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RESEARCH

Plane Stress Problems for Isotropic Incompressible Hyperelastic Materials

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Abstract
The analysis of plane stress problems has long been a topic of interest in linear elasticity. The corresponding problem for non-linearly elastic materials is considered here within the context of homogeneous incompressible isotropic elasticity. It is shown that when the problem is posed in terms of the Cauchy stress, a semi-inverse approach must be employed to obtain the displacement of a typical particle. If however the general plane stress problem is formulated in terms of the Piola-Kirchhoff stress, the deformation of a particle requires the solution of a non-linear partial differential equation for both simple tension and simple shear, the trivial solution of which yields a homogeneous deformation. It is also shown that the general plane stress problem can be solved for the special case of the neo-Hookean material.

Keywords Plane stress · Piola-Kirchhoff stress · Incompressible isotropic hyperelastic materials · Simple tension and shear · Neo-Hookean material

Mathematics Subject Classification 74B20 · 74G55

1 Motivation

Classical non-linear incompressible homogeneous isotropic hyperelasticity as formulated by Rivlin (see, for example Rivlin [1]) assumes that the Cauchy stress $\boldsymbol{T}$ can be determined as a function of the left Cauchy-Green deformation tensor $\boldsymbol{B}$ so that

$$\boldsymbol{T} = f(\boldsymbol{B}), \tag{1.1}$$

where, using the so-called semi-inverse approach, $\boldsymbol{B}$ is determined from a given displacement field. This is precisely what is needed if accurate and reliable predictions of the stress

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in a deformed body are needed as displacement, but not stress, can be measured experimentally. There is the dual problem however, that has been largely ignored in the literature, of inverting (1.1) to determine $\boldsymbol{B}$ from a known stress distribution, with the expectation that this might be most useful in experiments where the stress can be assumed to be homogeneous and is specified on the boundary of the deformed body. There are two significant impediments to a satisfactory solution to this dual problem: first, the stress-strain relation (1.1) is not in general invertible. Secondly, even if invertibility can be achieved, the existence and uniqueness of a regular deformation generating a pre-assigned left strain-tensor field $\boldsymbol{B}$ is an open problem (Blume [2]). Blume [2] has shown, however, that if a regular deformation results in a constant $\boldsymbol{B}$, then the deformation is homogeneous. This is obviously a very useful result when determining the existence of a deformation if a given Cauchy stress is assumed homogeneous.

Batra's ostensibly simple theorem (Batra [3]) yields insight into this inverse problem for perhaps the simplest problem that can be posed in this context, i.e., what can be deduced about the nature of the deformation for simple tension if the Cauchy stress $\boldsymbol{T}$ has the form

$$\boldsymbol{T} = T\boldsymbol{e}_x \otimes \boldsymbol{e}_x, \tag{1.2}$$

where T is a positive constant? Here the standard notation is used to denote unit vectors for a Cartesian co-ordinate system in the deformed configuration. Batra showed that, if the Empirical Inequalities hold, the left Cauchy-Green tensor $\boldsymbol{B}$ must have the form

$$\boldsymbol{B} = diag(\lambda^2, \lambda^{-1}, \lambda^{-1}), \quad \text{constant, positive } \lambda, \tag{1.3}$$

for the incompressible materials of interest here. The issue of determining a corresponding deformation was not considered by Batra. Trivially one such homogeneous deformation, and the deformation must be homogeneous (Blume [2]), has the form

$$x = \lambda X, \quad y = \lambda^{-\frac{1}{2}} Y, \quad z = \lambda^{-\frac{1}{2}} Z, \tag{1.4}$$

denoting the Cartesian coordinates of a typical particle before and after deformation by (X, Y, Z) and (x, y, z) respectively, thus resolving the issue of existence. However the issue of uniqueness remains unresolved, even in this, the simplest of problems.

The corresponding problem for simple shear was first considered by Moon and Truesdell [4] who investigated the consequences of assuming a Cauchy stress of the form

$$\boldsymbol{T} = \tau(\boldsymbol{e}_x \otimes \boldsymbol{e}_y + \boldsymbol{e}_y \otimes \boldsymbol{e}_x), \qquad \text{constant } \tau. \tag{1.5}$$

This study and subsequent work by, amongst others, Mihai and Goriely [5], Destrade *et al.* [6] and Thiel *et al.* [7] shows that the left Cauchy-Green deformation tensor for incompressible materials must have the form

$$(\mathbf{B})_{ij} = \begin{bmatrix} B_{xx} & B_{xy} & 0 \\ B_{xy} & B_{xx} & 0 \\ 0 & 0 & \frac{1}{B_{xx}^2 - B_{xy}^2} \end{bmatrix}. \tag{1.6}$$

Again, as for simple tension, the deformation field cannot be derived from knowledge of $\boldsymbol{B}$ but rather it can be simply shown that (1.6) is compatible with a simple shear superposed on a triaxial stretch so that, for example,

$$x = \lambda_1 X + \lambda_2 \sqrt{1 - \lambda_1^{-1}\lambda_2^{-1}}\, Y, \quad y = \lambda_2 Y, \quad z = \lambda_1^{-1}\lambda_2^{-1} Z, \tag{1.7}$$

with the uniqueness issue again remaining unresolved.

It should be pointed out that the general issue of constitutive modeling for hyperelastic materials using the inverse approach of considering $\boldsymbol{B} = \boldsymbol{g}(\boldsymbol{T})$ has been the subject of several studies in the literature (see, e.g., [8, 9] and references cited therein). The point of view proposed in these works is that such an approach is more realistic physically since one expects deformation to be the result of applied forces rather than the converse. The results to be presented below may be viewed as specific explicit illustrations of this new approach in the context of plane stress. In addition to this issue for the Cauchy stress, we also consider its counterpart for the first Piola-Kirchhoff stress. See also [10] for a recent treatment of simple shear resulting from applied Piola-Kirchhoff stress.

