $\mathcal{F}i_{\vdash}^{\mathbf{A}}$. A logic $\vdash$ *has a conjunction* (is *conjunctive*) provided that there exists a term-definable binary operation $\wedge$ such that, for any algebra $\mathbf{A}$ of the same type, and $a, b \in A$:

$$\mathrm{Fg}_{\vdash}^{\mathbf{A}}(a, b) = \mathrm{Fg}_{\vdash}^{\mathbf{A}}(a \wedge b),$$

namely if it satisfies the classical introduction and elimination inference rules for conjunction with respect to $\wedge$ (cf. Font, 2016, Definition 5.16).

A distinctive class of g-models of a logic $\vdash$ is the so-called class of *basic full g-models*, namely the g-matrices of the form $\langle \mathbf{A}, \mathcal{F}i_{\vdash}^{\mathbf{A}} \rangle$ for an algebra $\mathbf{A}$. We denote this class by $\mathsf{BFGMod}(\vdash)$. A congruence θ on $\mathbf{A}$ is compatible with a closure system $\mathscr{C}$ on A when for all $a, b \in A$, $a\theta b$ entails $\mathrm{C}_g^{\mathbf{A}}(a) = \mathrm{C}_g^{\mathbf{A}}(b)$. The set of congruences on $\mathbf{A}$ that are compatible with a closure system $\mathscr{C}$ is a complete lattice and has a maximum, called the *Tarski congruence* of the g-matrix, which we denote by $\tilde{\Omega}^{\mathscr{C}}$. Given a logic $\vdash$, we set

$$\mathsf{Alg}(\vdash) = \{\mathbf{A} : \text{there exists a g-model} \langle \mathbf{A}, \mathscr{C} \rangle \text{ s.t. } \tilde{\Omega}^{\mathscr{C}} = \Delta\}.$$

When the Tarski congruence of a g-matrix is the identity relation, we say that the g-matrix is *reduced*. The class of reduced full g-models of a logic is defined as

$$\mathsf{FGMod}^{\star}(\vdash) = \{\langle \mathbf{A}, \mathscr{C} \rangle : \mathbf{A} \in \mathsf{Alg}(\vdash), \mathscr{C} = \mathcal{F}i_{\vdash}^{\mathbf{A}}\}.$$

A logic $\vdash$ is said to be *filter-distributive* if $\mathcal{F}i_{\vdash}^{\mathbf{A}}$ is a distributive lattice, for every algebra $\mathbf{A}$ of the same type (see Czelakowski, 1984, Definition II.1).

The logic of paradox (LP for short) is the logic defined by the the g-matrix $\langle \mathbf{SK}, \{\{1,n\},\{0,n,1\}\}\rangle$, where **SK** is the algebra corresponding to the (Cayley) Strong Kleene tables depicted below:

$\wedge$	0	n	1
0	0	0	0
n	0	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	1
1	1	1	1

$\neg$	
1	0
n	n
0	1

The intended algebraic semantics associated with LP, namely Alg(LP), is the variety of Kleene lattices ($\mathcal{KL}$), whose definition is the following:

Definition 1 A *Kleene lattice* is a bounded distributive lattice with an additional unary operation $\neg$ satisfying the following conditions:[1]

(i) $x \vee y \approx \neg(\neg x \wedge \neg y)$
(ii) $x \wedge y \approx \neg(\neg x \vee \neg y)$
(iii) $\neg\neg x \approx x$
(iv) $x \wedge \neg x \leq y \vee \neg y$.

In this paper, we will work within the signature of Kleene lattices and, if not stated otherwise, we always assume algebras to be non-trivial.

It is well known (see e.g. Albuquerque et al., 2017; Font, 2016) that given $\mathbf{A} \in \mathcal{KL}$, the LP-filters on $\mathbf{A}$ are nothing but lattice filters containing all elements of the form $a \vee \neg a$, for every $a \in A$. In fact, since $\mathcal{LP}$ has a conjunction (Font, 2016, Definition 5.16), this fact follows by Albuquerque et al. (2017, Theorem 2.13.(ii), Theorem 3.4.(ii)) and Font (2016, Definition 2.18).

Remark 2 As a consequence, one has that, for any $\mathbf{A} \in \mathcal{KL}$ and $X \subseteq A$, $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(X) = \{a \vee \neg a : a \in A\} \cup \{b \in A : a_1 \wedge \cdots \wedge a_n \leq b, \text{ for some } a_1, \ldots, a_n \in X\}$.

The set of all LP-filters on $\mathbf{A}$ will be denoted by $\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$.

The reason why we assume Kleene lattices as our semantic ground relies on the perspective introduced in Fazio and Baldi (2021). The idea is that, when checking the validity of the AGM postulates in LP, the appropriate semantic framework to consider is the class FGMod*(LP). Now, as already noticed, the intended algebraic counterpart of LP is Alg(LP) $= \mathcal{KL}$[2]. Therefore, since the algebraic reducts of the models belonging to FGMod*(LP) are nothing but the algebras $\mathbf{A} \in \mathcal{KL} =$ Alg(LP), we can safely assume that Kleene lattices are precisely the semantics to work with for modeling belief revision in LP. Moreover, the closure system $\mathcal{F}i^{\mathbf{A}}_{\vdash}$ perfectly

[1] In this paper, we assume that the LP language contains constants symbols 0, 1, which when interpreted in a Kleene lattice, play the role of maximum and minimum elements, respectively. This is not an uncommon choice (see Albuquerque et al., 2017), though not necessary for the LP presentation (see Pynko, 2000).

[2] see Font (2016, Sec 5.4)

translates into a semantic perspective the original AGM purpose of working with the set of logical theories over Fm.

It is easy to see that LP is a finitary, disjunctive logic in the sense of Czelakowski (2001, §2.5.1), so, by Czelakowski (2001, Thm. 2.5.8), the following holds:

Theorem 3 LP *is a filter-distributive logic*.

3 Abstract Algebraic Expansion and Contraction Operators

In this section, we will summarize basic definitions and facts concerning abstract algebraic (logic) AGM expansion and contraction operators from Fazio and Baldi (2021). As remarked above, this approach is fully semantic. AGM expansion is meant to be an operation $\oplus$ taking as input a logical filter F and an element a of a basic full g-model $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle$ of a given finitary logic $\vdash$ and returning as output an $F' \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$ extending F, while AGM contraction associates to any pair $(F, a) \in \mathcal{F}i^{\mathbf{A}}_{\vdash} \times A$ a non-empty family $\mathscr{C}$ of sub-filters of F.

The following definition provides a set of postulates for abstract algebraic AGM expansion.

Definition 4 Consider $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$. Then $\oplus : \mathcal{F}i^{\mathbf{A}}_{\vdash} \times A \to \mathcal{F}i^{\mathbf{A}}_{\vdash}$ is an *expansion operator* over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle$ if it satisfies the following postulates, for any $a \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$:

($\oplus$1). $F \oplus a \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$;
($\oplus$2). $a \in F \oplus a$;
($\oplus$3). $F \subseteq F \oplus a$;
($\oplus$4). If $a \in F$, then $F = F \oplus a$;
($\oplus$5). For any $F' \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$, if $F' \subseteq F$, then $F' \oplus a \subseteq F \oplus a$;
($\oplus$6). $F \oplus a$ is the smallest set satisfying $\oplus 1 - \oplus 5$.

It can be seen that, given $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$, an expansion operator $\oplus$ over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle$ satisfies the postulates of Definition 4 if and only if $F \oplus a = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(F, a) = F \vee^{\mathcal{F}i^{\mathbf{A}}_{\vdash}} \mathrm{Fg}^{\mathbf{A}}_{\vdash}(a)$, for any $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$, $a \in A$. Moreover, note that if $\vdash = \vdash_{\mathrm{LP}}$, then by Theorem 3 and (Fazio & Baldi, 2021, Proposition 7), we have the following:

Proposition 5 *Consider* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash_{\mathrm{LP}}} \rangle \in \mathsf{BFGMod}(\vdash_{\mathrm{LP}})$ *and let* $\oplus$ *be the expansion operator over* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash_{\mathrm{LP}}} \rangle$. *Then the following holds, for any* $F, G \in \mathcal{F}i^{\mathbf{A}}_{\vdash_{\mathrm{LP}}}$ *and* $a \in A$:

$$(F \cap G) \oplus a = (F \oplus a) \cap (G \oplus a). \tag{dist}$$

We now consider contraction operators by first recalling a semantic, multiple-output generalisation of AGM postulates in classical logic.

In the abstract algebraic framework we consider in this paper, classical AGM contractions are meant to be operators defined over basic full g-models of $\vdash_{\mathrm{CL}}$. Apart from its intrinsic semantic nature, the notion of contraction we consider differs from the original one (see e.g. Gärdenfors, 1992) since it is *non-deterministic*. Given $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle \in \mathsf{BFGMod}(\vdash_{\mathrm{CL}})$, a contraction operator $\ominus$ takes a pair $(F, a) \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \times A$, and returns a (possibly infinite!) family $\mathscr{C}$ of sub-filters of F. The next definition is (Fazio & Baldi, 2021, Definition 4).

Definition 6 Let $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle \in \mathsf{BFGMod}(\vdash_{\mathrm{CL}})$. Then $\ominus : \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \times A \to \mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}})$ is a multiple-conclusion classical AGM *contraction operator* over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle$ if it satisfies the following postulates for every $a, b \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}}$:

($\ominus$1). $F \ominus a \subseteq \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}}$
($\ominus$2). For any $K \in F \ominus a$, $K \subseteq F$;
($\ominus$3). If $a \notin F$ then $F \ominus a = \{F\}$;
($\ominus$4). If $a \notin \mathrm{Fg}^{\mathbf{A}}_{\mathrm{CL}}(\emptyset)$ then $a \notin K$, for any $K \in F \ominus a$;
($\ominus$5). If $a \in F$ then, for any $K \in F \ominus a$, $K \oplus a = F$;
($\ominus$6). If $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{CL}}(a) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{CL}}(b)$ then $F \ominus a = F \ominus b$;
($\ominus$7). For any $K_1 \in F \ominus a$ and $K_2 \in F \ominus b$ there exists $H \in F \ominus a \wedge b$ such that $K_1 \cap K_2 \subseteq H$;
($\ominus$8). If $a \notin K \in F \ominus a \wedge b$ then $K \subseteq H$, for some $H \in F \ominus a$.

We now introduce some general notions concerning AGM theory. For the purpose of this paper, it is sufficient for the reader to internalize a restricted version of Definitions 7 and 9, which is summarized in Definition 13.

In passing from the classical setting to the fully general framework of abstract algebraic AGM contraction, a further step of generalisation is needed. Abstract algebraic contraction is not only semantic and non-deterministic but also *non-prioritised* (see, e.g. Fermé & Hansson, 2018, p. 68; Rott, 1992). Alternatively, in contracting a given belief set F by a proposition a, we consider the possibility for an epistemic agent $\mathscr{A}$ to express a *preference* concerning beliefs in F, which must be considered undeniable truths knowing that a must be rejected. The next definition simplifies (Fazio & Baldi, 2021, Definition 8).

Definition 7 Consider $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$. A *preference function* over $\mathcal{F}i^{\mathbf{A}}_{\vdash}$ is a mapping $\tau : \mathcal{F}i^{\mathbf{A}}_{\vdash} \times A \to \mathcal{F}i^{\mathbf{A}}_{\vdash}$ such that $\tau(F, a) = F$, if $a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset)$, and $\tau(F, a) = K$, where K is an arbitrary sub-filter of F, otherwise.

