



On Equivalence Relations Between Interpreted Languages, with an Application to Modal and First-Order Language

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Received: 24 December 2019 / Accepted: 6 July 2020 / Published online: 20 March 2021
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Abstract

I examine notions of equivalence between logics (understood as languages interpreted model-theoretically) and develop two new ones that invoke not only the algebraic but also the string-theoretic structure of the underlying language. As an application, I show how to construe modal operator languages as what might be called typographical notational variants of *bona fide* first-order languages.

1 Introduction

The present essay has two principal parts. The first, consisting of Sect. 2, endeavors to contribute to the study of ways in which two logics can be equivalent. In the second (Sect. 3), I define novel first-order systems that are equivalent, in the strongest non-trivial sense identified in Sect. 2, to various modal operator languages.

In the recent literature on translational equivalence and notational variance, a *logic* has typically been understood as a pair consisting of a language and a consequence relation, where consequence may be characterized in an abstract fashion or defined either proof-theoretically or model-theoretically. Authors who have provided systematic discussions of ways in which two logics, so conceived, may be equivalent include Wojcicki (1988), Epstein (1990), Pelletier and Urquhart (2003), Caleiro and Gonçalves (2007), Kocurek (2017) and French (2019).

By contrast, I here construe logics as *interpreted languages*, which are taken to be pairs consisting of a set L of expressions and a function $\lambda\mathfrak{M}.\lambda\phi.\llbracket\phi\rrbracket_{\mathfrak{M}}$ mapping models \mathfrak{M} to functions $\lambda\phi.\llbracket\phi\rrbracket_{\mathfrak{M}}$ that assign semantic values to expressions relative

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to models.¹ This is very roughly the notion of a logic common in the abstract model theory literature since at least Lindström (1969). As Ebbinghaus (1985, 28) points out, Feferman's (1974) term *model-theoretic language* seems more appropriate for the concept than *logic*, and my terminology of "interpreted languages" is intended to reflect this view.

Generally speaking, two interpreted languages are considered "equivalent" just in case there are maps between their sets of expressions that preserve semantic values relative to each model (and perhaps satisfy certain other constraints). Thus where consequence relations are defined model-theoretically, equivalence results in our sense will typically yield equivalence results in the consequence-focused sense as well, with our analysis being more fine-grained since model-theoretic consequence generalizes over all models, while we examine meaning-preservation in a pointwise, model-by-model fashion.

That said, the model-focused approach has its drawbacks, too; most significantly, it is inapplicable to language pairs that do not share a common model space (such as, say, classical and intuitionistic propositional logic). Nevertheless, from the point of view of truth-conditional semantics, if the truth conditions of a sentence are cashed out as the class of models in which it is true, notions of equivalence for interpreted languages in our sense are clearly more immediately relevant than those for languages endowed with consequence relations.

To a large extent, however, the actual innovations proposed in this paper are independent of one's construal of a logic, since they simply consist in two constraints, which give rise to what I call *word-word variance* and *character-word variance*, that maps between languages may or may not satisfy and which can thus be imposed regardless of the approach one follows.

A perhaps interesting feature of these novel equivalence relations between interpreted languages is that they marry two distinct construals of languages. These might be called the free algebraic (or Polish) view of languages as free algebras generated from a set of atoms by various operations (such as the forming of negations, conjunctions, necessitations, etc.) on the one hand, and the semigroup or formal-language theoretic view of languages as sets of strings over an alphabet endowed with the operation of concatenation, on the other.² Since the usual logical languages are plausibly regarded as exhibiting both kinds of structure, it seems apposite to consider structural aspects of both sorts in translating between them.³

As an application of our classification of translational equivalence relations in Sect. 2, I show that the basic propositional modal language is a character-word variant of a *bona fide* quantificational language, relative to models with *arbitrary* (i.e. not necessarily universal) accessibility relations. Thus, for example, the basic modal logic **K** turns out to be a notational variant, in a strong sense, of a certain first-order logic. The novel universal quantifier required for this result provides an exact first-order

¹ The way I use the term "interpreted language" (basically: formal language with a model theory) thus differs from another popular usage, where it is assumed that every lexical item of the language in question has one fixed interpretation in a single model.

² See Humberstone (2011, 100) for additional discussion.

³ For the purposes of this paper, we will assume that the relevant algebras are one-sorted and total. A more general approach would allow the algebras to either be many-sorted or partial.

analog of the box operator in modal logic. I further show how to generalize this result to polymodal and quantified modal languages, as well as modal languages containing the actuality operator.

2 Equivalence Relations Between Interpreted Languages

Given an alphabet A , we let A^* be the set of all finite strings over A , including the empty string ϵ . The set A^+ is $A^* \setminus \{\epsilon\}$, i.e. the set of all non-empty finite strings over A . Where $n \geq 1$ and $a_1, \dots, a_n \in A$, we write $a_1 \dots a_n$ for the string whose i -th entry is a_i , for $1 \leq i \leq n$. For $\sigma_1, \dots, \sigma_n \in A^*$, we let $\sigma_1 \frown \dots \frown \sigma_n$ be the concatenation of $\sigma_1, \dots, \sigma_n$, in that order. A language over the alphabet A is a non-empty subset of A^+ ; a language is a language over some alphabet. We typically refer to the members of a language L as the L -formulas.

If L is a language, a semantics for L is a family $\mathcal{S} = (\llbracket \cdot \rrbracket_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}}$ of functions $\llbracket \cdot \rrbracket_{\mathfrak{M}}$ with some (not necessarily proper) superset of L as their common domain, where \mathbf{M} is a non-empty class whose members are called models for L . For $\phi \in L$ and $\mathfrak{M} \in \mathbf{M}$, $\llbracket \phi \rrbracket_{\mathfrak{M}}$ is the semantic value of ϕ relative to \mathfrak{M} (according to \mathcal{S}). An interpreted language is a pair (L, \mathcal{S}) consisting of a language L and a semantics \mathcal{S} for L .

Definition 1 (Translation) For $i \in \{0, 1\}$, let L_i be a language, and let $\mathcal{S}_i = (\llbracket \cdot \rrbracket_{\mathfrak{M}}^i)_{\mathfrak{M} \in \mathbf{M}}$ be a semantics for L_i (note that both semantics must be based on the same model class \mathbf{M}). A function $t : L_0 \rightarrow L_1$ is a translation from the interpreted language (L_0, \mathcal{S}_0) to the interpreted language (L_1, \mathcal{S}_1) if, for every $\phi \in L_0$ and every $\mathfrak{M} \in \mathbf{M}$, $\llbracket \phi \rrbracket_{\mathfrak{M}}^0 = \llbracket t(\phi) \rrbracket_{\mathfrak{M}}^1$. □

When it is obvious from context what the respective semantics are, we sometimes speak simply of a translation from L_0 to L_1 .⁴

One obvious notion of equivalence between interpreted languages is mutual translatability, or bi-translatability as we shall call it.

Definition 2 (Bi-translation; bi-translatability) When for $i \in \{0, 1\}$, t_i is a translation from $(L_i, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^i)_{\mathfrak{M} \in \mathbf{M}})$ into $(L_{1-i}, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^{1-i})_{\mathfrak{M} \in \mathbf{M}})$, the pair (t_0, t_1) is a bi-translation from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$. $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are bi-translatable if there exists a bi-translation from the former to the latter. □

Remark 1 If (t_0, t_1) is a bi-translation from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$, it follows that $\llbracket t_1(t_0(\phi)) \rrbracket_{\mathfrak{M}}^0 = \llbracket \phi \rrbracket_{\mathfrak{M}}^0$ and $\llbracket t_0(t_1(\psi)) \rrbracket_{\mathfrak{M}}^1 = \llbracket \psi \rrbracket_{\mathfrak{M}}^1$ for all $\phi \in L_0$, $\psi \in L_1$, and $\mathfrak{M} \in \mathbf{M}$. □

Example 1 Let 1QL be the language of monadic predicate logic, whose syntax is given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall x\phi$, where p ranges over a countably infinite set S (whose members we will in this example construe as one-place predicate symbols)

⁴ In the abstract model theory literature mentioned in the Introduction, the existence of a translation from L_0 to L_1 , given the axiom of choice, amounts to L_1 being at least as strong as L_0 (often written $L_0 \leq L_1$). Similarly, bi-translatability, defined below, corresponds to L_0 and L_1 being equally strong, $L_0 \equiv L_1$.

and x ranges over the countably infinite set $\text{Var} = \{v_0, v_1, \dots, v_i, \dots\}$ of individual variables.

Let δ and Δ be fresh symbols, and let the languages 1QL_δ and 1QL_Δ be given by $\phi ::= px \mid p\delta \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall x\phi$ and $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall x\phi \mid \Delta x\phi$. The new symbols δ and Δ thus present syntactically as an individual constant and a quantifier, respectively. The model class \mathbf{M} consists of structures $\mathfrak{M} = (M, d^\mathfrak{M}, (p^\mathfrak{M})_{p \in S})$ where $d^\mathfrak{M} \in M$ and $p^\mathfrak{M} \subseteq M$ for each $p \in S$.