There is a rich tradition of exploring the consequences of assuming plane stress conditions in *linear* elasticity (see, for example, Timoshenko and Goodier [11]). However the corresponding problem for nonlinear elasticity is rarely considered, complicated as it is by the two difficulties mentioned previously for general dual problems. This is the main concern in this paper. If the plane stress problem is formulated in terms of the Cauchy stress, then

$$T_{xz} = T_{yz} = T_{zz} = 0, \tag{1.8}$$

using an obvious notation for the Cartesian components of the stress tensor. It will also be assumed that the remaining in-plane stresses are homogeneous so that the equations of equilibrium are satisfied identically. Trivially then the dual problem for plane stress includes the dual problems of simple tension considered by Batra [3] and simple shear considered by Moon and Truesdell [4] and their approach based on the classical constitutive law of expressing the Cauchy stress in terms of the left Cauchy-Green tensor and its invariants will first be adopted, where the problems of invertibility and existence become immediately apparent. This is described in Sect. 2.

The problem of compatibility is significantly reduced however by assuming plane stress conditions in terms of the Piola-Kirchhoff stress $\boldsymbol{P}$, so that

$$P_{xZ} = P_{yZ} = P_{zX} = P_{zY} = P_{zZ} = 0. \tag{1.9}$$

It is shown in Sects. 3 and 4 that the corresponding deformation field *must* be a plane deformation accompanied by a constant out-of-plane stretch for all incompressible isotropic hyperelastic materials for which the Empirical Inequalities hold. These results are made more explicit for the case of simple shear in Sect. 5. It is shown that the deformation for simple shear requires the solution of a non-linear partial differential equation, a particular solution of which yields a homogeneous deformation. In Sect. 6, it is shown that the general plane stress problem can be solved explicitly for the neo-Hookean material. The problem of simple tension is considered in Sect. 7. It is shown that in addition to the non-linear partial differential equation required for simple shear, the deformation also must satisfy Laplace's equation. A particular solution involving a homogeneous deformation is obtained.

2 Plane Cauchy Stress Problems

The mechanical response of a homogeneous isotropic incompressible hyperelastic solid for which $I_3 = \det \boldsymbol{B} = \det \boldsymbol{C} \equiv 1$ is completely determined by specification of the strain energy per unit undeformed volume $W = W(I_1, I_2)$ where

$$I_1 = \operatorname{tr} \boldsymbol{B}, \quad I_2 = \operatorname{tr} \boldsymbol{B}^{-1}. \tag{2.1}$$

Here $\boldsymbol{B} \equiv \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{C} \equiv \boldsymbol{F}^T\boldsymbol{F}$ are the left and right deformation tensors respectively and $\boldsymbol{F}$ is the deformation gradient tensor. The constitutive law in terms of the Cauchy stress is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\frac{\partial W}{\partial I_1}\boldsymbol{B} - 2\frac{\partial W}{\partial I_2}\boldsymbol{B}^{-1}, \tag{2.2}$$

where p is an arbitrary scalar field. The Empirical Inequalities

$$\frac{\partial W}{\partial I_1} > 0, \quad \frac{\partial W}{\partial I_2} \geq 0, \tag{2.3}$$

will be assumed to hold throughout. The universal relation

$$\boldsymbol{T}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{T}, \tag{2.4}$$

follows immediately from (2.2). For the Cauchy plane stress conditions (1.8) this universal relation yields

$$B_{xy}(T_{xx} - T_{yy}) = T_{xy}(B_{xx} - B_{yy}), \tag{2.5}$$

and

$$B_{xz}T_{xx} + B_{yz}T_{xy} = 0, \quad B_{xz}T_{xy} + B_{yz}T_{yy} = 0. \tag{2.6}$$

The first of these has an immediate interpretation in terms of simple shear. For example, Rivlin's formulation of plane stress simple shear (Rivlin [1]) specifies $B_{xx} = 1 + K^2$, $B_{yy} = 1$, $B_{xy} = K$, where K is the amount of shear. Equation (2.5) then yields the classical universal relation

$$T_{xx} - T_{yy} = KT_{xy}. \tag{2.7}$$

Murphy *et al.* [12] observed that an inverted form of Rivlin's universal relation (2.7) can be obtained by assuming a stress boundary problem for which $T_{xy} \neq 0$ so that

$$B_{xx} - B_{yy} = \frac{T_{xx} - T_{yy}}{T_{xy}}B_{xy}. \tag{2.8}$$

This result also follows directly from (2.5).

Now consider the second set of universal relations (2.6). It follows that there are two categories of plane stress boundary value problems. The first is the set of problems for which

$$T_{xx}T_{yy} - T_{xy}^2 \neq 0, \tag{2.9}$$

exemplified by simple shear for which $T_{xx} = T_{yy} = 0$, $T_{xy} = T \neq 0$. The second category is the set which satisfy the singular condition

$$T_{xx}T_{yy} - T_{xy}^2 = 0,$$

exemplified by simple tension for which $T_{xx} = T \neq 0$, $T_{yy} = T_{xy} = 0$. Counterintuitively then, simple shear could therefore be considered to be a more regular stress boundary value than simple tension, at least within the context of plane Cauchy stress.