If τ is such that $\tau(F, a) = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset)$, for any $a \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset)$, then τ is said to be the *absolutely skeptical* preference function, and it is denoted by τ_0.

Remark 8 It is easily seen that

$$a \in \tau_0(F, a) \text{ if and only if } a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset) \text{ and } \tau_0(F, a) = F,$$

for any $F \in \mathcal{F}i_{\vdash}^{\mathbf{A}}$ and $a \in A$.

The meaning of absolutely skeptic preference functions is clear. They reflect the behaviour of an epistemic agent willing to reject any belief which is not a tautology. In other words, to say that $b \notin \tau(F, a)$ means that b is rejectable when revising F with a new evidence a.

We are now ready to introduce postulates for contraction in the setting of abstract algebraic AGM theory.

Definition 9 Consider a logic $\vdash$, $\langle \mathbf{A}, \mathcal{F}i_{\vdash}^{\mathbf{A}} \rangle \in \mathsf{BFGMod}(\vdash)$, and a preference function $\tau : \mathcal{F}i_{\vdash}^{\mathbf{A}} \times A \to \mathcal{F}i_{\vdash}^{\mathbf{A}}$ over $\mathcal{F}i_{\vdash}^{\mathbf{A}}$. Then an operator $\ominus_{\tau} : \mathcal{F}i_{\vdash}^{\mathbf{A}} \times A \to \mathscr{P}(\mathcal{F}i_{\vdash}^{\mathbf{A}})$ is a τ *-contraction operator* over $\langle \mathbf{A}, \mathcal{F}i_{\vdash}^{\mathbf{A}} \rangle$ if it satisfies the following postulates for every $a_1, \dots, a_n, a, b \in A$, $F \in \mathcal{F}i_{\vdash}^{\mathbf{A}}$, and $\gamma, \varphi \in Fm$:

($\ominus_{\tau}$1). $F \ominus_{\tau} a \subseteq \mathcal{F}i_{\vdash}^{\mathbf{A}}$

($\ominus_{\tau}$2). For any $K \in F \ominus_{\tau} a$, $\tau(F, a) \subseteq K \subseteq F$;

($\ominus_{\tau}$3). If $a \notin F$, then $F \ominus_{\tau} a = \{F\}$;

($\ominus_{\tau}$4). If $a \notin \tau(F, a)$, then $a \notin K$, for any $K \in F \ominus_{\tau} a$;

($\ominus_{\tau}$5). If $a \in F$ and $a \notin \tau(F, a)$, then for any $K \in F \ominus_{\tau} a$, $F = K \oplus a$;

($\ominus_{\tau}$6). If $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(a) = \mathrm{Fg}_{\vdash}^{\mathbf{A}}(b)$ then $F \ominus_{\tau} a = F \ominus_{\tau} b$;

($\ominus_{\tau}$7). If $\mathrm{Fg}_{\vdash}^{\mathbf{A}}(a_1, \dots, a_n) = \mathrm{Fg}_{\vdash}^{\mathbf{A}}(b)$ then, for every family $K_i \in F \ominus_{\tau} a_i$ (for $1 \leq i \leq n$) there exists $H \in F \ominus_{\tau} b$ such that

$$\bigcap_{i \in I} K_i \subseteq H;$$

($\ominus_{\tau}$8). If $\gamma \vdash \varphi$, then for any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$, and for any $K \in F \ominus_{\tau} h(\gamma)$, if $h(\varphi) \notin K$, there exists $H \in F \ominus_{\tau} h(\varphi)$ such that $K \subseteq H$.

In what follows, to have an easier and smoother description of contractions outputs, we will consider τ_0-contraction operators. Indeed, for such types of contractions, if a logic $\vdash$ has a conjunction $\wedge$, then $\ominus_{\tau}1 - \ominus_{\tau}8$ from Definition 9 are equivalent to $\ominus 1 - \ominus 8$ from Definition 6.

Theorem 10 *Let* $\vdash$ *be a logic with conjunction* $\wedge$ *and consider* $\langle \mathbf{A}, \mathcal{F}i_{\vdash}^{\mathbf{A}} \rangle \in \mathsf{BFGMod}(\vdash)$. *Consider an operator* $\ominus : \mathcal{F}i_{\vdash}^{\mathbf{A}} \times A \to \mathscr{P}(\mathcal{F}i_{\vdash}^{\mathbf{A}})$. *The following are equivalent*:

(i) $\ominus$ *is a* τ_0*-contraction operator over* $\langle \mathbf{A}, \mathcal{F}i_{\vdash}^{\mathbf{A}} \rangle$;

(ii) $\ominus$ *satisfies postulates* $\ominus 1 - \ominus 8$ *of Definition* 6, *for any* $a, b \in A$ *and* $F \in \mathcal{F}i_{\vdash}^{\mathbf{A}}$.

Proof See (Fazio & Baldi, 2021, Theorem 17). □

We now consider the abstract algebraic AGM version of one of the most important and debated contraction operators, namely *maxichoice contraction*. Before giving an explicit definition thereof regarding absolutely skeptic preference functions, let us recall its building blocks.

The next definition is an easy adaptation of (Fazio & Baldi, 2021, Definition 18) to the case $\tau = \tau_0$.

Definition 11 Let $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$. For every $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$ and $a \in A$, we define $S_{\tau_0(F,a)} := \{X \subseteq F : a \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash}(X, \tau_0(F,a))\}$.

Note that, by Remark 8 and Definition 11, for any $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$, $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$ and $a \in A$, one has that

$$S_{\tau_0(F,a)} = \begin{cases} \emptyset & \text{if } a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset) \\ \{X \subseteq F : a \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash}(X), \} & \text{otherwise} \end{cases} \tag{1}$$

In order to simplify notation, since we will deal with absolutely skeptic preference functions only, we will frequently omit the subscript "τ_0". This convention will not be adopted in Sect. 5 , where we will deal with other contraction operators. Furthermore, due to the definition of τ_0 and since $\vdash$ is finitary, we have (see Fazio & Baldi, 2021, Remark 19):

Remark 12 For any $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$,

(i) $S_{(F,a)} \neq \emptyset$ if and only if it has maximal elements, for every $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$ and $a \in A$;
(ii) For any $H \in S_{(F,a)}$, there exists a maximal $K \in S_{(F,a)}$ such that $H \subseteq K$;

Let us denote by $\mathrm{Max}(S_{(F,a)})$ the set of maximal elements in $S_{(F,a)}$. Since $\mathrm{Fg}^{\mathbf{A}}_{\vdash}$ is a closure operator and by Definition 11, if $\mathrm{Max}(S_{(F,a)}) \neq \emptyset$, then any $K \in \mathrm{Max}(S_{(F,a)})$ is a *sub-filter* of F, which is maximal for not containing a.

Definition 13 Let $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$. The τ_0*–maxichoice contraction operator* over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle$ is the operator $\ominus^m_{\tau_0} : \mathcal{F}i^{\mathbf{A}}_{\vdash} \times A \to \mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\vdash})$ such that, for any $a \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$

$$F \ominus^m_{\tau_0} a := \begin{cases} \{\tau_0(F,a)\}, \text{ if } a \in \tau_0(F,a) \\ \{\mathrm{Fg}^{\mathbf{A}}_{\vdash}(H, \tau_0(F,a)) : H \in \mathrm{Max}(S_{(F,a)})\}, \text{ otherwise} \end{cases}$$

Until Sect. 5, we will exclusively deal with the maxichoice contraction based on the absolutely skeptic preference function. Therefore, we will unambiguously denote $\ominus^m_{\tau_0}$ with the symbol $\ominus$.

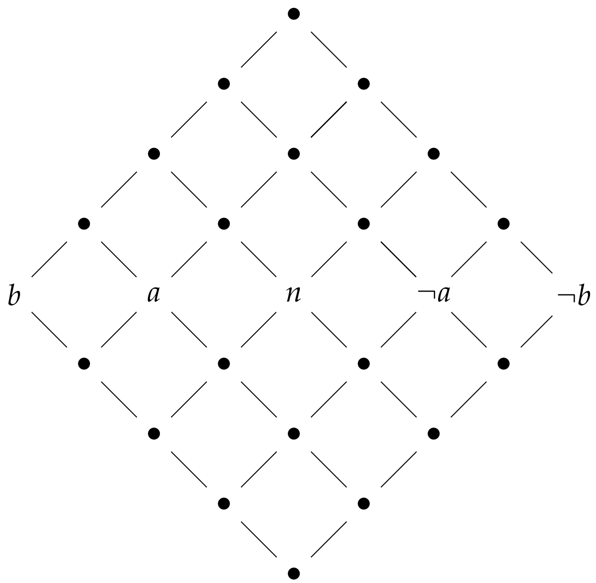
In light of the above considerations, it is easy to see that, for any $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\vdash} \rangle \in \mathsf{BFGMod}(\vdash)$, $F \in \mathcal{F}i^{\mathbf{A}}_{\vdash}$ and $a \in A$, $F \ominus a$ returns $\{F\}$, if $a \notin F$ or $a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\emptyset)$, and $F \ominus a$ yields the set of sub-filters of F, which are maximal for not containing a, otherwise.

Remark 14 Since LP has a conjunction, by (Fazio & Baldi, 2021, Example 15, Theorem 41) and Theorem 10, it happens that the τ_0-maxichoice contraction operator over an arbitrary $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{BFGMod}(\vdash_{\mathrm{LP}})$ satisfies postulates $\ominus 1 - \ominus 4, \ominus 6 - \ominus 8$.

However, the next remark shows that $\ominus 5$, i.e. the recovery postulate might fail.

Remark 15 Consider the 25-element Kleene lattice $\mathbf{A}$ depicted below, with n standing for the fixed point for negation. One has that $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}}) \subseteq \mathsf{BFGMod}(\vdash_{\mathrm{LP}})$. Fix the LP-filter $F = \uparrow b \wedge \neg a$. We have that $F \ominus a \wedge n = \{\uparrow \neg a \wedge n, \uparrow a \wedge b\}$ and, moreover, $\uparrow \neg a \wedge n \oplus a \wedge n = \uparrow a \wedge \neg a \neq F$, as suggested.

A 25-elements Kleene lattice



4 The Maxichoice-Based Revision Operator for LP

In this section, we introduce a revision operator for LP and we prove that it satisfies all the AGM postulates.

Let us begin by defining a multiple conclusion revision over models of classical logic.

Definition 16 Set $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{CL}})$. Then $\otimes : \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \times A \to \mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}})$ is a *multiple-conclusion classical AGM revision operator* over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle$ if it satisfies the following postulates for every $a, b \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}}$:

($\otimes$1). $F \otimes a \subseteq \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}}$;
($\otimes$2). For any $K \in F \otimes a$, $a \in K$;
($\otimes$3). For any $K \in F \otimes a$, $K \subseteq F \oplus a$;
($\otimes$4). If $\neg a \notin F$, then $F \otimes a = \{F \oplus a\}$;
($\otimes$5). If $A \in F \otimes a$, then $a = 0$;
($\otimes$6). If $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{CL}}(a) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{CL}}(b)$, then $F \otimes a = F \otimes b$.
($\otimes$7). For any $K \in F \otimes a \wedge b$, there exists $H \in F \otimes a$ such that $K \subseteq H \oplus b$;
($\otimes$8). For any $K \in F \otimes a$, if $\neg b \notin K$, then there exists $H \in F \otimes a \wedge b$ such that $K \oplus b \subseteq H$.