Let \mathfrak{M} be such a model. A variable assignment in \mathfrak{M} is a function $g : \text{Var} \rightarrow M$, and for $x \in \text{Var}$ and $m \in M$, $g[x := m]$ is the variable assignment in \mathfrak{M} that agrees with g on $\text{Var} \setminus \{x\}$ and maps x to m . For 1QL_δ -formulas ϕ and 1QL_Δ -formulas ψ , we define the local values $\|\phi\|_{\mathfrak{M}}^g, \langle\psi\rangle_{\mathfrak{M}}^g \in \{0, 1\}$ relative to \mathfrak{M} and g in familiar Tarskian fashion as follows:

1. $\|px\|_{\mathfrak{M}}^g = 1$ iff $g(x) \in p^\mathfrak{M}$
2. $\|p\delta\|_{\mathfrak{M}}^g = 1$ iff $d^\mathfrak{M} \in p^\mathfrak{M}$
3. $\|\neg\theta\|_{\mathfrak{M}}^g = 1 - \|\theta\|_{\mathfrak{M}}^g$
4. $\|(\theta \rightarrow \chi)\|_{\mathfrak{M}}^g = \max\{1 - \|\theta\|_{\mathfrak{M}}^g, \|\chi\|_{\mathfrak{M}}^g\}$
5. $\|\forall x\theta\|_{\mathfrak{M}}^g = \min\{\|\theta\|_{\mathfrak{M}}^{g[x:=m]} \mid m \in M\}$
6. $\langle px \rangle_{\mathfrak{M}}^g = 1$ iff $g(x) \in p^\mathfrak{M}$
7. $\langle \Delta x\theta \rangle_{\mathfrak{M}}^g = \langle \theta \rangle_{\mathfrak{M}}^{g[x:=d^\mathfrak{M}]}$
8. $\langle \neg\theta \rangle_{\mathfrak{M}}^g = 1 - \langle \theta \rangle_{\mathfrak{M}}^g$
9. $\langle (\theta \rightarrow \chi) \rangle_{\mathfrak{M}}^g = \max\{1 - \langle \theta \rangle_{\mathfrak{M}}^g, \langle \chi \rangle_{\mathfrak{M}}^g\}$
10. $\langle \forall x\theta \rangle_{\mathfrak{M}}^g = \min\{\langle \theta \rangle_{\mathfrak{M}}^{g[x:=m]} \mid m \in M\}$

For models \mathfrak{M} and $\phi \in 1\text{QL}_\delta, \psi \in 1\text{QL}_\Delta$, let $\|\phi\|_{\mathfrak{M}} := \lambda g. \|\phi\|_{\mathfrak{M}}^g$ and $\langle\psi\rangle_{\mathfrak{M}} := \lambda g. \langle\psi\rangle_{\mathfrak{M}}^g$. Then the following recursively defined functions $\tau : 1\text{QL}_\delta \rightarrow 1\text{QL}_\Delta$ and $\sigma : 1\text{QL}_\Delta \rightarrow 1\text{QL}_\delta$ translate between the interpreted languages $(1\text{QL}_\delta, (\|\cdot\|_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$ and $(1\text{QL}_\Delta, (\langle\cdot\rangle_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$.⁵

- (τ 1) $\tau(px) = px$
- (τ 2) $\tau(p\delta) = \Delta v_0 p v_0$
- (τ 3) $\tau(\neg\theta) = \neg\tau(\theta)$
- (τ 4) $\tau((\theta \rightarrow \chi)) = (\tau(\theta) \rightarrow \tau(\chi))$
- (τ 5) $\tau(\forall x\theta) = \forall x\tau(\theta)$
- (σ 1) $\sigma(px) = px$
- (σ 2) $\sigma(\Delta x\theta) = \sigma(\theta)_x[\delta]$
- (σ 3) $\sigma(\neg\theta) = \neg\sigma(\theta)$
- (σ 4) $\sigma((\theta \rightarrow \chi)) = (\sigma(\theta) \rightarrow \sigma(\chi))$
- (σ 5) $\sigma(\forall x\theta) = \forall x\sigma(\theta)$

The interpreted languages $(1\text{QL}_\delta, (\|\cdot\|_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$ and $(1\text{QL}_\Delta, (\langle\cdot\rangle_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$ are thus bi-translatable. □

Bi-translatability is arguably the weakest relation of semantic equivalence between languages. We can obtain stricter notions by imposing constraints on the translations.

⁵ In clause (σ 2) below, $\sigma(\theta)_x[\delta]$ is the result of replacing, in $\sigma(\theta)$, every free occurrence of x with δ . Note that, in clause (τ 2), we pick one particular variable, v_0 .

Such constraints typically require an additional assumption about the languages at hand; to wit, that they are freely generated by a set of primitive syntactic operations from a set of atoms. This is the case for all the particular languages we will consider. Henceforth it will be assumed in our definitions that a language L_i is freely generated by the primitive operations in O_i from the atoms in S_i .

Definition 3 (*Compositionality; Atomic Faithfulness; Definitionality*) A translation t from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ into $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is *compositional* if, for every n -ary operation $\circ \in O_0$ there is an n -place L_1 -context⁶ $\psi^\circ(*_1, \dots, *_n)$, called the *L_1 -definition* of \circ , such that for all $\phi_1, \dots, \phi_n \in L_0$: $t(\circ(\phi_1, \dots, \phi_n)) = \psi^\circ(t(\phi_1), \dots, t(\phi_n))$.

A translation t from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ into $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is *atomically faithful* if it maps S_0 into S_1 ; it is *definitional* if it is compositional and atomically faithful.

A bi-translation (t_0, t_1) from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is *definitional* if both t_0 and t_1 are definitional. The interpreted languages $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *definitionaly equivalent* if there exists a definitional bi-translation between them. □

Example 2 For $\circ \in \{\wedge, \vee\}$, let $PL_{\neg, \circ}$ be the language of propositional logic over the set S (whose members we construe, for the purposes of this example, as sentence letters) with the syntax $\phi ::= p \mid \neg\phi \mid (\phi \circ \phi)$. Let $\delta : PL_{\neg, \wedge} \rightarrow PL_{\neg, \vee}$ rewrite propositional formulas in \neg and \wedge into formulas in \neg and \vee using the equivalence of $(\phi \wedge \psi)$ with $\neg(\neg\phi \vee \neg\psi)$. That is, let δ be defined recursively by the following equations:

- $\delta(p) = p$ when $p \in S$
- $\delta(\neg\phi) = \neg\delta(\phi)$
- $\delta((\phi \wedge \psi)) = \neg(\neg\delta(\phi) \vee \neg\delta(\psi))$

Let the model class be the set 2^S of all Boolean valuations $v : S \rightarrow \{0, 1\}$ and let $\llbracket \phi \rrbracket_v$ be the truth value of the propositional formula ϕ relative to the Boolean valuation v . Then δ is a translation from $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ to $(PL_{\neg, \vee}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$, since $\llbracket \phi \rrbracket_v = \llbracket \delta(\phi) \rrbracket_v$ for all $PL_{\neg, \wedge}$ -formulas ϕ and all valuations v . Obviously δ is atomically faithful, since its restriction to S is the identity function. Moreover, δ is compositional, the $PL_{\neg, \vee}$ -definition for negation being $\neg*_1$ and the $PL_{\neg, \vee}$ -definition for conjunction being $\neg(\neg*_1 \vee \neg*_2)$. So δ is a definitional translation.

Conversely, let $\gamma : PL_{\neg, \vee} \rightarrow PL_{\neg, \wedge}$ rewrite propositional formulas in \neg and \vee into formulas in \neg and \wedge using the equivalence of $(\phi \vee \psi)$ with $\neg(\neg\phi \wedge \neg\psi)$. That is, let γ be defined recursively by the following equations:

- $\gamma(p) = p$ for $p \in S$
- $\gamma(\neg\phi) = \neg\gamma(\phi)$
- $\gamma((\phi \vee \psi)) = \neg(\neg\gamma(\phi) \wedge \neg\gamma(\psi))$

Then γ is a translation from $(PL_{\neg, \vee}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ to $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$. It is obviously atomically faithful and compositional, the $PL_{\neg, \wedge}$ -definition for negation being $\neg*_1$

⁶ For our purposes, we can take an n -place L_1 -context to be a non-empty string θ over the alphabet $A_1 \cup \{*_1, \dots, *_n\}$ such that, for any $\phi_1, \dots, \phi_n \in L_1$, the result $\theta(\phi_1, \dots, \phi_n)$ of simultaneously substituting ϕ_i for $*_i$ in θ is an L_1 -formula. We sometimes suggestively write $\theta(*_1, \dots, *_n)$ for such n -place L_1 -contexts θ .

and the $PL_{\neg, \wedge}$ -definition for disjunction being $\neg(\neg *1 \wedge \neg *2)$. So γ is also a definitional translation, (δ, γ) is a definitional bi-translation, and the interpreted languages $(PL_{\neg, \vee}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ and $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ are definitionally equivalent. \square

Remark 2 Bi-translatability does not entail definitional equivalence: As we saw in Example 1, $1QL_\delta$ and $1QL_\Delta$ are bi-translatable. They are not, however, definitionally equivalent. Indeed, it can be shown that there is no atomically faithful translation from $1QL_\delta$ to $1QL_\Delta$. To see this, suppose that $t : 1QL_\delta \rightarrow 1QL_\Delta$ is an atomically faithful translation. By atomic faithfulness, $t(p_0\delta)$ must be an atom of $1QL_\Delta$, i.e. some formula of the form qy with $q \in S$ and $y \in \text{Var}$. Moreover, since t is a translation, we must have $\llbracket p_0\delta \rrbracket_{\mathfrak{M}}^g = \llbracket qy \rrbracket_{\mathfrak{M}}^g$ for all \mathfrak{M} and g . But then q must be p_0 , since otherwise there are models \mathfrak{M} with $p_0^{\mathfrak{M}} = M$ and $q^{\mathfrak{M}} = \emptyset$. So $t(p_0\delta) = p_0y$. Now consider any \mathfrak{M} with $\emptyset \neq p_0^{\mathfrak{M}} \neq M$. We must have $\llbracket p_0\delta \rrbracket_{\mathfrak{M}}^g = \llbracket p_0y \rrbracket_{\mathfrak{M}}^g$ for all g . However, $\llbracket p_0\delta \rrbracket_{\mathfrak{M}}^g$ is independent of g , while there are h and k such that $\llbracket p_0y \rrbracket_{\mathfrak{M}}^h \neq \llbracket p_0y \rrbracket_{\mathfrak{M}}^k$, contradiction.

It is natural to consider a strengthening of the compositionality constraint on translations that requires the target language definition of a primitive operation to be itself provided by a primitive operation:

Definition 4 (*Homomorphism; Bi-Homomorphism; Isomorphism*) An atomically faithful translation t from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ into $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is *homomorphic* if there exists a function $\circ \mapsto \circ^t$ from O_0 into O_1 such that, for any n -ary operation $\circ \in O_0$, the operation $\circ^t \in O_1$ is n -ary and satisfies the condition that for any ϕ_1, \dots, ϕ_n in L_0 , $t(\circ(\phi_1, \dots, \phi_n)) = \circ^t(t(\phi_1), \dots, t(\phi_n))$.

