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RESEARCH

Renormalized Energy of a Dislocation Loop in a 3D Anisotropic Body

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For Roger Fosdick

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Abstract
The paper presents a rigorous analysis of the singularities of elastic fields near a dislocation loop in a body of arbitrary material symmetry that extends over the entire three-space. Explicit asymptotic formulas are given for the stress, strain and the incompatible distortion near the curved dislocation. These formulas are used to analyze the main object of the paper, the renormalized energy. The core-cutoff method is used to introduce that notion: first, a core in the form of a curved tube along the dislocation loop is removed; then, the energy of the complement is determined (= the core-cutoff energy). As in the case of a straight dislocation, the core-cutoff energy has a singularity that is proportional to the logarithm of the core radius. The renormalized energy is the limit, as the radius tends to 0, of the core-cutoff energy minus the singular logarithmic part. The main result of the paper are novel formulas for the coefficient of logarithmic singularity (the 'prelogarithmic energy factor') and for the renormalized energy.

Keywords Dislocations · Incompatible distortion field · Regularized energy · Prelogarithmic energy factor · Rigorous asymptotic analysis

Mathematics Subject Classification (2020) 35J50 · 35Q74 · 74G70

1 Introduction

A dislocation is an imperfection in the lattice structure of the crystal. At the macroscopic level, a dislocation is modeled as a defective, incompatible deformation in a continuous body. The paper deals with a linearly elastic body of arbitrary symmetry that contains a dislocation loop. The body occupies the whole three-dimensional space $\mathbf{R}^3$ and is free from external forces. The goal of the paper is to analyze rigorously the asymptotics of deformation and energy near the dislocation.

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The dislocation loop is represented by a closed non-intersecting curve $\mathscr{C} \subset \mathbf{R}^3$ (see Sect. 3) and by the Burgers vector $b \in \mathbf{R}^3$ (see Sect. 4). It is assumed that $\mathscr{C}$ has a twice continuously differentiable arc-length parametrization. We give $\mathscr{C}$ one of the two orientations and denote by $\tau(x)$ the unit tangent vector to $\mathscr{C}$ at $x \in \mathscr{C}$. A defective deformation is described by an incompatible distortion field $H : \mathbf{R}^3 \setminus \mathscr{C} \to \mathrm{Ten}^2$ i.e., H is not the gradient of a globally defined deformation $u : \mathbf{R}^3 \to \mathbf{R}^3$. This distinguishes the elastic dislocation theory from the classical elasticity, where H coincides with the displacement gradient ∇u of a globally defined deformation $u : \mathbf{R}^3 \to \mathbf{R}^3$. In the elastic dislocation theory H is a 'primitive' to be determined from the balance and constitutive equations.

Conventions on tensors We denote by Ten^l the space of tensors of order l on $\mathbf{R}^3$ (for any nonnegative integer l). Only orders from 0 to 4 are needed in the succeeding treatment. The spaces Ten^0 and Ten^1 have the standard representations. The space Ten^2 is identified with the space of linear transformations from $\mathbf{R}^3$ into itself. $\mathrm{Ten}^2_{\mathrm{sym}} \subset \mathrm{Ten}^2$ is the subspace of symmetric second-order tensors. The space Ten^3 is identified with the space of linear transformations T from $\mathbf{R}^3$ into Ten^2; the value of T on a vector $b \in \mathbf{R}^3$ is denoted by $\mathsf{T}[b]$. The space Ten^4 is identified with the space of linear transformations C from Ten^2 into itself; we again use square brackets to denote the linear argument of C, i.e., $\mathsf{C}[A]$ is the value of C on $A \in \mathrm{Ten}^2$.

The material is characterized by the fourth-order elasticity tensor C which has the major symmetry

$$\mathsf{C}[H_1] \cdot H_2 = H_1 \cdot \mathsf{C}[H_2]$$

for every $H_1, H_2 \in \mathrm{Ten}^2_{\mathrm{sym}}$, the minor symmetries

$$\mathsf{C}[H] = \mathsf{C}[H]^{\mathrm{T}}, \quad \mathsf{C}[H] = \mathsf{C}[H^{\mathrm{T}}], \tag{1}$$

for every $H \in \mathrm{Ten}^2$, and is positive definite on symmetric tensors

$$\mathsf{C}[E] \cdot E > 0$$

for every nonzero $E \in \mathrm{Ten}^2_{\mathrm{sym}}$. For notational convenience we allow non-symmetric arguments of C, but Equation $(1)_2$ shows that $\mathsf{C}[H]$ depends only on the symmetric part of H.

The strain E, stress T, and the stored energy W corresponding to an incompatible distortion H are given by

$$E = \frac{1}{2}(H + H^{\mathrm{T}}), \quad T = \mathsf{C}[H], \tag{2}$$

and

$$W(H) = \frac{1}{2}\mathsf{C}[H] \cdot H.$$

Given a dislocation loop $\mathscr{C}$ with the Burgers vector $b \in \mathbf{R}^3$, we seek to determine the distortion tensor field $H \in L^1_{\mathrm{loc}}(\mathbf{R}^3, \mathrm{Ten}^2)$ satisfying the system

$$\left.\begin{array}{ll} \operatorname{curl} H = -b \otimes \tau\, \delta_{\mathscr{C}} & \text{in } \mathbf{R}^3, \\ \operatorname{div} \mathsf{C}[H] = 0 & \text{in } \mathbf{R}^3, \\ H(x) \to 0 & \text{as } |x| \to \infty. \end{array}\right\} \tag{3}$$

Here the curl and divergence are interpreted in the sense of distributions (see Sect. 2), τ is the unit tangent to the curve $\mathscr{C}$, $\delta_{\mathscr{C}}$ is the length measure restricted to $\mathscr{C}$ (i.e., the 'line analog' of Dirac's delta function, see Sect. 4).

Theorem 1.1 *The system* (3) *has a unique solution* $H \in L^1_{\text{loc}}(\mathbf{R}^3, \text{Ten}^2)$, *given by*

$$H(x) = \int_{\mathscr{C}} \frac{\mathsf{K}(x-y)\big[b \otimes \tau(y)\big]}{|x-y|^2}\, dl(y) \tag{4}$$

$x \in \mathbf{R}^3 \setminus \mathscr{C}$, *where* $\mathsf{K} : \mathbf{R}^3 \setminus \{0\} \to \text{Ten}^4$ *is an infinitely differentiable degree* 0 *homogeneous function*.

Here dl is the length element along $\mathscr{C}$ and $y \in \mathscr{C}$ is the integration variable. The function K is said to be degree 0 homogeneous if $\mathsf{K}(\lambda r) = \mathsf{K}(r)$ for every non-zero $r \in \mathbf{R}^3$ and every $\lambda > 0$ (i.e., $\mathsf{K}(r)$ depends on r only through $r/|r|$). Theorem 1.1 is a particular case of [6, Theorem 4.1], which proves the existence and uniqueness of the system (3) with a general divergence-free measure μ in place of the measure $-b \otimes \tau\, \delta_{\mathscr{C}}$. Nevertheless, we give a proof in Sect. 5, based on the Fourier transformation that is similar to that of [6, Theorem 4.1].

The function K in Theorem 1.1 is determined completely by the tensor of elastic constants C, i.e., K is independent of the shape of $\mathscr{C}$. In the construction of the solution in (4), a use will be made of the well-known line integral of the 'Biot–Savart type' (see (36)).

In view of the linear relations (2), the strain and stress are given by equations qualitatively similar to (4), with degree 0 homogeneous functions easily derivable from K.

The form of the right-hand side of (4) shows that H has a singularity at the points of $\mathscr{C}$, i.e., $|H(x)| \to \infty$ as x approaches $\mathscr{C}$.

Sets with positive reach We now describe the singularity of H quantitatively. If $0 < \delta \le \infty$, the tubular δ-neighborhood of a subset M of $\mathbf{R}^n$ is defined by

$$U(M, \delta) = \big\{x \in \mathbf{R}^n : \text{dist}(x, M) < \delta\big\}$$

where

$$\text{dist}(x, M) := \inf\big\{|x-y| : y \in M\big\}$$

is the distance of the point $x \in \mathbf{R}^n$ from M. Following Federer [8], we say that M is a set with positive reach if there exists $\delta \in (0, \infty]$ with the property that for each $x \in U(M, \delta)$ there exists a *unique* closest point x^* on M, i.e., a unique point $x^* \in M$ such that

$$|x - x^*| \le |x - y| \quad \text{for all} \quad y \in M.$$

The supremum of all δ with the just described uniqueness property is referred to as the reach of M and denoted by $\text{reach}(M)$. The map $x \mapsto x^*$, defined on $U\big(M, \text{reach}(M)\big)$, is called the metric projection onto M. Every compact embedded manifold of class C^2 in $\mathbf{R}^n$ is a set with positive reach [8, p. 432]; thus $\mathscr{C}$ is a set with positive reach. The metric projection $x \mapsto x^*$, defined for any $x \in U\big(\mathscr{C}, \text{reach}(\mathscr{C})\big)$, plays a central role in the subsequent considerations.

Let $\mathsf{L} : \text{dom}\,\mathsf{L} \to \text{Ten}^3$ be a function defined on the set

$$\text{dom}\,\mathsf{L} = \big\{(\rho, \sigma) \in \mathbf{R}^3 \times \mathbf{R}^3 : \rho \ne 0,\ |\sigma| = 1,\ \rho \cdot \sigma = 0\big\}$$

by

$$\frac{\mathsf{L}(\rho,\sigma)[b]}{|\rho|}=\int_{\mathbf{R}}\frac{\mathsf{K}(\rho+t\sigma)[b\otimes\sigma]}{|\rho+t\sigma|^2}\,dt \tag{5}$$

for every $(\rho,\sigma)\in\operatorname{dom}\mathsf{L}$ and every $b\in\mathbf{R}^3$. Since K is bounded and $|\rho+t\sigma|^2=|\rho|^2+t^2$ for every $(\rho,\sigma)\in\operatorname{dom}\mathsf{L}$, the integral in (5) converges; moreover, the function $\mathsf{L}(\cdot,\sigma)$ is degree 0 homogeneous for each $\sigma\in\mathbf{S}^2$, as a simple scaling of the variable t in (5) shows.