The previous definition has a double goal: first, it lifts the usual notion of classical AGM revision operator up to the semantics of basic reduced full g-models of classical logic; second, it provides a multiple-conclusion version of the AGM postulates.

In the next definition and the subsequent lemma we introduce the notion of *trivialiser*. The intuitive role of trivialisers can be explained as follows: suppose we consider a certain belief set F on A and to reach a specific epistemic stage where we are forced to revise F with an element a (ideally not already contained in F), then the set of trivialisers identifies the elements of F whose logical closure with a coincides with the whole set A. A much more precise characterization of trivializers can be provided as follows. Let $\mathbf{A} \in \mathcal{KL}$ and let $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$. Note that, for any $X \subseteq A$, $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(X) = A$ if and only if $0 \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(X)$, and $0 = a \vee \neg a$ only in case $\mathbf{A}$ is trivial. Therefore, by Remark 2, for any $x \in F$, one has that $0 \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(x, a)$ if and only if $x \wedge a = 0$. The above reasoning motivates the next

Definition 17 Let $\mathbf{A}$ be a Kleene lattice and $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$. Given $a \in A$, we define:

$$F^{\perp}_{a} := \{x \in F \, : \, x \wedge a = 0\}.$$

The elements of $F^{\perp}_{a}$ are called the *trivialisers* of a with respect to F.

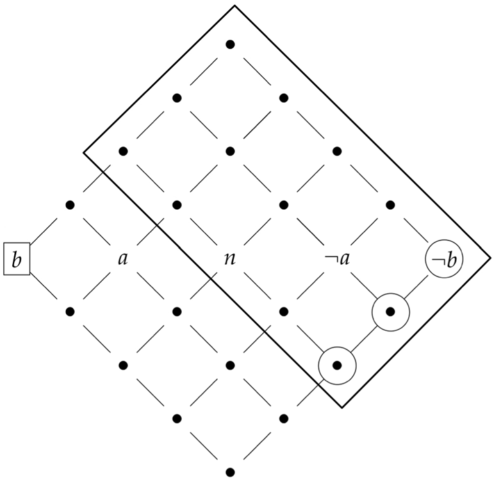
Remark 18 Note that, for any Kleene lattice $\mathbf{A}$, $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$, and $a \in A$, $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(F, a) \neq A$ if and only if F does not contain trivializers. In fact one has that

$$\begin{aligned}
\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(F, a) = A &\iff 0 \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(F, a) \\
&\iff \text{for some } a_1, \dots, a_n \in F, \; a_1 \wedge \cdots \wedge a_n \wedge a = 0 \\
&\iff F^{\perp}_{a} \neq \emptyset,
\end{aligned}$$

since F is closed under $\wedge$.

Let us provide a concrete example in order to illustrate the role of trivializers.

Example 19 In the following picture, let us denote by F the elements in the bigger rectangular shape. The trivializers of F with respect to b, in symbols $F_b^{\perp}$, are the elements within circles.



Let $\mathbf{A}$ be a Kleene lattice and $X \subseteq A$. We will denote by $I^{\mathbf{A}}(X)$ the lattice-ideal in $\mathbf{A}$ generated by X. The following facts provide a deeper insight into the algebraic properties of trivialisers.

Lemma 20 *Let* $\mathbf{A}$ *be a Kleene lattice and* $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$.

(i) *if* $F_a^{\perp} \neq \emptyset$, *then it is a sublattice of* $\mathbf{A}$;
(ii) *if* $b \in F_a^{\perp}$, *then* $b \leq \neg a$;
(iii) *if* $a \wedge \neg a = 0$, *then* $F_a^{\perp} = I^{\mathbf{A}}(\neg a) \cap F$.

Proof

(i). If $x, y \in F_a^{\perp}$, then $x \wedge a = 0$ and $y \wedge a = 0$ entail $(x \wedge y) \wedge a = 0$. By distributivity, we also have $(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) = 0$, as desired.
(ii). Clearly we have $a \vee \neg a \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(b)$ and $(a \vee \neg a) \wedge b = \neg a \wedge b$, by distributivity. By the notion of LP-filter generation, this entails that $b \wedge \neg a = c \vee \neg c$ for some $c \in A$ or that $b \leq b \wedge \neg a$. The first case cannot occur, for otherwise, $c \vee \neg c \in F_a^{\perp}$, so $b \leq \neg a$.
(iii). Note that, by (ii), one has $F_a^{\perp} \subseteq I^{\mathbf{A}}(\neg a) \cap F$. Moreover, if $\neg a \wedge a = 0$, then $\neg a \in F_a^{\perp}$. So, $I^{\mathbf{A}}(\neg a) \cap F \subseteq F_a^{\perp}$.

□

As an easy consequence of the above lemma, we have

Lemma 21 *Let* $\mathbf{A}$ *be a Kleene lattice and* $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$. *Given* $a \in A$, *if* $F^{\perp}_{a} \neq \emptyset$, *then* $\neg a \in F$.

Thanks to the acquired knowledge concerning trivialisers, we are now ready to provide a notion of revision operator for LP, where inconsistency and triviality are no longer identical.

Definition 22 Consider $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{FGMod}^{*}(\vdash_{\mathrm{LP}})$. A map $\otimes : \mathcal{F}i^{A}_{\mathrm{LP}} \times A \to \mathcal{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}})$ is an AGM LP-revision operator over $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle$ if it satisfies the following postulates, for any $a, b \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$:

- (⊗1) $F \otimes a \subseteq \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$;
- (⊗2) For any $K \in F \otimes a$, $a \in K$;
- (⊗3) For any $K \in F \otimes a$, $K \subseteq F \oplus a$;
- (⊗4) If $F^{\perp}_{a} = \emptyset$, then $F \otimes a = \{F \oplus a\}$;
- (⊗5) If $A \in F \otimes a$, then $a = 0$;
- (⊗6) If $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(a) = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(b)$ then $F \otimes a = F \otimes b$;
- (⊗7) For any $K \in F \otimes a \wedge b$, there exists $H \in F \otimes a$ such that $K \subseteq H \oplus b$;
- (⊗8) For any $K \in F \otimes a$, if $K^{\perp}_{b} = \emptyset$, then there exists $H \in F \otimes a \wedge b$ such that $K \oplus b \subseteq H$.

The simple, but crucial difference between the usual formulation of the AGM postulates for revision and the above one is highlighted in (⊗4) and (⊗8), where trivialisers come into play. A successful revision may now contain contradictory formulas, as it is expected when handling a paraconsistent logic. The connection between the role of trivialisers and contradictory formulas in classical logic is described by Theorem 24, whose proof essentially relies on the following lemma.

Lemma 23 *Let* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle \in \mathsf{FGMod}^{*}(\vdash_{\mathrm{CL}})$. *Then for any* $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}}$ *and* $a \in A$, $F^{\perp}_{a} = \emptyset$ *if and only if* $\neg a \notin F$.

Proof The right-to-left direction follows by Lemma 20 upon noticing that $\mathbf{A}$ is a Boolean algebra (and so, it is also a Kleene lattice) and F is a lattice filter of $\mathbf{A}$. Concerning the converse direction, just note that if $F^{\perp}_{a} = \emptyset$, then $\neg a \notin F$, since $a \wedge \neg a = 0$. □

Theorem 24 *Consider* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \rangle \in \mathsf{FGMod}^{*}(\vdash_{\mathrm{CL}})$ *and let* $\otimes : \mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}} \times A \to \mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{CL}})$ *be a map. Then the following are equivalent*:

(i) ⊗ *is a multiple-conclusion classical AGM revision operator*;

(ii) $\otimes$ *satisfies postulates* $\otimes 1 - \otimes 8$ *of Definition* 22.

Proof It follows directly by Lemma 23. □

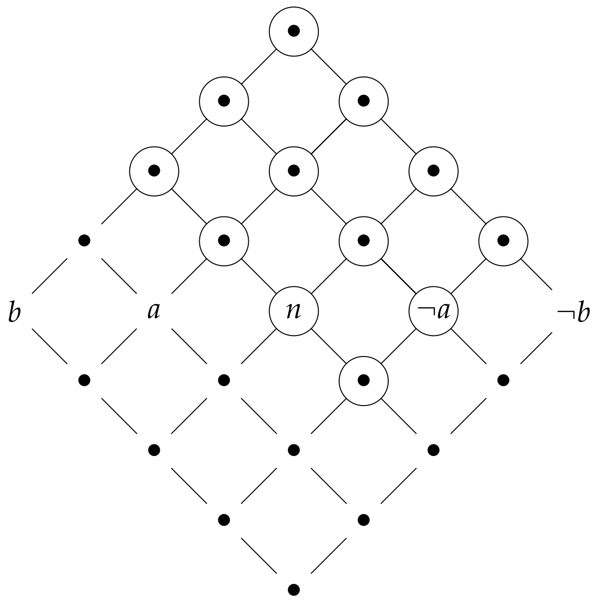
We now introduce the concrete construction of an AGM revision operator for LP. The crucial step is to determine how to remove the set of trivialisers of a filter by means of a contraction operation. Lemma 27 describes how to do so.

Definition 25 Let $\mathbf{A}$ be a Kleene lattice, $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$ and $a \in A$. We define:

$$F \ominus F^{\perp}_{a} := \begin{cases} \bigcup\{F \ominus a_i : a_i \in F^{\perp}_{a}\}, \text{ if } F^{\perp}_{a} \neq \emptyset, a \neq 0 \\ \{F\} \text{ otherwise} \end{cases}$$

The above definition can be better internalized by considering the following example.

Example 26 Consider the setting described in Example 19. The following picture describes the result of performing $F \ominus F^{\perp}_{b}$.



Lemma 27 *Let* $\mathbf{A}$ *be a Kleene lattice,* $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$ *and* $0 \neq a \in A$*. Then there exist* $H \in F \ominus F^{\perp}_{a}$ *without trivialisers.*

Proof Set $F_a^{\perp} = \{a_i\}_{i\in I}$, for some non-empty set of indexes I. Let us consider $X = \{a \vee a_i : i \in I\}$. Clearly, $X \cap F_a^{\perp} = \emptyset$. Now, take $\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(X)$. Observe that $b \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(X)$ if and only if $b = c \vee \neg c$ (for some $c \in A$) or there exist $a \vee x_1, \dots, a \vee x_n \in X$ ($n > 0$) such that

$$\bigwedge_{i=1}^{n}(a \vee x_i) = a \vee \bigwedge_{i=1}^{n} x_i = a \vee a_k,$$

by Lemma 20(i). Clearly, for any $i \in I$, $a_i \notin \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(X)$, since otherwise $a_i = c \vee \neg c$, which is impossible, or $a \vee a_j \leq a_i$, for some $j \in I$, and $a \leq a_i$, i.e. $a = 0$, against our assumptions. Now, let a_i be an arbitrary trivialiser. By Remark 12, $\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(X)$ can be extended to $H \in \mathrm{Max}(S_{(F,a_i)})$. Clearly, $H \in F \ominus a_i$. Let us show that $H \cap F_a^{\perp} = \emptyset$. Indeed, if there exists a trivializer $a_k \in H \cap F_a^{\perp}$, then $a_k \vee a_i \in H$. So, since $a \vee a_i \in H$, one has

$$a_i = a_i \vee (a \wedge a_k) = (a_i \vee a) \wedge (a_i \vee a_k) \in H,$$

a contradition. We conclude that H is a solution in $F \ominus a_i$ without trivialisers. $\square$

In light of Lemma 27, the following definitions make sense.