A *bi-homomorphism* from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is a pair (h_0, h_1) , where each h_i is a homomorphism from from $(L_i, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^i)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_{1-i}, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^{1-i})_{\mathfrak{M} \in \mathbf{M}})$.

The languages $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *homomorphically bi-translatable* if there exists a bi-homomorphism from the former to the latter.

A homomorphic translation t from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is an *isomorphic* translation if t is a bijection from L_0 onto L_1 . $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *isomorphic languages* just in case there is an isomorphic translation from the former to the latter. \square

Remark 3 Obviously any homomorphic translation is definitional, and hence homomorphically bi-translatable languages are definitionally equivalent. There are, however, definitional translations that are not homomorphisms, e.g. the functions δ and γ of Example 2. Indeed, $PL_{\neg, \wedge}$ and $PL_{\neg, \vee}$, while definitionally equivalent, are not homomorphically bi-translatable, since there are no homomorphic translations between $PL_{\neg, \wedge}$ and $PL_{\neg, \vee}$. For suppose $f : PL_{\neg, \wedge} \rightarrow PL_{\neg, \vee}$ were a homomorphic translation. Since it is then in particular an atomically faithful translation, it must map each sentence letter to itself.⁷ Now homomorphism requires that $f((p \wedge q)) = (f(p) \vee f(q))$ since disjunction is the only primitive binary operation available in $PL_{\neg, \vee}$. So

⁷ Atomic faithfulness requires that sentence letters be mapped to sentence letters; translation then requires that each sentence letter be mapped to itself.

$f((p \wedge q)) = (p \vee q)$, which contradicts f being a translation. The same argument shows that there are no homomorphic translations in the reverse direction. Thus homomorphic bi-translatability implies definitional equivalence but not conversely. \square

Remark 4 The inverse of an isomorphism is again an isomorphism; thus if f is an isomorphism, the pair (f, f^{-1}) is a bi-homomorphism. Therefore isomorphic interpreted languages are homomorphically bi-translatable. The converse fails, since the languages $PL_{\neg, \wedge}$ and $PL_{\neg, \wedge, \&}$, with \wedge and $\&$ given the same truth-tabular interpretation, are homomorphically bi-translatable but not isomorphic: Any homomorphism from $PL_{\neg, \wedge, \&}$ to $PL_{\neg, \wedge}$ must map both $(p \wedge q)$ and $(p \& q)$ to $(p \wedge q)$. \square

Remark 5 As the example in the previous remark shows, in order for t to be a homomorphism from L_0 to L_1 , the function $\circ \mapsto \circ^t$ from O_0 to O_1 is not required to be 1-1; nor is it required to be surjective: In the case of the homomorphism from $PL_{\neg, \wedge}$ to $PL_{\neg, \wedge, \&}$, surjectivity fails, and in the case of the homomorphism from $PL_{\neg, \wedge, \&}$ to $PL_{\neg, \wedge}$, injectivity fails. If, however, f is an isomorphic translation from L_0 to L_1 , the function $\circ \mapsto \circ^f$ from O_0 to O_1 is easily seen to be a bijection.

Example 3 Let $PPL_{\neg, \wedge}$ be the set of propositional formulas in \neg and \wedge over S written in Polish notation. That is, let the syntax of $PPL_{\neg, \wedge}$ be given by $\phi ::= p \mid \neg\phi \mid \wedge\phi\phi$. Then the function $\pi : PL_{\neg, \wedge} \rightarrow PPL_{\neg, \wedge}$ defined by the recursion equations

- $\pi(p) = p$
- $\pi(\neg\phi) = \neg\pi(\phi)$
- $\pi((\phi \wedge \psi)) = \wedge\pi(\phi)\pi(\psi)$

is an isomorphic translation from $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ to $(PPL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$, and hence these interpreted languages are isomorphic. \square

We next introduce a novel (as far as we're aware) equivalence relation that is stricter than isomorphism.

Definition 5 (*n-Word Basis; n-WW-Replacement; WW-Variance; WW-Variants*) Where L is a language over the alphabet A and $n \geq 1$, a subset Λ of A^+ is an *n-word basis for L over A* if every member of Λ is of a length $\leq n$, and for every $\phi \in L$ there is a unique finite sequence l_0, \dots, l_{n_ϕ} of elements of Λ such that ϕ is the concatenation $l_0 \frown \dots \frown l_{n_\phi}$ of l_0, \dots, l_{n_ϕ} . (So A , or more precisely, the set of length 1 words over A , is always a 1-word basis for L over A .)

For $i \in \{0, 1\}$, let L_i be a language over A_i . An *n-WW-replacement* from A_0 to A_1 relative to L_0 is a pair (Λ, r) , where Λ is an *n-word basis for L₀ over A₀* and $r : \Lambda \rightarrow A_1^*$ (so r replaces certain Words over A_0 by Words over A_1).

A function $f : L_0 \rightarrow L_1$ is *induced* by the *n-WW-replacement* (Λ, r) from A_0 to A_1 relative to L_0 if, whenever $\phi \in L_0$ and ϕ is the concatenation $l_0 \frown \dots \frown l_m$ of $l_0, \dots, l_m \in \Lambda$, $f(\phi)$ is $r(l_0) \frown \dots \frown r(l_m)$.

An isomorphism f from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is a *WW-variance* if there are m and n such that f is induced by an *m-WW-replacement* and its inverse f^{-1} is induced by an *n-WW-replacement* (equivalently, if there is an n such that f and f^{-1} are each induced by *n-WW-replacements*). The interpreted languages $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *WW-variants* if there exists a *WW-variance* from the former to the latter. \square

Remark 6 We require a finite bound n in the definition of WW-variance in order not to trivialize the definition: If strings of arbitrary lengths were allowed in the basis, any isomorphism f , which is trivially induced by (L, f) as an (unbounded) WW-replacement, would be a WW-variance. \square

The following example shows that isomorphism does not imply WW-variance.

Example 4 The interpreted languages $PL_{\neg, \wedge}$ and $PPL_{\neg, \wedge}$ aren't WW-variants (though they are isomorphic, as shown in Example 3). Indeed no homomorphic translation from $PPL_{\neg, \wedge}$ to $PL_{\neg, \wedge}$ is induced by an n -WW-replacement, for any n : Suppose that some n -WW-replacement (Λ, r) from $S \cup \{\neg, \wedge\}$ to $S \cup \{\neg, \wedge\} \cup \{(,)\}$ relative to $PPL_{\neg, \wedge}$ induces a homomorphic translation $f : PPL_{\neg, \wedge} \rightarrow PL_{\neg, \wedge}$. Clearly the only segmentation of an atomic formula p in terms of the n -word basis Λ is as p itself, so $S \subseteq \Lambda$ (more precisely, every singleton string over S is in Λ). Let ϕ_0 be p_0 and ϕ_{k+1} be $\wedge \phi_k p_{k+1}$. Then ϕ_n is $\wedge \dots \wedge p_0 \dots p_n$, with an initial string of n conjunction symbols. If the unique segmentation of ϕ_n in terms of Λ is $\sigma_0, \dots, \sigma_m$ with σ_m a singleton string consisting only of p_n , then the function induced by (Λ, r) maps ϕ_n to an expression that ends in p_n (this is because f is an atomically faithful translation, so that $r(p) = f(p) = p$ for $p \in S$). But if the induced function were a homomorphic translation, it would have to map ϕ_n to a (non-degenerate) conjunction and hence to an expression ending in a closing parenthesis, so this is impossible. Now the only other *prima facie* possible segmentations of ϕ_n in terms of Λ are of the form $\sigma_0, \dots, \sigma_m$ with σ_m being the length $k + 1$ string $p_{n-k} \dots p_n$ for some $1 \leq k < n$, since the segmenting words cannot be longer than n . But as each singleton string π_i consisting of a single sentence letter p_i belongs to Λ , this contradicts the uniqueness requirement of Λ 's being an n -word basis, for we could then also segment ϕ_n into $\sigma_0, \dots, \sigma_{m-1}, \pi_{n-k}, \dots, \pi_n$. \square

There are, however, non-identical languages that are WW-variants.

Example 5 Consider the language 1QL of Example 1 whose syntax is given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall x\phi$, as well as the language 1QL* that is exactly like 1QL except that it puts decorative parentheses around its quantifier prefixes. More precisely, the syntax of 1QL* is given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid (\forall x)\phi$.

For both languages, models are standard first-order structures $\mathfrak{M} = (M, (p^{\mathfrak{M}})_{p \in S})$, and the semantic value $\llbracket \phi \rrbracket_{\mathfrak{M}}$ (respectively, $^* \llbracket \phi \rrbracket_{\mathfrak{M}}$) of a 1QL-formula ϕ (respectively, 1QL*-formula ϕ) relative to such a structure \mathfrak{M} is the function that maps any variable assignment g over \mathfrak{M} to the usual Tarskian value $\llbracket \phi \rrbracket_{\mathfrak{M}}^g$ (respectively $^* \llbracket \phi \rrbracket_{\mathfrak{M}}^g$) of the formula ϕ in the structure \mathfrak{M} relative to g .

Let $t : 1QL \rightarrow 1QL^*$ be the function that changes nothing except that it encloses quantifier prefixes in parentheses. Obviously t is an isomorphism, and its inverse $t^{-1} : 1QL^* \rightarrow 1QL$ changes nothing except that it erases decorative parentheses around quantifier prefixes. The translation t is induced by the 2-WW-replacement whose 2-word basis contains the singleton strings of every primitive symbol except \forall and the length 2 strings $\forall x$ whenever $x \in \text{Var}$, and whose replacement function maps each singleton string in the basis to itself, and each string $\forall x$ to $(\forall x)$. Similarly, t^{-1} is induced by the 4-WW-replacement whose 4-word basis contains the singleton strings

of every primitive symbol except \forall as well as, for each $x \in \text{Var}$, the length 4 string $(\forall x)$, and whose replacement function maps each singleton string in the basis to itself, and each string $(\forall x)$ to $\forall x$. \square

We obtain a strengthening of WW-variance if we require that the translations must in fact be 1-WW-replacements; in other words, that translations proceed by replacing individual symbols of the source language’s alphabet by finite strings over the target alphabet. This leads to the notion of CW-variance.