Note The definition (5) shows that the function $\mathsf{L}(\rho,\sigma)/|\rho|$ is the one-dimensional Radon transform of the function $\mathsf{K}(r)/|r|^2$ (also termed X-ray transform in [11, Chapter I, § 6]).

Theorem 1.2 *Let* $H^*:U\big(\mathscr{C},\operatorname{reach}(\mathscr{C})\big)\setminus\mathscr{C}\to\operatorname{Ten}^2$ *be given by*

$$H^*(x)=\frac{\mathsf{L}\big(x-x^*,\tau(x^*)\big)[b]}{|x-x^*|} \tag{6}$$

$x\in U\big(\mathscr{C},\operatorname{reach}(\mathscr{C})\big)\setminus\mathscr{C}$. *Then there exist* $c>0$ *and* $\delta\in\big(0,\operatorname{reach}(\mathscr{C})\big]$ *such that*

$$\big|H(x)-H^*(x)\big|\le c\log\frac{1}{|x-x^*|}l \tag{7}$$

for every $x\in U(\mathscr{C},\delta)\setminus\mathscr{C}$.

Remarks (i) Equation (6) shows that the qualitative growth of H^* near $\mathscr{C}$ is

$$|H^*(x)|\sim\frac{1}{|x-x^*|}\quad\text{as}\quad\operatorname{dist}(x,\mathscr{C})\to 0;$$

the field H has the same qualitative growth (by Inequality (7)).

(ii) The logarithm on the right-hand side of (7) is not essential (in contrast to the logarithm in (15), below). The following weaker estimate suffices for the proof of Theorem 1.3:

$$\big|H(x)-H^*(x)\big|\le f(x)$$

for every $x\in U(\mathscr{C},\delta)\setminus\mathscr{C}$, where $f\in L^2\big(U(\mathscr{C},\delta)\big)$.

The asymptotics of H implies that H is not square integrable; thus the total energy of a dislocation loop is infinite,

$$\int_{\mathbf{R}^3}W(H)\,dv=\infty, \tag{8}$$

as is well known. Here dv is the volume element in $\mathbf{R}^3$. The infinite contribution to the integral in (8) comes from the singularity of H at $\mathscr{C}$: one has

$$\int_U W(H)\,dv=\infty \tag{9}$$

for any neighborhood U of $\mathscr{C}$. On the other hand, the energy of the complement of U is finite

$$\int_{\mathbf{R}^3\setminus U}W(H)\,dv<\infty. \tag{10}$$

In view (8)–(10), we apply the finite-core regularization of the energy functional. Namely, one removes a tubular core $U(\mathscr{C}, r)$ $(0 < r < \delta)$ along the dislocation loop and determines the energy of the complement $\mathbf{R}^2 \setminus U(\mathscr{C}, r)$. That energy is finite, but has a singularity that is proportional to the $\log(1/r)$. The regularized energy is the limit, as $r \to 0$, of the core-cutoff energy minus the singular logarithmic part. The following theorem gives formulas for the coefficient of logarithmic singularity (the 'prelogarithmic energy factor') and for the regularized energy.

To this end, we define the normal space of $\mathscr{C}$ at $y \in \mathscr{C}$ by

$$\operatorname{Nor}(\mathscr{C}, y) := \{\rho \in \mathbf{R}^3 : \rho \cdot \tau(y) = 0\}$$

and the circle of radius $r > 0$ in the normal space by

$$\operatorname{Circ}(\mathscr{C}, y, r) := \{\sigma \in \operatorname{Nor}(\mathscr{C}, y) : |\sigma| = r\}. \tag{11}$$

Furthermore, we denote by $\kappa(y)$ the curvature vector of $\mathscr{C}$ at the point $y \in \mathscr{C}$, i.e., the second derivative of the position on $\mathscr{C}$ with respect to arc-length parameter (see Sect. 3), and introduce the function $J : U(\mathscr{C}, \operatorname{reach}(\mathscr{C})) \to \mathbf{R}$ by

$$J(x) = \big(1 - \kappa(x^*) \cdot (x - x^*)\big)^{-1} \tag{12}$$

for every $x \in U(\mathscr{C}, \operatorname{reach}(\mathscr{C}))$. It will be shown in Sect. 3 that the denominator in (12) is positive for all indicated x. It will be also shown that J is the jacobian of the map $x \mapsto x^*$. Let δ be as in Theorem 1.2.

Theorem 1.3 (i) *The nested integrals in the formula*

$$\Theta = \int_{\mathscr{C}} \int_{\operatorname{Circ}(\mathscr{C}, y, 1)} W\big(\mathsf{L}(\sigma, \tau(y))\big)\, dl(\sigma)\, dl(y) \tag{13}$$

absolutely converge; *for any* $r \in (0, \delta)$ *the integrals in the formula*

$$\Phi = \int_{U(\mathscr{C}, r)} \big(W(H) - J W(H^*)\big)\, dv + \int_{\mathbf{R}^3 \setminus U(\mathscr{C}, r)} W(H)\, dv - \Theta \log \frac{1}{r} \tag{14}$$

absolutely converge and the value of the right-hand side of (14) *is independent of the choice of* r.

(ii) *We have*

$$\int_{\mathbf{R}^3 \setminus U(\mathscr{C}, \epsilon)} W(H)\, dv = \Theta \log \frac{1}{\epsilon} + \Phi + \varphi(\epsilon) \tag{15}$$

for every $0 < \epsilon \le \delta$, *where* $\varphi : (0, \delta) \to \mathbf{R}$ *satisfies*

$$\varphi(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{16}$$

The quantity Φ is the renormalized energy and Θ is the prelogarithmic energy factor. Both Φ and Θ are functions of the Burgers vector and the shape of the curve $\mathscr{C}$.

Future papers will treat dislocations in bounded domains, the variation of Φ under variations of the shape of $\mathscr{C}$, the Peach–Köhler force, and the particular case of isotropic materials.

This research was motivated by the paper [4] by Cermelli and Leoni and the subsequent paper [3] by Blass and Morandotti. The authors consider point dislocations in a bounded region in the two-dimensional space. The region is interpreted as the cross section of a cylindrical region in $\mathbf{R}^3$ containing straight dislocations. The cited papers prove, among other things, the existence of the renormalized energy and show that the derivative of the regularized energy with respect to the position of the dislocation can be identified with the Peach–Köhler force.

After the research of the present paper was completed, a recent paper by Fonseca, Ginster and Wojtowytsch [10] came to my attention. The paper deals with motion of dislocations in a simplified elasticity, where the elastic energy depends quadratically on the full displacement gradient rather than its symmetrized version. Thus the elasticity tensor is given by $\mathsf{C}[H] = H$ for all $H \in \mathrm{Ten}^2$ (and the minor symmetry requirement (1) is dropped). In [10, Theorem 4.5] the authors determine the prelogarithmic factor for their choice of C, and it can be shown that the above formula (13) yields this result of Fonseca, Ginster and Wojtowytsch in the particular case of their choice of C. (There is no counterpart in [10] of the formula (14) for the renormalized energy.)

This article is organized as follows. Section 2 collects some preliminaries, such as some notations and the distributional versions of divergence and curl. Section 3 describes basic geometric properties of the closed curve $\mathscr{C}$. First, a formula is given for the gradient of the metric projection onto $\mathscr{C}$. Further, the coarea formula is used to prove an equation that replaces the volume integration over $U\big(\mathscr{C}, \mathrm{reach}(\mathscr{C})\big)$ by the area integration over the cross sections of $U\big(\mathscr{C}, \mathrm{reach}(\mathscr{C})\big)$ followed by the length integration along $\mathscr{C}$. Finally, it is proved that the arc-length distance is majorized by a multiple of the euclidean length of the secant. Section 4 introduces the Burgers vector of the dislocation loop via the properties of normal currents [9]. Further, it is proved that the distortion tensor is the sum of a divergence-free term and the gradient of a globally defined displacement. Section 5 solves the equilibrium equations by the Fourier transformation. Section 6 proves the asymptotic form of the distortion field near the dislocation. Sect. 7 describes the summability properties of the equilibrium distortion field. Section 8 proves the above theorem on renormalization of the energy. Finally, Appendix summarizes some properties of the Fourier transformation used in the proofs.

Throughout the proofs in the succeeding sections, c denotes a “generic” constant that may change from line to line.

2 Distributional Gradient, Divergence and Curl

2.1 Vector Product

The vector product of u, and $v \in \mathbf{R}^3$ is denoted by $w = u \times v$; in components $w_i = \epsilon_{ijk} u_j v_k$, where ϵ_{ijk} is the alternating tensor and u_i, v_i and w_i are the components of u, v and w. The suffixes $i, j, k, \ldots$ range from 1 to 3 and the summation convention is used. The vector product of a second-order tensor A with a vector u is a second-order tensor $B = u \times A \in \mathrm{Ten}^2$ with the components $B_{ij} = \epsilon_{jkl} A_{il} u_k$. Note the identity

$$u \times (v \times A) = (Au) \otimes v - (u \cdot v) A, \tag{17}$$

which is an analog of $u \times (v \times w) = v(u \cdot w) - w(u \cdot v)$.

2.2 Gradient, Divergence and Curl for Smooth Functions

If $\mathbf{R}^3$ is an open subset of $\mathbf{R}^3$ and u a continuously differentiable function on $\mathbf{R}^3$ with values in $\mathbf{R}^3$, we define the divergence, curl, and gradient of u as functions on $\mathbf{R}^3$ with values in $\mathbf{R}$, $\mathbf{R}^3$ and Ten^2, respectively, given by

$$\operatorname{div} u = u_{i,i},$$

$$(\operatorname{curl} u)_i = \epsilon_{ijk} u_{k,j}$$

$$(\nabla u)_{ij} = u_{i,j}.$$

If A is a continuously differentiable function on $\mathbf{R}^3$ with values in Ten^2, we define the divergence and curl of A as functions on $\mathbf{R}^3$ with values in $\mathbf{R}^3$ and Ten^2, respectively, given by

$$(\operatorname{div} A)_i = A_{ij,j},$$

$$(\operatorname{curl} A)_{ij} = \epsilon_{jkl} A_{il,k}.$$

2.3 Gradient, Divergence and Curl for Distributions

We now introduce the scalar- vector- and tensor-valued distributions and the corresponding gradient, divergence and curl in the sense of distributions. To unify the treatment, we introduce distributions with values in a finite dimensional real inner product space Y. To this end, we define the Schwartz testfunction space $\mathscr{D}(\mathbf{R}^3, Y)$ as the set of all infinitely differentiable functions $f : \mathbf{R}^3 \to Y$ with compact support. We denote by $\mathscr{D}'(\mathbf{R}^3, Y)$ the set of all real linear functionals on $\mathscr{D}(\mathbf{R}^3, Y)$ which are continuous under the Schwartz topology in $\mathscr{D}(\mathbf{R}^3, Y)$. The elements T of $\mathscr{D}'(\mathbf{R}^3, Y)$ are called Y-valued distributions on $\mathbf{R}^3$. We denote the value of T on $f \in \mathscr{D}(\mathbf{R}^3, Y)$ by $\langle T, f \rangle$.