Definition 28 Let $\mathbf{A}$ be a Kleene lattice, $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in A$. We define:

$$\mathcal{S}_a^{+} := \begin{cases} \{G \in F \ominus F_a^{\perp} : G \cap F_a^{\perp} = \emptyset\}, \text{ if } F_a^{\perp} \neq \emptyset, a \neq 0 \\ \{F\} \text{ otherwise} \end{cases}$$

Of course, the notation introduced above might look like imprecise, since the set $\mathcal{S}_a^{+}$ is always defined for a given filter F. However, in the sequel, the filter w.r.t. the above set is introduced will be always deducible from the context. Intuitively, the members of $\mathcal{S}_a^{+}$ are precisely the filters belonging to $F \ominus a$ that do no contain trivializers. In other words $\mathcal{S}_a^{+}$ selects only the "good" solutions of a contraction.

Lemma 29 *Let $\mathbf{A}$ be a Kleene lattice. Then for any $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in A$ such that $a \wedge \neg a = 0$, one has*

$$\mathcal{S}_a^{+} = F \ominus \neg a.$$

Proof If $a = 0$ resp. $F_a^{\perp} = \emptyset$, then $\neg a = 1 \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(\emptyset)$ and $\neg a \notin F$ (since $\neg a \in F_a^{\perp}$), respectively. Hence, in both cases $F \ominus \neg a = \{F\} = \mathcal{S}_a^{+}$. Otherwise, suppose that $a \neq 0$ and $F_a^{\perp} \neq \emptyset$. By the definition of $\mathcal{S}_a^{+}$ and Lemma 20(ii), since $a \wedge \neg a = 0$, it is easy to see that $\mathcal{S}_a^{+} \subseteq F \ominus \neg a$. Conversely, if $H \in F \ominus \neg a$, then clearly $H \in F \ominus F_a^{\perp}$ and $H \cap F_a^{\perp} = \emptyset$, again by Lemma 20(ii). So $H \in \mathcal{S}_a^{+}$. $\square$

Lemma 30 *Let $\mathbf{A}$ be a Kleene lattice. Then for any $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in A$:*

$$\mathcal{S}_a^{+} = \{H \in F \ominus F_a^{\perp} : \{a \vee a_i : a_i \in F_a^{\perp}\} \subseteq H\}.$$

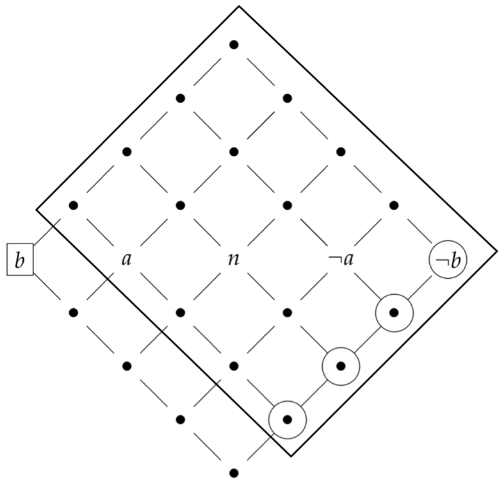
Proof Set $X = \{H \in F \ominus F_a^{\perp} : \{a \vee a_i : a_i \in F_a^{\perp}\} \subseteq H\}$. If $a = 0$, then $F_a^{\perp} = F$ and $\{a \vee a_i : a_i \in F_a^{\perp}\} = F$, namely $X = \{H \in F \ominus F : F \subseteq H\} = \bigcup\{F \ominus b : b \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(\emptyset)\} = \{F\} = \mathcal{S}_a^{+}$. So, let us assume $a \neq 0$. If $F_a^{\perp} = \emptyset$, then $\{a \vee a_i : a_i \in F_a^{\perp}\} = \emptyset$ and $X = \{H \in F \ominus F : F \subseteq H\} = \{H \in \{F\} : \emptyset \subseteq H\} = \{F\} = \mathcal{S}_a^{+}$. Finally, suppose that $F_a^{\perp} \neq \emptyset$. Adapting the argument from the proof of Lemma 27, it is easily seen that $X \subseteq \mathcal{S}_a^{+}$. Now, consider $G \in \mathcal{S}_a^{+} = \{G \in F \ominus F_a^{\perp} : G \cap F_a^{\perp} = \emptyset\}$. Let us suppose for contradiction that there exists $a_i \in F_a^{\perp}$ such that $a \vee a_i \notin G$. Since $G \in F \ominus F_a^{\perp}$, there exist $x_1, \ldots, x_n \in G$ such that $(\bigwedge_{i=1}^{n} x_i) \wedge (a \vee a_i) \leq a_k$, for some $a_k \in F_a^{\perp}$. Set $\bigwedge_{i=1}^{n} x_i = c$. One has $(c \wedge a) \vee (c \wedge a_i) = c \wedge (a \vee a_i) \leq a_k$ and so $(c \wedge a) \leq a_k$. Therefore, $c \wedge a = 0$ and $c \in G \cap F_a^{\perp}$, contradicting $G \cap F_a^{\perp} = \emptyset$. We conclude that $\{a \vee a_i : a_i \in F_a^{\perp}\} \subseteq G$ and $G \in X$, i.e. $\mathcal{S}_a^{+} = X$. □

Definition 31 Let $\mathbf{A}$ be a Kleene lattice, $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in A$. The *maxichoice-revision* of F with respect to a is defined as:

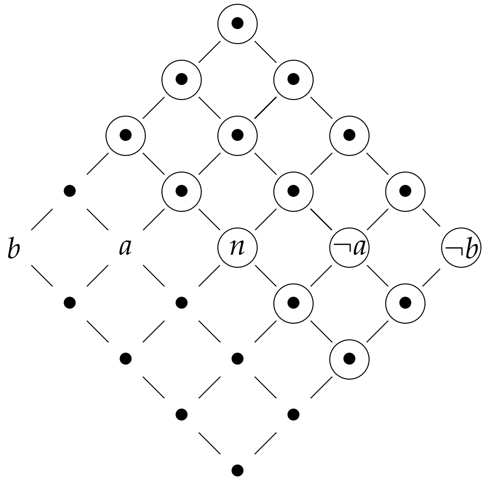
$$F \otimes_{\mathrm{LP}} a := \{\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, a) : H \in \mathcal{S}_a^{+}\}$$

The above definition of $\otimes_{\mathrm{LP}}$ allows to represent with an example the whose process of revision, which may be visualized as a 3-stages action.

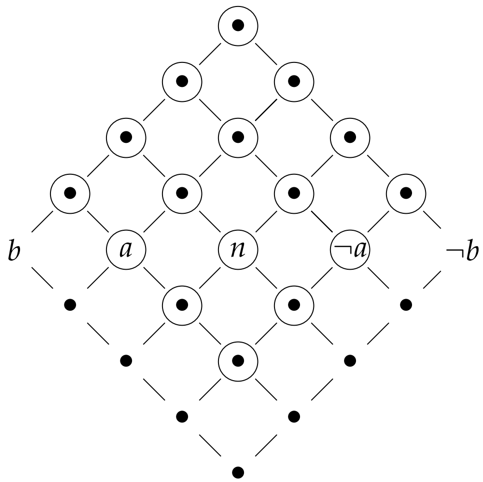
Example 32 We now graphically represent a concrete example of the 3 steps process of revision. Consider the following initial state. According to the previously introduced notation, the bigger rectangle identifies the filter F, the circled elements are trivializers, while the small rectangle highlights the element b.



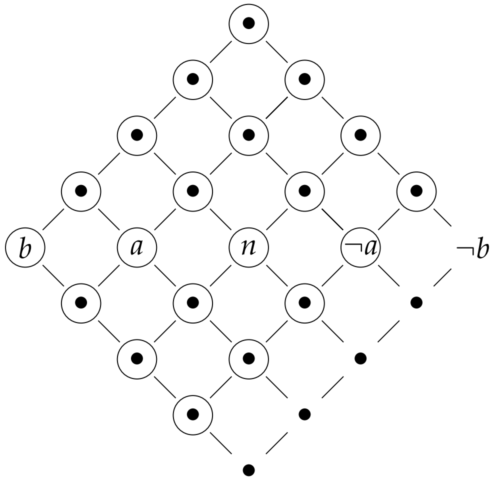
The next two pictures provide the outcomes of the contraction $F \ominus F_b^{\perp}$.



Among these two, only the one below belongs to $\mathcal{S}_b^+$, as the other one still contains trivializers.



The final result of the revision process, namely $F \otimes_{\mathsf{LP}} b$, is displayed here:



The next step is to show that the above defined operator fully copes with the AGM account of revision for LP (see Definition 22).

Theorem 33 *The revision operator $\otimes_{\mathrm{LP}}$ satisfies all the abstract AGM postulates.*

Proof ($\otimes$1)-($\otimes$3) are obviously satisfied by $\otimes_{\mathrm{LP}}$. ($\otimes$4) holds, since $F \ominus F_a^{\perp} = \{F\} = \mathcal{S}_a^{+}$ when $F_a^{\perp} = \emptyset$. Let now $a \neq 0$. If $F_a^{\perp} = \emptyset$, then $F \neq A$ and, by construction, $F \otimes_{\mathrm{LP}} a = \{\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(F, a)\} \neq \{A\}$. If $F_a^{\perp} \neq \emptyset$, Definition 22 entails that every solution $K \in F \otimes_{\mathrm{LP}} a$ is of the form $K = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, a)$ for $H \in \mathcal{S}_a^{+}$. Since, by Lemma 27, H does not contain trivialisers, $\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, a) = K$ is non trivial, as required by ($\otimes$5). If $\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(a) = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(b) = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(\emptyset)$, then $F \otimes_{\mathrm{LP}} a = \{F \oplus a\} = \{F\} = \{F \oplus b\}$. Otherwise, the antecedent of ($\otimes$6) entails $a = b$ and therefore ($\otimes$6) holds.