Definition 6 (*CW-Replacement; CW-Variance; CW-Variant*) For $i \in \{0, 1\}$, let L_i be a language over the alphabet A_i . A *CW-replacement* from A_0 to A_1 is a map from A_0 into the set A_1^* of finite strings (including the empty string ϵ) over A_1 (so that it replaces Characters from A_0 by Words over A_1). A function $f : L_0 \rightarrow L_1$ is *induced* by the CW-replacement r if, whenever $c_0, \dots, c_n \in A_0$ and the length $n + 1$ string $c_0 \dots c_n$ is in L_0 , $f(c_0 \dots c_n)$ is $r(c_0) \frown \dots \frown r(c_n)$. An isomorphism f from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is a *CW-variance* if both it and its inverse f^{-1} are induced by CW-replacements. The interpreted languages $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *CW-variants* if there exists a CW-variance from the former to the latter. \square

Remark 7 There are isomorphisms that are induced by CW-replacements but whose inverses are not so induced. Recall the isomorphism $\pi : \text{PL}_{\neg, \wedge} \rightarrow \text{PPL}_{\neg, \wedge}$ of Example 3. It is induced by the CW-replacement that maps every primitive symbol except the opening and closing parentheses and the conjunction symbol to itself, that maps the closing parenthesis as well as the conjunction symbol to the empty string, and the opening parenthesis to the conjunction symbol. The inverse π^{-1} , however, is not induced by any CW-replacement; indeed no atomically faithful translation $f : \text{PPL}_{\neg, \wedge} \rightarrow \text{PL}_{\neg, \wedge}$ can be induced by a CW-replacement r . Otherwise we would have $p = f(p) = r(p)$ for each $p \in S$ and hence $f(\wedge pq) = r(\wedge) \frown pq$, but no $\text{PL}_{\neg, \wedge}$ -formula has adjacent sentence letter occurrences.

Remark 8 Obviously CW-variance implies WW-variance, since CW-variances are essentially 1-WW-variances. The converse fails, since 1QL and 1QL* are WW-variants (as shown in Example 5) but not CW-variants.⁸ Indeed there cannot be any atomically faithful translation $f : 1\text{QL} \rightarrow 1\text{QL}^*$ that is induced by a CW-replacement r . Otherwise, by atomic faithfulness, for each $p \in S$ and $x \in \text{Var}$, $f(px)$ must be an atom of 1QL*, and by translation, $f(px)$ must be px . But of course we must also have $f(px) = r(p) \frown r(x)$. It follows that, for every $p \in S$ and every $x \in \text{Var}$, $r(p) = p$ and $r(x) = x$.⁹ It then follows that $f(\forall xpx) = r(\forall) \frown xpx$. But there are no 1QL*-formulas in which a variable x occurs directly before a predicate symbol $p \in S$.

The next example shows that there are non-trivial cases of CW-variant languages. Since the example provides the motivation for the discussion in the following section, we work through it in some detail.

⁸ Here we are assuming that, in 1QL, the string $\forall x$ is not an atom but consists of two characters, \forall and x , and that in 1QL*, the string $(\forall x)$ analogously isn’t atomic but consists of four characters.

⁹ If for some p_0 , $r(p_0)$ were not p_0 , it would have to be either ϵ or identical to p_0x for every $x \in \text{Var}$. The latter is plainly impossible. In the former case $r(y)$ would be p_0y for every $y \in \text{Var}$; but then $qx = f(qx) = r(q) \frown r(x) = r(q) \frown p_0x$ for every $q \in S$, which is impossible.

Example 6 The primitive symbols of the *basic modal language* ML are the members of the set S (which in the context of ML we'll again call *sentence letters*), the connectives \neg and \rightarrow , the unary operator \Box , as well as parentheses. The syntax of ML is given by $\phi ::= p \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \Box\phi$, where p ranges over the set S . The ML-formulas are interpreted in models $\mathfrak{M} = (W, R, (p^{\mathfrak{M}})_{p \in S})$, where W is a non-empty set (whose members may be called *worlds* or *indices*), R is a binary relation on W (the *accessibility relation*), and $(p^{\mathfrak{M}})_{p \in S}$ is an S -indexed family of sets $p^{\mathfrak{M}} \subseteq W$.

The local value $\llbracket \phi \rrbracket_{\mathfrak{M}}^w \in \{0, 1\}$ of an ML-formula ϕ at a world w relative to a model \mathfrak{M} is defined recursively as follows:

1. $\llbracket p \rrbracket_{\mathfrak{M}}^w = 1$ if $w \in p^{\mathfrak{M}}$
2. $\llbracket \neg\phi \rrbracket_{\mathfrak{M}}^w = 1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^w$
3. $\llbracket (\phi \rightarrow \psi) \rrbracket_{\mathfrak{M}}^w = \max\{1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^w, \llbracket \psi \rrbracket_{\mathfrak{M}}^w\}$
4. $\llbracket \Box\phi \rrbracket_{\mathfrak{M}}^w = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^v \mid wRv\}$

The semantic value $\llbracket \phi \rrbracket_{\mathfrak{M}}$ of ϕ in a model \mathfrak{M} is the function $\lambda w. \llbracket \phi \rrbracket_{\mathfrak{M}}^w$.

When attention is restricted to models $\mathfrak{M} = (W, R, (p^{\mathfrak{M}})_{p \in S})$ in which R is universal (i.e. $R = W \times W$), clause 4 can be simplified to $\llbracket \Box\phi \rrbracket_{\mathfrak{M}}^w = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^v \mid v \in W\}$. No reference is then made to the accessibility relation in any of the semantic clauses, which therefore idles as a component of the model. When this restriction is in place, we may thus take models to be pairs $\mathfrak{M} = (W, (p^{\mathfrak{M}})_{p \in S})$, which we call *universal models*. For the remainder of this example we impose this universality restriction.

Recall that 1QL is the first-order language over the signature given by the set S of sentence letters, reconstrued as one-place predicate symbols. Any universal model $\mathfrak{M} = (W, (p^{\mathfrak{M}})_{p \in S})$ for ML is at the same time a first-order structure for 1QL.

The local values $\langle \phi \rangle_{\mathfrak{M}}^g$ and the semantic values $\langle \phi \rangle_{\mathfrak{M}} := \lambda g. \langle \phi \rangle_{\mathfrak{M}}^g$ of 1QL-formulas ϕ in universal models \mathfrak{M} were defined in Example 1 along familiar Tarskian lines.

Now let the v_0 -fragment $1QL_{v_0}$ of 1QL be the set of all 1QL-formulas in which no variable other than v_0 occurs. Though defined here as a fragment of 1QL, $1QL_{v_0}$ is an inductively defined first-order language in its own right: With p ranging over S , its syntax can be given as $\phi ::= pv_0 \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall v_0\phi$.

The semantics for $1QL_{v_0}$ is that of 1QL restricted to formulas in which the only variable is v_0 . In evaluating a $1QL_{v_0}$ -formula relative to an assignment, we obviously needn't take into account the values of any variables other than v_0 ; so an assignment is simply the specification of a value w for the variable v_0 . But then we might as well say that $1QL_{v_0}$ -formulas have local values $\langle \phi \rangle_{\mathfrak{M}}^w$ relative to models \mathfrak{M} and individual members w of \mathfrak{M} 's domain, instead of local values $\langle \phi \rangle_{\mathfrak{M}}^{\{v_0, w\}}$ relative to \mathfrak{M} and assignment functions $\{v_0, w\}$. So we have the following semantic clauses for $\langle \phi \rangle_{\mathfrak{M}}^w$:

1. $\langle pv_0 \rangle_{\mathfrak{M}}^w = 1$ if $w \in p^{\mathfrak{M}}$
2. $\langle \neg\phi \rangle_{\mathfrak{M}}^w = 1 - \langle \phi \rangle_{\mathfrak{M}}^w$
3. $\langle (\phi \rightarrow \psi) \rangle_{\mathfrak{M}}^w = \max\{1 - \langle \phi \rangle_{\mathfrak{M}}^w, \langle \psi \rangle_{\mathfrak{M}}^w\}$
4. $\langle \forall v_0\phi \rangle_{\mathfrak{M}}^w = \min\{\langle \phi \rangle_{\mathfrak{M}}^v \mid v \in W\}$

Now consider the functions $\lambda : ML \rightarrow 1QL_{v_0}$ and $\mu : 1QL_{v_0} \rightarrow ML$ defined by the following recursion equations:

- $\lambda(p) = pv_0$ for $p \in S$
- $\lambda(\neg\phi) = \neg\lambda(\phi)$
- $\lambda((\phi \rightarrow \psi)) = (\lambda(\phi) \rightarrow \lambda(\psi))$
- $\lambda(\Box\phi) = \forall v_0 \lambda(\phi)$
- $\mu(pv_0) = p$ for $p \in S$
- $\mu(\neg\phi) = \neg\mu(\phi)$
- $\mu((\phi \rightarrow \psi)) = (\mu(\phi) \rightarrow \mu(\psi))$
- $\mu(\forall v_0\phi) = \Box\mu(\phi)$

Informally, λ appends v_0 to every sentence letter and replaces \Box with $\forall v_0$, while μ deletes v_0 and replaces \forall with \Box . It is easy to see that λ and μ are isomorphic translations that are inverses of each other. Indeed, both λ and μ are induced by CW-replacements: λ is induced by the CW-replacement that maps every $p \in S$ to the string pv_0 , the box \Box to the string $\forall v_0$, and every other primitive symbol of ML to itself. Its inverse μ is induced by the CW-replacement that maps v_0 to the empty string ϵ , \forall to \Box , and every other primitive character of $1QL_{v_0}$ to itself. Thus λ is a CW-variance, and the interpreted languages ML and $1QL_{v_0}$ are CW-variants of each other—*provided that the model class is restricted to universal models.*¹⁰ □

There is an obvious strengthening of the notion of CW-variance that can be obtained by requiring of the relevant translations that they be induced by replacement functions that swap primitive symbols of the source language for *primitive symbols* of the target language (so that single Characters are replaced by single Characters).