The singular simple tension problem was considered in [13] and attention will be focused here instead on the problem of simple shear. It follows immediately from (2.6) than

$$B_{xz} = B_{yz} = 0. \tag{2.10}$$

Solving for the pressure term using the remaining plane stress condition $T_{zz} = 0$ and substitution into the constitutive law (2.2) yields the following in-plane stresses:

$$\begin{aligned} T_{xx} &= 2\frac{\partial W}{\partial I_1}(B_{xx} - B_{zz}) + 2\frac{\partial W}{\partial I_2}\left(\frac{1}{B_{zz}} - B_{zz}B_{yy}\right), \\ T_{yy} &= 2\frac{\partial W}{\partial I_1}\left(B_{yy} - B_{zz}\right) + 2\frac{\partial W}{\partial I_2}\left(\frac{1}{B_{zz}} - B_{zz}B_{xx}\right), \\ T_{xy} &= 2B_{xy}\left(\frac{\partial W}{\partial I_1} + B_{zz}\frac{\partial W}{\partial I_2}\right). \end{aligned} \tag{2.11}$$

This set of equations and the incompressibility constraint $\det \boldsymbol{B} = 1$, which reduces to

$$B_{zz} = \frac{1}{B_{xx}B_{yy} - B_{xy}^2}, \tag{2.12}$$

constitute a system of four equations in the four unknowns B_{xx}, B_{xy}, B_{yy}, B_{zz} for stress boundary value problems. However, inversion of (2.11) in order to obtain the components of $\boldsymbol{B}$ in terms of the in-plane stresses is challenging, even for the simplest of materials. For example, for the neo-Hookean material

$$W = \frac{\mu}{2}(I_1 - 3), \tag{2.13}$$

where μ is the infinitesimal shear modulus, invertibility requires the solving of a cubic equation since substitution of (2.13) into (2.11) yields

$$\begin{aligned} &B_{xy} = \hat{T}_{xy}, \quad B_{yy} = B_{xx} + \hat{T}_{yy} - \hat{T}_{xx}, \\ &B_{xx}^3 + B_{xx}^2(\hat{T}_{yy} - 2\hat{T}_{xx}) + B_{xx}(\hat{T}_{xx}^2 - \hat{T}_{xy}^2 - \hat{T}_{xx}\hat{T}_{yy}) + \hat{T}_{xx}\hat{T}_{xy}^2 - 1 = 0, \end{aligned} \tag{2.14}$$

where the hat notation denotes non-dimensionalisation with respect to μ.

The compatability problems follow immediately from (2.10). In terms of the components of the deformation gradient tensor $\boldsymbol{F}$, these equations have the form

$$F_{xX}F_{zX} + F_{xY}F_{zY} + F_{xZ}F_{zZ} = 0, \quad F_{yX}F_{zX} + F_{yY}F_{zY} + F_{yZ}F_{zZ} = 0. \tag{2.15}$$

Noting that $F_{iJ} \equiv \frac{\partial x_i}{\partial X_J}$, where (X, Y, Z), (x, y, z) are the Cartesian coordinates of a typical particle before and after deformation respectively, it is easily seen that (2.15) constitute an underdetermined system of non-linear coupled partial differential equations in (x, y, z). A semi-inverse approach is typically adopted (Moon and Truesdell [4], Mihai and Goriely [5], Destrade *et al.* [6]) to solve this system in which it is assumed that

$$x = x(X, Y), \quad y = y(X, Y), \quad z = z(Z). \tag{2.16}$$

The incompressibility constraint then yields

$$z = \lambda Z, \quad \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial x}{\partial Y}\frac{\partial y}{\partial X} = \frac{1}{\lambda}. \tag{2.17}$$

Trivially then

$$F_{zX} = F_{zY} = F_{xZ} = F_{yZ} = 0, \tag{2.18}$$

and the equations (2.15) are identically satisfied.

Assume now that the constitutive law is invertible. For deformations of the form (2.16), the non-identically zero components of $\boldsymbol{B}$ can be expressed in terms of the components of the deformation gradient tensor as follows:

$$B_{xx} = F_{xX}^2 + F_{xY}^2, \quad B_{yy} = F_{yY}^2 + F_{yX}^2, \quad B_{xy} = F_{xX}F_{yX} + F_{yY}F_{xY}, \quad B_{zz} = \lambda^2. \tag{2.19}$$

Classically the ultimate goal in the analysis of plane stress boundary value problems is the determination of the displacement field of a typical particle. Therefore $(2.19)_{1,2,3}$ constitute an over-determined system of non-linear partial differential equations for the in-plane coordinates (x, y) of a typical particle in the deformed configuration. When the applied stress field is homogeneous, and therefore $\boldsymbol{B}$ is homogeneous, another semi-inverse approach is usually adopted with deformation tensors of the form (2.19) being interpreted as describing a simple shear superposed on a triaxial stretch (Moon and Truesdell [4], Mihai and Goriely [5], Destrade *et al.* [6]) so that the deformation gradient tensor is assumed to be of the form

$$(\mathbf{F})_{ij} = \begin{bmatrix} \lambda_1 & \lambda_2\sqrt{1-\lambda_1^2\lambda_2^{-2}} & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix}. \tag{2.20}$$

In contrast, it is shown later that for Piola-Kirchhoff plane stress problems a semi-inverse approach leading to (2.16) is not required and the non-uniqueness in the determination of the deformation gradient tensor can be more prescribed. This might suggest that the Piola-Kirchhoff formulation of plane stress is a more natural choice.