To prove ($\otimes$7), first consider the case that $\mathcal{S}_{a \wedge b}^{+} = F$, i.e. (1) $a \wedge b = 0$ or (2) $F_{a \wedge b}^{\perp} = \emptyset$. If (1), then $F \otimes_{\mathrm{LP}} (a \wedge b) = \{A\}$ and, since $a \in H$ for every $H \in F \otimes_{\mathrm{LP}} a$, we obtain $a \wedge b \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, b) = A$, as desired. If (2), then clearly $F_a^{\perp} = F_b^{\perp} = \emptyset$. Therefore, if $K \in F \otimes_{\mathrm{LP}} (a \wedge b)$, then $K = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(F, a \wedge b)$ and if $H \in F \otimes_{\mathrm{LP}} a$, then $H = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(F, a)$. These observations entail that $K = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(F, a \wedge b) = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(F, a, b) = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, b)$, as required. The only remaining case to consider is $\mathcal{S}_{a \wedge b}^{+} \neq F$, i.e. $a \wedge b \neq 0$ and $F_{a \wedge b}^{\perp} \neq \emptyset$. We claim that if $P \in \mathcal{S}_{a \wedge b}^{+}$, then there exists $Q \in \mathcal{S}_a^{+}$ such that $P \subseteq Q$. To this end, let $P \in \mathcal{S}_{a \wedge b}^{+}$. By Lemma 30, $x_i \vee (a \wedge b) \in P$ for every $x_i \in F_{a \wedge b}^{\perp}$. Since $F_a^{\perp} \subseteq F_{a \wedge b}^{\perp}$, $y_j \vee (a \wedge b) \in P$ for every $y_j \in F_a^{\perp}$. By distributivity, we have $(y_j \vee a) \wedge (y_j \vee b) \in P$ and, since P is an LP-filter, this entails $y_j \vee a \in P$. Therefore, we conclude $\{y_j \vee a : y_j \in F_a^{\perp}\} \subseteq P$. By noticing that $P \cap F_a^{\perp} = \emptyset$, we can extend P to $Q \in F \ominus y_k$, for an arbitrary $y_k \in F_a^\perp$. By Lemma 27, $Q \in \mathcal{S}_a^+$, and this proves the claim. Suppose now $K = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(P, a \wedge b) \in F \otimes_{\mathrm{LP}} (a \wedge b)$ and consider $H = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(Q, a) \in F \otimes_{\mathrm{LP}} a$. By the previous observation $K \subseteq \mathrm{F}_g^{\mathbf{A}}(Q, a, b) = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, b) = H \oplus b$, as desired.

Finally, to prove (⊗8), consider $K \in F \otimes_{\mathrm{LP}} a$ and assume $K_b^\perp = \emptyset$, which entails $K \neq A, \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(K, b) \neq A$ and $a, b, a \wedge b \neq 0$. First, if $F_a^\perp = \emptyset$, then $K = F \oplus a$ and, since $(F \oplus a)_b^\perp = \emptyset$, one has $F_{a\wedge b}^\perp = \emptyset$ and $F \otimes_{\mathrm{LP}} a \wedge b = \{F \oplus a \wedge b\}$. So, we conclude $K \oplus b = (F \oplus a) \oplus b = F \oplus a \wedge b$. Therefore, let us assume w.l.o.g. that $F_a^\perp \neq \emptyset$ which entails that $F_{a\wedge b}^\perp \neq \emptyset$. Fix $K = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(Q, a)$ for $Q \in \mathcal{S}_a^+$. Consider $X = \{(a \wedge b) \vee x_i : x_i \in F_{a\wedge b}^\perp\}$. Since $a \wedge b \in K \oplus b$, we have $X \subseteq K \oplus b$. Moreover, $X \subseteq F$ and so $X \subseteq (K \oplus b) \cap F = D$. Now, since $K_b^\perp = \emptyset$ entails $D \cap F_{a\wedge b}^\perp = \emptyset$, we can extend D to $H \in F \ominus x_i$, for some $x_i \in F_{a\wedge b}^\perp$. By Lemma 30, $H \in \mathcal{S}_{a\wedge b}^+$ and so $H \oplus a \wedge b \in F \otimes_{\mathrm{LP}} a \wedge b$. By distributivity, we have $K \oplus b = K \oplus a \wedge b = (K \oplus a \wedge b) \cap (F \oplus a \wedge b) = (K \cap F) \oplus a \wedge b \subseteq H \oplus a \wedge b$, as desired. □

Remark 34 Observe that, in some cases, the above described operator $\otimes_{\mathrm{LP}}$, when applied to the formula algebra, always coincides with the expansion operator $\oplus$. Indeed, let $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{Fm}}$ be a non trivial theory, and consider $\varphi \in Fm$. If $\varphi = 0$, then clearly $F_\varphi^\perp = \{F\} = \mathcal{S}_\varphi^+$ and so $F \otimes_{\mathrm{LP}} \varphi = \mathrm{F}_{g\mathrm{LP}}^{\mathbf{Fm}}(F, \varphi) = F \oplus \varphi = Fm$. If $\varphi \neq 0$, then $F_\varphi^\perp = \emptyset$, as it is immediate to verify that, for every $\psi \in F \neq Fm$, we have $\psi, \varphi \nvdash_{\mathrm{LP}} 0$. Therefore $F \ominus F_\varphi^\perp = \{F\}$, and consequently $F \otimes_{\mathrm{LP}} \varphi = \{\mathrm{F}_{g\mathrm{LP}}^{\mathbf{Fm}}(F, \varphi)\} = \{F \oplus \varphi\}$, as desired.

The previous remark highlights the advantages of a semantic-oriented interpretation of the revision operator, in particular when the underlying logic is paraconsistent. Indeed, switching from formulas to the intended algebraic semantics allows for a more fine-grained treatment of the revision operator, avoiding its collapse to the expansion operator. A further important difference between the classical and the paraconsistent case is described in the following Remark 36 and Proposition 37.

Definition 35 Let $\langle \mathbf{A}, \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}})$. A filter $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ is *complete* provided that, for any $a \in A$, $a \in F$ or $\neg a \in F$.

Remark 36 Observe that the revision operator $\otimes_{\mathrm{LP}}$ does not fall prey to the usual problem of the classical revision operator based on a maxichoice contraction, namely that it forces a sort of "logical omniscience" (see e.g. Lévy (1994)). Indeed, given a filter F, the fact that $\neg a \in F$ does not entail, in general, that solutions of $F \otimes_{\mathrm{LP}} a$ are complete filters. As an example, consider the figure depicted in Remark 15 and fix $F = \uparrow a \wedge n$. Clearly, $a \vee \neg a \in F$. However, it is easy to verify that $F \otimes_{\mathrm{LP}} a \wedge \neg a = \{\uparrow a \wedge \neg a\}$ and $b, \neg b \notin \uparrow a \wedge \neg a$.

Proposition 37 *Let* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}\rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}})$ *and* $a \in A$ *such that* $a \wedge \neg a = 0$. *Then for any* $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$, $\neg a \in F$ *entails that* K *is a complete* LP-*filter of* $\mathbf{A}$, *for any* $K \in F \otimes_{\mathrm{LP}} a$.

Proof Assume that $a \wedge \neg a = 0$, let F be an arbitrary LP-filter such that $\neg a \in F$. Clearly $\neg a \in F^{\perp}_{a}$. Now, if $a = 0$, then $F \otimes_{\mathrm{LP}} a = \{A\}$ and our statement trivially holds. So, assume that $a \neq 0$. Note that $\neg a \in F$ and $a \wedge \neg a = 0$ entail that $F^{\perp}_{a} \neq \emptyset$. Let K be an arbitrary filter in $F \otimes_{\mathrm{LP}} a$. Then, by Definition 31, there exists $H \in \mathcal{S}^{+}_{a}$ such that $K = \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(H, a)$. Let us show that for any $b \in A$, one has $\neg a \vee b \in H$ or $\neg a \vee \neg b \in H$. Suppose for contradiction that $\neg a \vee b, \neg a \vee \neg b \notin H$. This means that there exist $c_1, c_2 \in H$ such that $c_1 \wedge (\neg a \vee b) \leq \neg a$ and $c_2 \wedge (\neg a \vee \neg b) \leq \neg a$. By distributivity, one has that $c_1 \wedge b, c_2 \wedge \neg b \leq \neg a$. Now, set $d = c_1 \wedge c_2 \in H$. We have that $\neg a \geq (d \wedge b) \vee (d \wedge \neg b) = d \wedge (b \vee \neg b) \in H$ and so $\neg a \in H$, a contradiction, since $H \in \mathcal{S}^{+}_{a}$. We conclude that $\neg a \vee b \in H$ or $\neg a \vee \neg b \in H$. Finally, note that $\neg a \vee b \in H$ entails that $b \in K$, because of $b \geq b \wedge a = (\neg a \vee b) \wedge a$ (by distributivity). By an analogous argument we have that $\neg a \vee \neg b \in H$ entails that $\neg b \in K$. □

Observe that the converse of Proposition 37 is false. This can be easily seen just by considering the algebra **SK** (the algebra corresponding to the Strong Kleene tables) and letting $F = \{1, n\}$, $a = 1$.

The next proposition shows that our $\otimes_{\mathrm{LP}}$ operator always subsumes any possible LP revision operator.

Proposition 38 *Let* $\otimes : \mathcal{F}i^{A}_{\mathrm{LP}} \times A \to \mathcal{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}})$ *be an arbitrary AGM* LP-*revision operator over a reduced basic full g-model* $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}\rangle$. *Then for any* $F \in \mathcal{F}i^{A}_{\mathrm{LP}}$, $a \in A$, $G \in F \otimes a$, *there exists* $H \in F \otimes_{\mathrm{LP}} a$ *such that* $G \subseteq H$.

Proof If $F^{\perp}_{a} = \emptyset$, then by Theorem 33, $F \otimes_{\mathrm{LP}} a = \{F \oplus a\}$. Therefore, by ($\otimes$4) our conclusion trivially follows. Hence, let us assume w.l.o.g. that $F^{\perp}_{a} \neq \emptyset$. Consider $D = G \cap F$. If $F^{\perp}_{a} \cap D \neq \emptyset$, then $G = A$ and so, if $A \in F \otimes a$ then $a = 0$. Hence, $F \otimes a = \{A\} = F \otimes_{\mathrm{LP}} a$.

So, we can assume w.l.o.g. that $F^{\perp}_{a} \cap D = \emptyset$. Since, by ($\otimes$1) and ($\otimes$2), $G \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$ and $a \in G$, one has that $X = \{a \vee a_i : a_i \in F^{\perp}_{a}\} \subseteq G$. Hence, $X \subseteq D$. Now, extend D to a maximal $H \in F \ominus a_i$, for some $a_i \in F^{\perp}_{a}$. By Lemma 30, $H \in \mathcal{S}^{+}_{a}$ and $P = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(H, a) \in F \otimes_{\mathrm{LP}} a$. Therefore, $G \cap F = D \subseteq P$ and Theorem 3 entail that $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(a) \vee^{\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}} (G \cap F) = (\mathrm{Fg}^{\mathbf{A}}_{\vdash}(a) \vee^{\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}} G) \cap (\mathrm{Fg}^{\mathbf{A}}_{\vdash}(a) \vee^{\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}} F) = G \cap (F \oplus a) \subseteq P \vee^{\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}} a = P$. However, by ($\otimes$3), one has that $G \subseteq F \oplus a$. So, we conclude that $G \subseteq P$. □

We close this section with a remark concerning the possibility of modelling complex epistemic processes by means of the machinery introduced above. As it has been pointed out in the literature (see Mares, 2002), a suitable account of revision should allow to represent the behaviour of agents which might hold contradictory beliefs, but also refuse some contradiction although it does not cause their belief set to be trivial. In what follows, we show that, in our framework, in order to model complex epistemic processes, we do not need more (complex) operators than those already encountered in the general theory developed above. To this aim, we first observe that contractions and revisions produce *families* of closed subsets as outputs, while expansions take closed sets as inputs. Therefore, to represent 'compound' epistemic processes, we have to introduce *generalised* expansion, contraction and revision operators over a given $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}})$. This task can be accomplished by representing epistemic states as *sets* of LP-filters, i.e. as given $\mathscr{C} \subseteq \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$, and epistemic operators as mappings $\mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}) \times A \to \mathscr{P}(\mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}})$ e.g. by setting

$$\mathscr{C} \boxplus a = \{H \oplus a : H \in \mathscr{C}\},$$
$$\mathscr{C} \boxminus^m_{\tau_0} a = \bigcup \{H \ominus a : H \in \mathscr{C}\} \text{ and}$$
$$\mathscr{C} \boxtimes_{\mathrm{LP}} a = \bigcup \{H \otimes_{\mathrm{LP}} a : H \in \mathscr{C}\},$$

for any $\mathscr{C} \subseteq \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$ and $a \in A$ (cf. (Fazio and Baldi 2021, p. 927)). Consider the following example.