Definition 7 (*CC-Replacement; CC-Variance; CC-Variant*) For $i \in \{0, 1\}$, let L_i be a language over the alphabet A_i . A *CC-replacement* from A_i to A_{1-i} is a function from the former to the latter. A function $f : L_i \rightarrow L_{1-i}$ is *induced* by a CC-replacement r from A_i to A_{1-i} if, whenever $c_0, \dots, c_n \in A_i$ and the length $n + 1$ string $c_0 \dots c_n$ is in L_i , $f(c_0 \dots c_n)$ is the length $n + 1$ string $r(c_0) \dots r(c_n)$. An isomorphism f from $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ to $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ is a *CC-variance* if f is induced by a CC-replacement, and its inverse f^{-1} is also induced by a CC-replacement. The interpreted languages $(L_0, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^0)_{\mathfrak{M} \in \mathbf{M}})$ and $(L_1, (\llbracket \cdot \rrbracket_{\mathfrak{M}}^1)_{\mathfrak{M} \in \mathbf{M}})$ are *CC-variants* if there exists a CC-variance from the former to the latter. □

Example 7 Let ρ be the function from $PL_{\neg, \wedge}$ to $PL_{\sim, \&}$ that is induced by the CC-replacement mapping every primitive symbol of $PL_{\neg, \wedge}$ except \neg and \wedge to itself, \neg to \sim , and \wedge to $\&$. The inverse of ρ is induced by the CC-replacement that maps every primitive symbol of $PL_{\sim, \&}$ except \sim and $\&$ to itself, \sim to \neg , and $\&$ to \wedge . Then ρ is a CC-variance from $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ to $(PL_{\sim, \&}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$, and so $(PL_{\neg, \wedge}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ and $(PL_{\sim, \&}, (\llbracket \cdot \rrbracket_v)_{v \in 2^S})$ are CC-variants. □

Remark 9 Obviously every CC-variance is a CW-variance. The converse fails, for (relative to the class of universal models) the translation $\lambda : ML \rightarrow 1QL_{v_0}$ is a CW-variance (as shown in Example 6) but not a CC-variance. Indeed no function from ML to $1QL_{v_0}$ is induced by a CC-replacement: If $f : ML \rightarrow 1QL_{v_0}$ were induced by the CC-replacement r , $f(p) = r(p)$ would have to be a $1QL_{v_0}$ -formula consisting of a single character; but no such $1QL_{v_0}$ -formulas exist. □

¹⁰ We shall return to the universality constraint in Sect. 3.1.

3 Boxes Versus Quantifiers: Arbitrary Accessibility Relations

3.1 The Basic Modal Language

Once we permit models with non-universal accessibility relations, the functions λ and μ between ML and $1QL_{v_0}$ defined in Example 6 are no longer translations: They match $\Box p$ with $\forall v_0 p v_0$, but the former will be true at w just in case p is true at all worlds *accessible from* w , whereas the latter will be satisfied by the assignment of w to v_0 just in case $p v_0$ is satisfied by *all* assignments (not just those that map v_0 to a world accessible from w).

In this section we will see that even relative to the class of *all* possible-world models it is possible to identify a *bona fide* first-order, monadic language whose one-variable fragment is a CW-variant of ML. Before we turn to this task, let us briefly recall a folklore result, if only to realize that it does not help in the search for a first-order language that is *CW-variant* with ML.

Let $1QLR$ be the extension of $1QL$ by a single two-place predicate symbol R , so that its formulas are given by $\phi ::= px \mid Rxy \mid \neg\phi \mid (\phi \rightarrow \psi) \mid \forall x\phi$, where p ranges over S , and x and y range over Var . For any ML-formula ϕ , define auxiliary translations ϕ^{v_0} and ϕ^{v_1} like so:

- $p^{v_i} ::= p v_i$
- $(\neg\phi)^{v_i} ::= \neg\phi^{v_i}$
- $(\phi \wedge \psi)^{v_i} ::= (\phi^{v_i} \wedge \psi^{v_i})$
- $(\Box\phi)^{v_i} ::= \forall v_{1-i} (R v_i v_{1-i} \rightarrow \phi^{v_{1-i}})$

The so-called *standard translation* from ML into $1QLR$ is then the function st that maps each ML-formula ϕ to ϕ^{v_0} .

Given any object w , let \mathbf{w} be the constant function on Var with value w . Define the semantic value $\langle\!\langle\phi\rangle\!\rangle_{\mathfrak{M}}$ of a $1QLR$ -formula ϕ in a model $\mathfrak{M} = (W, R, (p^{\mathfrak{M}})_{p \in S})$ to be the function that maps any $w \in W$ to the local value $\langle\!\langle\phi\rangle\!\rangle_{\mathfrak{M}}^{\mathbf{w}}$. Then st is indeed a translation from $(ML, (\llbracket \cdot \rrbracket_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$ to $(1QLR, (\langle\!\langle \cdot \rangle\!\rangle_{\mathfrak{M}})_{\mathfrak{M} \in \mathbf{M}})$, where \mathbf{M} is the class of *all* models. Since st is injective, it follows that ML is bi-translatable with the range of the standard translation.

That, however, is about as much as we can get out of the standard translation, for it is clearly *not* an isomorphism, since it is not compositional: Suppose there were a $1QLR$ -formula context $\chi(\cdot)$ such that $\forall v_1 (R v_0 v_1 \rightarrow p v_1) = st(\Box p) = \chi(st(p)) = \chi(p v_0)$. Then, since $p v_0$ does not occur in $\forall v_1 (R v_0 v_1 \rightarrow p v_1)$, $\chi(\cdot)$ would have to be $\forall v_1 (R v_0 v_1 \rightarrow p v_1)$ itself—a context with no gap. But then $st(\Box\Box p) = \forall v_1 (R v_0 v_1 \rightarrow \forall v_0 (R v_1 v_0 \rightarrow p v_0))$ would have to be $\chi(st(\Box p))$, i.e. just $\forall v_1 (v_0 R v_1 \rightarrow p v_1)$, contradiction.¹¹

Moreover, the range of the standard translation is not freely generated from any subset of the $1QLR$ -atoms by any subset of the primitive $1QLR$ -operations, i.e. not

¹¹ If we let $[y|x]\phi$ be the result of switching all occurrences (free and bound) of y and x in ϕ , we can define st very simply by means of the clauses $st(p) = p v_0$, $st(\neg\phi) = \neg st(\phi)$, $st((\phi \wedge \psi)) = (st(\phi) \wedge st(\psi))$, and $st(\Box\phi) = \forall v_1 (R v_0 v_1 \rightarrow [v_1|v_0]st(\phi))$. It is tempting to conclude that st is compositional after all, the required $1QLR$ -context for \Box being $\forall v_1 (R v_0 v_1 \rightarrow [v_1|v_0]*_1)$. Note, however, that the latter is *not* a $1QLR$ -context, since $[x|y]$ is not an object-linguistic $1QLR$ -operator but a metalinguistically defined function on $1QLR$ -formulas.

a well-behaved fragment of IQLR. This follows immediately from the fact that the range of st is not closed under the subformula-relation of IQLR: While the formula $\forall v_1 (Rv_0v_1 \rightarrow pv_1) = st(\Box p)$ is in the range, its IQLR-subformulas Rv_0v_1 , pv_1 , and $(Rv_0v_1 \rightarrow pv_1)$ are not.

Nor is st induced by a CW-replacement: Suppose that st were induced by such a replacement function r . Then we would have $pv_0 = st(p) = r(p)$, and the formula $\forall v_1 (Rv_0v_1 \rightarrow pv_1) = st(\Box p)$ would have to be the concatenation of $r(\Box)$ and $r(p)$, i.e. of $r(\Box)$ and pv_0 . This is clearly not the case.

So the standard translation merely shows ML to be bi-translatable with a subset of IQLR that does not present itself, in any natural way, as a sublanguage of IQLR in its own right. This is of no help to us.¹²

But now consider the language IQL^R , which is very similar to IQL except for the unusual syntax of its quantifier. Its formulas are given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall yx\phi$, where p ranges over S and x and y range over Var .

Call expressions of the form $\forall yx$ *accessibility quantifiers*.¹³

The IQL^R -formulas are interpreted in models $\mathfrak{M} = (W, R, (p^{\mathfrak{M}})_{p \in S})$, understood as first-order structures for IQL equipped with an additional relation $R \subseteq W \times W$; call such structures *accessibility structures*.¹⁴ The local value $\llbracket \phi \rrbracket_{\mathfrak{M}}^g$ of a IQL^R -formula ϕ in such a structure \mathfrak{M} relative to a variable assignment g is defined recursively much as one would expect. The only noteworthy clause is the one for the new quantifier.¹⁵

$$(\forall yx\phi)_{\mathfrak{M}}^g = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{g[x:=w]} \mid g(y)Rw\}.$$

So g satisfies $\forall yx\phi$ in \mathfrak{M} just in case for every w accessible from $g(y)$, the x -variant $g[x := w]$ of g satisfies ϕ in \mathfrak{M} .¹⁶ It follows that we may define the set $FV(\phi)$ of variables occurring free in an IQL^R -formula ϕ recursively as follows: $FV(px) = \{x\}$;

¹² This observation should not be taken as criticism of the standard translation. Quite to the contrary, the standard translation is an immensely useful metatheoretical tool, as is evident e.g. from van Benthem's (1976) characterization theorem, according to which a first-order formula is bisimulation-invariant just in case it is logically equivalent to the standard translation of a modal formula. See also van Benthem (1985) as well as the seminal Andréka et al. (1998), in which it is shown that the range of st is included in the (syntactically well-behaved and decidable) *guarded fragment* of first-order logic.