3 Piola-Kirchhoff Plane Stress Problems

Let $\boldsymbol{P}$ denote the first Piola-Kirchhoff stress tensor so that in general

$$\boldsymbol{T} = \boldsymbol{P}\boldsymbol{F}^T, \tag{3.1}$$

noting that for the incompressible materials of interest here $J \equiv \det \boldsymbol{F} \equiv 1$. The symmetry of the Cauchy stress tensor follows from conservation of angular momentum and therefore

$$\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T. \tag{3.2}$$

If the Piola-Kirchhoff plane stress conditions (1.9) are assumed then (3.2) yields

$$F_{xX}P_{yX} + F_{xY}P_{yY} = F_{yX}P_{xX} + F_{yY}P_{xY}, \tag{3.3}$$

and

$$F_{zX}P_{xX} + F_{zY}P_{xY} = 0, \quad F_{zX}P_{yX} + F_{zY}P_{yY} = 0, \tag{3.4}$$

which could be viewed as the Piola-Kirchhoff equivalents of the Cauchy stress universal relations (2.5), (2.6). Note however that the Piola-Kirchhoff relations hold *for all deformable solids*, whereas the Cauchy relations (2.4) are only valid for isotropic materials.

As for plane Cauchy stress problems, there are therefore two classes of plane stress problems for deformable solids when Piola-Kirchhoff stress is being considered. The first is that class for which

$$P_{xX}P_{yY} - P_{xY}P_{yX} \neq 0, \tag{3.5}$$

exemplified by the problem of simple shear for which

$$P_{xX} = P_{yY} = 0, \quad P_{xY} \neq 0, P_{yX} \neq 0. \tag{3.6}$$

The second is the seemingly singular class for which

$$P_{xX}P_{yY} - P_{xY}P_{yX} = 0. \tag{3.7}$$

This class can be considered to describe simple tension since trivially classical simple tension

$$P_{xX} = P \neq 0, \quad P_{yY} = P_{xY} = P_{yX} = 0, \tag{3.8}$$

satisfies (3.7) as does the stress distribution

$$P_{xX} = P_{yY} = P_{xY} = P_{yX} \equiv P \neq 0, \tag{3.9}$$

which describes the experiment in which equal and opposite forces are applied to opposing vertices of a cuboid specimen (Murphy [13]).

First consider the class of simple shear plane stress problems characterised by (3.5), with the simple tension class defined by (3.7) considered later. It follows from (3.4) that

$$F_{zX} = F_{zY} = 0, \tag{3.10}$$

and therefore

$$F_{zZ} = F_{zZ}(Z) \iff z = z(Z). \tag{3.11}$$

The deformation gradient tensor therefore has the form

$$(\mathbf{F})_{iJ} = \begin{bmatrix} F_{xX} & F_{xY} & F_{xZ} \\ F_{yX} & F_{yY} & F_{yZ} \\ 0 & 0 & F_{zZ}(Z) \end{bmatrix}, \tag{3.12}$$

and therefore for incompressible materials

$$\det \boldsymbol{F} = (F_{xX}F_{yY} - F_{xY}F_{yX})F_{zZ} = 1. \tag{3.13}$$

It follows from (3.12) that

$$(\mathbf{F}^{-T})_{iJ} = \begin{bmatrix} F_{yY}F_{zZ} & -F_{yX}F_{zZ} & 0 \\ -F_{xY}F_{zZ} & F_{xX}F_{zZ} & 0 \\ F_{xY}F_{yZ} - F_{yY}F_{xZ} & F_{yX}F_{xZ} - F_{xX}F_{yZ} & \frac{1}{F_{zZ}} \end{bmatrix}, \tag{3.14}$$

$$(\mathbf{C})_{IJ} = \begin{bmatrix} F_{xX}^2 + F_{yX}^2 & F_{xY}F_{xX} + F_{yX}F_{yY} & F_{xZ}F_{xX} + F_{yZ}F_{yX} \\ F_{xY}F_{xX} + F_{yX}F_{yY} & F_{xY}^2 + F_{yY}^2 & F_{xZ}F_{xY} + F_{yZ}F_{yY} \\ F_{xZ}F_{xX} + F_{yZ}F_{yX} & F_{xZ}F_{xY} + F_{yZ}F_{yY} & F_{xZ}^2 + F_{yZ}^2 + F_{zZ}^2 \end{bmatrix}, \tag{3.15}$$

$$(\mathbf{B})_{ij} = \begin{bmatrix} F_{xX}^2 + F_{xY}^2 + F_{xZ}^2 & F_{yX}F_{xX} + F_{xY}F_{yY} + F_{xZ}F_{yZ} & F_{xZ}F_{zZ} \\ F_{yX}F_{xX} + F_{xY}F_{yY} + F_{xZ}F_{yZ} & F_{yX}^2 + F_{yY}^2 + F_{yZ}^2 & F_{yZ}F_{zZ} \\ F_{xZ}F_{zZ} & F_{yZ}F_{zZ} & F_{zZ}^2 \end{bmatrix}, \tag{3.16}$$

and

$$I_1 = tr\,(\boldsymbol{C}) = tr\,(\boldsymbol{B}) = F_{xX}^2 + F_{yY}^2 + F_{zZ}^2 + F_{yX}^2 + F_{xY}^2 + F_{xZ}^2 + F_{yZ}^2.$$

4 Hyperelastic Incompressible Isotropic Solids

To make further progress in the analysis of Piola-Kirchhoff plane stress problems, the constitutive law must be specified. The constitutive law for homogeneous isotropic incompressible materials in terms of the Piola-Kirchhoff stress can be written in the form

$$\boldsymbol{P} = -p\boldsymbol{F}^{-T} + 2\frac{\partial W}{\partial I_1}\boldsymbol{F} + 2\frac{\partial W}{\partial I_2}\left(I_1\boldsymbol{F} - \boldsymbol{FC}\right), \tag{4.1}$$

where p is the undetermined hydrostatic pressure. It follows from the results of the last section that

$$\begin{aligned} P_{xZ} &= 2F_{xZ}\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(F_{yY}^2 + F_{yX}^2)\right) - 2F_{yZ}\frac{\partial W}{\partial I_2}(F_{xX}F_{yX} + F_{xY}F_{yY}), \\ P_{yZ} &= 2F_{yZ}\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(F_{xX}^2 + F_{xY}^2)\right) - 2F_{xZ}\frac{\partial W}{\partial I_2}(F_{xX}F_{yX} + F_{xY}F_{yY}). \end{aligned} \tag{4.2}$$