Example 39 Consider $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}})$. Let $\mathscr{A}$ be an agent having an epistemic state $\mathscr{C} = \{F\}$ such that $a \in F$ and $a \neq 0$. Now, suppose $\mathscr{A}$ recognises that $\neg a$ must be accepted and a must be rejected (hence, $a \notin \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$). Moreover, assume that, for any $K \in F \otimes_{\mathrm{LP}} \neg a$, $a \in K$. This means that $a \wedge \neg a$ can be considered as a true contradiction. We show that

$$(\mathscr{C} \boxtimes_{\mathrm{LP}} \neg a) \boxminus^m_{\tau_0} a$$

produces a new epistemic state $\mathscr{C}'$ such that, for any $H \in \mathscr{C}'$, $\neg a \in H$ but $a \notin H$. Alternatively, we show that $(\mathscr{C} \boxtimes_{\mathrm{LP}} \neg a) \boxminus^m_{\tau_0} a$ is an epistemic process that accepts $\neg a$ and rejects a (and so $a \wedge \neg a$) successfully.

We first note that

$$\mathscr{C}' = (\mathscr{C} \boxtimes_{\mathrm{LP}} \neg a) \boxminus^m_{\tau_0} a = \bigcup \{K \ominus a : K \in F \otimes_{\mathrm{LP}} \neg a\}.$$

Consider $K \in \mathscr{C}'$, then there exists $H \in F \otimes_{\mathrm{LP}} \neg a$ such that $K \in H \ominus a$. By Theorem 33, $\neg a \in H$. Moreover, by Remark 14, $a \notin K$. Now, let us suppose for contradiction that $\neg a \notin K$. By Definition 13, this entails that $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(K, \neg a)$. Therefore, (*i*) $\neg a \leq a$, but this is impossible, since, otherwise, $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$, or (*ii*) there exist $x_1 \dots x_n \in K$ such that $x_1 \wedge \cdots \wedge x_n \wedge \neg a \leq a$. Hence, setting $c = x_1 \wedge \cdots \wedge x_n$, one has $a = a \vee (c \wedge \neg a) = (a \vee c) \wedge (a \vee \neg a) \in K$. Hence, $a \in K$, a contradiction. We conclude that $\neg a \in K$.

5 On the Levi and Harper Identities

In the classical AGM framework, Levi and Harper's identities capture mutual interdefinability relationships between basic belief change operators. In particular, due to Levi's identity, any (single-output) revision operator $\otimes$ can be mimicked by subsequent applications of a suitable contraction $\ominus$ and expansion as follows, for any $\varphi \in Fm_{\mathrm{CL}}$ and $F \in \mathcal{F}i_{\mathrm{CL}}^{\mathbf{Fm}_{\mathrm{CL}}}$:

$$F \otimes \varphi = (F \ominus \neg\varphi) \oplus \varphi. \tag{L}$$

Recall that, throughout the present section, the symbol $\ominus$ does not denote the maxichoice contraction based on the absolutely skeptic preference function, but an arbitrary contraction. Moreover, any operator defined by (L) with respect to a contraction function $\ominus$ happens to be indeed a revision operator (cf. Gärdenfors, 1988, Theorem 3.2, Theorem 3.3). Hence, due to (L), belief revision can always serve as a secondary notion constructed via the primitive operations of belief expansion and belief contraction (cf. Nayak et al., 2006). This fact has suggested the following *Decomposition principle* due to Fuhrmann Fuhrmann (1989) and Hansson (2003).

> Every legitimate belief change is decomposable into a sequence of contractions and expansions.

Similarly, Harper's identity guarantees that any (single-output) contraction operation $\ominus$ can be constructed using a suitable revision $\otimes$ as follows, for any $\varphi \in Fm_{\mathrm{CL}}$ and $F \in \mathcal{F}i_{\mathrm{CL}}^{\mathbf{Fm}_{\mathrm{CL}}}$:

$$F \ominus \varphi = (F \otimes \neg\varphi) \cap F. \tag{H}$$

Moreover, any operation defined by (H) with respect to a revision function $\otimes$ is indeed a contraction (cf. Gärdenfors, 1988, Theorems 3.4, 3.5). Now, in our context, given the algebraic and multiple-output nature of our belief change operators, (L) and (H) can be rephrased, for any reduced basic full g-model $\langle \mathbf{A}, \mathcal{F}i_{\mathrm{CL}}^{\mathbf{A}} \rangle$, $F \in \mathcal{F}i_{\mathrm{CL}}^{\mathbf{A}}$ and $a \in A$, as

$$F \otimes a = \{K \oplus a : K \in F \ominus \neg a\}$$

and

$$F \ominus a = \{K \cap F : K \in F \otimes \neg a\}.$$

We will refer to the latter versions of Levi and Harper's identities by (L) and (H) as well.

First, we observe that due to our construction of $\otimes_{\mathrm{LP}}$, we could express the connection between contraction and revision provided by (L) for classical logic at a greater general level. Indeed, the next proposition shows explicitly that our account of revision just replaces consistency preservation by avoiding triviality. Alternatively, in our paraconsistent setting, the revision of an LP-filter F by a new belief a comprises: 1) removing from F any information which is "incompatible" with our new evidence, 2) adding a to obtain a set of non-trivial theories containing a.

Proposition 40 *Let* $\mathbf{A}$ *be a Kleene lattice. Then for any* $a \in A$ *and* $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$, *the following hold*:

(i) $F \otimes_{\mathrm{LP}} a = \{G \oplus a : G \in \mathcal{S}_a^+\}$;
(ii) $\mathcal{S}_a^+ = \{G \cap F : G \in F \otimes_{\mathrm{LP}} a\}$.

Proof

(i) trivially follows by the definition of $\otimes_{\mathrm{LP}}$.
(ii) If $a = 0$ or $F_a^\perp = \emptyset$, then $F \otimes_{\mathrm{LP}} a = \{F \oplus a\}$ and so $\mathcal{S}_a^+ = \{F\} = \{(F \oplus a) \cap F\}$. Otherwise, suppose that $a \neq 0$ and $F_a^\perp \neq \emptyset$. If $K \in \mathcal{S}_a^+$, then clearly $K \subseteq G \cap F$, for some $G \in F \otimes_{\mathrm{LP}} a$, by the definition of $\otimes_{\mathrm{LP}}$. Moreover, suppose for contradiction that there exists $c \in (G \cap F) \smallsetminus K$. This means that $a_i \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(K, c)$, for some $a_i \in F_a^\perp$ and so $G = A$, i.e. $a = 0$, by Theorem 33 ($\otimes$5), a contradiction. We conclude that $K = G \cap F$ and so $\mathcal{S}_a^+ \subseteq \{G \cap F : G \in F \otimes_{\mathrm{LP}} a\}$. Conversely, consider $G \cap F$, for some $G = \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(K, a) \in F \otimes_{\mathrm{LP}} a$ with $K \in \mathcal{S}_a^+$. Then reasoning as above we have $G \cap F = K$ and so $\mathcal{S}_a^+ = \{G \cap F : G \in F \otimes_{\mathrm{LP}} a\}$.

□

Therefore, if the core meaning of (L) is to add new beliefs avoiding triviality, then it still allows the definability of a suitable (at least for the AGM account) revision operator since it does not rely on any particular feature of classical logic. Indeed, although its formulation in our context depends on algebraic properties of Kleene lattices, a closer look shows that the notion of trivialiser is independent both from the language and algebraic semantics of logics it is conceived for. Hence, we argue that a reformulation of (L) that perfectly fits with an AGM-friendly paraconsistent revision operator (if adequately generalised up to arbitrary vocabularies starting from Definition 22) should be the following:

$$F \otimes a = (F \ominus F_a^\perp)^s \oplus a, \tag{PL}$$

where $F \in \mathcal{F}i_{\vdash}^{\mathbf{A}}$, $a \in A$, $F_a^\perp$ is the set of b's in F such that $\mathrm{F}_{g\,\vdash}^{\mathbf{A}}(b, a) = A$ and $(F \ominus F_a^\perp)^s$ is a set of subfilters of F resulting from removing trivialisers of a from F by a suitable abstract algebraic contraction $\ominus$ (see Definition 9). Here we do not mean that (PL) ensures the definability of a revision operator satisfying our postulates for *any* contraction operator $\ominus$ (since e.g. $(F \ominus F_a^\perp)^s$ might be empty). However, nothing prevents us from arguing that any suitable revision should encode, in some respects, (PL) as its key ingredient. We leave the verification of the above conjecture and its eventual full development to future studies.

As stated above, (L) can be easily generalised to fully cope with a paraconsistent setting. However, the same cannot be argued for (H). Revising a filter F by $\neg a$ need not be sufficient for getting rid of a, since the latter might not contradict the former. Indeed, as shown by Proposition 40, $F \ominus a$ need not be fully recoverable using $\otimes_{\mathrm{LP}}$, since $\mathcal{S}_{\neg a}^+$ might not coincide with $F \ominus_{\tau_0}^m a$. We conclude that, while the regulative ideal underlying (L) transcends classical logic, (H) strongly relies on one of its pillars: the non-contradiction principle. So, it cannot be considered as an identity which, in general, make sense within a paraconsistent setting.

The remaining part of this section is devoted to motivating why we believe that (L) and (H), formulated as they are, fit well within belief revision theory only if its underlying logic is, in some sense, classical. We will outline to what extent (L) and (H) can still be considered reasonable defining identities for revision and contraction, respectively, in the LP setting. Let us summarise the basic achievements to be found below.

Concerning Levi's identity, we will show that, under some conditions (see Lemma 43), any output given by (L) extends to a solution of $\otimes_{\mathrm{LP}}$. However, this need not hold in general (cf. Remark 41). Consequently, whenever $\mathbf{A}$ is such that there exists an element a and an LP-filter F such that $a \leq \neg a$ but $F_a^{\perp} \neq \emptyset$, outputs of (L) cannot be extended to solutions of a suitable revision operator $\otimes$ over $\langle \mathbf{A}, \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}} \rangle$, by Remark 41 and Proposition 38. Furthermore, in the (not at all rare) case in which one is willing to revise a belief set F with an element a such that $\neg a \in F \smallsetminus \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(\emptyset)$, the definability of $\otimes_{\mathrm{LP}}$ by means of $\ominus_{\tau_0}^{m}$ conveys the very strong requirement that $a \wedge \neg a = 0$, i.e a is Boolean[3], by Theorem 48. Therefore, our framework does justice to, and (formally) clarifies the reasoning behind the common insight according to which consistency preservation cannot be considered, in general, as the core concept of belief revision, when a paraconsistent (here LP) framework is considered. Indeed, it can be regarded as a leading idea in prototyping a revision operator satisfying generalised AGM postulates, and so, the informational economy principle they encode, on the heavy constraint that the underlying logic has a classical negation. Therefore, from our perspective, Levi's identity, as it is, should be mostly considered as an accident rather than the substance of revision processes.