Two particular cases of Y-valued distributions to be used below are the distribution T_g corresponding to a locally integrable function $g : \mathbf{R}^3 \to Y$ and the distribution T_μ corresponding to an Y-valued measure μ on $\mathbf{R}^3$. These are given by

$$\langle T_g, f \rangle = \int_{\mathbf{R}^3} f(x) \cdot g(x)\, dv(x), \quad \langle T_\mu, f \rangle = \int_{\mathbf{R}^3} f(x) \cdot d\mu(x),$$

$f \in \mathscr{D}(\mathbf{R}^3, Y)$. We denote by $L^1_{\mathrm{loc}}(\mathbf{R}^3, Y)$ the set of all locally integrable Y-valued functions on $\mathbf{R}^3$ and by $\mathscr{M}(\mathbf{R}^3, Y)$ the set of all (finite) Y-valued measures on $\mathbf{R}^3$.

We define the distributional versions of differential operators ∇, div and curl by formal integration by parts. Let $v \in \mathscr{D}'(\mathbf{R}^3, \mathbf{R}^3)$ and $B \in \mathscr{D}'(\mathbf{R}^3, \mathrm{Ten}^2)$ be vector- and tensor-valued distributions. We define the gradient of v as a distribution $\nabla v \in \mathscr{D}'(\mathbf{R}^3, \mathrm{Ten}^2)$ by

$$\langle \nabla v, A \rangle = -\langle v, \operatorname{div} A \rangle,$$

for any $A \in \mathscr{D}(\mathbf{R}^3, \mathrm{Ten}^2)$. Similarly, we define the divergence of B as a distribution $\operatorname{div} B \in \mathscr{D}'(\mathbf{R}^3, \mathbf{R}^3)$ by

$$\langle \operatorname{div} B, u \rangle = -\langle B, \nabla u \rangle$$

for any $u \in \mathscr{D}(\mathbf{R}^3, \mathbf{R}^3)$, as above. Finally, we define the curls of v and B as distributions $\operatorname{curl} v \in \mathscr{D}'(\mathbf{R}^3, \mathbf{R}^3)$ and $\operatorname{curl} B \in \mathscr{D}'(\mathbf{R}^3, \mathrm{Ten}^2)$ by

$$\big\langle \operatorname{curl} v, u \big\rangle = \big\langle v, \operatorname{curl} u \big\rangle,$$
$$\big\langle \operatorname{curl} B, A \big\rangle = \big\langle B, \operatorname{curl} A \big\rangle$$

for any u and A as above.

Throughout the rest of the paper, the operators ∇, div and curl are interpreted in the distributional sense (unless stated otherwise).

3 Geometry of Closed Curves

The purpose of this section is to summarize some properties of closed curves that will be used in the proof in the subsequent sections. Proposition 3.1 determines the gradient ∇P of the metric projection P onto $\mathscr{C}$. Proposition 3.2 determines the formula for the replacement of the volume integration over a tube $U(\mathscr{C}, \epsilon)$ by a successive integration over the perpendicular cross-sections of $U(\mathscr{C}, \epsilon)$ followed by a line integration along $\mathscr{C}$. The formula involves the jacobian which is equal to $|\nabla P|$. Proposition 3.3 estimates the arc-length distance along segments on $\mathbf{S}^3$ and on $\mathscr{C}$ by a multiple of the euclidean distance of the endpoints of the segment. This will be used in Sect. 6 to estimate the difference of the function $H^*(x)$ at two points by integrating its gradient along a curve that avoids the singularity at the origin (rather than along the segment connecting these points).

By a loop we mean the range $\mathscr{C}$ of a twice continuously differentiable map $\gamma : [a, b] \to \mathbf{R}^3$ $(-\infty < a < b < \infty)$ which satisfies

(i) $\gamma(a) = \gamma(b)$, $\dot\gamma(a) = \dot\gamma(b)$ and $\ddot\gamma(a) = \ddot\gamma(b)$;
(ii) $|\dot\gamma(t)| = 1$ for every $t \in [a, b]$;
(iii) the restriction of γ to $[a, b)$ is injective (i.e., if $t, s \in [a, b)$ satisfy $\gamma(t) = \gamma(s)$ then $t = s$).

Any map γ with these properties is called a parametrization of $\mathscr{C}$. The tangent and curvature vectors are maps $\tau : \mathscr{C} \to \mathbf{R}^3$ and $\kappa : \mathscr{C} \to \mathbf{R}^3$ given by

$$\tau\big(\gamma(t)\big) = \dot\gamma(t), \quad \kappa\big(\gamma(t)\big) = \ddot\gamma(t), \quad t \in [a, b];$$

clearly,

$$\tau \cdot \kappa = 0 \ \text{ everywhere on } \ \mathscr{C}.$$

Recall that $\mathscr{C}$ is a set with finite reach and that the metric projection associates with any point x in the tubular neighborhood $U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$ its projection $x^* \in \mathscr{C}$, which we alternatively denote by $P(x)$.

Proposition 3.1 *The metric projection P is continuously differentiable on $U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$; for every $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$ we have*

$$1 - \kappa(x^*) \cdot (x - x^*) > 0 \tag{18}$$

and

$$\nabla P(x) = \big[1 - \kappa(x^*) \cdot (x - x^*)\big]^{-1} \tau(x^*) \otimes \tau(x^*). \tag{19}$$

Proof Let $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. Then

$$|x - x^*|^2 \le |x - y|^2 \quad \text{for all} \quad y \in \mathscr{C}.$$

Let $x^* = \gamma(t_0)$ where $t_0 \in (a, b)$ and let $\varphi : (a, b) \to \mathbf{R}$ be given by $\varphi(t) := |x - \gamma(t)|^2$, $t \in (a, b)$. The function φ has a minimum at t_0. We have

$$\dot\varphi(t) = -2(x - \gamma(t)) \cdot \dot\gamma(t),$$

$$\ddot\varphi(t) = 2|\dot\gamma(t)|^2 - 2(x - \gamma(t)) \cdot \ddot\gamma(t) = 2(1 - (x - x^*) \cdot \ddot\gamma(t)).$$

The conditions $\dot\varphi(t_0) = 0$, $\ddot\varphi(t_0) \ge 0$ yield

$$\tau(x^*) \cdot \big(x - x^*\big) = 0, \quad 1 - \kappa(x^*) \cdot (x - x^*) \ge 0. \tag{20}$$

Let us show that we have the strict inequality sign in $(20)_2$ for every point $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. Assume, to the contrary that

$$1 - \kappa(x_0^*) \cdot (x_0 - x_0^*) = 0 \tag{21}$$

for some $x_0 \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. Let $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$ be such that $x^* = x_0^*$. Then (21) gives

$$1 - \kappa(x^*) \cdot (x - x^*) = -\kappa(x_0^*) \cdot (x - x_0)$$

and as (21) requires $\kappa(x_0^*) \ne 0$, we see that there are points x in the vicinity of x_0 for which the left-hand side of $(20)_2$ is negative. This contradiction shows that we have the strict inequality sign in $(20)_2$ for every point $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. This also completes the proof of (18).

We now proceed to the proof of (19). By [7, Theorem 4.1 and Corollary 4.5], P is continuously differentiable. Formula (19) is a particular case of the derivative of the metric projection onto a C^2 manifold $M \subset \mathbf{R}^n$ given in a coordinate form in [1, Theorem 4.1] and in the coordinate-free form from [16, Theorem 2.3.4(ii)]. However, we give a direct derivation here. Let $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. Since the function is constant on the set $x^* + \operatorname{Nor}(\mathscr{C}, x^*)$, we see that the kernel of $\nabla P(x)$ contains $\operatorname{Nor}(\mathscr{C}, x^*)$. Further, since the values of the map P are constrained to belong to $\mathscr{C}$, the range of $\nabla P(x)$ is contained in the tangent space $\operatorname{Tan}(\mathscr{C}, x^*)$. The described properties of the kernel and range of $\nabla P(x)$ imply that $\nabla P(x)$ is of the form

$$\nabla P(x) = m(x)\tau(x^*) \otimes \tau(x^*) \tag{22}$$

where m is a scalar-valued function on $U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big)$. We rewrite $(20)_1$ as

$$\tau(P(x)) \cdot (x - P(x)) = 0$$

and differentiate in the direction $a \in \mathbf{R}^3$ to obtain

$$m\kappa \cdot (x - x^*)(\tau \cdot a) + \tau \cdot (a - m(\tau \cdot a)\tau) = 0,$$

where we write κ and τ for $\kappa(x^*)$ and $\tau(x^*)$. As a is arbitrary, this simplifies to

$$\big(\kappa \cdot (x - x^*) - 1\big)m + 1 = 0,$$

i.e., $m = \big(1 - \kappa \cdot (x - x^*)\big)^{-1}$ and (22) gives (19). $\square$

For any numbers r, s satisfying $0 \le r < s \le \operatorname{reach}(\mathscr{C})$ and any $y \in \mathscr{C}$ we put

$$U(\mathscr{C}, r, s) = \{x \in \mathbf{R}^3 : r \le \operatorname{dist}(x, \mathscr{C}) < s\}, \tag{23}$$

$$\operatorname{Ann}(y, r, s) = \{\rho \in \operatorname{Nor}(\mathscr{C}, y) : r \le |\rho| < s\}. \tag{24}$$

Finally, define the function $J : U(\mathscr{C}, \operatorname{reach}(\mathscr{C})) \to \mathbf{R}$ by (12), where we recall (18).

