



Mid-Surface Scaling Invariance of Some Bending Strain Measures

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Abstract

The mid-surface scaling invariance of bending strain measures proposed in (Int. J. Solids Struct. 37(39):5517–5528, 2000) is discussed in light of the work of (J. Elast. 146(1):83–141, 2021).

Keywords Nonlinear shell theory · Bending strain · Scaling invariance

Mathematics Subject Classification 74K25

1 Introduction

This brief note discusses the mid-surface scaling invariance of three nonlinear measures of pure bending strain, as introduced in [1] and physically motivated therein more than 20 years ago, in light of the recent work of [3] where the said invariance is introduced.

It is shown that one of the strain measures introduced in [1] possesses scaling invariance, and the other two are easily modified to have the invariance as well. There has been a recent surge of interest in such matters, as can be seen from the works of [3, 5, 6].

We use the notation of [1]: a shell mid-surface is thought of as a 2D surface in ambient 3D space (the qualification ‘mid-surface’ will not be used in all instances; it is hoped that the meaning will be clear from the context). Both the reference and deformed shells are parametrized by the same coordinate system $((\xi^\alpha, \alpha = 1, 2)$ (convected coordinates). Points on the reference geometry are denoted generically by \mathbf{X} and on the deformed geometry by \mathbf{x} . The reference unit normal is denoted by \mathbf{N} and the unit normal on the deformed geometry by \mathbf{n} . A subscript comma refers to partial differentiation, e.g. $\frac{\partial \Omega}{\partial \xi^\alpha} = (\cdot)_{,\alpha}$. Summation over repeated indices will be assumed. The convected coordinate basis vectors in the reference geometry will be referred to by the symbols (\mathbf{E}_α) and those in the deformed geometry by (\mathbf{e}_α) , $\alpha = 1, 2$, with corresponding dual bases (\mathbf{E}^α) , (\mathbf{e}^α) , respectively. A suitable number of dots placed between two tensors represent the operation of contraction, while the symbol \otimes will represent a tensor product. The deformation gradient will be denoted by $\mathbf{f} = \mathbf{e}_\alpha \otimes \mathbf{E}^\alpha$ and admits the right polar decomposition $\mathbf{f} = \mathbf{r} \cdot \mathbf{U}$, where $\mathbf{U}(\mathbf{X}) : T_{\mathbf{X}} \rightarrow T_{\mathbf{X}}$ and $\mathbf{r}(\mathbf{X}) :$

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$T_X \rightarrow T_x$, where T_c represents the tangent space of the shell at the point c . The curvature tensor on the deformed shell is denoted as $\mathbf{b} = \mathbf{n}_{,\beta} \otimes \mathbf{e}^\beta$ and that on the undeformed shell as $\mathbf{B} = \mathbf{N}_{,\beta} \otimes \mathbf{E}^\beta$.

2 Some Measures of Pure Bending and Their Invariance Under Mid-Surface Scaling

In [1] three measures of bending strain were proposed, given by

$$\tilde{\mathbf{K}} = (\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^T \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta}) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta = \mathbf{f}^T \cdot \mathbf{b} \cdot \mathbf{f} - \mathbf{U} \cdot \mathbf{B}, \tag{1a}$$

$$\begin{aligned} \check{\mathbf{K}} &= \left(\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^T \cdot \mathbf{n}_{,\beta} - \frac{1}{2} (\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} + \mathbf{E}_\beta \cdot \mathbf{U} \cdot \mathbf{N}_{,\alpha}) \right) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \\ &= \mathbf{f}^T \cdot \mathbf{b} \cdot \mathbf{f} - (\mathbf{U} \cdot \mathbf{B})_{sym}, \end{aligned} \tag{1b}$$

$$\bar{\mathbf{K}} = (\mathbf{E}_\alpha \cdot \mathbf{r}^T \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{N}_{,\beta}) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta = \mathbf{r}^T \cdot \mathbf{b} \cdot \mathbf{f} - \mathbf{B}. \tag{1c}$$

Equation (1c) was unnumbered in that work, as the main emphasis was to obtain a non-linear generalization of the Koiter-Sanders-Budiansky bending strain measure [2, 4]; $\tilde{\mathbf{K}}$ is introduced as Equation (8) and $\check{\mathbf{K}}$ as Equation (10) in [1].