A similar argument (*mutatis mutandis*) applies to (H). Although one can find sufficient conditions under which, for a given filter F and element a, outputs of $F \ominus_{\tau_0}^{m} a$ can be extended to solutions of $\mathcal{S}_{\neg a}^{+} = \{H \cap F : F \otimes_{\mathrm{LP}} \neg a\}$ (by Proposition 40 and Lemma 43 below), this need not hold in general. As for (L), the possibility of defining the maxichoice contraction $\ominus_{\tau_0}^{m}$ by (H) when $a \in F \smallsetminus \mathrm{F}_{g_{\mathrm{LP}}}^{\mathbf{A}}(\emptyset)$, conveys the very strong requirement that $a \wedge \neg a = 0$ holds, by Theorem 45.

Considering the above discussion, next we investigate the relationship between basic AGM operators introduced so far to highlight sufficient and necessary conditions under which (L) and (H) grant the *interdefinability* of $\ominus_{\tau_0}^{m}$ and $\otimes_{\mathrm{LP}}$.

Consider $(\mathbf{A}, \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}) \in \mathsf{FGMod}^{*}(\vdash_{\mathrm{LP}})$, $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in A$. Let us denote by $(F \ominus_{\tau_0}^{m} \neg a) \oplus a$ resp. $(F \otimes_{\mathrm{LP}} a) \cap F$ the sets $\{\mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(H, a) : H \in F \ominus_{\tau_0}^{m} \neg a\}$ and $\{H \cap F : H \in F \otimes_{\mathrm{LP}} a\}$, respectively.

Remark 41 In general, (L) does not yield, using $\ominus_{\tau_0}^{m}$, a reliable revision operator for LP. Indeed, suppose $\mathbf{A} \in \mathcal{KL}$ is nontrivial and assume $\neg a \geq a \neq 0$ for some $a \notin F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ such that $F_a^{\perp} \neq \emptyset$. By the characterization of LP filters, $\neg a = \neg a \vee a \in \mathrm{Fg}_{\mathrm{LP}}^{\mathbf{A}}(\emptyset)$. Therefore, $F \ominus_{\tau_0}^{m} \neg a = \{F\}$. This, together with $F_a^{\perp} \neq \emptyset$ entails $(F \ominus_{\tau_0}^{m} \neg a) \oplus a = \{A\}$, showing that Levi's identity can induce trivial results even when starting from a nontrivial setting. A concrete example of this situation

[3] Indeed, it is well known that, given a Kleene lattice $\mathbf{A}$, the set $S \subseteq A$ such that, for any $a \in S$, $a \wedge \neg a = 0$, forms a Boolean subalgebra of $\mathbf{A}$ (the *center* of $\mathbf{A}$), see e.g. Giuntini et al. (2016).

can be obtained, e.g. from the 25-elements Kleene lattice from Remark 15, by letting $F = \uparrow b \wedge n$, $a = \neg b \wedge n$, $\neg a = n \vee b$.

In the remaining part of this section, we fix $\langle \mathbf{A}, \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}} \rangle \in \mathsf{FGMod}^*(\vdash_{\mathrm{LP}})$. The next easy lemma will be expedient for developing our arguments.

Lemma 42 *For any $a \in A$ and $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$, if $a \nleq \neg a$ and $\neg a \in F$, then, for any $H \in F \ominus^m_{\tau_0} \neg a$, $\neg a \notin \mathrm{Fg}^{\mathbf{A}}_{\vdash}(H, a)$.*

Proof If $a \nleq \neg a$, then $\neg a \neq b \vee \neg b$, for any $b \in A$. Therefore, $F \ominus^m_{\tau_0} \neg a \neq \{F\}$. Now, consider $G \in F \ominus^m_{\tau_0} \neg a$. Obviously, $\neg a \notin G$. If $\neg a \in \mathrm{Fg}^{\mathbf{A}}_{\vdash}(G, a)$, then there exist $x_1, \dots, x_n \in G$ $(n \geq 1)$ such that $x_1 \wedge \cdots \wedge x_n \wedge a \leq \neg a$. Set $\bigwedge_{i=1}^n x_i = c$. One has $c \wedge a \leq \neg a$. So $(c \wedge a) \vee \neg a = \neg a$. However, this entails that $\neg a = (c \vee \neg a) \wedge (a \vee \neg a)$ and, since $(c \vee \neg a), (a \vee \neg a) \in G$, one has $\neg a \in G$. A contradiction. □

The following result summarises some preliminary basic facts concerning the interdefinability of $\otimes_{\mathrm{LP}}$ and $\ominus^m_{\tau_0}$ by (L) and (H). As it will be clear, apart from "trivial" cases, if $\neg a$ resp. a is not a tautology, then any solution of $(F \ominus^m_{\tau_0} \neg a) \oplus a$ resp. $F \ominus^m_{\tau_0} a$ can be extended to a suitable output of $F \otimes_{\mathrm{LP}} a$ resp. $(F \otimes_{\mathrm{LP}} \neg a) \cap F$.

Lemma 43 *Let $a \in A$ and $F \in \mathcal{F}i^{A}_{\mathrm{LP}}$. The following hold:*

(i) *If $a \nleq \neg a$ then, for any $H \in (F \ominus^m_{\tau_0} \neg a) \oplus a$, there exists $H' \in F \otimes_{\mathrm{LP}} a$ such that $H \subseteq H'$.*
(ii) *If $\neg a \notin F$ or $\neg a = 1$, then $(F \ominus^m_{\tau_0} \neg a) \oplus a = F \otimes_{\mathrm{LP}} a$*
(iii) *If $\neg a \nleq a$, then for any $H \in F \ominus^m_{\tau_0} a$, there exists $D \in (F \otimes_{\mathrm{LP}} \neg a) \cap F$ such that $H \subseteq D$.*
(iv) *If $a \notin F$ or $a = 1$, then $F \ominus^m_{\tau_0} a = (F \otimes_{\mathrm{LP}} \neg a) \cap F$.*

Proof

(i) First, since $a \nleq \neg a$, then $\neg a \neq b \vee \neg b$, for any $b \in A$. Now, if $F^{\perp}_a = \emptyset$, then our conclusion follows trivially since $F \otimes_{\mathrm{LP}} a = \{F \oplus a\}$, by ⊗4. Therefore, suppose that $F^{\perp}_a \neq \emptyset$ and so $\neg a \in F$ (by Lemma 20(ii)). Let G be an arbitrary element in $F \ominus^m_{\tau_0} \neg a$. Set $X = \{a \vee a_i : a_i \in F^{\perp}_a\}$. We show that $X \subseteq G$. If $a \vee a_i \notin G$, for some $a_i \in F^{\perp}_a$, then, since $G \in \mathrm{Max}(S_{\tau_0(F, \neg a)})$, one has $\neg a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(G, a \vee a_i)$. Hence, there exist $x_1, \dots, x_n \in G$ $(n \geq 1)$ such that $x_1 \wedge \cdots \wedge x_n \wedge (a \vee a_i) \leq \neg a$. Put $\bigwedge_{i=1}^n x_i = c$. Then $c \wedge (a \vee a_i) = (c \wedge a) \vee (c \wedge a_i) \leq \neg a$, namely $c \wedge a \leq \neg a$. Therefore, we reach a contradiction since, by Lemma 42, $\neg a \notin \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(G, a)$. We conclude that $X \subseteq G$. Moreover, by Lemma 20(ii), $F^{\perp}_a \cap G = \emptyset$. Now, consider an arbitrary $a_i \in F^{\perp}_a$ and extend G to $H \in F \ominus^m_{\tau_0} a_i$. By Lemma 30, $H \in \mathcal{S}^{+}_a$ and so $\mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(G, a) \subseteq \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(H, a) \in F \otimes_{\mathrm{LP}} a$.

(ii) If $\neg a \notin F$, then $F_a^{\perp} = \emptyset$, by Lemma 21. Therefore $F \ominus_{\tau_0}^m F_a^{\perp} = F \ominus_{\tau_0}^m \neg a = \{F\}$ and so $(F \ominus_{\tau_0}^m \neg a) \oplus a = \{F \oplus a\} = F \otimes_{\mathrm{LP}} a$ (by $\otimes$4). Furthermore, if $\neg a = 1$, then $a = 0$ and so $(F \ominus_{\tau_0}^m \neg a) \oplus a = \{A\} = F \otimes_{\mathrm{LP}} a$.

(iii) The case $F_{\neg a}^{\perp} = \emptyset$ is clear, since this implies that $F \otimes_{\mathrm{LP}} \neg a = \{F \oplus \neg a\}$ and so $H \subseteq F = F \cap (F \oplus a)$, for any $H \in F \ominus_{\tau_0}^m a$. So, we assume w.l.o.g. that $F_{\neg a}^{\perp} \neq \emptyset$. Suppose that $H \in F \ominus_{\tau_0}^m a$. Note that, since $\neg a \nleq a$, $a \notin \mathrm{F}_{g\mathrm{LP}}^{\mathbf{A}}(\emptyset)$. Therefore, $a \notin H$ and so $H \cap F_{\neg a}^{\perp} = \emptyset$, by Lemma 20(ii). Now, we show that $\{\neg a \vee a_i : a_i \in F_{\neg a}^{\perp}\} \subseteq H$. So, suppose for contradiction that this is not the case. Hence, there exists $a_j \in F_{\neg a}^{\perp}$ such that, for some $c \in H$, $c \wedge (\neg a \vee a_j) \leq a$. Then, by Lemma 20(ii), $(\neg a \vee a_j) \vee a = (a \vee a_j) \vee \neg a = a \vee \neg a \in H$ entails that

$$a \wedge c = ((\neg a \vee a_j) \wedge c) \vee (a \wedge c) = ((\neg a \vee a_j) \vee a) \wedge c \in H.$$

So, we have $a \in H$. However, since $H \in F \ominus_{\tau_0}^m a$, this means that $a \in \mathrm{F}_{g\mathrm{LP}}^{\mathbf{A}}(\emptyset)$ (by $\ominus$4). A contradiction. Therefore $\{\neg a \vee a_i : a_i \in F_{\neg a}^{\perp}\} \subseteq H$. Let us extend H to a closed subset $G \in F \ominus_{\tau_0}^m a_i$, for some $a_i \in F_{\neg a}^{\perp}$. By Lemma 30, $G \in \mathcal{S}_{\neg a}^{+}$. Hence, $H \subseteq D = \mathrm{F}_{g\mathrm{LP}}^{\mathbf{A}}(G, \neg a) \in F \otimes_{\mathrm{LP}} \neg a$ and so $H \subseteq F \cap D$.

(iv) Just note that, if $a \notin F$ or $a = 1$, then $F \ominus_{\tau_0}^m a = \{F\}$. Now, concerning the first case, it is easily seen that, by Lemma 20(ii), $F_{\neg a}^{\perp} = \emptyset$ and so $F \otimes_{\mathrm{LP}} \neg a = \{F \oplus \neg a\}$ and our conclusion easily follows. In the latter case, since $\neg a = 0$, then $F \otimes_{\mathrm{LP}} \neg a = \{A\}$ and so the desired result is obtained by noticing that $F = F \cap A$.