Proposition 3.2 *Let $0 \le r < s \le \operatorname{reach}(\mathscr{C})$, and let $f : U(\mathscr{C}, r, s) \to \mathbf{R}$ be a Lebesgue measurable function satisfying*

$$\int_{U(\mathscr{C},r,s)} |f| J \, dv < \infty.$$

Then

$$\int_{U(\mathscr{C},r,s)} f J \, dv = \int_{\mathscr{C}} \int_{\operatorname{Ann}(y,r,s)} f(y+\rho) \, da(\rho) \, dl(y) \tag{25}$$

da is the area element of the plane $\operatorname{Nor}(\mathscr{C}, y)$.

Proof We use the coarea formula for maps with values in manifolds [9, Theorem 3.2.22] to the map metric projection P from $U(\mathscr{C}, \operatorname{reach}(\mathscr{C}))$ into the manifold $\mathscr{C}$. The application to the situation of the present proposition gives

$$\int_{U(\mathscr{C},r,s)} f J \, dv = \int_{\mathscr{C}} \int_{U(\mathscr{C},r,s)\cap P^{-1}(y)} f(x) \, da(x) \, dl(y) \tag{26}$$

where J is the jacobian of the map P. In the present case of the unidimensional target manifold $\mathscr{C}$ we have $J = |\nabla P|$ and hence (19) provides (12). Further, observing that

$$U(\mathscr{C}, r, s) \cap P^{-1}(y) = U(\mathscr{C}, r, s) \cap \big(y + \operatorname{Nor}(\mathscr{C}, y)\big) = y + \operatorname{Ann}(y, r, s),$$

we see that (26) reduces to

$$\int_{U(\mathscr{C},r,s)} f J \, dv = \int_{\mathscr{C}} \int_{y+\operatorname{Ann}(y,r,s)} f(x) \, da(x) \, dl(y)$$

and the substitution $x \mapsto \rho = x - y$ in the inner integral yields (25). □

Recall that the arc-length distance $d(p, q)$ of two points p and q on a compact manifold $M \subset \mathbf{R}^n$ is the length of the shortest curve on M that connects p and q. Clearly, the euclidean distance of p and q in $\mathbf{R}^n$ is majorized by the arc-length distance:

$$|p - q| \le d(p, q).$$

We now show that for a sphere in $\mathbf{R}^n$ and for $\mathscr{C}$ conversely the arc-length distance is majorized by a constant multiple of the euclidean distance in $\mathbf{R}^3$.

Proposition 3.3 (i) *Let M be a sphere in $\mathbf{R}^n$, $n \ge 2$. Then the arc-length distance d_0 on M satisfies*

$$d_0(p, q) \le \frac{1}{2}\pi |p - q| \tag{27}$$

for any $p, q \in M$.

(ii) *There exists* $c > 0$ *such that the arc-length distance* $d_{\mathscr{C}}$ *on* $\mathscr{C}$ *satisfies*

$$d_{\mathscr{C}}(y, z) \le c|y - z| \tag{28}$$

for any $y, z \in \mathscr{C}$.

Proof (i): Since (27) is invariant under scaling and translation, it suffices to consider $M = \mathbf{S}^{n-1}$. Let c be the shortest arc on M that connects p and q, let $\Pi \subset \mathbf{R}^n$ be the plane in $\mathbf{R}^n$ that contains p, q and 0, and let $\Sigma := M \cap \Pi$ (i.e., Σ is the great circle containing p and q). It is well-known that c is the shorter of the two segments on Σ with endpoints p and q. We identify Π with $\mathbf{R}^2$, Σ with the unit circle in $\mathbf{R}^2$, and the points with $p = (1, 0)$ and $q = (\cos\varphi, \sin\varphi)$, where $\varphi \in [0, \pi]$. Then $d_0(p, q) = \varphi$; further, denoting $l := |p - q|$, we have

$$l^2 = (\cos\varphi - 1)^2 + \sin^2\varphi = 2 - 2\cos\varphi,$$

i.e.,

$$l = 2\sqrt{\frac{1 - \cos\varphi}{2}} = 2\sin(\varphi/2).$$

Consider l as a function of φ on the interval $(0, \pi]$ and put $f(\varphi) := l/\varphi$. We have

$$f'(\varphi) = \frac{\varphi\cos(\varphi/2) - 2\sin(\varphi/2)}{\varphi^2}.$$

The numerator on the right-hand side is non-positive: indeed, a differentiation shows that the numerator is a decreasing function of φ on $[0, \pi]$ and its value at 0 is 0. Thus $f' \le 0$ and hence f is a decreasing function. Its minimum value is $f(\pi) = 2/\pi$ since $l(\pi) = 2$. Hence $f(\varphi) \ge 2/\pi$, i.e., (27) holds.

(ii): The length of $\mathscr{C}$ is $L = b - a$. Let M be the circle in $\mathbf{R}^2$ with the center at the origin and of the circumference L, and denote the arc-length distance on M by d_0. It will be shown below that there exists a continuously differentiable map $G : \mathbf{R}^3 \to \mathbf{R}^2$ with bounded gradient that maps $\mathscr{C}$ isometrically onto the circle M (under the arc-length distances on $\mathscr{C}$ and M), i.e., $G(\mathscr{C}) = M$ and

$$d_{\mathscr{C}}(y, z) = d_0(G(y), G(z)) \tag{29}$$

for all $y, z \in \mathscr{C}$. By (27),

$$d_0(G(y), G(z)) \le \frac{1}{2}\pi|G(y) - G(z)|; \tag{30}$$

note also that

$$|G(y) - G(z)| \le k|y - z| \tag{31}$$

for every $y, z \in \mathbf{R}^3$, where $k = \max\{|\nabla G(x)| : x \in \mathbf{R}^3\}$. Relations (29), (30), and (31) give (28) with $c = \pi k/2$.

The proof is now concluded with the construction of the map G. Let $\omega : \mathscr{C} \to M$ be given by

$$\omega\bigl(\gamma(t)\bigr) = (L/2\pi)(\cos t, \sin t), \quad t \in [a, b).$$

Then ω is an isometry under the arc-length distances on $\mathscr{C}$ and M. Let further $\theta : \mathbf{R} \to [0, 1]$ be a continuously differentiable function such that $\theta = 1$ on $\bigl(-\infty, \mathrm{reach}(\mathscr{C})/2\bigr]$ and $\theta = 0$ on $\bigl[\mathrm{reach}(\mathscr{C}), \infty\bigr)$. Finally, let $G : \mathbf{R}^3 \to \mathbf{R}^2$ be given by

$$G(x) = \begin{cases} \theta\bigl(\mathrm{dist}_{\mathscr{C}}(x, \mathscr{C})\bigr)\omega\bigl(P(x)\bigr) & \text{if} \quad x \in U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})\bigr), \\ 0 \in \mathbf{R}^2 & \text{if} \quad x \in \mathbf{R}^3 \setminus U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})\bigr). \end{cases}$$

One finds that G is continuously differentiable on $U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})\bigr)$ and its derivative is bounded and that G vanishes on $U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})\bigr) \setminus U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})/2\bigr)$. Thus the extension by 0 outside $U\bigl(\mathscr{C}, \mathrm{reach}(\mathscr{C})\bigr)$ results in a continuously differentiable function on $\mathbf{R}^3$ with bounded derivative. Finally, since P reduces to the identity map on $\mathscr{C}$, we have $G(x) = \omega(x)$ for all $x \in \mathscr{C}$. As ω is an isometry under the arc-length distances on $\mathscr{C}$ and M, we have (29). Thus G has all the required properties. □

4 Dislocation Density Tensor. Burgers Vector

In the present approach to dislocations, a deformation with defects is described by a distortion tensor field $H \in L^1_{\mathrm{loc}}(\mathbf{R}^3, \mathrm{Ten}^2)$ such that the dislocation density tensor

$$\alpha = \mathrm{curl}\, H$$

is a measure in $\mathscr{M}(\mathbf{R}^3, \mathrm{Ten}^2)$. The definition gives that α is divergence-free:

$$\mathrm{div}\, \alpha = 0.$$

The distortion tensor field H is said to be defect-free if there exists a function $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^3, \mathbf{R}^3)$, called the displacement, such that

$$H = \nabla u.$$

We now analyze defective deformations for which the dislocation density is a measure that is supported on a loop $\mathscr{C}$ in $\mathbf{R}^3$. We denote by $\delta_{\mathscr{C}} \in \mathscr{M}(\mathbf{R}^3, \mathbf{R})$ the length l measure (= the 1-dimensional Hausdorff measure) restricted to $\mathscr{C}$. Thus $\delta_{\mathscr{C}}$ is defined as to satisfy

$$\int_{\mathbf{R}^3} f\, d\delta_{\mathscr{C}} = \int_a^b f\bigl(\gamma(t)\bigr)\, dt$$

for any continuous scalar-valued function f on $\mathbf{R}^3$, where $\gamma : [a, b] \to \mathbf{R}^3$ is a parametrization of $\mathscr{C}$.

Proposition 4.1 *If $\alpha \in \mathscr{M}(\mathbf{R}^3, \mathrm{Ten}^2)$ is a measure supported on a loop $\mathscr{C} \subset \mathbf{R}^3$ then*

$$\mathrm{div}\, \alpha = 0 \tag{32}$$

if and only if there exists a (constant) vector $b \in \mathbf{R}^3$ such that

$$\alpha = -b \otimes \tau\, \delta_{\mathscr{C}}. \tag{33}$$

The vector b in (33) is the Burgers vector corresponding to the dislocation density α. The form of α in equation (33) is traditionally postulated; here it will be shown to be a consequence of the tangentiality of Federer–Fleming's normal currents [9]. The reader is referred to [5], [6] and [13], [14] for earlier usages of currents in the theories of dislocations.