Since the determinant of the coefficients of the F_{xZ}, F_{yZ} terms

$$4\left(\frac{\partial W}{\partial I_1}\right)^2 + 4\frac{\partial W}{\partial I_1}\frac{\partial W}{\partial I_2}(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2) + 4\left(\frac{\partial W}{\partial I_2}\right)^2(F_{xX}F_{yY} - F_{xY}F_{yX})^2 > 0,$$

by virtue of the Empirical Inequalities (2.3), it follows that the plane stress conditions $P_{xZ} = P_{yZ} = 0$ can be satisfied if and only if

$$F_{xZ} = F_{yZ} = 0 \iff x = x(X, Y), \quad y = y(X, Y). \tag{4.3}$$

It then follows from this and (3.11) that the incompressibility condition now yields

$$z = \lambda Z, \tag{4.4}$$

so that

$$F_{xX} F_{yY} - F_{xY} F_{yX} = \frac{1}{\lambda}. \tag{4.5}$$

Thus if the plane *shear* stress conditions

$$P_{xZ} = P_{yZ} = P_{zX} = P_{zY} = 0, \tag{4.6}$$

hold then the resulting deformation for all incompressible isotropic materials must be a plane deformation (4.3) restricted by (4.5) accompanied by a uniform out-of-plane contraction ($\lambda < 1$) or expansion ($\lambda > 1$).

The deformation tensors now have the following simplified forms:

$$(\mathbf{F})_{iJ} = \begin{bmatrix} F_{xX} & F_{xY} & 0 \\ F_{yX} & F_{yY} & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad (\mathbf{F}^{-T})_{iJ} = \begin{bmatrix} \lambda F_{yY} & -\lambda F_{yX} & 0 \\ -\lambda F_{xY} & \lambda F_{xX} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}, \tag{4.7}$$

$$(\mathbf{C})_{IJ} = \begin{bmatrix} F_{xX}^2 + F_{yX}^2 & F_{xY} F_{xX} + F_{yX} F_{yY} & 0 \\ F_{xY} F_{xX} + F_{yX} F_{yY} & F_{xY}^2 + F_{yY}^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}, \tag{4.8}$$

$$(\mathbf{C}^{-1})_{IJ} = \begin{bmatrix} \lambda^2 (F_{xY}^2 + F_{yY}^2) & -\lambda^2 (F_{xY} F_{xX} - F_{yX} F_{yY}) & 0 \\ -\lambda^2 (F_{xY} F_{xX} + F_{yX} F_{yY}) & \lambda^2 (F_{xX}^2 + F_{yX}^2) & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}. \tag{4.9}$$

The corresponding invariants are therefore given by

$$I_1 = F_{xX}^2 + F_{yY}^2 + F_{yX}^2 + F_{xY}^2 + \lambda^2, \quad I_2 = \lambda^2 (F_{xX}^2 + F_{yY}^2 + F_{yX}^2 + F_{xY}^2) + \frac{1}{\lambda^2}. \tag{4.10}$$

It remains to satisfy the normal plane stress condition

$$P_{zZ} = 0, \tag{4.11}$$

which is satisfied if and only if

$$p = 2\lambda^2 \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} (F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2) \right). \tag{4.12}$$

The in-plane stress components for plane stress conditions therefore have the form

$$P_{xX} = 2\frac{\partial W}{\partial I_1}(F_{xX} - \lambda^3 F_{yY}) + 2\frac{\partial W}{\partial I_2}\left(\lambda^2 F_{xX} + \frac{F_{yY}}{\lambda} - \lambda^3 F_{yY}(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2) \right),$$

$$P_{yY} = 2\frac{\partial W}{\partial I_1}(F_{yY} - \lambda^3 F_{xX}) + 2\frac{\partial W}{\partial I_2}\left(\lambda^2 F_{yY} + \frac{F_{xX}}{\lambda} - \lambda^3 F_{xX}(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2) \right),$$

$$P_{xY} = 2\frac{\partial W}{\partial I_1}(F_{xY} + \lambda^3 F_{yX}) + 2\frac{\partial W}{\partial I_2}\left(\lambda^2 F_{xY} - \frac{F_{yX}}{\lambda} + \lambda^3 F_{yX}(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2) \right),$$

$$P_{yX} = 2\frac{\partial W}{\partial I_1}(F_{yX} + \lambda^3 F_{xY}) + 2\frac{\partial W}{\partial I_2}\left(\lambda^2 F_{yX} - \frac{F_{xY}}{\lambda} + \lambda^3 F_{xY}(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2)\right). \quad (4.13)$$

It is easy to check that the relation (3.3) is satisfied. Assume here and in what follows that

$$F_{xX} > 0, \; F_{yY} > 0. \quad (4.14)$$

Two approaches will be adopted here to make further progress. The first is to consider those problems for which two of the stresses are specified to be identically zero. In that special case a universal relation valid for all materials can be obtained. The second, which is considered in Sect. 6, is to specify the strain energy and for ease of exposition only the neo-Hookean material will be considered.