□

Lemma 44 *Suppose that $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$, $a \in A$, $a \wedge \neg a \neq 0$ and $F_a^{\perp} \neq \emptyset$. Then there exist $K \in \mathcal{S}_a^{+}$ (and so $G \in F \otimes_{\mathrm{LP}} a$) such that $\neg a \in K$.*

Proof If $F_a^{\perp} \neq \emptyset$, then $\neg a \in F$ (by Lemma 20) and it is easily seen that the closed subset $V = \mathrm{F}_{g\mathrm{LP}}^{\mathbf{A}}(\neg a, \{a \vee a_i : a_i \in F_a^{\perp}\}) \cap F_a^{\perp} = \emptyset$, otherwise (as in the proof of Lemma 27) there exist $a_k, a_i \in F_a^{\perp}$ such that $\neg a \wedge (a \vee a_i) \leq a_k$ and so $\neg a \wedge a = 0$, which is impossible. Let us extend V to some $H \in F \ominus F_a^{\perp}$. By Lemma 30, $a \in H \in \mathcal{S}_a^{+}$. □

The next theorem clarifies sufficient and necessary conditions under which $\ominus_{\tau_0}^m$ is definable using $\otimes_{\mathrm{LP}}$ and (H) provided that the proposition a one aims to reject is not a tautology. Indeed, this happens if and only if $a \wedge \neg a$ cannot be considered as a true contradiction.

Theorem 45 *Let $F \in \mathcal{F}i_{\mathrm{LP}}^{\mathbf{A}}$ and $a \in (F \smallsetminus \mathrm{F}_{g\mathrm{LP}}^{\mathbf{A}}(\emptyset)) \cup \{1\}$. The following are equivalent:*

(i) $F \ominus^m_{\tau_0} a = (F \otimes_{\mathrm{LP}} \neg a) \cap F$;
(ii) $a \wedge \neg a = 0$.

Proof We first consider the case $a = 0$ ($\neg a = 1$). If $F \neq A$, then $a \notin F$ and so $F \ominus^m_{\tau_0} 0 = \{F\} = F \otimes_{\mathrm{LP}} 1$ and we are done. Moreover, if $F = A$, by $1 \nleq 0$ and Lemma 43(iii), for any $H \in A \ominus^m_{\tau_0} 0$ there exists $D \in \{G \cap A : G \in A \otimes_{\mathrm{LP}} 1\} = A \otimes_{\mathrm{LP}} 1$ such that $H \subseteq D$. Since $0 \notin D$, by the maximality of H, one has $H = D$. Hence, $A \ominus^m_{\tau_0} 0 \subseteq A \otimes_{\mathrm{LP}} 1$. The converse inclusion can be proven similarly and so our statement is vacuously true. Furthermore, if $a = 1$, then our statement holds trivially by Lemma 43(iv). Therefore, in the rest of the proof, we can safely assume that $a \notin \{0\} \cup \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$.

$(i) \Rightarrow (ii)$. Let us reason by contraposition. Assume that $a \wedge \neg a \neq 0$. If $F^{\perp}_{\neg a} = \emptyset$, then $F \otimes_{\mathrm{LP}} \neg a = \{F \oplus a\}$. So $F \ominus^m_{\tau_0} a \neq \{F \cap (F \oplus a)\} = \{F\}$, otherwise $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$ (by $\ominus$4) which is impossible by hypothesis. Also, if $F^{\perp}_{\neg a} \neq \emptyset$, then by Lemma 44, there exists $H \in \mathcal{S}^{+}_{\neg a}$ such that $a \in H$. Since, by Proposition 40, $\{G \cap F : G \in F \otimes_{\mathrm{LP}} \neg a\} = \mathcal{S}^{+}_{\neg a}$, one must have that $F \ominus^m_{\tau_0} a \neq \mathcal{S}^{+}_{\neg a}$, otherwise $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$ (again by $\ominus$4), as desired.

$(ii) \Rightarrow (i)$. It directly follows from Lemma 29 and Proposition 40 □

Some remarks clarifying the importance of assumptions in the above theorem come next.

Remark 46 Observe that condition $a \in (F \setminus \mathrm{F}^{\mathbf{A}}_{g_{\mathrm{LP}}}(\emptyset)) \cup \{1\}$ cannot be dropped from assumptions of Theorem 45. In fact, if $a \notin F$, then item (1) of the above result trivially holds true even when $a \wedge \neg a \neq 0$, by Lemma 43(iv). Furthermore, note that condition $a \notin \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset) \setminus \{1\}$ cannot be eliminated as well. Indeed, taking the 25-elements Kleene lattice from Remark 15, and setting $F = \uparrow a \wedge \neg a$, $b = a \vee \neg a$, we have $F \ominus^m_{\tau_0} b = \{F\} = F \otimes_{\mathrm{LP}} \neg b = \{G \cap F : G \in F \otimes_{\mathrm{LP}} \neg a\} = (F \otimes_{\mathrm{LP}} \neg a) \cap F$ but $\neg b = b \wedge \neg b \neq 0$. However, even if $a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$, $(ii) \Rightarrow (i)$ of the above proposition still holds. In fact, $a \wedge \neg a = 0$ entails that $a = 1$ and the claim follows by Lemma 43(iv).

Remark 47 It is easy to see that assumptions of Thereom 45 cannot be applied to Levi's identity. It is indeed possible to verify that, in the figure of Remark 15, the filter $\uparrow a \wedge n$ and $\neg b \wedge n$ satisfy all the assumptions of the proposition,

$$(\uparrow a \wedge n \ominus^m_{\tau_0} b \vee n) \oplus \neg b \wedge n = \uparrow a \wedge n \otimes_{\mathrm{LP}} \neg b \wedge n$$

but

$$(\neg b \wedge n) \wedge (b \vee n) \neq 0.$$

The counterexample provided by the above remark strongly relies on the fact that in $\mathbf{A}$ one might have, for some $a \in A$, $\neg a \in \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)$ but $a \wedge \neg a \neq 0$. Therefore, to prove an analogous of Theorem 45 for (L), some specific assumptions are needed.

Theorem 48 *Let $F \in \mathcal{F}i^{\mathbf{A}}_{\mathrm{LP}}$ and $a \in A$ with $\neg a \in (F \smallsetminus \mathrm{Fg}^{\mathbf{A}}_{\mathrm{LP}}(\emptyset)) \cup \{1\}$. The following are equivalent*

(i) $(F \ominus^{m}_{\tau_0} \neg a) \oplus a = F \otimes_{\mathrm{LP}} a$;
(ii) $a \wedge \neg a = 0$.

Proof First, let us note that if $\neg a = 1$, then our statement is trivially true by Lemma 43(ii). Furthermore, we observe that if $F^{\perp}_{a} = \emptyset$, then $F \otimes_{\mathrm{LP}} a = \{F \oplus a\}$. However, by Lemma 42, for any $H \in F \ominus^{m}_{\tau_0} \neg a$, $H \oplus a \neq F \oplus a$. We conclude that (i) does not hold. Moreover, since $\neg a \in F$ and $F^{\perp}_{a} = \emptyset$, one must have $a \wedge \neg a \neq 0$. Hence, (ii) does not hold as well. So, also in this case, our statement trivially holds true.

In light of the above arguments, in the rest of the proof, we can assume that $\neg a \neq 1$ and $F^{\perp}_{a} \neq \emptyset$.

$(i) \Rightarrow (ii)$. Suppose, by contraposition, that $a \wedge \neg a \neq 0$. By Lemma 44, there exists $G \in F \otimes_{\mathrm{LP}} a$ such that $\neg a \in G$. However, since $a \nleq \neg a$ and $\neg a \in F$, for any $K \in (F \ominus^{m}_{\tau_0} \neg a) \oplus a$, $\neg a \notin K$ (Lemma 42). We conclude that $(F \ominus^{m}_{\tau_0} \neg a) \oplus a \neq F \otimes_{\mathrm{LP}} a$, as desired.

$(ii) \Rightarrow (i)$. It follows from Lemma 29 and Proposition 40. □

6 Conclusions

In this paper, we developed a logico-algebraic analysis of AGM, a tool able to provide a unique framework for studying contraction in non-classical logics. Specifically, we showed how this framework can be applied to the Logic of Paradox (LP). We have given an answer to the following question: How is a paraconsistent account of belief revision compatible with the AGM perspective? We answered arguing that, once the AGM framework is appropriately generalised, there is room for treating a paraconsistent revision operator. Specifically, once the classical interpretation of *negation* was replaced with a *trivialiser* (Definition 17), we showed that a large part of AGM perfectly fits with a paraconsistent account of revision.

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References

Albuquerque, H., Přenosil, A., & Rivieccio, U. (2017). An algebraic view of super-Belnap logics. *Studia Logica, 105*(6), 1051–1086.
Alchourrón, C. E., Gärdenfors, P., & Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *The Journal of Symbolic logic, 50*(2), 510–530.
Alchourrón, C., & Makinson, D. (1982). On the logic of theory change: Contraction functions and their associated revision functions. *Theoria, 48*(1), 14–37.
Bueno, O., & da Costa, N. (1998). Belief change and inconsistency. *Logique et analyse*, 50(2), 41.161/163, 31–56.
Burris, S., & Sankappanavar, H. P. (1981). *A course in universal algebra*. Springer.
Czelakowski, J. (1984). Filter distributive logics. *Studia Logica, 43*, 353–377.
Czelakowski, J. (2001). *Protoalgebraic logics*. Dordrecht: Springer.
Fazio, D., & Baldi, M. P. (2021). On a logico-algebraic approach to AGM-belief contraction theory. *Journal of Philosophical Logic*, 50(5), 911–938.
Fermé, S. E., & Hansson, D.O. (2018). *Belief change: Introduction and overview*. Springer.
Font, J. (2016). *Abstract algebraic logic: An introductory textbook*. College Publications.
Fuhrmann, A. (1989). Reflective modalities and theory change. *Synthese, 81*, 115–134.
Giuntini, R., Ledda, A., & Paoli, F. (2016). A new view of effects in a Hilbert space. *Studia Logica, 104*, 1145–1177.
Gärdenfors, P. (1988). *Knowledge in flux: Modeling the dynamics of epistemic states.* The MIT Press.
Gärdenfors, P. (1992). Belief revision: An introduction. In P. Gärdenfors (Ed.), *Belief revision*, Cambridge tracts in theoretical computer science (pp. 1–28). Cambridge University Press.
Hansson, S. O. (2003). Ten philosophical problems in belief revision. *Journal of Logic and Computation, 13*(1), 37–49.
Lévy, F. (1994). A survey of belief revision and updating in classical logic. *International Journal of Intelligent Systems, 9*(1), 29–59.
Mares, E. D. (2002). A paraconsistent theory of belief revision. *Erkenntnis, 56*, 229–246.
Nayak, A., Goebel, R., Orgun, M., & Pham, T. (2006). Taking Levi Identity Seriously: A Plea for Iterated Belief Contraction in J. Lang, F. Lin, and J. Wang (Eds.), *Knowledge Science, Engineering and Management,* Proceedings of the First International Conference, KSEM 2006, Guilin, China, August 5-8, 2006, LNCS 4092, 305–317, Springer-Verlag.
Priest, G. (2001). Paraconsistent belief revision. *Theoria, 67*, 214–228.
Pynko, A. (2000). Subprevarieties versus extensions. Applications to the logic of paradox. *Journal of Symbolic Logic, 65*(2), 756–766.
Rott, H. (1992). Preferential belief change using generalised epistemic entrenchment. *Journal of Logic, Language and Information, 1*(1), 45–78.
Testa, R. R., Coniglio, M. E., & Ribeiro, M. M. (2017). AGM-like paraconsistent belief change. *Logic Journal of the IGPL, 25*(4), 632–672.