Proof We prove preliminarily the following assertion about a vector-valued measures: if $\mu \in \mathscr{M}(\mathbf{R}^3, \mathbf{R}^3)$ is a measure supported on a loop $\mathscr{C} \subset \mathbf{R}^3$ then

$$\operatorname{div} \mu = 0 \tag{34}$$

if and only if there exists a constant $c \in \mathbf{R}$ such that

$$\mu = -c\tau\, \delta_{\mathscr{C}}. \tag{35}$$

The necessity of (35): Assume that μ satisfies (34). In the language of the geometric measure theory μ is a normal 1-dimensional current [9, Chapter Four] with the support contained in $\mathscr{C}$. We first note that μ represents a 1-dimensional flat chain and invoke [15, Proposition 4.1] to learn that μ is absolutely continuous with respect to l and the corresponding the density a is parallel with τ, i.e., $a = g\tau$, where $g : \mathscr{C} \to \mathbf{R}$. In terms of the parametrization, Equation (34) reads

$$\int_a^b g\big(\gamma(t)\big)\dot{\gamma}(t) \cdot \nabla f\big(\gamma(t)\big)\, dt = 0$$

for every $f \in \mathscr{D}(\mathbf{R}^3, \mathbf{R})$. This is rewritten as

$$\int_a^b g\big(\gamma(t)\big) \frac{d}{dt} f\big(\gamma(t)\big)\, dt = 0,$$

and the Du Bois–Reymond lemma implies that g is constant. The sufficiency of (35) follows by reversing the last few steps of the preceding part. This completes the proof of the preliminary assertion.

To complete the proof of Proposition 4.1, we apply the preliminary assertion to the measures $\mu_i = \alpha^{\mathrm{T}} e_i$, $i = 1, 2, 3$, where $\{e_i : i = 1, 2, 3\}$ is the standard basis in $\mathbf{R}^3$. If α satisfies (32), then each μ_i satisfies (34) and thus $\mu_i = -c_i \tau_i\, \delta_{\mathscr{C}}$ with some constants $c_i \in \mathbf{R}$. Then, if we define $b = (c_1, c_2, c_3)$, the measure α satisfies (33). This completes the proof of the direct implication in Proposition 4.1. The converse implication follows from the converse implication in the preliminary assertion. □

The following proposition determines the distortion field H corresponding to the dislocation density tensor α of the form (33). It will be apparent from the proof that the result can be generalized to the dislocation density represented by any divergence-free tensor-valued measure α.

Proposition 4.2 *Let $M : \mathbf{R}^3 \setminus \mathscr{C} \to \mathrm{Ten}^2$ be given by*

$$M(x) = \frac{1}{4\pi} \int_{\mathscr{C}} \frac{b \otimes \big((x - y) \times \tau(y)\big)}{|x - y|^3}\, dl(y) \tag{36}$$

for each $x \in \mathbf{R}^3 \setminus \mathscr{C}$. Then M is locally integrable and

$$\operatorname{curl} M = -b \otimes \tau\, \delta_{\mathscr{C}}. \tag{37}$$

A field $H \in L^1_{\mathrm{loc}}(\mathbf{R}^3, \mathrm{Ten}^2)$ *is a solution of the equation*

$$\operatorname{curl} H = -b \otimes \tau\, \delta_{\mathscr{C}} \tag{38}$$

if and only if

$$H = M + \nabla v$$

where v *is some function in* $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^3, \mathbf{R}^3)$.

Proof To prove the local integrability of M, it suffices to prove the local integrability of the function m, given by

$$m(x) = \int_{\mathscr{C}} \frac{(x-y) \times \tau(y)}{|x-y|^3}\, dl(y),$$

$x \in \mathbf{R}^3 \setminus \mathscr{C}$. Then

$$|m(x)| \le \int_{\mathscr{C}} \frac{1}{|x-y|^2}\, dl(y),$$

and hence the integration over the ball $B(x_0, r)$ of center $x_0 \in \mathbf{R}^3$ and radius $r > 0$ gives

$$\int_{B(x_0,r)} |m(x)|\, dv(x) \le \int_{\mathscr{C}} \int_{B(x_0,r)} \frac{1}{|x-y|^2}\, dv(x)\, dl(y). \tag{39}$$

We now choose and fix an arbitrary value of the radius $r > 0$ (e.g., $r = 1$) and prove that there exists a $c < \infty$ such that

$$\int_{B(x_0,r)} \frac{1}{|x-y|^2}\, dv(x) \le c \tag{40}$$

for all x_0 and y in $\mathbf{R}^3$. This can be proved elementarily, but it also follows by using an estimate for the Riesz potentials. Indeed, the integrand in (40) is the Riesz potential $I_1(x-y)$ (up to the Riesz normalization constant, which is not essential here). Inequality (40) then follows from [2, Proposition 3.1.2(a)] with $\alpha = p = 1$ and $f =$ the characteristic function of $B(x_0, r)$. (The constant c then turns to be equal to $c_1 r$ where c_1 is independent of r, which is not needed here.) Inequalities (39) and (40) then give

$$\int_{B(x_0,r)} |m(x)|\, dv(x) \le c l(\mathscr{C}),$$

i.e., m (and hence M) is locally integrable.

Thus the theory of Fourier transformations of tempered distributions can be used to prove (37). Appendix summarizes the properties of the Fourier transformation that will be needed in the proof. We first determine the Fourier transform $\hat{M}$ of M. To this end, we rewrite (36) in the form

$$M(x) = \int_{\mathbf{R}^3} g(x-y) \times dC(y) \tag{41}$$

where g is the function

$$g(r) = (4\pi)^{-1} r/|r|^3, \qquad 0 \ne r \in \mathbf{R}^3$$

and C is the measure

$$C = b \otimes \tau \, \delta_{\mathscr{C}}.$$

The right-hand side of (41) is the convolution of g with C. By Property 3 in Appendix, $\hat{M}$ is the product of the Fourier transforms $\hat{g}$ and $\hat{C}$ of g and C, i.e.,

$$\hat{M} = \hat{g} \times \hat{C}. \tag{42}$$

To determine $\hat{g}$, we note that $g = \nabla f$, where $f(r) = -(4\pi)^{-1}|r|^{-1}$ is Newton's potential. The Fourier transform $\hat{f}$ of f is given by $\hat{f}(\xi) = -|\xi|^{-2}$. This follows, e.g., from the general formula [12, Equation (25.25)]. Here and below we assume that $\xi \neq 0$. Thus

$$\hat{g}(\xi) = \mathrm{i}\xi/|\xi|^2$$

by (81). The Fourier transform of C is represented by a bounded continuous Ten^2-valued function. Then (42) gives that $\hat{M}$ is represented by a locally integrable function

$$\hat{M}(\xi) = \mathrm{i}\xi \times \hat{C}(\xi)/|\xi|^2. \tag{43}$$

By (85) and (17),

$$(\operatorname{curl} M)\hat{} = \xi \times \big(\xi \times \hat{C}(\xi)\big)/|\xi|^2 = \big(\hat{C}(\xi)\xi\big) \otimes \xi/|\xi|^2 - \hat{C}(\xi).$$

By Proposition 4.1 we have $\operatorname{div} C = 0$ and hence $\hat{C}(\xi)\xi = 0$ by (84). Thus

$$(\operatorname{curl} M)\hat{} = -\hat{C}(\xi)$$

and the inverse Fourier transformation yields (37).

To complete the proof of Proposition 4.2, we note that S is a solution of Equation (38) if and only if

$$\operatorname{curl}(S - M) = 0$$

and the last equation is satisfied if and only if

$$S - M = \nabla v$$

for some distribution $v \in \mathscr{D}'(\mathbf{R}^3, \mathbf{R}^3)$; the requirement $S \in L^1_{\mathrm{loc}}(\mathbf{R}^3, \mathrm{Ten}^2)$ gives that ∇v is represented by a locally integrable function; hence $v \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^3, \mathbf{R}^3)$. $\square$

5 The Solution of Equilibrium Equations (Proof of Theorem 1.1)

We first assume that the system has a solution H and prove that then H is given by (4) with K having the properties described in Theorem 1.1. This will prove the uniqueness. The existence will follow by showing that (4) with the just constructed function K gives the solution.

By Proposition 4.2, any solution of $(3)_1$ is given by

$$H = M + \nabla v \tag{44}$$

where M is given by (36) and v is some function in $W^{1,1}_{\text{loc}}(\mathbf{R}^3, \mathbf{R}^3)$. Thus the goal is to determine v. If we insert H from (44) into $(3)_1$, we obtain the equation

$$\operatorname{div} \mathsf{C}[\nabla v] = -\operatorname{div} \mathsf{C}[M]. \tag{45}$$

We employ the Fourier transformation of tempered distributions to describe the solution v of (45). In the notation of the proof of Proposition 4.2, the Fourier transform $\hat{M}$ is given by (43) and consequently, the Fourier transform of $\mathsf{C}[M]$ is

$$\mathrm{i}\mathsf{C}[\xi \times \hat{C}(\xi)]/|\xi|^2.$$

By (84), $-\operatorname{div} \mathsf{C}[M]$ is transformed into

$$-\mathsf{C}[\xi \times \hat{C}(\xi)]\xi/|\xi|^2. \tag{46}$$

Let A be the acoustic tensor of the material, i.e., the function $A : \mathbf{R}^3 \to \mathrm{Ten}^2$ defined uniquely by the equation

$$A(\xi)a \cdot b = \mathsf{C}[a \otimes \xi, b \otimes \xi]$$

for every $\xi, a, b \in \mathbf{R}^3$. The tensor $A(\xi)$ is symmetric and positive semidefinite for all $\xi \in \mathbf{R}^3$ and positive definite if $\xi \neq 0$. If $\xi \neq 0$, we denote by $B(\xi)$ the inverse of $A(\xi)$. Since $A(\cdot)$ is 2-homogeneous, its inverse $B(\cdot)$ is degree -2 homogeneous and infinitely differentiable function on $\mathbf{R}^3 \setminus \{0\}$.