¹³ For practical purposes, it seems advisable to notate accessibility quantifiers as $\forall^y x$ or even $\forall^y x$ to avoid confusion with the more familiar meanings "for all y and x " or "for all pairs $\langle y, x \rangle$ " of the symbol combination $\forall yx$. I nevertheless choose the slightly awkward latter notation here simply because superscripts do not sit all that well with the string-theoretic construal of expressions.

¹⁴ Note that unlike in the case of the standard translation, R is not considered to be the interpretation of a relation symbol in the language's signature.

¹⁵ More generally, in languages whose signatures may include individual constants and function symbols, we would have $\llbracket \forall tx\phi \rrbracket_{\mathfrak{M}}^g = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{g[x:=w]} \mid \llbracket t \rrbracket_{\mathfrak{M}}^g R w\}$, where $\llbracket t \rrbracket_{\mathfrak{M}}^g$ is the local value of the term t relative to the assignment g in \mathfrak{M} . Here, of course, $\llbracket c \rrbracket_{\mathfrak{M}}^g$ is the interpretation $c^{\mathfrak{M}}$ of c in \mathfrak{M} if c is a constant symbol, and $\llbracket f t_1 \dots t_n \rrbracket_{\mathfrak{M}}^g$ is $f^{\mathfrak{M}}(\llbracket t_1 \rrbracket_{\mathfrak{M}}^g, \dots, \llbracket t_n \rrbracket_{\mathfrak{M}}^g)$ for n -place function symbols f and terms t_1, \dots, t_n .

¹⁶ The accessibility quantifier is perhaps not as unusual as it at first seems. Under certain circumstances, free logic's inner and outer quantifiers \forall^i and \forall^o can be regarded as instances of accessibility quantifiers: Let a free-logical model \mathfrak{M} have inner domain M_i and outer domain $M_o \supseteq M_i$, and suppose $c_i^{\mathfrak{M}} \in M_i$ and $c_o^{\mathfrak{M}} \in M_o \setminus M_i$. In \mathfrak{M} , \forall^i ranges over M_i and \forall^o ranges over M_o . Recarve \mathfrak{M} into a model \mathfrak{N} suitable for interpreting accessibility quantifiers as follows: \mathfrak{N} 's domain is M_o , and $R := (\{c_i^{\mathfrak{M}}\} \times M_i) \cup (\{c_o^{\mathfrak{M}}\} \times M_o)$. Now the inner quantifier $\forall^i x$ is the accessibility quantifier $\forall c_i x$ and the outer quantifier $\forall^o x$ is $\forall c_o x$.

$FV(\neg\phi) = FV(\phi)$; $FV((\phi \rightarrow \psi)) = FV(\phi) \cup FV(\psi)$; $FV(\forall yx\phi) = (FV(\phi) \setminus \{x\}) \cup \{y\}$. Thus in $\forall yx\phi$, the leading accessibility quantifier binds any free occurrences of x in ϕ but introduces a new free occurrence of y .

The accessibility quantifier may initially seem to be nothing but a restricted quantifier, with $\forall yx\phi$ simply a fancy notation for $\forall x(Ryx \rightarrow \phi)$. That would be a misconstrual. In the case where x and y are the same variable, the standard restricted quantifier $\forall x(Rxx \rightarrow \phi)$ overwrites the value of x everywhere whereas $\forall xx\phi$ does not overwrite the value of the first displayed occurrence of x . As a consequence, relative to an assignment g , $\forall xx\phi$ generalizes over all points accessible from $g(x)$, whereas $\forall x(Rxx \rightarrow \phi)$ ends up generalizing over self-accessible points. This difference is crucial for the modal language's translatability into the one-variable fragment of the first-order language $1QL^R$.

For let $1QL_{v_0}^R$ be the v_0 -fragment of $1QL^R$. With only one variable in the language, we may again write $(\phi)_M^w$ instead of $(\phi)_M^g$, construing w as the g -value of the language's sole variable. We then have that $(ML, \llbracket \cdot \rrbracket_M)$ and $(1QL_{v_0}^R, (\cdot)_M)$ are CW-variants: The translation from ML to $1QL_{v_0}^R$ is induced by the CW-replacement that maps \Box to $\forall v_0 v_0$, sentence letters p to pv_0 , and every other primitive symbol of ML to itself. Its inverse is induced by the CW-replacement that maps v_0 to ϵ , \forall to \Box , and every other primitive symbol of $1QL_{v_0}^R$ to itself. Thus, in the general setting too, we have found a monadic first-order language, viz. $1QL_{v_0}^R$, of which the modal operator language ML is a CW-variant.

To be sure, the accessibility quantifiers are not the standard first-order quantifiers, but they are in a precise sense generalizations of them: The class of standard first-order structures for a given signature can be construed as that subclass of the accessibility structures for the same signature in which the accessibility relation is universal. Over this subclass, the accessibility quantifiers coincide with the standard quantifiers. In this sense the CW-variance of Example 6 is a special case of our general result.

3.2 Temporal Language

The CW-variance established in the previous section generalizes easily to polymodal languages, as long as the first-order language is given a separate accessibility quantifier $\forall^i yx$ for each of the accessibility relations R_i .¹⁷ As a case in point, we will look briefly at the bi-modal language TL of propositional tense logic.

The syntax of TL is given by $\phi ::= p \mid \neg\phi \mid (\phi \rightarrow \phi) \mid H\phi \mid G\phi$. The models for TL are the possible-world models $\mathfrak{M} = (W, R, (p^{\mathfrak{M}})_{p \in S})$ for ML . Of course, in the context of TL one usually speaks of the members of W not as worlds but as times, and notates the accessibility relation as $<$ rather than R .

The local value $\llbracket \phi \rrbracket_M^w \in \{0, 1\}$ of a TL -formula ϕ at a time w in a model $\mathfrak{M} = (W, <, (p^{\mathfrak{M}})_{p \in S})$ is defined by recursion on ϕ in the familiar way:

Obviously, the standard universal quantifier can similarly be construed as an instance of the accessibility quantifier: Let \mathfrak{M} be a model and define R as $M \times M$. Then $\forall x$ is just $\forall xx$.

¹⁷ Alternatively: As long as the accessibility quantifier is endowed with an additional parameter i and the semantic clause $(\forall^i tx\phi)_M^g = \min\{(\phi)_M^{g[x:=w]} \mid \llbracket t \rrbracket_M^g R_i w\}$.

- $\llbracket p \rrbracket_{\mathfrak{M}}^w = 1$ if $w \in p^{\mathfrak{M}}$
- $\llbracket \neg\phi \rrbracket_{\mathfrak{M}}^w = 1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^w$
- $\llbracket (\phi \rightarrow \psi) \rrbracket_{\mathfrak{M}}^w = \max\{1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^w, \llbracket \psi \rrbracket_{\mathfrak{M}}^w\}$
- $\llbracket H\phi \rrbracket_{\mathfrak{M}}^w = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^v \mid v < w\}$
- $\llbracket G\phi \rrbracket_{\mathfrak{M}}^w = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^v \mid v > w\}$

Let \mathfrak{T} be the first-order language whose primitive symbols are the variables, the parentheses, the members of S *qua* one-place predicate symbols, \neg and \rightarrow , the *retrograde accessibility quantifier* \forall and the *forward-looking accessibility quantifier* $\bar{\forall}$. The \mathfrak{T} -formulas are given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall yx\phi \mid \bar{\forall}yx\phi$, where p ranges over S and x and y range over Var .¹⁸

The local value $\llbracket \phi \rrbracket_{\mathfrak{M}}^g$ of a \mathfrak{T} -formula ϕ in a model \mathfrak{M} relative to a variable assignment g is defined recursively, the interesting clauses being those for \forall and $\bar{\forall}$:

- $\llbracket (\forall yx\phi) \rrbracket_{\mathfrak{M}}^g = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{g[x:=w]} \mid w < g(y)\}$
- $\llbracket (\bar{\forall}yx\phi) \rrbracket_{\mathfrak{M}}^g = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{g[x:=w]} \mid g(y) < w\}$

The one-variable fragment \mathfrak{T}_{v_0} of \mathfrak{T} is a CW-variant of TL, since the function \mathcal{I} from TL to \mathfrak{T}_{v_0} that is induced by the CW-replacement mapping sentence letters p to pv_0 , the retrograde temporal operator H to $\forall v_0 v_0$, the forward-looking temporal operator G to $\bar{\forall} v_0 v_0$, and every other primitive symbol to itself, is a CW-variance (its inverse is induced by the CW-replacement that maps v_0 to ϵ , \forall to H, $\bar{\forall}$ to G, and every other primitive symbol to itself).

3.3 Quantified Modal Logic

For each $n \geq 1$, let \mathcal{P}^n be a (possibly empty) set of n -ary predicate symbols, and let \mathcal{P} be the union of the \mathcal{P}^n . Let \mathcal{C} be a (possibly empty) set of individual constants. Let Σ be the signature $(\mathcal{P}, \mathcal{C})$. Then the primitive symbols of the language QML_{Σ} of quantified modal logic over Σ are the members of \mathcal{P} and \mathcal{C} , the members of the countably infinite set $Z = \{z_0, z_1, z_2, \dots\}$ of individual variables, the propositional connectives \neg and \rightarrow , the necessity operator \Box , the actualist individual quantifier \forall , the possibilist individual quantifier Π , as well as opening and closing parentheses.¹⁹ The QML_{Σ} -terms are the members of $\mathcal{C} \cup Z$. The QML_{Σ} -formulas are generated inductively as follows: Whenever $P \in \mathcal{P}^n$ and t_1, \dots, t_n are QML_{Σ} -terms, $Pt_1 \dots t_n$ is an atomic QML_{Σ} -formula. Whenever ϕ and ψ are QML_{Σ} -formulas and $r \in Z$, all of $\neg\phi$, $(\phi \rightarrow \psi)$, $\Box\phi$, $\forall r\phi$, and $\Pi r\phi$ are QML_{Σ} -formulas.