5 Simple Shear

Now consider the problem of pure simple shear (3.6). It follows from the Empirical Inequalities (2.3) and $(4.13)_{1,2}$ that $(3.6)_{1,2}$ are satisfied if and only if

$$\left(F_{xX}^2 - F_{yY}^2\right)\left(\frac{1}{\lambda} + \lambda^5 - \lambda^3(F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2)\right) = 0. \quad (5.1)$$

Consider first the solution for which $F_{xX} = F_{yY} > 0$ or alternatively that

$$\frac{\partial x}{\partial X} = \frac{\partial y}{\partial Y}.$$

Assuming that $x = x(X, Y)$, $y = y(X, Y)$ are sufficiently regular, this condition is equivalent to the existence of a deformation potential function $\psi(X, Y)$ such that

$$x = \frac{\partial \psi}{\partial Y}, \quad y = \frac{\partial \psi}{\partial X}, \quad \frac{\partial^2 \psi}{\partial X \partial Y} > 0. \quad (5.2)$$

Enforcing the incompressibility condition (4.5) yields the determining non-linear partial differential equation for ψ,

$$\left(\frac{\partial^2 \psi}{\partial X \partial Y}\right)^2 - \frac{\partial^2 \psi}{\partial X^2}\frac{\partial^2 \psi}{\partial Y^2} = \frac{1}{\lambda}. \quad (5.3)$$

Only the consequences of the particular solution to (5.3),

$$\frac{\partial^2 \psi}{\partial X \partial Y} = c_1 > 0, \quad \frac{\partial^2 \psi}{\partial X^2} = c_2, \quad \frac{\partial^2 \psi}{\partial Y^2} = c_3, \quad (5.4)$$

where c_i are constants satisfying $c_1^2 - c_2 c_3 > 0$, are explored here. The consequences of more general solutions to the nonlinear hyperbolic Monge-Ampère partial differential equation (5.3) will be examined elsewhere. Solving (5.4) and ignoring constant and translational terms yields

$$\psi = c_1 XY + \frac{1}{2}c_2 X^2 + \frac{1}{2}c_3 Y^2. \quad (5.5)$$

The deformation of a typical particle then follows from (4.4) and (5.2) and is given by

$$x = c_1 X + c_3 Y, \quad y = c_1 Y + c_2 X, \quad z = \lambda Z, \tag{5.6}$$

where, to satisfy (5.3),

$$c_1^2 - c_2 c_3 = \frac{1}{\lambda}. \tag{5.7}$$

The in-plane normal stress conditions $P_{xX} = P_{yY} = 0$ and $(4.13)_{1,2}$ then yield

$$0 = c_1 \left(\frac{\partial W}{\partial I_1}(1 - \lambda^3) + \frac{\partial W}{\partial I_2} \left(\frac{1}{\lambda} - \lambda^2 - \lambda^3 (c_2 + c_3)^2 \right) \right), \tag{5.8}$$

on using the incompressibility condition. Since c_1 has been assumed positive, it therefore follows that

$$\frac{\partial W}{\partial I_1}(1 - \lambda^3) + \frac{\partial W}{\partial I_2} \left(\frac{1}{\lambda} - \lambda^2 - \lambda^3 (c_2 + c_3)^2 \right) = 0, \tag{5.9}$$

where now

$$I_1 = \lambda^2 + \frac{2}{\lambda} + (c_2 + c_3)^2, \quad I_2 = 2\lambda + \frac{1}{\lambda^2} + \lambda^2 (c_2 + c_3)^2.$$

The universal relation (3.3) now yields

$$P_{yX} = P_{xY}.$$

Adding $(4.13)_{3,4}$ then yields

$$\begin{aligned} P_{xY} = P_{yX} &= (c_2 + c_3) \left(\frac{\partial W}{\partial I_1}(1 + \lambda^3) + \frac{\partial W}{\partial I_2}(3\lambda^2 - \frac{1}{\lambda} + \lambda^3 (c_2 + c_3)^2) \right), \\ &= 2(c_2 + c_3) \left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2} \right), \end{aligned} \tag{5.10}$$

on using (5.9). To ensure a non-zero shear stress, it must be assumed that

$$c_2 + c_3 \neq 0. \tag{5.11}$$

The consequences of the second branch of the solution to (5.1) are now explored. Assume then that

$$\frac{1}{\lambda} + \lambda^5 = \lambda^3 (F_{xX}^2 + F_{yY}^2 + F_{xY}^2 + F_{yX}^2). \tag{5.12}$$

The in-plane normal stresses are then given by

$$0 = (F_{xX} - \lambda^3 F_{yY}) \left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2} \right), \quad 0 = (F_{yY} - \lambda^3 F_{xX}) \left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2} \right), \tag{5.13}$$

which, on using the Empirical Inequalities, yield

$$0 = F_{xX} - \lambda^3 F_{yY} = F_{yY} - \lambda^3 F_{xX}. \tag{5.14}$$

If $\lambda \neq 1$, then $F_{xX} = F_{yY} = 0$, contradicting the assumption (4.14). Assume then that $\lambda = 1$ and therefore that $F_{xX} = F_{yY}$, from (5.14). The governing equation (5.12) and the incompressibility constraint (4.5) then take the respective forms

$$2 = 2F_{xX}^2 + F_{xY}^2 + F_{yX}^2, \quad F_{xX}^2 - F_{xY}F_{yX} = 1,$$

which yield

$$F_{xY} = -F_{yX}.$$

Given that the shear stresses are now identically zero, the conclusion is that (5.12) is not a valid potential solution branch for (5.1) and therefore we have the unique solution that $F_{xX} = F_{yY} > 0$.