By (82) and (84), $\operatorname{div} \mathsf{C}[\nabla v]$ is transformed into $-A(\xi)\hat{v}$ where $\hat{v}$ is the Fourier transform of v. Hence (45) transforms into

$$A(\xi)\hat{v} = \mathsf{C}[\xi \times \hat{C}(\xi)]\xi/|\xi|^2$$

by (46). Thus

$$\hat{v}(\xi) = B(\xi)\mathsf{C}[\xi \times \hat{C}(\xi)]\xi/|\xi|^2.$$

By (82), the Fourier transform of the gradient of v is given by

$$(\nabla v)\hat{}\,(\xi) = \frac{1}{\mathrm{i}} B(\xi)\mathsf{C}\big[\xi \times \hat{C}(\xi)\big]\xi \otimes \xi/|\xi|^2. \tag{47}$$

Let $D \in \mathrm{Ten}^2$. The second-order tensor-valued function

$$Z(\xi, D) := \frac{1}{\mathrm{i}} B(\xi)\mathsf{C}\big[\xi \times D\big]\xi \otimes \xi/|\xi|^2$$

is linear in D and degree -1 homogeneous in ξ (recall that $B(\cdot)$ is degree -2 homogeneous). Hence there exists a function $\mathsf{G}_1 : \mathbf{R}^3 \setminus \{0\} \to \mathrm{Ten}^4$ such that $Z(\xi, D) = \mathsf{G}_1(\xi)[D]$ for all $\xi \neq 0$ and $D \in \mathrm{Ten}^2$; Equation (47) is then rewritten as

$$(\nabla v)\hat{}\,(\xi) = \mathsf{G}_1(\xi)[\hat{C}(\xi)]$$

for all $\xi \neq 0$. Similarly, (43) can be rewritten as

$$\hat{M}(\xi) = \mathsf{G}_2(\xi)[\hat{C}(\xi)]$$

where $\mathsf{G}_2 : \mathbf{R}^3 \setminus \{0\} \to \mathrm{Ten}^4$ degree -1 homogeneous. Thus the Fourier transform $\hat{H}$ of H is given by

$$\hat{H}(\xi) = \mathsf{G}(\xi)[\hat{C}(\xi)] \tag{48}$$

where $\mathsf{G} = \mathsf{G}_1 + \mathsf{G}_2$.

The inverse Fourier transform $\check{\mathsf{G}}$ of the function G is therefore degree -2 homogeneous by Property 2 in Appendix, and hence of the form

$$\check{\mathsf{G}}(r) = \frac{\mathsf{K}(r)}{|r|^2}, \quad r \neq 0,$$

where $\mathsf{K} : \mathbf{R}^3 \setminus \{0\} \to \mathrm{Ten}^4$ is an infinitely differentiable degree 0 homogeneous function. The inverse Fourier transformation changes Equation (48) into Equation (4) by Property 3 in Appendix.

The proof is complete.

6 Asymptotics of the Solution Near $\mathscr{C}$ (Proof of Theorem 1.2)

We put

$$\mathsf{F}(r) := \frac{\mathsf{K}(r)}{|r|^2}, \tag{49}$$

$0 \neq r \in \mathbf{R}^3$, where K is as in Theorem 1.1. Since K is infinitely differentiable, $|\mathsf{K}|$ and $|\nabla \mathsf{K}|$ have finite maxima on the unit sphere $\mathbf{S}^2$ in $\mathbf{R}^3$; since K is degree 0 homogeneous function, the maximum of $|\mathsf{K}|$ on $\mathbf{S}^2$ is also the maximum of $|\mathsf{K}|$ on $\mathbf{R}^3 \setminus \{0\}$. Thus

$$m_0 := \max\left\{|\mathsf{K}(r)| : r \in \mathbf{R}^3,\ r \neq 0\right\} \text{ and } m_1 := \max\left\{|\nabla \mathsf{K}(r)| : r \in \mathbf{S}^2\right\}$$

are finite numbers.

Lemma 6.1 *There exists $m > 0$ such that*

$$|\mathsf{F}(r) - \mathsf{F}(s)| \leq m r_0^{-3} |r - s| \tag{50}$$

for every nonzero r, s in $\mathbf{R}^3$, where

$$r_0 = \min\{|r|, |s|\}.$$

Proof We shall prove (50) with

$$m = 2m_0 + \frac{1}{2}\pi m_1. \tag{51}$$

We put $\rho := |r|$ and $\sigma := |s|$ and assume, without loss in generality, that $\rho \leq \sigma$. We write

$$\mathsf{F}(r) - \mathsf{F}(s) = A + B,$$

where

$$A = \left[\frac{1}{\rho^2} - \frac{1}{\sigma^2}\right]\mathsf{K}(r), \quad B = \frac{\mathsf{K}(r) - \mathsf{K}(s)}{\sigma^2}.$$

We have

$$|A| \le m_0 \left| \frac{1}{\rho^2} - \frac{1}{\sigma^2} \right|.$$

We have

$$\left| \frac{1}{\rho^2} - \frac{1}{\sigma^2} \right| = \frac{1}{\rho^2} - \frac{1}{\sigma^2} \le \frac{2(\sigma - \rho)}{\rho^3} \le \frac{2|r-s|}{\rho^3}$$

where the first inequality follows from $\rho \le \sigma$ and the second from $\sigma - \rho = |\sigma - \rho| \le |r - s|$. Thus

$$|A| \le 2m_0 r_0^{-3} |r-s| \tag{52}$$

since $r_0 = \rho$ by $\rho \le \sigma$.

The quantity B will be estimated by integrating $\nabla \mathsf{K}$ along the shortest arc on $\mathbf{S}^2$ that connects the projections of r, s onto $\mathbf{S}^2$. Thus let $p := r/\rho$ and $q := s/\sigma$ be the projections of r and s onto $\mathbf{S}^2$, let c be the shortest arc on $\mathbf{S}^2$ that connects p with q (see Sect. 4) and let d be the length of c. Let us show that

$$|p-q| \le \frac{|r-s|}{\rho}. \tag{53}$$

Splitting the difference $r - s$ into the sum of $r - \rho s/\sigma$ and $\rho s/\sigma - s$, we obtain

$$|r-s|^2 = \left|r - \rho s/\sigma\right|^2 + \left|\rho s/\sigma - s\right|^2 + 2\left(r - \rho s/\sigma\right) \cdot \left(\rho s/\sigma - s\right); \tag{54}$$

using the identity

$$\left(r - \rho s/\sigma\right) \cdot \left(\rho s/\sigma - s\right) = \rho\left(1/\sigma - 1/\rho\right)(r \cdot s - \rho\sigma)$$

and the inequalities $1/\sigma - 1/\rho \le 0$ and $r \cdot s - \rho\sigma \le 0$, we see that the third term on the right-hand side of (54) is non-negative. Thus (54) provides

$$|r-s|^2 \ge \left|r - \rho s/\sigma\right|^2 = \rho^2 |p-q|^2$$

which proves (53). Let $\omega : [0, d] \to c$ be an arc-length parametrisation of c. By the homogeneity of degree 0 of K, we have $\mathsf{K}(r) - \mathsf{K}(s) = \mathsf{K}(p) - \mathsf{K}(q)$. Then

$$|\mathsf{K}(r) - \mathsf{K}(s)| = \left| \int_0^d \nabla \mathsf{K}(\omega)[\dot{\omega}]\, dt \right| \le \int_0^d |\nabla \mathsf{K}(\omega)||\dot{\omega}|\, dt \le m_1 d$$

and Inequalities (27) and by (53) yield

$$|\mathsf{K}(p) - \mathsf{K}(s)| \le \frac{1}{2}\pi m_1 |p-q| \le \frac{1}{2}\pi m_1 \frac{|r-s|}{\rho}.$$

Thus

$$|B| \le \frac{1}{2}\pi m_1 \frac{|r-s|}{r^3}. \tag{55}$$

Inequalities (52) and (55) give (50) with m as in (51). □

Proof of Theorem 1.2 Let H and H^* be as in Theorems 1.1 and 1.2. Let $x \in U\big(\mathscr{C}, \operatorname{reach}(\mathscr{C})\big) \setminus \mathscr{C}$ be fixed. To simplify the notation, we temporarily translate the coordinate system to achieve that $x^* = 0$. Accordingly, it is possible to choose a parametrization such that $\gamma : [-a, a] \to \mathbf{R}^3$, $\gamma(0) = 0$, and $|\dot\gamma(t)| = 1$ for all t. Throughout the proof, t denotes any element of $[-a, a]$ and c denotes a positive constant that is independent of x and t, but whose value changes from line to line. We also use the abbreviations

$$n(t) := \big(|x|^2 + t^2\big)^{1/2} \ \text{ and } \ \gamma^*(t) := t\tau^*.$$

Since τ^* is a unit vector orthogonal to x (the latter being a consequence of the fact that 0 is the closest point on $\mathscr{C}$ to x), we have

$$n(t) = |x - \gamma^*(t)|. \tag{56}$$

We rewrite (4) and (6) as

$$H(x) = \int_{-a}^{a} \mathsf{F}\big(x - \gamma(t)\big)\big[b \otimes \dot\gamma(t)\big]\, dt, \quad H^*(x) = \int_{\mathbf{R}} \mathsf{F}\big(x - \gamma^*(t)\big)\big[b \otimes \tau^*\big]\, dt,$$

where the second relation uses the definition of L in (5). Next we split the difference $H(x) - H^*(x)$ into the sum $A + B$ where

$$A = \int_{-a}^{a} \big(\mathsf{F}\big(x - \gamma(t)\big)\big[b \otimes \dot\gamma(t)\big] - \mathsf{F}\big(x - \gamma^*(t)\big)\big[b \otimes \tau^*\big]\big)\, dt, \tag{57}$$

$$B = -\int_{|t|>a} \mathsf{F}\big(x - \gamma^*(t)\big)\big[b \otimes \tau^*\big]\, dt.$$

We denote the integrand in (57) by $I(t)$ and write

$$I(t) = I_1(t) + I_2(t)$$

where

$$I_1(t) = \mathsf{F}\big(x - \gamma(t)\big)\big[b \otimes \dot\gamma(t)\big] - \mathsf{F}\big(x - \gamma^*(t)\big)\big[b \otimes \dot\gamma(t)\big],$$

$$I_2(t) = \mathsf{F}\big(x - \gamma^*(t)\big)\big[b \otimes (\dot\gamma(t) - \tau^*)\big].$$

We have

$$|I_1(t)| \le \big|\mathsf{F}\big(x - \gamma(t)\big) - \mathsf{F}\big(x - \gamma^*(t)\big)\big||b|$$

since $|b \otimes \dot\gamma(t)| = |b|$. By (50),

$$\big|\mathsf{F}\big(x - \gamma(t)\big) - \mathsf{F}\big(x - \gamma^*(t)\big)\big| \le A R_0^{-3}(t)|\gamma(t) - \gamma^*(t)|$$

where

$$r_0(t) = \min\big\{|x - \gamma(t)|, |x - \gamma^*(t)|\big\}.$$

Further, since K is bounded by m_0 and $m_0 \le m$, the definition (49) of F gives

$$|I_2(t)| \le m|b|\big|\dot\gamma(t) - \tau^*\big|/\big|x - \gamma^*(t)\big|^2 \le m|b|r_0^{-2}\big|\dot\gamma(t) - \tau^*\big|.$$