A model for QML_{Σ} is a quintuple $\mathfrak{M} = (W, (D_w)_{w \in W}, R, (P^{\mathfrak{M}})_{P \in \mathcal{P}}, (c^{\mathfrak{M}})_{c \in \mathcal{C}})$, where $W \neq \emptyset$ is called the set of worlds, $(D_w)_{w \in W}$ is a W -indexed family of sets D_w , not all of which are empty (so that the union D of all the D_w is non-empty), $R \subseteq W \times W$ is the accessibility relation, for each $P \in \mathcal{P}^n$, $P^{\mathfrak{M}}$ maps worlds to

¹⁸ Alternatively, in the spirit of footnote 17, we could have a single accessibility quantifier \forall in the lexicon but allow it to take an additional parameter from $\{f, b\}$ (mnemonic for “forward” and “backward”) as follows: $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall iyx\phi$, where i ranges over $\{f, b\}$ and $\forall fyx$ (resp. $\forall byx$) is given the semantic behavior of $\forall yx$ (resp. $\forall yx$).

¹⁹ To be sure, quantified modal languages do not typically contain both actualist and possibilist quantifiers. We include both here just to show that both varieties can be translated in the appropriate fashion.

subsets of D^n , and $(c^{\mathfrak{M}})_{c \in \mathcal{C}}$ is a \mathcal{C} -indexed family of members $c^{\mathfrak{M}}$ of the union D of all the D_w . An individual variable assignment in such a model \mathfrak{M} is a function $g : Z \rightarrow D$.

The local value $\llbracket t \rrbracket_{\mathfrak{M}}^{w,g}$ of a QML $_{\Sigma}$ -term t relative to a world w , an individual variable assignment g and a model \mathfrak{M} is $c^{\mathfrak{M}}$, if t is a constant symbol c , and is $g(r)$, if t is an individual variable r . The local value $\llbracket \phi \rrbracket_{\mathfrak{M}}^{w,g} \in \{0, 1\}$ of a QML $_{\Sigma}$ -formula ϕ relative to a world w , an individual variable assignment g , and a model \mathfrak{M} , is defined by recursion on ϕ as follows.

- $\llbracket Pt_1 \dots t_n \rrbracket_{\mathfrak{M}}^{w,g} = 1$ if $\langle \llbracket t_1 \rrbracket_{\mathfrak{M}}^{w,g}, \dots, \llbracket t_n \rrbracket_{\mathfrak{M}}^{w,g} \rangle \in P^{\mathfrak{M}}(w)$
- $\llbracket \neg \phi \rrbracket_{\mathfrak{M}}^{w,g} = 1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^{w,g}$
- $\llbracket (\phi \rightarrow \psi) \rrbracket_{\mathfrak{M}}^{w,g} = \max\{1 - \llbracket \phi \rrbracket_{\mathfrak{M}}^{w,g}, \llbracket \psi \rrbracket_{\mathfrak{M}}^{w,g}\}$
- $\llbracket \Box \phi \rrbracket_{\mathfrak{M}}^{w,g} = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{v,g} \mid wRv\}$
- $\llbracket \forall r \phi \rrbracket_{\mathfrak{M}}^{w,g} = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{w,g[r:=o]} \mid o \in D_w\}$
- $\llbracket \Pi r \phi \rrbracket_{\mathfrak{M}}^{w,g} = \min\{\llbracket \phi \rrbracket_{\mathfrak{M}}^{w,g[r:=o]} \mid o \in D\}$

The two-sorted first-order language Ω_{Σ} is based on the sorts i (individuals) and w (worlds). Its primitive symbols are:

- the members of the set Z of individual variables, which are of sort i
- the members of the set Var of world variables, which are of sort w
- for each $n \geq 1$, the members of the set \mathcal{P}^n , but construed as $n + 1$ -ary rather than n -ary predicate symbols, where the first argument place of $P \in \mathcal{P}^n$ is of sort w and the next n places are of sort i
- the members of the set \mathcal{C} , which are of sort i
- the propositional connectives \neg and \rightarrow
- the actualist individual quantifier \forall^i
- the possibilist individual quantifier Π^i
- the accessibility quantifier \forall^w
- opening and closing parentheses

The Ω_{Σ} -terms of sort i are the members of $Z \cup \mathcal{C}$; the Ω_{Σ} -terms of sort w are the members of Var . Whenever $P \in \mathcal{P}^n$, $x \in \text{Var}$, and t_1, \dots, t_n are Ω_{Σ} -terms of individual sort, $Pxt_1 \dots t_n$ is an atomic Ω_{Σ} -formula. Compound Ω_{Σ} -formulas are generated by way of negation and the conditional, as well as the following formation rules for the various quantifiers:

- Whenever ϕ is a Ω_{Σ} -formula, $x \in \text{Var}$, and $r \in Z$, $\forall^i x r \phi$ is a Ω_{Σ} -formula.
- Whenever ϕ is a Ω_{Σ} -formula and $r \in Z$, $\Pi^i r \phi$ is a Ω_{Σ} -formula.
- Whenever ϕ is a Ω_{Σ} -formula and $x, y \in \text{Var}$, $\forall^w y x \phi$ is a Ω_{Σ} -formula.

The local value $\llbracket t \rrbracket_{\mathfrak{M}}^{g,\omega}$ of a Ω_{Σ} -term t relative to a model $\mathfrak{M} = (W, (D_w)_{w \in W}, R, (P^{\mathfrak{M}})_{P \in \mathcal{P}}, (c^{\mathfrak{M}})_{c \in \mathcal{C}})$, an individual variable assignment g , and a world variable assignment $\omega : \text{Var} \rightarrow W$, is $c^{\mathfrak{M}}$ if t is the individual constant $c \in \mathcal{C}$, is $g(z)$ if t is the individual variable z , and is $\omega(x)$ if t is the world variable x . The local value $\llbracket \phi \rrbracket_{\mathfrak{M}}^{g,\omega} \in \{0, 1\}$ of a Ω_{Σ} -formula ϕ relative to g , ω , and \mathfrak{M} is defined by recursion on ϕ as follows.²⁰

²⁰ Note that the Ω_{Σ} -equivalent \forall^i of the actualist quantifier \forall of QML $_{\Sigma}$ naturally presents itself as the accessibility quantifier for the accessibility relation between worlds and individuals that holds between w and o just in case $o \in D_w$.

- $\langle Pxt_1 \dots t_n \rangle_{\mathfrak{M}}^{g,\omega} = 1$ if $\langle (t_1)_{\mathfrak{M}}^{g,\omega}, \dots, (t_n)_{\mathfrak{M}}^{g,\omega} \rangle \in P^{\mathfrak{M}}(\langle x \rangle_{\mathfrak{M}}^{g,\omega})$
- $\langle \neg\phi \rangle_{\mathfrak{M}}^{g,\omega} = 1 - \langle \phi \rangle_{\mathfrak{M}}^{g,\omega}$
- $\langle (\phi \rightarrow \psi) \rangle_{\mathfrak{M}}^{g,\omega} = \max\{1 - \langle \phi \rangle_{\mathfrak{M}}^{g,\omega}, \langle \psi \rangle_{\mathfrak{M}}^{g,\omega}\}$
- $\langle \forall^i x r\phi \rangle_{\mathfrak{M}}^{g,\omega} = \min\{\langle \phi \rangle_{\mathfrak{M}}^{g[r:=o],\omega} \mid o \in D_{\langle x \rangle_{\mathfrak{M}}^{g,\omega}}\}$
- $\langle \Pi^i r\phi \rangle_{\mathfrak{M}}^{g,\omega} = \min\{\langle \phi \rangle_{\mathfrak{M}}^{g[x:=o],\omega} \mid o \in D\}$
- $\langle \forall^w y x\phi \rangle_{\mathfrak{M}}^{g,\omega} = \min\{\langle \phi \rangle_{\mathfrak{M}}^{g,\omega[x:=w]} \mid \langle y \rangle_{\mathfrak{M}}^{g,\omega} R w\}$

Now let $\Omega_{\Sigma}^{v_0}$ be the one-variable fragment of Ω_{Σ} , and let $q : \text{QML}_{\Sigma} \rightarrow \Omega_{\Sigma}^{v_0}$ be the function induced by the CW-replacement that maps $P \in \mathcal{P}^n$ to Pv_0 , \Box to $\forall^w v_0 v_0, \forall$ to $\forall^i v_0$, Π to Π^i , and all other primitive symbols of QML_{Σ} to themselves. Then q is a CW-variance, its inverse being induced by the CW-replacement that maps v_0 to ϵ , \forall^i to \forall , Π^i to Π , \forall^w to \Box , and all other primitive Ω_{Σ} -symbols to themselves.²¹

3.4 Actuality

In this section we consider the generalization of our CW-variance result to the propositional modal language augmented with the actuality operator.

The language AML of actuality modal logic is the language ML extended by a single one-place operator A , the *actuality operator*, and therefore has the additional formation rule that A may be prefixed to any AML-formula: $\phi ::= p \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \Box\phi \mid A\phi$. These formulas are interpreted in “pointed” models, which are quadruples $\mathfrak{M} = (W, a^{\mathfrak{M}}, R, (p^{\mathfrak{M}})_{p \in \mathcal{S}})$, where $(W, R, (p^{\mathfrak{M}})_{p \in \mathcal{S}})$ is a model in our original sense, and $a^{\mathfrak{M}}$ is a distinguished member of W (the “actual world” of \mathfrak{M}). The recursive clauses defining local values at worlds in pointed models are as for ML, with the addition of a clause for formulas governed by A : $\llbracket A\phi \rrbracket_{\mathfrak{M}}^w = \llbracket \phi \rrbracket_{\mathfrak{M}}^{a^{\mathfrak{M}}}$; in other words, $A\phi$ is true at w just in case ϕ is true at $a^{\mathfrak{M}}$, the actual world of \mathfrak{M} .²²

Let $1\text{QL}\mathbb{A}^R$ be the extension of 1QL^R by a quantifier symbol \mathbb{A} that denotes, in each model $\mathfrak{M} = (W, a^{\mathfrak{M}}, R, (p^{\mathfrak{M}})_{p \in \mathcal{S}})$, the generalized quantifier $\{X \subseteq W \mid a^{\mathfrak{M}} \in X\}$; in other words, \mathbb{A} stands for the Montagovian individual obtained by type-lifting $a^{\mathfrak{M}}$ from type e to type $\langle\langle e, t \rangle, t\rangle$.²³

²¹ It should be noted that some of our equivalence relations are sensitive to quite subtle design decisions. For example, if we had constructed the Ω_{Σ} -atomic formulas from the QML_{Σ} -atomic formulas by adding the world argument place after all the individual argument places rather than before, we would not have gotten a CW-variant, and indeed, if $\mathcal{P}^n \neq \emptyset$ for infinitely many n , not even a WW-variant.