6 The Neo-Hookean Material

Let $\hat{P}_{iA} \equiv \frac{P_{iA}}{\mu}$. Then the in-plane stress equations (4.13) can be inverted for the neo-Hookean material (2.13) to obtain

$$\begin{aligned}
(1-\lambda^6)F_{xX} &= \hat{P}_{xX} + \lambda^3 \hat{P}_{yY}, \\
(1-\lambda^6)F_{yY} &= \hat{P}_{yY} + \lambda^3 \hat{P}_{xX}, \\
(1-\lambda^6)F_{xY} &= \hat{P}_{xY} - \lambda^3 \hat{P}_{yX}, \\
(1-\lambda^6)F_{yX} &= \hat{P}_{yX} - \lambda^3 \hat{P}_{xY}.
\end{aligned} \tag{6.1}$$

Substitution into the incompressibility relation (4.5) then yields

$$(\hat{P}_{xX}\hat{P}_{yY} - \hat{P}_{xY}\hat{P}_{yX})\lambda^6 + (\hat{P}_{xX}^2 + \hat{P}_{yY}^2 + \hat{P}_{xY}^2 + \hat{P}_{yX}^2)\lambda^3 + \hat{P}_{xX}\hat{P}_{yY} - \hat{P}_{xY}\hat{P}_{yX} = \frac{(1-\lambda^6)^2}{\lambda}. \tag{6.2}$$

Since it has been assumed that $\hat{P}_{xX}\hat{P}_{yY} - \hat{P}_{xY}\hat{P}_{yX} \neq 0$, first assume that

$$\hat{P}_{xX}\hat{P}_{yY} - \hat{P}_{xY}\hat{P}_{yX} > 0, \tag{6.3}$$

so that the normal stresses are dominant. Then there are two solutions to (6.2), one contractile with $\lambda < 1$ and one extensile with $\lambda > 1$. Now assume that the shear stresses are dominant with

$$\hat{P}_{xX}\hat{P}_{yY} - \hat{P}_{xY}\hat{P}_{yX} < 0. \tag{6.4}$$

Again there are two solutions to (6.2), one contractile and one extensile, with the exception of the unique solution $\lambda = 1$ when

$$\hat{P}_{xX} = -\hat{P}_{yY} \equiv T, \quad \hat{P}_{xY} = \hat{P}_{yX} \equiv S. \tag{6.5}$$

In this case it now follows from the plane stress constitutive law that

$$T = F_{xX} - F_{yY}, \quad S = F_{xY} + F_{yX}. \tag{6.6}$$

There is therefore a non-uniqueness in both the normal and shear components of the deformation gradient tensor for this class of plane stress boundary value problems.

It follows from (6.1) therefore that when $\lambda \neq 1$ the general plane stress deformation has the form

$$\begin{aligned} x &= \frac{\hat{P}_{xX} + \lambda^3 \hat{P}_{yY}}{1-\lambda^6} X + \frac{\hat{P}_{xY} - \lambda^3 \hat{P}_{yX}}{1-\lambda^6} Y, \\ y &= \frac{\hat{P}_{yX} - \lambda^3 \hat{P}_{xY}}{1-\lambda^6} X + \frac{\hat{P}_{yY} + \lambda^3 \hat{P}_{xX}}{1-\lambda^6} Y, \\ z &= \lambda Z, \end{aligned} \tag{6.7}$$

with λ determined from (6.2).

7 Simple Tension

Now consider the problem of determining the deformation for the class of plane stress simple tension problems defined by the condition that

$$P_{xX} P_{yY} - P_{xY} P_{yX} = 0. \tag{7.1}$$

The analysis of the general case is complicated by the fact the universal relations (3.3), (3.4) do not allow the simplification of the deformation gradient tensor that was possible for simple shear problems. Some progress can be made for some important special cases such as simple and biaxial tension and the case of uniform in-plane stress for which

$$P_{xX} = P_{yY} = P_{xY} = P_{yX}. \tag{7.2}$$

For illustrative purposes only simple tension is considered. Assume then that

$$P_{yY} = P_{xY} = P_{yX} = 0, \quad P_{xX} \neq 0, \tag{7.3}$$

which corresponds to Piola-Kirchhoff simple tension in the X-direction. Then (7.1) is trivially satisfied and the universal relations (3.3), (3.4) yield

$$F_{yX} = F_{zX} = 0. \tag{7.4}$$

The deformation gradient tensor now has the form

$$(\mathbf{F})_{iJ} = \begin{bmatrix} F_{xX} & F_{xY} & F_{xZ} \\ 0 & F_{yY} & F_{yZ} \\ 0 & F_{zY} & F_{zZ} \end{bmatrix}, \tag{7.5}$$

and therefore for incompressible materials

$$\det \boldsymbol{F} = F_{xX}(F_{yY} F_{zZ} - F_{yZ} F_{zY}) = 1. \tag{7.6}$$

Denoting partial differentiation using the comma notation, it follows from (7.6) that $F_{xX,X} = 0$ and therefore

$$x = A(Y, Z)X + B(Y, Z), \quad \text{arbitrary } A, B. \tag{7.7}$$

This is therefore the unique form of the x-component of the deformation field for all deformable incompressible solids *regardless of symmetry*.

It follows from (7.5) that

$$(\mathbf{F}^{-T})_{iJ} = \begin{bmatrix} \frac{1}{F_{xX}} & 0 & 0 \\ F_{xZ}F_{zY} - F_{xY}F_{zZ} & F_{xX}F_{zZ} & -F_{xX}F_{zY} \\ F_{xY}F_{yZ} - F_{yY}F_{xZ} & -F_{xX}F_{yZ} & F_{xX}F_{yY} \end{bmatrix}, \tag{7.8}$$

$$(\mathbf{C})_{IJ} = \begin{bmatrix} F_{xX}^2 & F_{xY}F_{xX} & F_{xZ}F_{xX} \\ F_{xY}F_{xX} & F_{xY}^2 + F_{yY}^2 + F_{zY}^2 & F_{xZ}F_{xY} + F_{yZ}F_{yY} + F_{zY}F_{zZ} \\ F_{xZ}F_{xX} & F_{xZ}F_{xY} + F_{yZ}F_{yY} + F_{zY}F_{zZ} & F_{xZ}^2 + F_{yZ}^2 + F_{zZ}^2 \end{bmatrix}. \tag{7.9}$$