Thus

$$|I(t)| \le m|b|r_0^{-3}(t)|\gamma(t) - \gamma^*(t)| + m|b|r_0^{-2}(t)|\dot\gamma(t) - \tau^*| \tag{58}$$

for any t. We estimate the right-hand side of (58). Prove first that

$$r_0(t) \ge cn(t) \tag{59}$$

for any t and some $c > 0$. By [8, Theorem 4.8, Assertion (13)], the map $\Lambda : U(\mathscr{C}, \operatorname{reach}(\mathscr{C})) \to \mathbf{R}^3 \times \mathbf{R}^3$, given by

$$\Lambda(x) = (x^*, x - x^*), \quad x \in U(\mathscr{C}, \operatorname{reach}(\mathscr{C})),$$

is lipschitzian on $U(\mathscr{C}, \epsilon')$ for every $\epsilon' < \operatorname{reach}(\mathscr{C})$. Thus there exists $c = c(\epsilon') > 0$ such that

$$|\Lambda(x) - \Lambda(y)| \le c|x - y|$$

for any $x, y \in U(\mathscr{C}, \epsilon')$; in particular,

$$|x^* - y| + |x - x^*| \le c|x - y| \tag{60}$$

for any $x \in U(\mathscr{C}, \epsilon')$ and $y \in \mathscr{C}$. Using $x^* = 0$, we see that Inequality (60) with $y = \gamma(t)$ provides

$$|x| + |\gamma(t)| \le c|x - \gamma(t)|. \tag{61}$$

Next we apply Inequality (28) with $y = \gamma(t)$ and $z = 0$ to obtain

$$d_{\mathscr{C}}(\gamma(t), 0) \le c|\gamma(t)|;$$

noting that $d_{\mathscr{C}}(\gamma(t), 0) = |t|$, this reduces to $|t| \le c|\gamma(t)|$. Hence (61) gives

$$n(t) \le |x| + |t| \le c|x - \gamma(t)|.$$

A combination with (56) establishes (59).

Finally, we estimate the differences $|\gamma(t) - \gamma^*(t)|$ and $|\dot\gamma(t) - \tau^*|$ in (58). Since the second derivative of the function $\delta(t) := \gamma(t) - \gamma^*(t)$ is bounded on $[-a, a]$ and $\delta(0) = \dot\delta(0) = 0$, we obtain by Taylor's expansion that there exists a constant c such that

$$|\gamma(t) - \gamma^*(t)| \le ct^2, \quad |\dot\gamma(t) - \tau^*| \le c|t|. \tag{62}$$

By (59) and (62), the two terms on the right-hand side of (58) are estimated by

$$ct^2 n(t)^{-3} \le c|t|n(t)^{-2} \ \text{ and } \ c|t|n(t)^{-2},$$

respectively. Here we have used the inequality $|t|/n(t) \le 1$, which is a direct consequence of the definition of $n(t)$. Hence $\left|I(t)\right| \le 2c|t|n(t)^{-2}$ and consequently

$$|A| \le 2\int_0^a |I(t)|\,dt \le \int_0^a 4ctn(t)^{-2}\,dt = 2c\log\left(1 + a^2/|x|^2\right). \tag{63}$$

Since $|\mathsf{K}|$ is bounded, we obtain from (49) the estimate

$$\left|\mathsf{F}\big(x-\gamma^*(t)\big)\big[b\otimes\tau^*\big]\right|\le c|b|n(t)^{-2}$$

and hence

$$|B|=c\int_{|t|>a}n(t)^{-2}\,dt=2c\int_a^\infty n(t)^{-2}\,dt\le 2c\int_a^\infty dt/t^2=2c/a. \tag{64}$$

Inequalities (63) and (64) provide

$$|H(x)-H^*(x)|\le|A|+|B|\le 2c\log\big(1+a^2/|x|^2\big)+2c/a. \tag{65}$$

Elementary properties of logarithm yield that there exist $c>0$ and $\delta\in\big(0,\operatorname{reach}(\mathscr{C})\big]$ (with c possibly larger than the current value of c) such that the last expression in (65) is majorized by $c\log 1/|x|$ for all $|x|<\delta$. Thus (65) reduces to

$$|H(x)-H^*(x)|\le c\log\frac{1}{|x|}.$$

This proves (7) in the particular case $x^*=0$. Returning to the original coordinate system (with x^* possibly $\ne 0$), we obtain (7) in full generality. □

7 Summability Properties of the Solution

This section is devoted to establishing the convergence of various integrals occurring in this paper. The following lemma will be used in Sect. 8 to establish the convergence of the integrals in the definition (14) of the renormalized energy and to prove Proposition 7.2 on the integrability properties of the distortion H.

Lemma 7.1 *Let $0<\delta<\operatorname{reach}(\mathscr{C})$ and let f be a measurable function on $U(\mathscr{C},\delta)$ such that*

$$|f(x)|\le g\big(|x-x^*|\big) \tag{66}$$

for all $x\in U(\mathscr{C},\delta)\setminus\mathscr{C}$, where the function $g:(0,\delta]\to[0,\infty)$ satisfies

$$\int_0^\delta tg(t)\,dt<\infty. \tag{67}$$

Then

$$\int_{U(\mathscr{C},\delta)}|f|\,dv<\infty. \tag{68}$$

Proof We apply Proposition 3.2 with $r=0$, $s=\delta$ and with f replaced by $|f|$. Formula (25) then takes the form

$$\int_{U(\mathscr{C},\delta)}|f|J\,dv=\int_{\mathscr{C}}\int_{\operatorname{Disc}(y,\delta)}|f(y+\rho)|\,da(\rho)\,dl(y),$$

where we put

$$\operatorname{Disc}(\mathscr{C},y,r):=\operatorname{Ann}(\mathscr{C},y,0,r)\equiv\big\{\rho\in\operatorname{Nor}(\mathscr{C},y):|\rho|<r\big\}.$$

Inequality (66) then reduces to $|f(y+\rho)| \le g(|\rho|)$; hence

$$\int_{\mathrm{Disc}(y,\delta)} |f(y+\rho)|\, da(\rho) \le \int_{\mathrm{Disc}(y,\delta)} g\big(|\rho|\big)\, da(\rho).$$

By Fubini's theorem the last integral is equal to $2\pi\int_0^\delta t g(t)\, dt$. Thus

$$\int_{U(\mathscr{C},\delta)} |f| J\, dv < \infty. \tag{69}$$

Since $\delta < \operatorname{reach}(\mathscr{C})$, Equation (12) shows that there exists a $c > 0$ such that $c < J(x) < c^{-1}$ for all $x \in U(\mathscr{C},\delta)$. Thus (69) implies (68). □

Proposition 7.2 *The solution H of the equilibrium equations* (3) *satisfies*

$$H \in \begin{cases} L^p_{\mathrm{loc}}(\mathbf{R}^3, \mathrm{Ten}^2) & \textit{if} \quad 1 \le p < 2, \\ L^p(\mathbf{R}^3, \mathrm{Ten}^2) & \textit{if} \quad 3/2 < p < 2, \\ L^2\big(\mathbf{R}^3 \setminus U(\mathscr{C},\epsilon), \mathrm{Ten}^2\big) & \textit{for every} \quad \epsilon > 0 \end{cases} \tag{70}$$

and

$$H \notin L^2(\mathbf{R}^3, \mathrm{Ten}^2). \tag{71}$$

Proof Let L and H^* be as in Theorem 1.2.

Inclusion $(70)_1$: since H is bounded on each compact subset of $\mathbf{R}^2 \setminus \mathscr{C}$, we have $H \in L^p(K, \mathrm{Ten}^2)$ for every $p \in [1,\infty]$ by continuity. We further prove that $H \in L^p(U(\mathscr{C},\delta), \mathrm{Ten}^2)$ for all $p \in [1,2)$. Indeed, Inequality (7) can be rewritten as

$$H(x) = H^*(x) + N(x)$$

for every $x \in U(\mathscr{C},\delta)$, where

$$|N(x)| \le c \log \frac{1}{|x - x^*|}. \tag{72}$$

Since L is bounded, we have

$$|H^*(x)| \le \frac{c}{|x - x^*|}.$$

Thus majorizing the right-hand side of (72) by $c/|x - x^*|$, we obtain

$$|H(x)|^p \le \frac{c}{|x - x^*|^p}.$$

Thus the function $f(x) := |H(x)|^p$ satisfies Inequality (66) with $g(t) = c/t^p$ and hence (67) reduces to

$$\int_0^\delta t^{1-p}\, dt < \infty,$$

which holds if and only if $p \in [1, 2)$. Thus

$$\int_{U(\mathscr{C},\delta)} |H(x)|^p \, dv(x) < \infty \tag{73}$$

by Lemma 7.1. The proof of $(70)_1$ is complete.

Inclusion $(70)_2$: It follows from (4) that if $\bar{B}(0, R)$ is a closed ball of sufficiently large radius R, there exists a constant c such that

$$|H(x)| \le \frac{c}{|x|^2} \quad \text{for all} \quad x \in \mathbf{R}^3 \setminus \bar{B}(0, R).$$

Hence

$$\int_{\mathbf{R}^3 \setminus \bar{B}(0,R)} |H(x)|^p \, dv(x) \le c \int_{\mathbf{R}^3 \setminus \bar{B}(0,R)} |x|^{-2p} \, dv(x)$$

and the last integral is finite if and only if $p > 3/2$. On the other hand,

$$\int_{\bar{B}(0,R)} |H(x)|^p \, dv(x) < \infty$$

if $1 \le p < 2$ by $(70)_1$. The proof of $(70)_2$ is complete.

Inclusion $(70)_3$: If $\bar{B}(0, R)$ is as above then

$$\int_{\mathbf{R}^3 \setminus \bar{B}(0,R)} |H(x)|^2 \, dv(x) < \infty$$

and

$$\int_{\bar{B}(0,R) \setminus U(\mathscr{C},\epsilon)} |H(x)|^p \, dv(x) < \infty$$

for every $p \in [1, \infty)$ and $\epsilon > 0$ since H is bounded on $\bar{B}(0, R) \setminus U(\mathscr{C}, \epsilon)$ by continuity. The proof of $(70)_3$ is complete.

Relation (71): it follows from the proof of $(70)_1$ that (73) does not hold for $p = 2$. □

8 Renormalization of the Energy (Proof of Theorem 1.3)

Let δ be as in Theorem 1.2.