²² The first appearance of the actuality operator is, as far as I am aware, in Kamp (1971), in the form of the *now* operator of tense logic. The *locus classicus* for the modal actuality operator is Crossley and Humberstone (1977). In a propositional environment, the actuality operator is eliminable relative to real-world validity (Hazen et al. 2013), though not relative to general validity. In a quantificational environment, it is not even eliminable relative to real-world validity (Kamp 1971; Hazen 1976; Hodes 1984; Wehmeier 2003).

²³ The possibility of replacing individuals with generalized quantifiers is familiar from Montague semantics; cf. e.g. Dowty et al. (1981: 193). The notation used here resembles more closely the one suggested by Barwise and Cooper (1981: 166). We’ve already encountered this type of quantifier in the shape of Δ in Example 1.

The $L^{\mathbb{A}}$ -formulas are, accordingly, given by $\phi ::= px \mid \neg\phi \mid (\phi \rightarrow \phi) \mid \forall yx\phi \mid \mathbb{A}x\phi$, where p ranges over S and x and y range over Var . The semantic clause for the new quantifier symbol \mathbb{A} reads:

$$(\mathbb{A}x\phi)_{\mathfrak{M}}^g = (\phi)_{\mathfrak{M}}^{g[x:=a^{\mathfrak{M}}]}$$

which, as mentioned, just means that the interpretation of \mathbb{A} is the distinguished world $a^{\mathfrak{M}}$, lifted to quantifier type.

Now let $1QL\mathbb{A}_{v_0}^R$ be the one-variable fragment of $1QL\mathbb{A}^R$. Then AML is a CW-variant of $1QL\mathbb{A}_{v_0}^R$ via the translations induced by the CW-replacements defined in the previous section, supplemented by the following clauses for \mathbb{A} and \mathbb{A} , respectively:

- The CW-replacement for the translation $AML \rightarrow 1QL\mathbb{A}_{v_0}^R$ maps \mathbb{A} to $\mathbb{A}v_0$.
- The CW-replacement for the translation $1QL\mathbb{A}_{v_0}^R \rightarrow AML$ maps \mathbb{A} to \mathbb{A} (and of course still maps v_0 to ϵ).

We thus translate from AML to $1QL\mathbb{A}_{v_0}^R$ by appending v_0 to every sentence letter and replacing sans-serif \mathbb{A} with $\mathbb{A}v_0$ and \square with $\forall v_0v_0$. Conversely, we translate from $1QL\mathbb{A}_{v_0}^R$ to AML by switching every \forall to \square , every \mathbb{A} to \mathbb{A} , and deleting every v_0 .²⁴

4 Conclusion

We’ve isolated seven levels of language equivalence that are strictly weaker than literal identity; and for each of these levels we’ve found pairs of languages that are semantically equivalent at that level but not at a higher one. The results of our investigation can be summed up in the following table, which lists the equivalence notions in order of strictly ascending strictness:

Equivalence relation	Representative pair
Bi-translatability	$1QL_{\delta}$ versus $1QL_{\Delta}$
Definitional equivalence	$PL_{\neg, \wedge}$ versus $PL_{\neg, \vee}$
Homomorphic bi-translatability	$PL_{\neg, \wedge}$ versus $PL_{\neg, \wedge, \&}$
Isomorphism	$PL_{\neg, \wedge}$ versus $PPL_{\neg, \wedge}$
WW-variance	$1QL$ versus $1QL^*$
CW-variance	ML versus $1QL_{v_0}$
CC-variance	$PL_{\neg, \wedge}$ versus $PL_{\sim, \&}$

I’ve deliberately chosen more or less neutral terms for these various equivalence relations so as not to intermingle, in the main body of the paper, purely technical observations with philosophical judgments. This concluding section, however, would seem to be an appropriate place to reflect on the intuitive significance of these equivalence relations, and tentatively to propose a more memorable nomenclature.

²⁴ Note that, in contradistinction to the languages $1QL^R$ and $1QL_{v_0}^R$ of the previous section, $1QL\mathbb{A}^R$ and $1QL\mathbb{A}_{v_0}^R$ do have closed formulas, since \mathbb{A} can bind variables without at the same time introducing new free occurrences.

Let us begin with CC-variance. CC-variant languages differ at most in trivial ways, namely in the *shapes* of the primitive symbols they employ (think English written in lower-case letters versus English written in upper-case letters). These are such uninteresting differences that some logic textbooks don't even bother to define languages more specifically than up to CC-variance (see e.g. Bell and Machover 1977, 8). It therefore seems appropriate to propose the more intuitive term *trivial notational variance* for CC-variance.

At the next level down there is CW-variance, which might also be called *strict* or, more descriptively, *typographical notational variance*. This designation seems apt because one can enable an agent to translate a language into a CW-variant by simply reprogramming the keys of their keyboard according to the relevant CW-replacement.²⁵ In our hierarchy, typographical notational variance would be the strictest non-trivial notion of notational variance.

Taking another step down the hierarchy, we have WW-variance, to which we might refer as *linear notational variance* since, in contradistinction to mere isomorphism (compare the case of $PL_{\neg, \wedge}$ and $PPL_{\neg, \wedge}$), no re-arranging of elements of the strings can occur. It should be noted that, unlike typographical notational variance and trivial notational variance, linear notational variance requires some parsing of the input string, if only into linear, contiguous segments.

This conveniently leaves the unqualified *notational variance* as an alternative term for what we've been calling language isomorphism. Our paradigmatic cases of the isomorphic languages $PL_{\neg, \wedge}$ and $PPL_{\neg, \wedge}$ on the one hand, and the merely homomorphically bi-translatable languages $PL_{\neg, \wedge}$ and $PL_{\neg, \wedge, \&}$ makes this usage plausible: Whether one employs ordinary, parenthesis-based notation or Polish notation is indeed a purely notational, non-substantive matter. On the other hand, a language like $PL_{\neg, \wedge, \&}$ that contains distinct but synonymous primitives differs not just notationally from the result of excising this synonymy, $PL_{\neg, \wedge}$; the languages have distinct conceptual structures, one containing a redundancy, the other not.

That talk of notational variance is inappropriate in the case of the definitionally equivalent languages $PL_{\neg, \wedge}$ and $PL_{\neg, \vee}$ is, I take it, even less controversial, since the adequacy of the target language definitions rests on De Morgan's laws, substantive principles of classical logic rather than merely conventional, notational choices.

In light of the fact that, in these latter cases, more than notation is at stake, it might make sense to use terms not containing "notational", like *homomorphic variants* and *definitional variants*, for homomorphically bi-translatable and definitionally equivalent language pairs, respectively. Bi-translatability, finally, seems sufficiently descriptive to warrant adoption as such.²⁶

²⁵ To be sure, we would have to allow for macros, i.e. for each single key to be reprogrammed so as to produce a fixed *finite string*, possibly the empty string, of characters from the target alphabet. If the keys can only be reprogrammed so as to produce a single character from the target alphabet, we end up with CC-variance, or as we now also call it, trivial notational variance.

²⁶ A reviewer for this journal has pointed out that, while I suggest to draw the line between notational and non-notational variance between homomorphic bi-translatability and language isomorphism, another natural cleavage point is between those equivalence relations that do not invoke string-theoretic structure and those that do, i.e. between language isomorphism and WW-variance. There is no doubt that this latter boundary is theoretically significant; nevertheless my own sense is that it does not coincide with the notational/non-notational divide, given pairs like $PL_{\neg, \wedge}$ and $PPL_{\neg, \wedge}$.

Besides developing the beginnings of a theory of model-focused language equivalence, we considered the special case of the basic modal language ML. It has been known for a long time that, relative to universal models, ML is a typographical notational variant (i.e. a CW-variant) of IQL's one-variable fragment. In the general case, where models with arbitrary accessibility relations are permitted, extant techniques do not establish more than mere bi-translatability of ML with a subset of IQLR that is neither freely generated nor closed under subformulas in the sense of IQLR. With the single conceptual innovation of the accessibility quantifier, we were able to exhibit a *bona fide* first-order, monadic language whose one-variable fragment is likewise a typographical notational variant of ML. Modal operator language is thus equivalent to explicitly quantificational language in the strictest non-trivial sense of semantic equivalence.

Acknowledgements This paper has its origins in conversations with my late colleague and friend Aldo Antonelli about the relationship between modal and first-order language. It would undoubtedly be a better paper had I had the chance to discuss it with him. I am grateful to Johan van Benthem for correspondence over an early version of a predecessor of this paper. Many thanks, moreover, to Rohan French, Allen Hazen, Maciej Kleczek, Jan Köpping, Fabio Lampert, Heinrich Wansing, Ede Zimmermann, and especially Lloyd Humberstone, for feedback on various drafts or presentations of the material. Audiences at the Frankfurt Semantics Colloquium and the workshop *Logic in Bochum V* in July of 2019 provided helpful comments and asked productive questions, for which I owe them thanks. Anonymous reviewers made several insightful suggestions that improved the paper, even if I was unable to accommodate them all. Funding was provided by the National Endowment for the Humanities (Grant No. FA-232235-16). Any views, findings, conclusions, or recommendations expressed in this article do not necessarily represent those of the National Endowment for the Humanities.

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