The constitutive law (4.1) now yields

$$P_{xY} = 2F_{xY}\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(F_{zZ}^2 + F_{yZ}^2)\right) - 2F_{xZ}\frac{\partial W}{\partial I_2}(F_{yY}F_{yZ} + F_{zZ}F_{zY}),$$
$$P_{xZ} = 2F_{xZ}\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}(F_{yY}^2 + F_{zY}^2)\right) - 2F_{xY}\frac{\partial W}{\partial I_2}(F_{yY}F_{yZ} + F_{zY}F_{zZ}). \tag{7.10}$$

The Empirical Inequalities (2.3) then yield that $P_{xY} = P_{xZ} = 0$ if and only if

$$F_{xY} = F_{xZ} = 0.$$

It follows from (7.7) that the x-component of the deformation field for incompressible, isotropic hyperelastic materials in simple tension must have the form

$$x = \lambda X, \quad \text{constant}\,\lambda. \tag{7.11}$$

The simple tension conditions $P_{yY} = P_{zZ} = 0$ now take the respective forms

$$0 = 2F_{yY}\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right) + F_{zZ}\left(-p\lambda + 2\frac{\partial W}{\partial I_2}(F_{yY}F_{zZ} - F_{yZ}F_{zY})\right),$$
$$0 = 2F_{zZ}\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right) + F_{yY}\left(-p\lambda + 2\frac{\partial W}{\partial I_2}(F_{yY}F_{zZ} - F_{yZ}F_{zY})\right). \tag{7.12}$$

Multiplying the first of these by F_{yY}, the second by F_{zZ} and subtracting yields

$$F_{yY} = \pm F_{zZ},$$

on using the Empirical Inequalities (2.3). Assume then that $F_{yY} = F_{zZ} \neq 0$. Then mirroring the analysis for simple shear it follows that there exists a potential function Ψ such that

$$y = \frac{\partial \Psi}{\partial Z}, \quad z = \frac{\partial \Psi}{\partial Y}, \tag{7.13}$$

assuming sufficient regularity of the displacement field, with the potential function again satisfying the Monge-Ampére partial differential equation (5.3) with the appropriate changes in independent variables. The analysis following (5.3) for simple shear is therefore also valid for simple tension and will not be repeated here.

To reflect the more singular nature of simple tension, a further restriction other than that necessary for simple shear must also be imposed here. First note that it follows from (7.12) that

$$p\lambda = 2\frac{\partial W}{\partial I_1} + 2\frac{\partial W}{\partial I_2}\left(\lambda^2 + F_{yY}F_{zZ} - F_{yZ}F_{zY}\right). \tag{7.14}$$

The remaining zero stress conditions $P_{yZ} = P_{zY} = 0$ then yield, on substitution from (7.14),

$$\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right)(F_{zY} + F_{yZ}) = 0, \tag{7.15}$$

so that

$$F_{zY} = -F_{yZ}. \tag{7.16}$$

on employing the Empirical Inequalities (2.3). The potential representation (7.13) then shows that Ψ must therefore also satisfy the two-dimensional Laplace's equation

$$\frac{\partial^2\Psi}{\partial Y^2} + \frac{\partial^2\Psi}{\partial Z^2} = 0, \tag{7.17}$$

in addition to the Monge-Ampére equation. Again only the trivial solution (5.5)

$$\Psi = c_1 YZ + \frac{1}{2}c_2 Y^2 + \frac{1}{2}c_3 Z^2 \tag{7.18}$$

to the Monge-Ampére equation will be considered. Substitution into Laplace's Equation then yields

$$c_2 + c_3 = 0,$$

in contrast to the condition (5.11) for simple shear. The displacement field is thus given by

$$x = \lambda X, \quad y = c_1 Y - c_2 Z, \quad z = c_1 Z + c_2 Y. \tag{7.19}$$

The incompressibility condition now simplifies to

$$c_1^2 + c_2^2 = \frac{1}{\lambda}. \tag{7.20}$$

Finally, the simple tension stress condition $P_{xX} = P$ yields

$$P = 2\frac{\partial W}{\partial I_1}\left(\lambda - \frac{1}{\lambda^2}\right) + 2\lambda\frac{\partial W}{\partial I_2}\left(\lambda + \frac{1}{\lambda^2} + F_{yY}^2 + F_{zZ}^2 + F_{yZ}^2 + F_{zY}^2)\right), \tag{7.21}$$

where

$$I_1 = \lambda^2 + F_{yY}^2 + F_{zZ}^2 + F_{zY}^2 + F_{yZ}^2, \quad I_2 = \frac{1}{\lambda^2} + \lambda^2(F_{yY}^2 + F_{zZ}^2 + F_{xY}^2 + F_{zY}^2 + F_{xZ}^2 + F_{yZ}^2).$$

8 Concluding Remarks

The objective of this work was to examine two different formulations of plane stress problems for incompressible isotropic hyperelastic materials. The first approach involves an initial prescription of a prescribed Cauchy stress field and an investigation of how much information can be deduced on the corresponding stretch tensor and deformation. Early work on this issue for special cases was carried out by Batra [3] for simple tension and by Moon and Truesdell [4] for simple shear. More recent work by other authors has been described in the Introduction. The second approach is concerned with prescription of a plane stress state in terms of the Piola-Kirchhoff stress and it is shown here that this formulation is more tractable analytically. All these developments have been carried out for a general incompressible isotropic hyperelastic material with strain-energy density expressed in terms of the two classical principal invariants. For the special case of a neo-Hookean material, an explicit representation for the general plane stress deformation in terms of the prescribed Piola-Kirchhoff stresses was obtained.

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