Proof of Theorem 1.3, Part (i) The convergence of the integrals in (13) is immediate since both the inner and outer integrals involve continuous functions on compact sets.

The convergence of the integrals in (14). To prove the convergence of the first integral, we let

$$f(x) = W(H) - JW(H^*).$$

Using the major symmetry of C, this can be rearranged as

$$f(x) = W(H - H^*) + H^* \cdot \mathsf{C}[H - H^*] + (J - 1)W(H^*).$$

We estimate each term on the right-hand side separately. By Inequality (7),

$$|W(H - H^*)| \le c \log^2 \frac{1}{|x - x^*|}. \tag{74}$$

Further, as a direct consequence of (6) and the properties of L, we have

$$|H^*| \le \frac{c}{|x - x^*|}$$

and hence

$$|H^* \cdot \mathsf{C}[H - H^*]| \le \frac{c}{|x - x^*|} \log \frac{1}{|x - x^*|}. \tag{75}$$

Finally, we have

$$J - 1 = -\frac{\kappa(x^*) \cdot (x - x^*)}{1 - \kappa(x^*) \cdot (x - x^*)};$$

and as the denominator is bounded by our choice of reach($\mathscr{C}$) and as κ is bounded, we have

$$|J - 1| \le c|x - x^*|.$$

Consequently,

$$|(J - 1) W(H^*)| \le \frac{c}{|x - x^*|}. \tag{76}$$

Hence, by (74), (75) and (76),

$$|f(x)| \le g\big(|x - x^*|\big),$$

where

$$g(t) = c\bigg(\log^2 \frac{1}{t} + \frac{1}{t} \log \frac{1}{t} + \frac{1}{t}\bigg).$$

Since g satisfies (67), Lemma 7.1 says that f is integrable, i.e., the first integral in (14) converges.

The second integral in (14) converges by $(70)_3$.

Finally, let us prove that the value of the right-hand side of (14) is independent of the choice or r. Let r, s satisfy $0 < r < s$ and denote by Δ_1 and Δ_2 the values of the right-hand side of (14) with $r = r$ and $r = s$, respectively. Then

$$\begin{aligned}\Delta_2 - \Delta_1 = &\int_{U(\mathscr{C},r,s)} \big(W(H) - J W(H^*)\big)\, dv \\ &- \int_{U(\mathscr{C},r,s)} W(H)\, dv - \Theta\bigg(\log \frac{1}{r} - \log \frac{1}{s}\bigg)\end{aligned}$$

where $U(\mathscr{C}, r, s)$ is defined in (23). The first integral can be split into the difference

$$\int_{U(\mathscr{C},r,s)} W(H)\, dv - \int_{U(\mathscr{C},r,s)} J W(H^*)\, dv$$

and a cancellation reduces the last equation to

$$\Delta_2 - \Delta_1 = -\int_{U(\mathscr{C},r,s)} JW(H^*)\,dv + \Theta\left(\log\frac{1}{r} - \log\frac{1}{s}\right).$$

By Proposition 3.2,

$$\int_{U(\mathscr{C},r,s)} JW(H^*)\,dv = \int_{\mathscr{C}}\int_{\mathrm{Ann}(y,r,s)} W(H^*)\,da(x)\,dl(y),$$

where $\mathrm{Ann}(y,r,s)$ is given by (24). By Fubini's theorem and (6),

$$\int_{U(\mathscr{C},r,s)} JW(H^*)\,dv = \int_{\mathscr{C}}\int_r^s \frac{1}{t^2}\int_{\mathrm{Circ}(\mathscr{C},y,t)} W\big(\mathsf{L}\big(\rho,\tau(y)\big)\big)\,dl(\rho)\,dt\,dl(y) \tag{77}$$

where $\mathrm{Circ}(\mathscr{C},y,t)$ is given by (11). Since L is degree 0 homogeneous in the first variable, the integrand in the inner integral in (77) is independent of t and thus the whole integral scales only due to the change of the radius of the circle. Thus

$$\begin{aligned}\int_{U(\mathscr{C},r,s)} JW(H^*)\,dv &= \int_{\mathscr{C}}\int_r^s \frac{1}{t}\int_{\mathrm{Circ}(\mathscr{C},y,1)} W\big(\mathsf{L}\big(\rho,\tau(y)\big)\big)\,dl(\rho)\,dt\,dl(y)\\ &= \int_{\mathrm{Circ}(\mathscr{C},y,1)} W\big(\mathsf{L}\big(\rho,\tau(y)\big)\big)\,dl(\sigma)\left(\log\frac{1}{r} - \log\frac{1}{s}\right)\\ &= \Theta\left(\log\frac{1}{r} - \log\frac{1}{s}\right),\end{aligned}$$

i.e.,

$$\int_{U(\mathscr{C},r,s)} JW(H^*)\,dv = \Theta\left(\log\frac{1}{r} - \log\frac{1}{s}\right). \tag{78}$$

Thus $\Delta_1 = \Delta_2$ and hence the value of the right-hand side of (14) is independent of the choice or r.

This completes the proof of Part (i) of Theorem 1.3. □

Proof of Theorem 1.3, Part (ii) Let $0 < \epsilon < \delta$ and let r be any number satisfying $\epsilon < r < \delta$. We write

$$\int_{\mathbf{R}^3\setminus U(\mathscr{C},\epsilon)} W(H)\,dv = \int_{U(\mathscr{C},\epsilon,r)} W(H)\,dv + \int_{\mathbf{R}^3\setminus U(\mathscr{C},r)} W(H)\,dv. \tag{79}$$

A rearrangement gives

$$\begin{aligned}\int_{U(\mathscr{C},\epsilon,r)} W(H)\,dv &= \int_{U(\mathscr{C},\epsilon,r)} \big(W(H) - JW(H^*)\big)\,dv + \int_{U(\mathscr{C},\epsilon,r)} JW(H^*)\,dv\\ &= \int_{U(\mathscr{C},\epsilon,r)} \big(W(H) - JW(H^*)\big)\,dv + \Theta\left(\log\frac{1}{\epsilon} - \log\frac{1}{r}\right),\end{aligned}$$

where we have used (78). Next, we split the integral

$$\int_{U(\mathscr{C},\epsilon,r)} \big(W(H) - JW(H^*)\big)\,dv$$

into the difference

$$\int_{U(\mathscr{C},r)} \big(W(H) - JW(H^*)\big)\,dv - \int_{U(\mathscr{C},\epsilon)} \big(W(H) - JW(H^*)\big)\,dv.$$

Thus

$$\begin{aligned}\int_{U(\mathscr{C},\epsilon,r)} W(H)\,dv = \int_{U(\mathscr{C},r)} \big(W(H) - JW(H^*)\big)\,dv - \Theta \log\frac{1}{r} \\ - \int_{U(\mathscr{C},\epsilon)} \big(W(H) - JW(H^*)\big)\,dv + \Theta \log\frac{1}{\epsilon}.\end{aligned} \tag{80}$$

A combination of (79) and (80) and simple rearrangements provide (15) with

$$\varphi(\epsilon) = -\int_{U(\mathscr{C},\epsilon)} \big(W(H) - JW(H^*)\big)\,dv.$$

Since the function $W(H) - JW(H^*)$ is integrable on $U(\mathscr{C}, r)$ by the preceding proof, we have (16). This completes the proof of Part (ii) of Theorem 1.3. □

Appendix: Fourier Transformation

Generally, if f is a function defined on $\mathbf{R}^3 \setminus \{0\}$ with values in a vector space and if z is a complex number, f is said to be degree z homogeneous if $f(\lambda r) = \lambda^z f(r)$ for every r and λ as in the preceding sentence.

If Y is a finite dimensional inner product space, $\mathscr{S}(\mathbf{R}^3, Y)$ denotes the space of rapidly decaying Y-valued testfunctions on $\mathbf{R}^3$ and $\mathscr{S}'(\mathbf{R}^3, Y)$ denotes the space of tempered Y-valued distributions on $\mathbf{R}^3$. The Fourier transform of a function $f \in \mathscr{S}(\mathbf{R}^3, Y)$ is the function $\hat{f} \in \mathscr{S}(\mathbf{R}^3, Y)$ defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^3} f(x)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,dx, \quad \xi \in \mathbf{R}^3.$$

The Fourier transform of $T \in \mathscr{S}'(\mathbf{R}^3, Y)$ is $\hat{T} \in \mathscr{S}'(\mathbf{R}^3, Y)$ defined by

$$\langle \hat{T}, f\rangle = \langle T, \hat{f}\rangle$$

for every $f \in \mathscr{S}(\mathbf{R}^3, Y)$.

Property 1 *Fourier transformation changes of the operator ∇ into multiplication by $-\mathrm{i}\xi$. This implies transformations of linear differential operators with constant coefficients. The following table reviews the transforms of gradients and divergences of scalar-, vector- and tensor-valued distributions $f \in \mathscr{S}'(\mathbf{R}^3, \mathbf{R})$, $u \in \mathscr{S}'(\mathbf{R}^3, \mathbf{R}^3)$ and $B \in \mathscr{S}'(\mathbf{R}^3, \mathrm{Ten}^2)$:*

$$\nabla f \quad \to \quad -\mathrm{i}\xi \hat{f}, \tag{81}$$

$$\nabla u \quad \to \quad -\mathrm{i}\hat{u} \otimes \xi, \tag{82}$$

$$\operatorname{div} u \quad \to \quad -\mathrm{i}\xi \cdot \hat{u}, \tag{83}$$

$$\operatorname{div} B \quad \to \quad -\mathrm{i}\hat{B}\xi, \tag{84}$$

$$\operatorname{curl} B \quad \to \quad -\mathrm{i}\xi \times \hat{B}. \tag{85}$$

Property 2 *Fourier transformation of homogeneous distributions. If $z > -3$, any degree z homogeneous function f on $\mathbf{R}^3 \setminus \{0\}$ represents a tempered distribution T. If additionally $z \le 0$, the Fourier transformation of T is represented by a $-3 - z$-homogeneous function $\hat{f}$. Moreover, if f is infinitely differentiable, then $\hat{f}$ is infinitely differentiable also; see* [17, *Chap*. 3, *Proposition* 8.1].

Property 3 *Fourier transformation changes the convolution of distributions into the product of their Fourier transforms.*

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