In [3] a physically natural requirement of invariance of bending strain measure under simple scalings of the form

$$\mathbf{x} \rightarrow a\mathbf{x}, \quad 0 < a \in \mathbb{R}$$

is introduced (for plates, but the requirement is natural for shells as well) and it is shown that the measures $\tilde{\mathbf{K}}$, $\check{\mathbf{K}}$ are not invariant under such a scaling. The measure $\bar{\mathbf{K}}$ is not discussed in [3].

It is straightforward to see that under the said scaling, the deformation gradient scales as

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{r} \cdot \mathbf{U} = \mathbf{f} \quad \rightarrow \quad a\mathbf{f} = \mathbf{r} \cdot (a\mathbf{U}) = a \frac{\partial \mathbf{x}}{\partial \mathbf{X}},$$

resulting in the bending measures scaling as

$$\tilde{\mathbf{K}} \rightarrow a\tilde{\mathbf{K}}; \quad \check{\mathbf{K}} \rightarrow a\check{\mathbf{K}}; \quad \bar{\mathbf{K}} \rightarrow \bar{\mathbf{K}}.$$

Thus, the bending strain measure $\bar{\mathbf{K}}$ from [1], not discussed by [3], is actually *invariant under scaling deformations of the deformed shell mid-surface*. Furthermore, the simple modifications of the measures $\tilde{\mathbf{K}}$, $\check{\mathbf{K}}$ to

$$\tilde{\mathbf{K}}_{mod} = \frac{1}{|\mathbf{U}|} (\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^T \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta}) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \tag{2a}$$

$$= (tr(\mathbf{f}^T \mathbf{f}))^{-\frac{1}{2}} (\mathbf{x}_{,\alpha} \cdot \mathbf{n}_{,\beta} - \mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta}) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta,$$

$$\check{\mathbf{K}}_{mod} = \frac{1}{|\mathbf{U}|} \left(\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{r}^T \cdot \mathbf{n}_{,\beta} - \frac{1}{2} (\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} + \mathbf{E}_\beta \cdot \mathbf{U} \cdot \mathbf{N}_{,\alpha}) \right) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta \tag{2b}$$

$$= (tr(\mathbf{f}^T \mathbf{f}))^{-\frac{1}{2}} \left(\mathbf{x}_{,\alpha} \cdot \mathbf{n}_{,\beta} - \frac{1}{2} (\mathbf{E}_\alpha \cdot \mathbf{U} \cdot \mathbf{N}_{,\beta} + \mathbf{E}_\beta \cdot \mathbf{U} \cdot \mathbf{N}_{,\alpha}) \right) \mathbf{E}^\alpha \otimes \mathbf{E}^\beta,$$

where

$$|\mathbf{U}| = \sqrt{\mathbf{U} : \mathbf{U}} = \sqrt{\text{tr}(\mathbf{f}^T \mathbf{f})},$$

make them mid-surface scaling invariant.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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References

1. Acharya, A.: A nonlinear generalization of the Koiter–Sanders–Budiansky bending strain measure. *Int. J. Solids Struct.* **37**(39), 5517–5528 (2000)
2. Budiansky, B., Sanders, J.L.: On the ‘Best’ first-order linear shell theory. In: *Progress in Applied Mechanics (the Prager Anniversary Volume)*, pp. 129–140 (1963)
3. Ghiba, I.D., Bîrsan, M., Lewintan, P., Neff, P.: A constrained Cosserat shell model up to order $O(h^5)$: modelling, existence of minimizers, relations to classical shell models and scaling invariance of the bending tensor. *J. Elast.* **146**(1), 83–141 (2021)
4. Koiter, W.T.: A consistent first approximation in the general theory of thin elastic shells. In: *Proceedings of the IUTAM Symposium on the Theory of Thin Shells*, pp. 12–33. North-Holland, Amsterdam (1960)
5. Virga, E.G.: Pure measures of bending for soft plates. *Soft Matter* **20**(1), 144–151 (2024)
6. Vitral, E., Hanna, J.A.: Dilation-invariant bending of elastic plates, and broken symmetry in shells. *J. Elast.* **153**(4–5), 571–579 (2023)

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